Contra Cantor Pro Occam - Proper constructivism with abstraction

Keywords: Logic, mathematics, constructivism, infinity, mathematics education

MSC2010 03E10 ordinal and cardinal numbers, 00A35 methodology and didactics

Thomas Colignatus
http://thomascool.eu
March 26 2012, January 23 2013, March 26 2013

Summary

> **Context** • In the philosophy of mathematics there is the distinction between platonism (realism), formalism, and constructivism. There seems to be no distinguishing or decisive experiment to determine which approach is best according to non-trivial and self-evident criteria. As an alternative approach it is suggested here that philosophy finds a sounding board in the didactics of mathematics rather than mathematics itself. Philosophers can go astray when they don't realise the distinction between mathematics (possibly pure modeling) and the didactics of mathematics (an empirical science). The approach also requires that the didactics of mathematics is cleansed of its current errors. Mathematicians are trained for abstract thought but in class they meet with real world students. Traditional mathematicians resolve their cognitive dissonance by relying on tradition. That tradition however is not targetted at didactic clarity and empirical relevance with respect to psychology. The mathematical curriculum is a mess. Mathematical education requires a (constructivist) re-engineering. Better mathematical concepts will also be crucial in other areas, such as e.g. brain research. > **Problem** • Aristotle distinguished between potential and actual infinite, Cantor proposed the transfinites, and Occam would want to reject those transfinites if they aren't really necessary. My book "A Logic of Exceptions" already refuted 'the' general proof of Cantor's Theorem on the power set, so that the latter holds only for finite sets but not for 'any' set. There still remains Cantor's diagonal argument on the real numbers. > **Results** • There is a 'bijection by abstraction' between ℕ and ℝ. Potential and actual infinity are two faces of the same coin. Potential infinity associates with counting, actual infinity with the continuum, but they would be 'equally large'. The notion of a limit in ℝ cannot be defined independently from the construction of ℝ itself. Occam's razor eliminates Cantor's transfinites. > **Constructivist content** • Constructive steps S_1, ..., S_5 are identified while S_6 gives nonconstructivism and the transfinites. Here S_3 gives potential infinity and S_4 actual infinity. The latter is taken as 'proper constructivism' and it contains abstraction. The confusions about S_6 derive rather from (bad) logic than from

infinity. > **Key words** • Logic, mathematics, constructivism, infinity, mathematics education

PM. I thank Richard Gill (infinitely much), Bart van Donselaar, Alain Schremmer, J. John C. Kuiper, Wilfrid Hodges, two readers who wish to remain anonymous, three anonymous referees and the editor Alexander Riegler for comments and suggestions for relevant literature.

1. Introduction

1.1 Constructivism

The following analysis may help the definition of constructivism and the delineation of views on constructivism. A very practical result also concerns highschool mathematics. It may be somewhat amazing how philosophical and methodological discussion can boil down to a course in highschool. The real challenge is to avoid rote learning and instead to rekindle the processes of wonder and insight.

A context for this discussion is the *Special Issue* of CF of March 2012 edited by Van Kerkhove and Van Bendegem: *Constructivism In and About Mathematics*. There is the eternal tension between mathematics and engineering. Mathematicians are trained for abstract thought and they may lose contact with limitations that are relevant for constructivism that people will tend to regard as rather practical. Consider the logic: I fit in my coat. My coat fits in my schoolbag. Thus I fit in my schoolbag. A mathematician may be perfectly happy with this since the propositions are abstract and need not concern a real world and might only concern some topology. For an engineer interested in an application to the real world the reasoning gives a problem. The assumptions seem true, the reasoning is sound, the conclusion is false, hence something is amiss. The correction is straightforward: If I wear it, I fit in my coat. If nobody wears it, the coat fits in my schoolbag. Conclusion: If I want to put the coat into the bag then I have to take it off. In the same way, the mathematical expositions on constructivism, and thus aforementioned *Special Issue*, may be misleading with regard to proper constructivism, since some mathematical assumptions may hang in the air. Let us coin the term *proper* constructivism indeed for what this present paper will try to clarify as well. This does not concern a new branch of constructivism but it is about what any branch would contain, namely a balance between abstraction and practical considerations. For example ‘strict finitism’ (Van Bendegem (2012)) might perhaps allow more abstraction since it by itself already implies some abstraction; and in other branches the abstraction might have to be reduced in some respects.

The following analysis concentrates on finitism, the continuum and (its) infinity. The quintessential notion to understand constructivism is this: what Aristotle called the difference between the potential versus the actual infinite. While Democritos held that division of matter eventually resulted into atoms, Aristotle held that division of space could be continued forever, and thus he helped Euclid in defining a point as location without size and a line as length without width. Some authors seem to hold that Aristotle rejected the actual infinite but it seems to me that he would not have rejected the actual infinity of the continuum, e.g. the interval [0, 1]. Classical or non-constructivist mathematics tends to allow relatively free assumptions on the continuum and even create higher forms of infinity, the transfinites. My suggestion is that mathematicians since Cantor have been ‘too abstract and unrealistic’ (in some sense) about those transfinites. A neoclassical - or *proper* constructivist approach, namely constructivism that looks for the balance of abstraction and practicality - can restore sense in many areas affected by mathematics, not the least in philosophical discourse and in the didactics of mathematics.

Quale (2012) mentions different forms of constructivism, next to solipsism, platonism / realism and relativism. Cariani (2012) uses the perhaps more traditional distinction between three approaches of platonism (realism), formalism and constructivism (including intuitionism and finitism). It is dubious whether there is a convincing experiment to distinguish the one from the other, so these labels are likely to refer to flavours in psychology. Indeed, see Cariani (2012:123, right column). Davis & Hersh (1980, 1983:358-359) regard these three different philosophies as the aspects of a multidimensional phenomenon that have to beconsidered all in order to arrive at a whole. They regard these three even as extremes in abstraction, and they hold that real (mental) activities by mathematicians are of a more practical kind. The words ‘matter’ and ‘mind’ could be overused, imprecise, unrealistic, to describe what is really happening, see Davis & Hersh (1980, 1983:410): “Mathematics does have a subject matter, and its statements are meaningful. The meaning, however, is to be found in the shared understanding of human beings, not in an external reality.” Compare the ‘average length of 10 cars’, which average may be taken to ‘exist’, though can get different values even when the cars are identified, still depending upon method of measurement and, say, temperature.

The discussion becomes even more complicated when mathematics can hang in the air. Perhaps one type of computer program can be seen as constructivist without any doubt but perhaps there are all kinds of variations in computer programs, not only with a common random generator but also with input from outside measurement instruments, that still might be seen as constructivist in some way or other. Rather than approach the issue head on, this paper proposes to take one step by another to identify (the) different views on constructivism that pertain to finitism, continuum and infinity. Below will define various steps.

Constructivist Foundations is an interdisciplinary scholarly journal and only a smaller part of its readers are professional mathematicians. Nevertheless, the very topic of this paper requires some mathematical insight. And, as said, this paper takes the approach to link up the philosophical discussion with a highschool course, and that ought to be an acceptable entry level. The level of math required is first-year mathematics and logic at a non-math-major level for a field that uses mathematics, such as economics, physics or biology. The following uses elementary mathematical concepts and notations from logic, set theory and functions, since these are purely on target with respect the subject matter. There are some key proofs in the body of the text since it is important that non-math majors can verify that professional mathematicians have gone astray in the most elementary manners. Constructivism has been burdened by irrational winds from mathematical quarters and it is essential to show how Occam helps to cut away the nonsense. Readers who have developed an aversion against mathematics may actually be drawn into the argument and discover how things finally make sense. Indeed, one target of the paper is to develop the outline of a course on real numbers in highschool. Most readers might want to hold that in mind, while some might perhaps want to skip some details about where Cantor went wrong.

The reader is invited to read Riedler (2005) again, the first editorial on the constructivist challenge, with the catalogue of ten points for the program to meet that challenge. Interdisciplinarity doesn't mean sacrificing the standards of quality of one field merely in order to create some umbrella for its own sake however vague it is. Instead, the standards of quality of all fields must be maintained if the results are to be useful. For some researchers perhaps like me it comes across as somewhat curious that other people divide up into disciplines while it ought to be clear that you need all to arrive at the best picture. A paper that closely matches those ten points is Colignatus (2011g), *Brain research and mathematics education: some comments,* that argues that brain research on mathematics that is intended to be used for its education may go astray when brain researchers do not see that many concepts in math are quite messed up. Mathematicians are trained for abstract thought but in class they meet with real world students. Traditional mathematicians resolve their cognitive dissonance by relying on tradition. That tradition however is not targetted at didactic clarity and empirical relevance with respect to psychology. The content in the mathematics curriculum has grown over the ages by conscious construction but also as waste flushed onto the shores. A quick example is that the Arabic numbers like 19 are written from right to left (as in Arabia) while the West commonly writes from the left to the right. In pronunciation there is even a switch in order, compare 19 and 29. Another quick example is that two-*and*-a-half is denoted as two-*times*-a-halve, namely $2\frac{1}{2}$. This kind of mathematical confusion applies to the finite, continuum and infinite as well. It even contains pure errors against logic. It is not only a question how the mind constructs those concepts but also whether our concepts are mathematically sound and not messed up like so much else. A first step towards clarity is to consider the educational context. At issue is not to educate what is in the books on mathematics but to discover what we really would like to teach. Math education requires a re-engineering, and likely in the constructivist manner. If we do this re-engineering, it helps to have some soundness in philosophy as well.

In the Riegler three-dimensional space of discipline, school and type of enquiry, this paper then can be located as follows. Discipline: (1) the education of mathematics, the philosophy of mathematics and mathematical foundations, with aspects on history, cognitive psychology and epistemology. School: linking up to the approach to constructivism in mathematics (not in the Riegler list). Type of enquiry: conceptual paper, to develop philosophical-argumentative support, though with the understanding that philosophy doesn't hang in the air but at least in this case deals with practical questions in epistemology and the didactics of mathematics.

1.2 Steps in constructivism

It is tempting to use the levels of measurement: commonly the nominal, ordinal, interval and ratio scales. It seems better however to mention that the discussion might be embedded in such a structure, but not try this at this very moment since it would introduce new areas of discussion. It suffices here that we consider the ordinal versus the ratio scale. The ordinal scale is given by the set of natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$ and the ratio scale is given by the set of real numbers $\mathbb{R} = \{x \mid x \text{ is a number with decimals}\}$. It suffices to look at the points in the inclusive interval [0, 1] since others could be found by $1 / x$ etcetera.

Thus, when an argument has a step S then some might hold that it is or isn’t constructivist. When there are steps $S_1, ..., S_n$ then there are the various permutations in views. For example, the present author thinks that abstraction is an activity of the human mind that can sometimes be seen as *proper constructivist*. For example, the natural numbers are figments of abstraction and don’t occur in empirical reality (in the standard sense, nonplatonic). Brain research suggests that at least the first digits are hard-wired in the brain, but can we agree that these are proper representations of the notion of ‘number’ ? Results of abstract thought can be put into computer programs that might not fully copy that abstraction (since computers cannot think (yet)) but they reproduce it to good effect. An example of this kind of dealing with abstraction is the ability to give a name or label to the infinite set of natural numbers without actually counting all of them, $\mathbb{N} = \{0, 1, 2, ... \}$. This can be done in computer algebra systems and it is unclear how the mind does it though we can presume that it doesn’t store all numbers. It is customary in math to use *mathematical induction* but the latter is procedural and doesn’t seem quite the same as human abstraction. With finite $\mathbb{N}[n] = \{0, 1, 2, ..., n\}$ then *mathematical induction* is: that for each $\mathbb{N}[n]$ there is an $\mathbb{N}[n+1]$. The procedure uses n to create $n+1$ and subsequently the set. Introspection suggests that this procedure differs from the mental act to grasp the whole $\mathbb{N}$. Perhaps one type of computer program can only count in actual numbers (printed on a paper trail), with only different instances and a specific value per instance, but another type of computer algebra system might use the symbol $\mathbb{N}$ as a representation for all natural numbers (with the associated algebra to make it work). While the specification $\mathbb{N} = \{0, 1, 2, ... \}$ suggests that we are hard-pressed to understand what infinity could be as a completed whole, the list $\{1, 1/2, 1/3, 1/4, ... \}$ clarifies that we only need the interval [0, 1] (with 0 and 1 included) to grasp that completed whole. Another example is to work with the real numbers $\mathbb{R}$ also using a calculator even though the calculator represents such numbers only in finite form up to a certain depth of digits. For example, for 1/9 = 0.111... the calculator screen may show only 8 digits, but on paper we can include the ellipsis (trailing dots) to indicate that the 1’s continue, while a computer algebra system may formalize that and only display 0.111... but continue internally to work with 1/9 till a final answer is required. In my mind I might think of a circle and a computer might print the mere word “Circle[r]” (with radius r). Some authors might hold that thinking about a number or circle is platonic but others might agree that ontology is a different subject (since people may also dream about ghosts). It seems that it suffices to hold that this kind of abstraction is precisely what we want to include in the constructivist view on mathematical activity.

Agreement and disagreement on this step-by-step approach will help to delineate what constructivism is, or what kind applies at a particular instance. Presumably a particular view is more efficient in terms of information processing in some cases than in others. This holds a fortiori with respect to points of view on volition, determinism or randomness, where we also lack distinguishing and decisive experiments, but where we can develop models that result in different successes and failures.

PM 1. Wittgenstein (1889-1951) used the term 'language game' to indicate that individuals have their own understandings and negotiate meanings with one another. This approach might reduce language to a soup. Mathematics educator Pierre van Hiele (1909-2010) allowed for levels in understanding or abstraction, which notion seems to have merit of itself and does not merely derive from the stratified language game in class. With words that have a different meaning depending upon the level of (mathematical) understanding, a language contains at least four sublanguages relating to these levels. Our reference to the ordinal and ratio scales implies a reference to such four levels, going from a child that is trying to master arithmetic to abstract axiomatics.

PM 2. Some readers may have a background in psychological constructivism and may observe that measurement tends to reduce to the use of a discrete grid since instruments or sense organs may never capture infinite accuracy. The point however is that repeated measurements can generate different values on the same phenomenon, so that the ratio scale or the set of real numbers has been developed to capture that very notion of the infinite accuracy of the underlying model for reality. The ancient Greeks used a theory of proportions to deal with geometric lengths but in the subsequent two millennia mathematicians have developed the theory of the real numbers or ratio scale to better handle these phenomena.

1.3 A simple core

When readers progress through the argument in this paper, they may think that it is quite complex, but in fact it is rather simple. There is only one major goal and that is to introduce a new definition, namely the notion of *'bijection by abstraction'*. Though the intention of this paper is a contribution to clarity about the definition of constructivism, that contribution quickly narrows down to emphasising a new definition for a minor though apparently key aspect, namely 'bijection by abstraction'.

A bijection is a one-to-one relationship. A simpler word is 'map'. If a merry-go-round has as many seats as children then it is possible to match each seat to a single kid and to match each kid to a single seat. The map explains where everyone sits. A bijection avoids empty seats (or one kid having more seats) and kids who cannot find a seat (or have to sit together). The 'bijection by abstraction' contains the notion that the human mind applies abstraction to create such a bijection between $\mathbb{N}$ and $\mathbb{R}$. We can denote $\mathbb{N} \sim \mathbb{R}$ to express that the sets are 'equally large' (though ordered differently). Since one can hardly object to a definition, the prime goal of this paper succeeds by itself. The only possible objection to a definition is that it is vacuous and has no application. A strong version of this rejection is that the definition is inconsistent and has no application by necessity.

If the reader keeps an eye on this major goal of this paper, to introduce this new definition, then the other aspects in this paper can be moved to the background, and then it would seem that the issue is rather simple. Readers of CF will emphasise the relevance for constructivism, namely that a procedural approach to ℕ and ℝ is insufficient for constructivism and that abstraction can be taken as an aspect in *proper constructivism*. Mathematicians might get lost in proof details but we don't focus on full formality. Purely mathematical papers may focus on full formality but other papers look at the 'intended application' of the mathematical model. This paper contains some proofs and in that respect it has features of a mathematical paper. Still, the major issue is the intended application to abstraction, and hence this paper wouldn't fit in a purely mathematical journal and might perhaps not even be understood by pure mathematicians. Mathematicians who stick to two-valued logic may not be aware that they can produce nonsense.

Georg Cantor (1845-1918) claimed to prove that there was no bijection between ℕ and ℝ, and he created a whole universe of 'transfinites' to deal with the consequences. The suggestion of this paper is that Cantor may not have appreciated what abstraction may entail. William of Ockham (1288-1348) held that complexities should not be increased without necessity, and this paper uses Occam's razor to cut away Cantor's universe as overly complex and without a base in necessity. Given the wide acceptance of Cantor's results, the opposition to this paper will be strong. A Cantorian will tend to hold that the definition of 'bijection by abstraction' is vacuous and irrelevant. The purpose of this paper is to at least present the definition, so that discussants know about its existence and possibility.

That said, a fairly quick consequence however is that readers may wish to understand more about the definition and its area of application. This is a somewhat dangerous consequence. At this moment the application is tentative and not fully established by itself. It seems relatively easy to generate all kinds of questions about what such abstraction does entail indeed. Such questions and uncertainty may easily cause the reader to reject this analysis. The reader is invited however to concentrate on understanding the definition, and suspend judgement till after subsequent discussions about the application.

A prime application is in the area of highschool education, where pupils could be presented with a clear and consistent theory of numbers and infinity without the convoluted Cantorian universe of the transfinites. Teachers of mathematics might feel guilty when they don't explain Cantor's universe but they might be happy if there is a sound alternative and when they can explain more about the wonders of abstraction itself.

1.4 A key in the education of mathematics

To aid the discussion, my proposal is to use evidence based didactics of mathematics as our anchor in a real world activity, to prevent that we get lost in mathematical inconsequentialism. This actually holds for the philosophical discussion about mathematics anyway. The didactics of mathematics are an excellent sounding board for philosophy, and it seems also a necessary sounding board. It would be somewhat curious to hold that the approaches to clarification would be entirely different for philosophy on the one hand and didactics (of mathematical concepts involved) on the other hand, when the subjects would still be the same. It is more reasonable to assume some overlap. The editor of *CF* suggested that my reference to the education of mathematics would be a distracting deviation to what the proper topic of this paper would be - constructivism with abstraction in the rejection of Cantor in favour of Occam - but this would be a misunderstanding. The misguidedness in mathematics and its application can be quite horrible and there is a huge need for anchors. In standard applications we can refer to engineering, and for the present discussion it is a key insight that we can refer to the education of mathematics. In a way, this very paper is a development of the didactics on the infinite. Also, when a philosopher would object to ‘abstraction’ as something quite undefined, then we can refer to the classroom situation, and refer to the Van Hiele (1973) levels of understanding, while Colignatus (2011g) contains some comments with respect to research on the brain.

The prime lesson is to beware of mathematical confusions. Apparently it cannot be emphasised enough how important that is. (Mathematical) philosophies for example relating to the Russell set paradox may be misguided, and this kind of misguidedness has been happening in mathematics overall. Given the suggestion above, the key reference is to education. A didactic reconstruction results into another curriculum. Already the term ‘didactic reconstruction’ causes the question how this is seen in terms of constructivism itself.

The discussion here selects for didactics the teaching for secondary education or for first year college and university students who will not be mathematics majors. The discussion in this present paper can be complex in itself, but it is directed at the more simplified content that is to be taught. The latter simplified world view would still be true (perhaps in quotes: ‘true’) but merely be less rich in complexity to allow easier understanding at a more basic level. My paper *Neoclassical mathematics for the schools* (NMS, 2011) uses the label ‘neoclassical’ but the approach may be understood as *proper* constructivist. That is, it is constructivism with some scope for human abstraction. It may help the discussion when such constructivism with abstraction is recognised. When a student can construct a path towards understanding then this will seem more attractive didactics than requiring them to merely ‘get it’ (or tell them to find another job).

There is an additional advantage of pointing to the mess in education. That mess in itself does not prove anything about particular topics in this paper (according to the title about Cantor’s results). When official dogma is that mathematics in the curricula is perfect, then it is less likely that mathematicians goof on Cantor as well. However, when it is called into attention that mathematics in the curricula is a mess, then the likelihood increases that this may also be with Cantor’s results.

1.5 A note on the references

In the following, I will phrase the argument such that its content can be understood, so that this article is self-contained. The references are only useful for further reading on that particular angle. This also holds for my own work that is generally selfpublished. My general position is that logicians and mathematicians may do fine work but also make serious errors. Refutation requires whole books for *reconstruction* and *re-engineering* rather than mere articles in journals. This present paper may be a unique presentation in a journal on my approach to these matters.

My original research in this area has shifted from the re-engineering of methodology, philosophy and the foundations of mathematics (in textbook format) in *A Logic of Exceptions* (ALOE, 1981, 2007, 2011) towards the re-engineering of explicit didactics of mathematics in *Elegance with Substance* (EWS, 2009) and *Conquest of the Plane* (COTP, 2011), and now returns to philosophy and the foundations, with work in progress *Contra Cantor Pro Occam* (CCPO - WIP, 2011, 2012) (also discussing infinitesimals and nonstandard analysis) and the present paper (CCPO - PCWA, 2013). Useful reviews of my books are by Gill (2008) and Gill (2012) in the journal of the Dutch mathematical society and Gamboa (2011) and Bradley (2012) at the website of the European Mathematical Society. These books on re-engineering logic and mathematics are targetted at non-math-majors at highschool and first year of college and university for fields that use math. The reader is advised to study these books and include them in first year courses in any case. This present paper is self-contained for its purposes but if you would wish to rekindle your logic and mathematics using the standard books then you still may be misled by the standard approaches.

DeLong (1971) is my standard and much praised undergraduate introduction into standard mathematical logic. My approach in ALOE, EWS, COTP and the present paper is much in contrast to this book yet also much indebted to it. The book mentions the infinite but further concentrates on logic. Davis & Hersh (1980, 1983) is an excellent study in many respects and rather accessible for a larger audience since it avoids formulae. It may be taken as the common point of departure though it is amazing how many confusions it contains compared to the approach in ALOE, EWS, COTP and the present paper. Aczel (2000) and Wallace (2003) are popular scientific expositions on the infinite that also contributed to this author’s understanding. Hart (2011) is a syllabus for the mathematical course at TU Delft but unfortunately in Dutch.

Hodges (1998), the section on Cantor’s proof, discusses submissions to the *Bulletin of Symbolic Logic* that claimed to refute Cantor but that failed on basic academic standards. This is indeed an area where intuition meets hard proof. Professor Hodges sent me an email (August 10 2012) that he allows me to quote from: “You are coming at Cantor’s proof from a constructivist point of view. That’s something that I didn’t consider in my paper, because all of the critics that I was reviewing there seemed to be attacking Cantor from the point of view of classical mathematics; I don’t think they knew about constructivist approaches. Since then some other people have written to me with constructivist criticisms of Cantor. There is not much I can say in general about this kind of approach, because constructivist mathematicians don’t always agree with each other about what is constructivist and what isn’t.” Let me emphasise again that the core of this present paper is the new definition of ‘bijection by abstraction’. This new definition should appeal to all those who have had intuitive misgivings about Cantor’s proof. The definition includes an aspect of completion that some may consider rather classical and non-traditional-constructivist. This paper also discusses where Cantor’s proof goes wrong. I suppose that there will be discussion about this but consider that of secondary value. It is more important to improve the didactics in highschool than getting lost in discussions with mathematicians who are stuck in two-valued logic and who have no feeling for the intended application (and who conclude that they can fit in their bag).

In said email, professor Hodges also recalls that intuitionistic L.E.J. Brouwer "certainly thought that Cantor's argument can be read so that it applies to potentially infinite sets too". In my recollection, but I have no direct reference, Brouwer rejected the general Cantor Theorem on the power set but accepted the proof on the difference between $\mathbb{N}$ and $\mathbb{R}$. However, $\mathbb{N} \sim \mathbb{R}$ also eliminates that prospect for the potential infinite.

A referee pointed to Brady & Rush (2008). I was surprised to see that they also drop the law of the excluded middle (LEM) and then in their section 7 come to a rejection of Cantor's proof. LEM is that propositions are only True or False. Intuitionistic Brouwer would say Proven or Refuted, showing some confusion between truth and having a proof, and allowing for a third case of Undecided. My approach in ALOE is to have *truth, falsehood* and *nonsense* for logic, and in particular for statements about reality, and *proven, refuted* and *undecided* for abstract mathematical systems. When a mathematical system is used as a model for reality it may show up nonsense. While two-valuedness would apply for (models for) physical reality, nonsense could apply to statements that refer to language itself, such as in the liar paradox. Brady & Rush (2008:201) conclude that Meaning Containment (MC) does not warrant that LEM holds for Cantor's diagonal, so that it hangs in the air whether it exists or not. There is some parallelism to the present approach here: (1) ALOE already rejects Cantor's general theorem on the power set, (2) here we show that his diagonal argument on $\mathbb{N}$ and $\mathbb{R}$ is invalid. The following reproduces (1) but concentrates on (2). I haven't further checked the MC approach though it has attractive features.

1.6 Reductio ad absurdum

The *reductio ad absurdum* format of proof seems to be a convenient way for the human mind to reason. This convenience may derive from cultural convention, there further doesn't seem to be anything special about it.

A *reductio ad absurdum* format of proof assumes hypotheses, deduces a contradiction, and concludes to the falsity of one of the hypotheses. For example, define a *squircle* as a shape in Euclidean space that is both square and circular. A theorem is that it cannot exist. If it is square then the distance to the center will differ for corners and other points, and this contradicts the property of being circular. If it is circular, then it cannot have right angles, and this contradicts the property of being square. Hence a squircle does not exist in Euclidean space. This is a fine proof.

Now consider the theorem that squares cannot exist in Euclidean space. We use the same definition of squircles. There is a lemma that any square associates with a squircle, e.g. the squircle with a circle with the same area as the square. The proof then is: Take a square, find its associated squircle, and deduce a contradiction as done above. Square implies falsehood. Ergo, squares don't exist. QED.

We know that squares exist in Euclidean space, so something must be wrong. To pinpoint where it goes wrong may be less clear. After careful study we may conclude that the proof uses the existence of squircles as a hidden assumption. The lemma is false. Once this is spelled out, it is rather clear for this example.

It appears to be a bit more complex for Cantor's diagonal argument. What is his hidden assumption ?

1.7 The structure of the paper

We will first look at the context of ALOE that generates the three-valued logic and approach to set theory, and then discuss the general argument in Cantor's Theorem. Subsequently, we develop an approach on the natural and real numbers and the 'bijection by abstraction' between them, such that $\mathbb{N} \sim \mathbb{R}$. Potential and actual infinity are two faces of the same coin, where the potential $\mathbb{N}[n]$ with $n \to \infty$ might be considered as procedural only and differing from the abstractly completed actuality of $\mathbb{N}$. Subsequently we can show where Cantor went wrong in the diagonal argument on $\mathbb{R}$.

2. The context of ALOE

2.1 An approach to epistemology

A proposal is the 'definition & reality methodology'. Youngsters grow up in a language and culture and learn to catalogue events using particular terms. The issue of matching an abstract idea (circle) with a concrete case (drawn circle) is basic to thought itself. For circles we can find stable definitions and this might hold more in general. Questions like "all swans are white" can be resolved by defining swans to be white. The uncertainty then is shifted from the definition to the process of cataloguing. A black swanlike bird may be important enough to revise the definition of a swan. See *Definition & Reality in the General Theory of Political Economy* (DRGTPE, 2011). In the case of space, my suggestion in COTP is that the human concept of space is Euclidean, so that we don't have the liberty to redefine it. Einstein's redefinition of space-time may be a handy way to deal with measurement errors but could be inappropriate in terms of our understanding. Definitions in economic models may restrict outcomes which other models may not observe that don't maintain those definitions.

With respect to consciousness, language is a bit tricky here. As people experience consciousness, and this experience is created by (what some models call) atoms and energy in the universe, apparently consciousness is a phenomenon created by the universe as well, and in this sense consciousness is as real as those atoms and energy or the universe itself. While atoms and energy seem to be dead categories without pleasure and pain it is strange that there can be a mind that experiences pleasure and pain. One way to approach this is to say that sound, sight, smell and touch are the senses, but that consciousness then is a 'sense' too. See Colignatus (2011g) and Davis & Hersh (1980, 1983:349). This is vague and speculative and not directly relevant for this present paper but it seems relevant enough to at least mention it. The point namely is that abstraction takes place in consciousness or that consciousness might actually be composed of abstractions.

2.2 Three-valued logic

It seems that (constructivist) Brouwer mixed up the notions of truth and proof. It might be that his interpretation of a double negation might differ from twice a single negation. "Not-not-*A*" might mean "There isn't a proof that *A* is not the case" which differs from *A*. It is somewhat of a miracle that Heyting succeeded in finding apparently consistent axioms. Eventually there might be an interpretation in terms of truth and proof but for now these intuitionistic axioms are difficult to interpret.

My preference is for three-valued logic that better suits common understandings of logic, with True, False and Indeterminate, where the latter can also be seen as Nonsensical. This logic has a straightforward interpretation and allows the solution of the Liar paradox and Russell's paradox, while the Gödeliar collapses to the Liar in a sufficiently strong system. See ALOE. Russell's solution with the Theory of Types outlawed selfreferential terms, and implicitly declared such forms as nonsensical. The proposal of a three-valued logic thus only makes explicit what Russell left implicit, while it actually allows useful forms of selfreference. Gödel's uncertainty due to incompleteness is replaced by an epistemological uncertainty for selfreferential forms that some day an inconsistency might turn up that shows some assumptions to be nonsensical.

While this paper will rephrase arguments in terms of two-valued logic, it will allow some selfreference in the definition of sets, and thus has to rely on some form of solution where such selfreference would cause nonsense. It will also be useful to be able to make the distinction between existence and non-existence of sensical notions versus nonsense itself. A common way of expression is to say that nonsensical things cannot exist but that might also cause the confusion that the nonexistence makes the notions involved sensical.

Axiomatics may create (seemingly) consistent systems that don't fit an intended interpretation. See the example of the coat and schoolbag above. Van Bendegem (2012:143) gives the example that (a) 1 is small, (b) for each *n,* if *n* is small then *n*+1 is small, (c) hence all *n* are small. The quick fix is to hold that "small" can be nonsensical when taken absolutely, and that (a') 1 is smaller than 100, (b') for each *n,* if *n* is smaller than 100 *n,* then *n*+1 is smaller than 100 (*n*+1), (c') hence for all *n, n* is smaller than 100 *n*. The conclusion is that not all concepts or axiomatic developments are sensical in terms of the intended interpretation even though they may seem so.

2.3 Set theory

Next to an axiomatic system we recognize the 'intended interpretation'. In this paper the discussion about set theory is *within* the 'intended interpretation' and doesn't rely on an axiomatic base. If we arrive at some coherent view then it will be up to others to see whether they can create an axiomatic system.

Set theory belongs to logic because of the notions of *all, some* and *none*, and it belongs to mathematics once we start counting and measuring. Cantor's Theorem on the power set somewhat blurs that distinction since the general proof uses logical methods while it would also apply to infinity - and the latter notion applies to the set of natural numbers $\mathbb{N}$ = {0, 1, 2, ... } and the set of reals $\mathbb{R} = 2^{\mathbb{N}}$. The continuum finds an actual infinity in the interval [0, 1].

Kauffman (2012) gives a modern perspective on set theory, that still results into a Theory of Types, but he does not mention the view from three-valued logic used in this paper, and explained in ALOE.

2.4 Russell's paradox

Russell's set is $R \equiv \{x \mid x \notin x\}$. This definition can be diagnosed as self-contradictory, whence it is decided that the concept is nonsensical. Using a three-valued logic, the definition is still allowed, i.e. not excluded by a Theory of Types, but statements using it receive a truthvalue Indeterminate. An example of a set similar to Russell's set but without contradiction is the set $S = \{x \neq S \mid x \notin x\}$. This applies self-reference but in a consistent manner.

Above construction of S might seem arbritray since it is explicitly imposed that $x \neq S$. However, consider $V = \{x \mid x \notin x \land x \in V\}$, which definition uses a small consistency condition, taken from Paul of Venice (1368-1428), see ALOE:127-129. It follows that $V \notin V$. The exclusion is not an arbitrary choice but derives from logic.

ALOE:127-129 actually uses a longer form. Above V causes an infinite regress for $x \neq V$ so the full form is $S = \{x \neq S \mid x \notin x\} = \{x \mid ((x \neq S) \Rightarrow (x \notin x)) \lor ((x = S) \Rightarrow (x \notin x \land x \in S))\}$.

Thus the form $S = \{x \neq S \mid x \notin x\}$ might convey the impression that $x \neq S$ would be a matter of choice, while it isn't. Hence, if in need of a short expression, we might adopt the V shorthand, but this comes with the risk that readers unfamiliar with this analysis might grow confused about the infinite regress.

2.5 Caveat

The literature on number theory and the infinite is huge, and my knowledge is limited to only a few pages (that summarize some points of that huge literature). My only angle for this present paper is the insight provided in ALOE (1981, 2007, 2011) on some logical relationships, plus two new books EWS (2009) and COTP (2011) that focus on mathematics and its education. Given the existence of that huge part of the literature that is still unknown to me I thus have my hesitations about expressing my thoughts on this subject. When I read those summaries then it might be considered valid however that I do so, since in essence I only express this logical angle. This first resulted in CCPO (2011) and now this present paper.

2.6 A note on reductio ad absurdum

W.r.t. section 1.6, the following may be added. Let q = "Squircles exist." Then we find q

$\Rightarrow \neg q$. Trivially $\neg q \Rightarrow \neg q$. The Law of the Excluded Middle (LEM) is that $q \vee \neg q$. Hence in all cases $\neg q$, or that squircles do not exist.

Consider however the proof that squares don't exist. Let p = "Squares exist." Using the definition of the squircle and the lemma that each square generates a squircle, we find $p \Rightarrow \neg p$. Trivially $\neg p \Rightarrow \neg p$. The Law of the Excluded Middle (LEM) is that $p \vee \neg p$. Hence in all cases $\neg p$, or that squares do not exist.

However, with three-valued logic we must allow that there can be nonsense. Thus $p \vee \neg p \vee \dagger p$, where the dagger indicates nonsense. Is it possible to construct the argument that $\dagger p \Rightarrow \neg p$ as well ? According to the truth-table (ALOE:183) an implication from nonsense is only true if the consequence is true or again nonsense, and it is false when the consequence is false. If we want squares to truly exist, the implication $\dagger p \Rightarrow \neg p$ must be false, and then we cannot use $p \vee \neg p \vee \dagger p$ to conclude that $\neg p$. What happens is that the definition of squircles and the lemma that each square causes a squircle actually start to make the notion of a square nonsensical itself too. In this simple case the conclusion is clear that the lemma is false, or that $p \Rightarrow q$ is false, since an implication is false if the antecedens is true and the consequence is false.

3. Cantor's Theorem in general

As with Russell's set, using a similar consistency criterion for Cantor's Theorem on the power set we find that its proof collapses. This allows us to speak about a 'set of all sets' (unless we would find some other contradiction). Below we also reject the diagonal argument on the real numbers. ALOE in 2007 rejected the general theorem but still allowed the diagonal argument for the reals only. In 2011 I found an argument that the set of real numbers ℝ is as large as the set of natural numbers ℕ. My knowledge about Cantor's transfinites is limited to DeLong (1971) and popular discussion like Wallace (2003), and see **Appendix A**. Nevertheless it seems possible (see below) to reject the theorem on which those transfinites are based. See ALOE p238-240 for the context.

Cantor's Theorem holds that there is no bijection between a set and its power set (the set of all its subsets). For finite sets this is easy to show (by mathematical induction). The problem now is for infinite set A such as the natural or real numbers. The proof (in Wallace (2003:275)) is as follows. Let f: $A \to 2^A$ be the hypothetical bijection between (vaguely defined 'infinite') A and its power set. Let $\Phi = \{x \in A \mid x \notin f[x]\}$. Clearly Φ is a subset of A and thus there is a $\varphi = f^{-1}[\Phi]$ so that $f[\varphi] = \Phi$. The question now arises whether $\varphi \in \Phi$ itself. We find that $\varphi \in \Phi \Leftrightarrow \varphi \notin f[\varphi] \Leftrightarrow \varphi \notin \Phi$ which is a contradiction. Ergo, there is no such f. This completes the current proof of Cantor's Theorem. The subsequent discussion is to show that this proof cannot be accepted.

In the same line of reasoning as with Russell's set paradox, we might hold that above Φ is badly defined, since its definition is self-contradictory under the hypothesis that there *is* a bijection. A badly defined set cannot be a subset of something. We see the same structure of proof as the example of the squircle in section 1.6, where we 'proved' that squares don't exist. This 'proof' implictly used the existence of squircles that actually don't exist. Cantor's Φ is like a squircle. Under the assumption that there *is* a bijection it cannot be defined as suggested that it is.

A test on this line of reasoning is to insert the similar small consistency condition, $\Phi = \{x \in A \mid x \notin f[x] \land x \in \Phi\}$ (see above note on the infinite regress and the full form). It will be useful to reserve the term Φ for the latter and use Φ' for the former inconsistent definition. Now we conclude that $\varphi \notin \Phi$ since it cannot satisfy the condition for membership, i.e. we get $\varphi \in \Phi \Leftrightarrow (\varphi \notin f[\varphi] \land \varphi \in \Phi) \Leftrightarrow (\varphi \notin \Phi \land \varphi \in \Phi) \Leftrightarrow$ *falsum.* There is no contradiction and no reason (yet) to reject the (assumed) existence of the bijection *f*. Puristically speaking, the earlier defined Φ' differs lexically from the later defined Φ, the first expression being nonsensical and the latter consistent. Φ' refers to the lexical description but not meaningfully to a set. Using this, we can also use $\Phi^* = \Phi \bigcup \{\varphi\}$ and we can express consistently that $\varphi \in \Phi^*$. So the earlier 'proof' above can be seen as using a confused mixture of Φ and Φ*. (And, to avoid the infinite regress like with the Russell paradox, a puristically proper form is $\Phi = \{x \in A \land \ x \neq f^{-1}[\Phi] \mid x \notin f[x]\}$, and now we have the explanation why $f^{-1}[\Phi] \notin \Phi$.)

It follows:

1. that the current proof for Cantor's Theorem for infinite sets is based upon a badly defined and inherently paradoxical construct, and that the proof evaporates once a sound construct is used.
2. that the theorem is still unproven for (vaguely defined) infinite sets (that is, I am not aware of other proofs). We could call it "Cantor's Impression" (rather than "Cantor's Conjecture" since Cantor might not have conjectured it if he had been aware of above rejection).
3. that it becomes feasible to speak again about the 'set of all sets'. This has the advantage that we do not need to distinguish 'any' versus 'all' sets. And neither between sets versus classes.
4. that the transfinites that are defined by using Cantor's Theorem evaporate with it.
5. that the distinction between ℕ and ℝ rests (only) upon the specific diagonal argument (that differs from the general proof) (and it will be discussed below).
6. that there is a switch point here. Since bijection *f* in the approach above is merely assumed and not constructed, it will be a lure to constructivist mathematicians to conclude that *f* doesn't exist indeed. They may be less sensitive to the logic that if *f* is assumed then Φ' is nonsense. Constructivists who are open to that approach might see to their horror that a whole can of worms of nonconstructivist 'set of sets' and such monsters is opened. It might be a comfort though that this seems to be the most logical and simplest solution.

When we consider the diagonal argument on ℝ then it appears that we may reject it as well.

4. Abstraction on numbers

4.1 Abstraction on the natural numbers

Aristotle's distinction between the potential and the actual infinite is a superb common sense observation on the workings of the human mind. Elements of ℕ and the notion of repetition or recursion allow us to develop the potential infinite. The actual infinite is developed (a) via abstraction with associated 'naming' or (b) the notion of continuity of space (rather than time as Brouwer does), or intervals in ℝ. While we use the symbol ℕ to denote the natural numbers, this not merely means that we can give a program to construct integer values consecutively but at the same moment our mind leaps to the idea of the completed whole (represented by the symbol ℕ or the phrase "natural numbers"), even though the latter seems as much a figment of the imagination as the idea of an infinite line. The notion of continuity however for say the interval [0, 1] would be a close encounter with the actual infinite. In the same way it is OK to use the mathematical construct that the decimal expansion of $\Theta = 2\,\pi$ has an infinity of digits, which is apparently the conclusion when we use such decimals.

We can present this argument without the term 'infinity'.

(1) Potential form: $\mathbb{N}[n] = \{0, 1, 2, ..., n\}$ (The human ability to count. The successor function.)

(2) Actual form $\mathbb{N} = \{0, 1, 2, ...\}$ (The ability to give a name to some totality. The ability to measure in [0, 1].)

(3) $\mathbb{N}[n]$ @ ℕ

The @ can be read as 'abstraction'. It records that (1) and (2) are related in their concepts and notations. In the potential form for each n there is an n+1, in the actual form there is a conceptual switch to some totality caught in the label ℕ. The switch can be interpreted as the change from counting to measuring, as we will later see that there is a sense in which $\mathbb{N} \sim \mathbb{R}$, or that both are 'equally large'.

PM. In CCPO-WIP there is also a text that uses the terms 'limit' and 'bijection in limit'. The mathematical notion of a limit can be used to express the leap from the potential to the actual, though the use and precise definition of that notion of a limit also appears to depend upon context, e.g. with a distinction between 'up to but not including' and 'up to and including'. To avoid confusion this present paper uses only the notion of abstraction as defined here.

4.2 Steps in construction and abstraction

Steps (1) and (2) may be too large and we can try to find intermediate steps. This is tricky since shifts are gradual. At a lower level of abstraction you can be blind to the larger implications and higher levels. At a higher level of abstraction you might think that this might be logically be included in the lower level though you didn't see it. (This is one link between philosophy and didactics.)

One approach is to distinguish numbers 0, 1, 2, ... from lists of numbers {0}, {0, 1}, ... Supposedly the notion of 'listing the numbers' generates the notion of a 'whole' which might be absent from the numbers themselves. But in the intended interpretation the numbers are supposed to count something and thus include some notion of 'whole' anyway. Perhaps the successor function might be used without the notion of a 'whole', but when used for counting as generally understood it has that notion. Thus we proceed as follows:

(S_1) ℕ[0], ℕ[1], ... for concrete numbers only. (They just 'are'. This might be seen as the platonic case, where there is no invention but discovery. In strict finitism there might even be a biggest number.)

(S_2) $\mathbb{N}[n] \Rightarrow \mathbb{N}[n+1]$. This would be an algorithm that generates the numbers consecutively. Given some n, it has the ability to calculate n+1 and include it in a list. There is no recognition of a variable n yet however. (This cannot be Aristotle's potential infinite. Though Aristotle didn't explicitly use the modern notion of a variable his reasoning anticipated it.) (This may also be represented by the successor function.)

(S_3) $\mathbb{N}[n] = \{0, 1, 2, ..., n\}$ as an abstraction of S_2. The variable n is identified explicitly. (The neoclassical form of Aristotle's potential infinite.)

Mathematical induction is at the level of S_3 because of the abstract use of the variable n. Namely, for predicate P the application of $P[n] \Rightarrow P[n+1]$ is mathematical induction only if there is explicit understanding of the necessary link via n. A computer program that for each $P[n]$ subsequently prints $P[n+1]$ merely shows the execution of a mathematical proposition (S_2), but does not provide a proof that something would hold for all n (S_3). (A student might continue to work at level S_2 before it dawns that this could potentially continue ad infinitum, S_3.)

(S_4) $\mathbb{N} = \{n \mid n = 0 \lor (n - 1) \in \mathbb{N}\}$. The abstraction of S_3 that it could continue for any n, but then generate a completed whole. This uses the recursive procedure written as up to n, but note that any n still transforms into all n. (The neoclassical form of Aristotle's actual infinite.)

As Kronecker is reported to have said "God made the integers" the subsequent question is: "Really all of them ? He didn't forget a single one ?" The crux in S_4 lies in the symbol ℕ that captures the "all", and a consistent Kronecker thus would accept S_4. (But it seems that he wanted to remain in S_3.) Mathematical induction is often understood to be relevant for this level. In that case it might be useful to speak about 'basic' m.i. for S_3 and 'full' m.i. for S_4.

(S_5) $\mathbb{N} = \{0, 1, 2, ...\}$. The reformulation of S_4 in the format of S_3 with an ellipsis, to emphasize the shift from finite n to a completed whole. (This is merely a matter of notation. The dots now are used within the notation and not at the meta level. There will be some students who will have a problem to shift from the procedural form S_4 to the more abstract form S_5, but it shows mathematical maturity to see that the forms are equivalent.)

(S_6) The next step would leave the realm of constructivism. The intellectual movement towards constructivism might become so popular that all want to join up, also users of nonconstructive methods. But a line may be drawn and this line will actually define constructivism. An example of a nonconstructive method is Cantor’s manipulation of the diagonal element, see below, where he assumes some positional number C so that there is a digit $d_{C,C}$ on the diagonal, but we find that this number is undefined, so that actually $C = \infty$. Drawing the line here, allows us to express that S_5 with its abstraction still belongs to (traditional) constructivism.

The distinctions between these S_i would be crucial if we would deal with inflexible intelligences who cannot get used to some forms. For those who can use all forms, the distinctions may seem somewhat arbitrary, because they will wonder: don’t the simpler forms invite the abstractions to the higher levels ?

My suggestion is that S_1 and S_2 aren’t relevant for mathematics (except for the engineering of calculators), and that Aristotle was right that the interesting question concerns the distinction between S_3 and S_4 (or the form S_5) (which also could apply to the engineering of computer algebra languages).

4.3 No need for strict finitism

Cariani (2012) summarizes his result: “If we want to avoid the introduction of entities that are ill-defned and inaccessible to verification, then formal systems need to avoid introduction of potential and actual infnities. If decidability and consistency are desired, keep formal systems finite. Infnity is a useful heuristic concept, but has no place in proof theory.”

I don’t think that is true. The issues by Cantor and Gödel rather seem issues of logic than of infinity. If the number of bits in the universe is limited and we stick to such an empirical representation then there follows an empirically biggest number. But the mind would allow the imagination of two universes and thus a number twice as big. My suggestion is to resolve the logical conundrums. See ALOE for Gödel while the present paper summarizes CCPO for Cantor.

Cariani (2012:120) quotes Hilbert 1964: “We have already seen that the infinite is nowhere to be found in reality, no matter what experiences, observations, and knowledge are appealed to.” This is a curious statement given the continuum, or interval [0, 1], and its actual infinity of points (locations). Also, Hilbert wanted to maintain “Cantor’s paradise” while the transfinites are rather a horror-show. Cariani: “Radical constructivist thinking about mathematical foundations might likely depart from Hilbert’s program on two grounds: because of its end goal of justifying and rationalizing infinitistic entities and because of its abandonment of the construction of mathematical objects.” Instead, there is value in maintaining the potential and actual infinite, and via abstraction we can find that $\mathbb{N} \sim \mathbb{R}$.

4.4 Definition of ℝ

Let us define the real numbers in a variant of Gowers (2003), leaving out some of his algebra. It suffices to look at the points in [0, 1] (and others could be found by 1 / x etcetera). Thus ℝ is the set of numbers from 0 to 1 inclusive. A number between 0 and 1 is an infinite sequence of digits not ending with only 9's; if it ends with only 0's we call it terminating. Rather than defining ℝ independently it is better to create it simultaneously with a map (bijection) with ℕ, to account for the otherwise hidden dependence.

4.5 A map between ℕ and ℝ

First, let d be the number of digits:

For $d = 1$, we have 0.0, 0.1, 0.2, ..., 0.9, 1.0.

For $d = 2$, we have 0.00, 0.01, 0.02, ...0.09, 0.10, 0.11, 0.12, ..., 0.98, 0.99, 1.00.

For $d = 3$, we have 0.000, 0.001, ..., 1.000

Etcetera. Thus for each d we have ℝ[d].

Values in ℕ can be assigned to these, using this algorithm: For $d = 1$ we assign numbers 0, ... , 10. For $d = 2$ we find that 0 = 0.0 = 0.00 and thus we assign 11 to 0.01, 12 to 0.02, etcetera, skipping 0.10, 0.20, 0.30, ... since those have already been assigned. Thus the rule is that an assignment of 0 does not require a new number from ℕ. Thus for real numbers with a finite number of digits d in ℝ we associate a finite list of $1 + 10^d$ numbers in ℕ, or ℕ[10^d]. (Some might want to do a full recount and that is fine too.)

Subsequently, ℕ[10^d] @ ℕ. This creates both ℝ and a map between that ℝ and ℕ.

PM. Observe that ℝ conventionally has surprising properties. Regard for example α = 0.9999... and β = 1.000.... It is common to conclude that $\alpha = \beta$. Notably, with 1 / 3 = 0.3333... we want 1 = 3 * 1/ 3 = 3 * 0.3333... = 0.9999....

4.6 Definition of bijection by abstraction

This approach simply defines away Cantor's problem. The state of paradox is turned into a definition. The intention of these terms is to only *capture* what we have been doing in mathematics for ages. It is not intended to present something horribly new. It only describes what we have been doing, but what has not been described in these terms before. It is a new photograph but at higher resolution, and it allows to see where Cantor was too quick.

The created map is better called not merely a 'bijection' but rather 'bijection by abstraction'. A common bijection should allow us to identify the index of say 1 / 3 = 0.333.... while we lose that ability both in potential infinity (S_3) and actual infinity (S_4). In an overview, our procedure thus is:

(1) Potential form: $\mathbb{N}[n] = \{0, 1, 2, ..., n\}$

(2) Actual form $\mathbb{N} = \{0, 1, 2, ...\}$

(3) For all $n \in \mathbb{N}$, including variables $n \in \mathbb{N}$: $\mathbb{N}[n]$ @ $\mathbb{N}$

(4) The definition of $\mathbb{R}[d]$, and subsequently the creation of $\mathbb{R}$ via ($\mathbb{N}[d]$ @ $\mathbb{N}$) $\Rightarrow$ ($\mathbb{R}[d]$ @ $\mathbb{R}$). Check that indeed $\mathbb{R}$ arises: no holes.

(5) We construct the bijection $b[d]$: $\mathbb{N}[10^d] \leftrightarrow \mathbb{R}[d]$ for d a finite depth of digits.

(6) Definition of what it means to have a 'bijection by abstraction' between domain D and range R: this applies when these three properties are satisfied:

(a) there are a function $f[d]$: $\mathbb{N} \to \mathbb{N}$ and a bijection $b[d]$: $D[f[d]] \leftrightarrow R[d]$

(b) ($D[d]$ @ D) or ($D[f[d]]$ @ D)

(c) $R[d]$ @ R

Bijection by abstraction can be denoted b: $D \leftrightarrow R$ or $D \sim R$. In that sense D and R are equally large. When (6a) - (6c) are satisfied then this is also accepted as sufficient proof that there is a b, at constructive level S_4 even though that b no longer needs to be constructive in S_3.

Note that the function f also allows the use of binaries (0 and 1 only) and other formats.

(7) Then we get the scheme: on the left we use $\mathbb{N}\left[10^d\right]$ @ $\mathbb{N}$ and on the right simultaneously $\mathbb{R}[d]$ @ $\mathbb{R}$:

$$\begin{array}{cccc} b[d]: & \mathbb{N}\left[10^d\right] & \leftrightarrow & \mathbb{R}[d] \\ \downarrow & @ & \downarrow & @ \\ ?: & \mathbb{N} & ?? & \mathbb{R} \end{array}$$

(8) Hence: given its definition, there is a 'bijection by abstraction' between $\mathbb{R}$ and $\mathbb{N}$.

Our construction apparently is valid for the creation of $\mathbb{R}$. Since we have a map to $\mathbb{N}$ for each value of d, we find ourselves forced to the conclusion that with the creation of $\mathbb{R}$ there is *simultaneously* the creation of a map between $\mathbb{R}$ and $\mathbb{N}$.

$\mathbb{N}$ and $\mathbb{R}$ are abstract notions that may be understood by a mental act by a conscious brain. Nobody has ever seen a fully listed print of these numbers and it is physically inconceivable that this will ever happen. The above steps seem to properly capture what steps in abstractions are taken to handle these notions.

PM. In (4) for $m = 10^d$, $\mathbb{N}[m]$ @ $\mathbb{N}$ has the same portent as $\mathbb{N}[n]$ @ $\mathbb{N}$ in (3), and this has the same portent as $\mathbb{N}[d]$ @ $\mathbb{N}$. Thus ($\mathbb{N}\left[10^d\right]$ @ $\mathbb{N}$) $\Leftrightarrow$ ($\mathbb{N}[d]$ @ $\mathbb{N}$). Thus we can also use ($\mathbb{N}[d]$ @ $\mathbb{N}$) $\Rightarrow$ ($\mathbb{R}[d]$ @ $\mathbb{R}$). Currently (6) uses the function f but also the latter might be used.

4.7 On the interpretation of 'bijection by abstraction'

These are the two observations on the 'to' and 'from' relations:

(a) From ℕ to ℝ. Above scheme allows for each particular element in ℕ to determine what number in ℝ is associated with it (and it will have a finite number of digits).

(b) From ℝ to ℕ. The abstraction ℕ[*d*] @ ℕ appears to be vague and insufficiently constructive (in terms of S_3) to the effect we cannot pinpoint a particular value in ℕ associated with say 1 / 3 or a truly random sequence. It is paradoxical that we can decode a value in ℕ to a particular number in ℝ but that we cannot specify an algorithm to decode from 1 / 3 to a particular value in ℕ. The construction with ℕ[*d*] @ ℕ apparently introduces vagueness, even though we can infer that such a map *must have been created* since also ℝ has been created. Perhaps it is this very vagueness that causes that we have to distinguish between ℕ and ℝ, and make the distinction between counting and measuring.

This might also be summarised in this manner. Though the name ℕ suggests an actual infinite, and though the collection is an actual infinite, the natural numbers remain associated with *counting* and counting is always the potential infinite. Whence ℝ associates much better with the actual infinite given by the totality of ℕ, which is the continuum, which is *measuring*. If you look for something in a filing cabinet or encyclopedia, you might start with A, and step through all values, but it is smarter ('measuring') to jump to the appropriate first letter, etcetera.

An unrepenting constructivist (S_3) might want to see a constructive bijection between ℕ and ℝ and might reject the vagueness of the 'bijection by abstraction'. An eclectic and unrepenting Aristotelian (S_4) might be happy that both sets have the same 'cardinal number', namely infinity, and that there is no necessity for 'transfinites'.

4.8 A fallacy of composition

When we consider a real value with an infinite number of digits, like 1 / 3 or a truly random sequence, we employ the notion of the actual infinite. At the same time, in above definition and construction of ℝ we employ the potential infinite. When we combine these notions then we are at risk of making the *fallacy of composition.*

It is not quite proper to ask for the value in ℕ for 1 / 3 in the list generated for ℝ, if 1 / 3 is still in the process of being built up as an element in ℝ. By abstraction we get ℝ, including 1 / 3, but this apparently also means that we resign constructive specifics.

Stating 'ℕ[*d*] @ ℕ' means a 'leap of faith' or rather a *shift of perspective* from the potential to the actual infinite. Rather than counting 1, 2, 3, we shift to the set of natural numbers, ℕ (and the name 'the natural numbers' refers to that actual infinite). When we use that symbol then this does not mean that we actually have a full list of all the natural numbers. We only have the name. The shift in perspective is not per se 'constructive' in the sense of S_3 but can be accepted as 'constructive' in the sense of S_4.

4.9 A misunderstanding due to 'replacement'

One might see the step from finite d to infinity as a 'mere' replacement of d by the symbol ∞ (now not read as "undefined" but as "infinity"). This could be a form of algebra. Such mere replacement might be relevant for how our actual brains work. It might be relevant for didactics, something to suggest to some students who have difficulty understanding what is happening. However, at this point in the discussion there is no developed algebra on such methods, and the proper interpretation still is, only, the switch from the potential infinite to the actual infinite, which is a conceptual leap.

4.10 Properties of @ and ~

The symbols @ and ~ have been introduced for ℕ and ℝ specifically, and without claim for generality. Perhaps we can work towards some rules on those, such that we can assume those rules and some weaker property to arrive at the same outcome. This however is a tricky area.

(i) One reader argued:

(1) $\mathbb{N}[d]$ @ ℕ means that for every $n \in \mathbb{N}$ there is an m (say n+1) such that for $d > m$ we have $n \in \mathbb{N}[d]$.

(2) Then $\mathbb{R}[d]$ @ ℝ means that for every $r \in \mathbb{R}$ there is an m such that for $d > m$ we have $r \in \mathbb{R}[d]$.

(3) The latter however is not true. Trivially, 1/3 has no finite number of digits.

(4) Hence the meaning of $a[d]$ @ a differs for ℕ and ℝ and thus is not well defined.

In reply: Above, the symbol @ is <u>*not*</u> presented in a general format $a[d]$ @ a. Only the expressions $\mathbb{N}[d]$ @ ℕ and $\mathbb{R}[d]$ @ ℝ are defined separately, where it thus matters whether we look at ℕ or ℝ. The observation by the reader thus is partly accurate since there is indeed no general definition given for $a[d]$ @ a, but it is inaccurate since it wants to impose such a definition while it hasn't been given.

(ii) One reader wondered whether the expression $\pi[d]$ @ π would be meaningful. It is doubtful whether there is any value in looking in this kind of questions. Perhaps $\pi[d]$ might be defined as the number with the first d digits of π, and then what ? There is no meaningful way in how abstraction might cause one to get from such a value to the full value of π.

There is however a useful exposition on the irrational numbers in general, see **Appendix B**.

(iii) Some potential algebraic properties

Some rules in relation to 4.5 step (5) might be:

((A @ B) & ($A \sim C$)) ⇒ (C @ B) applied to ($\mathbb{R}[d]$ @ ℝ) & ($\mathbb{R}[d] \sim \mathbb{N}[m]$ for some $m = 10^d$) ⇒ ($\mathbb{N}[m]$ @ ℝ)

$((A\ @\ B)\ \&\ (A\ @\ C)) \Rightarrow (B \sim C)$ applied to $(\mathbb{N}[m]\ @\ \mathbb{N})\ \&\ (\mathbb{N}[m]\ @\ \mathbb{R}) \Rightarrow (\mathbb{N} \sim \mathbb{R})$

4.11 In sum

The interpretation is:

(i) The decimals in [0, 1] can be constructed via a loop on *d*, the depth of decimals, and then the infinite application using *countable* infinity. This is not radically novel. The distinction between potential and actual infinity is given by Aristotle, and everyone has been aware of a sense of paradox.

(ii) Due to Cantor people have started thinking that the loop would require ‘higher’ infinity. Cantor’s arguments however collapse in three-valued logic (and his universe has strange beasts anyway).

(iii) The concept of ‘bijection by abstraction’ helps to get our feet on the ground again. The potential infinite can be associated with counting and the actual infinite can be associated with the continuum. Two faces of the same infinity. Clarity restored.

(iv) The clarity actually arises by taking the paradox of the relation between the natural numbers and the continuum as the definition of ‘bijection via abstraction’. (The paradox is that for each *d* we have 10^d decimal numbers but for $\mathbb{N}[d]\ @\ \mathbb{N}$ we lose identification.)

(v) To avoid confusion in discussion: $\mathbb{N}$ is ‘countably infinite’ in all approaches, also via abstraction. $\mathbb{R}$ is ‘uncountably infinite’ in Cantor’s view but ‘countably infinite by abstraction’ according to this paper. For $\mathbb{N}$ we might drop the “via abstraction” but for $\mathbb{R}$ we might include it for clarity. We may also say that $\mathbb{R}$ is ‘Cantor uncountably infinite’ for clarity.

(vi) This discussion can also be held in secondary school, where pupils have to develop a number sense, except for perhaps some philosophical technicalities and use of language. The point of view deserves to be included in courses on set theory, also for math majors, since students ought to have a chance to occulate themselves against the transfinites.

5. Cantor’s diagonal argument for the real numbers

5.1 Occam’s razor

It is quite another thing to go from these considerations to conclusions on ‘transfinites’. I wholeheartedly agree with Cantor’s plea for freedom but mathematics turns to philosophy indeed if there is no *necessary* reason to distinguish different cardinalities for $\mathbb{N}$ and $\mathbb{R}$. See also Edwards (1988) and (2008). If there is no necessity, then Occam’s razor applies. Let us see whether there is necessity.

5.2 Restatement

Cantor's diagonal argument on the non-denumerability of the reals ℝ is presented in DeLong (1971:75&83) and Wallace (2003:254). We assume familiarity with it and quickly restate it. It suffices to assume a bijection between ℕ and ℝ that uses digits $d_{i,j}$: ($1 \sim 0.d_{1,1}d_{1,2}$....), ($2 \sim 0.d_{2,1}d_{2,2}$...) ... etcetera. The diagonal number is $n_D = 0.d_{1,1}d_{2,2}$.... taken from that list. The trick is to define a real number that will not be in the list. For example $n_C = 0.n_{C,1}n_{C,2}$...., where $n_{C,i} = 2$ iff $d_{i,i} = 1$, and $n_{C,i} = 1$ iff $d_{i,i} \neq 1$. If the position in the list would be C then $n_{C,C} = d_{C,C}$ by definition of the list and $n_{C,C} \neq d_{C,C}$ by definition of n_C, which is a contradiction. Nevertheless, n_C would be a true real number and thus should be in the list somewhere. (QED). PM. We can create an infinity of such points.

I've seen this argument in 1980 and considered it at some length, and have done so now again. In 1980-2010 I still accepted it. With some more maturity I can better appreciate some 'constructivist' views. One may observe that neither DeLong (1971) nor Wallace (2003) mentions those constructivist considerations on this proof. It would be better if those would be mentioned in summary statements since they better clarify what is at issue. Curiously though I have not found a direct counterargument yet, neither in papers by others on Kronecker, so the following are my own.

5.3 An aspect of selfreference

The diagonal argument might attract attention since there seems to be something fishy about taking an element $d_{C,C}$ and redefine it to have another value than it already has. This aspect of selfreference would be clearer if we could pinpoint a value for C.

5.4 C = ∞ ?

There is no algorithm to find the specific number C for the diagonal digit $d_{C,C}$. The reasoning is non-constructive in the sense that the number cannot be calculated. This might be clarified by writing $C = \infty$ so that we are discussing $n_{\infty,\infty}$ which may be recognized as rather awkward since the symbol ∞ generally stands for "undefined".

5.5 The reason why *C* is undefined

The diagonal argument apparently suffers from the fallacy of composition. The list of numbers in ℝ is created in the manner of a potential infinite but the diagonal proof suggests that they can be accessed as actual infinites.

Above, for each ℝ[d] in the list we might try to take a diagonal but the numbers are not long enough. For $d = 2$ we already get stuck at 0.01. The mutated number becomes 0.12 and when we move up the list we find it. Supposedly though we could extend the numbers with a sufficient length of zero's. Creating a new number based upon such a diagonal number would not be proper however since we are already creating ℝ in another fashion. Such diagonal number conflicts with the situation defined for that particular value of *d*. If we take ℕ[*d*] @ ℕ then the notion of a diagonal starts hanging in the air.

Cantor's argument has this structure: "Suppose that there is a list, then there is a diagonal, then a new number is created that cannot be on the list. Hence there is no such list, hence real numbers are not denumerable." But the above showed that there must be a list, that comes about alongside with the creation of ℝ itself. The alternative argument is rather: *Given the list and if we assume that such a notion of a diagonal is well defined, we apparently cannot find a value for C, whence such rules of creation like* n_C *are nonsense.* The mutation rule on the 'diagonal' stated in n_C is rather a waiting rule than a number creation rule. The numbers are in the list at some point, and do not have to be created anew. We only have to go from one value of *d* to another value of *d* to let the mutated number appear (up to the required value of *d*). And given the approach of abstraction this apparently also holds for completed ℕ and ℝ.

PM 1. Above we pointed to the fallacious lemma: Each square has an associated squircle. In the same way Cantor's argument uses a fallacious lemma: Each diagonal has an associated mutated number. The fallacy lies in the $n_{C,C}$ construction.

PM 2. The unrepenting constructivist (S_3) who rejects the usefulness of the 'bijection by abstraction' and who wants to see a constructive bijection such that we can calculate the proper number for 1 / 3, would also stick to a constructive approach for the diagonal, which is not what Cantor offers. In other words, S_3 regards it as a fallacy to suggest that there would be a 'diagonal' $n_D = 0.d_{1,1}d_{2,2}$.... Cantor's proof assumes a diagonal but rather that diagonal should be created. (While it is constructed, at the same time the mapped value of the diagonal is created, and then it appears that it could *not* be created since it is inconsistent that $n_{C,C} = d_{C,C}$ by definition of the list but $n_{C,C} \neq d_{C,C}$ by definition of n_C.)

PM 3. Readers who allow Cantor this freedom to be nonconstructive on diagonal digit $d_{C,C}$ should perhaps also allow for the 'nonconstructive' aspect in the 'bijection by abstraction' (namely that an index for 1/3 must exist but cannot be identified). Conversely, who allows for the 'nonconstructive' aspect in the 'bijection by abstraction' (S_4 instead of S_3) does not necessarily have to allow for the nonconstructive Cantorian handling of that diagonal digit $d_{C,C}$ (with $C = \infty$).

5.6 A formalization of the argument structure

Let us make the above a bit more formal. Let the proposition be p = "There is a (well-formed) diagonal element $d_{C,C}$". Cantor suggests the following scheme: $p \Rightarrow \neg p$ ergo $\neg p$. In the creation of ℕ and ℝ by abstraction, the diagonal element $d_{C,C}$ is not well-defined since the value of C remains vague so that the true form rather is $\neg p \Rightarrow$ (Cantor: $p \Rightarrow \neg p$), which is an instance of the 'ex falso sequitur quodlibet' (EFSQ) $\neg p \Rightarrow (p \Rightarrow q)$, now with $q = \neg p$.

There is a distinction between not-being-well-defined and non-existence. We can sensibly discuss the existence or non-existence of something when we know what we are speaking about. For well-defined topics we can accept the LEM $p \vee \neg p$ but it may be nonsense, $\dagger p$, so that in general only $p \vee \neg p \vee \dagger p$. When John exists, we can say that he is in the room or not. For squircles we may say that they don't exist but we actually mean to say that the notion isn't well-defined. We may say that elements $d_{i,i}$ exist but we cannot say that $d_{C,C}$ meaningfully exists.

The argument that $n_{C,C}$ only gets a *new* value is fallacious, since that assumes that there is a value $d_{C,C}$ to start with. It is too simple to say that $d_{C,C}$ must be a digit from 0,..., 9, and that each digit allows a redefinition. The latter only allows a conclusion that $n_{C,C}$ may have a value but it does not allow a conclusion on C. If $n_{C,C}$ would be on the diagonal as some $d_{C,C}$ then it would be redefined such that it is no longer $n_{C,C}$. Cantor implicitly uses that the diagonal element $d_{C,C}$ does not exist (he suggests to give it a value) to prove its nonexistence. Hence it is also 'petitio principii' or begging the question, $\neg p \Rightarrow \neg p$.

That this diagonal element $d_{C,C}$ is not well-defined does not prove that $\mathbb{R}$ is non-denumerable. When something is not well defined then it is tempting to conclude that it doesn't exist, and then Cantorian reasoning $p \Rightarrow \neg p$ takes off. It is better to hold on to the notion that it is not well defined what $d_{C,C}$ would be.

This paper comes close to generating a diagonal, via the bijection $b[d]$ and the step of abstraction. But that final step loses an index value C that Cantor wishes to use. When something cannot be identified then we should be cautious to use it. This is rather not an issue on the infinite but rather a point of logic.

5.7 Cantor's original argument of 1874

The syllabus on set theory by Hart (2011) opens on page 1 with Cantor's original argument on nondenumerability, which argument he later improved upon with the diagonal argument.

The original argument of 1874 suffers the same fallacy of composition. The formulation of the theorem assumes that $\mathbb{R}$ is built up in the manner of a potential infinite, but the proof uses that all elements are actual infinites. Instead, the proof can only use numbers up to a certain digital depth d, and create the full construction only alongside the construction of $\mathbb{R}$ itself.

See **Appendix A** for a longer discussion how Cantor went wrong in that original argument too.

6. The context of education

This paper is written in the context of education. This appears to cause misunderstandings amongst some readers, so it is useful to spend some attention to that context. This paper is not about education itself but about how to handle the infinite and the mathematics of the infinite, such that there is more scope for proper treatment in education. This paper does not develop a course on the infinite, but identifies essential issues, and creates scope for the development of such a course.

An essential feature in education is that we do not want to overburden students, even though they may think to the contrary. If a chapter would end with the statement “We didn’t tell you all yet, there are still things too complex for you”, then students might feel lost and cheated. They quite understand that there are many things that they do not understand, but, the closing statement of a chapter should be about what they have learned. Then a test to check this, then a new chapter.

The steps S_1, ..., S_6 allow an educational ladder, in which there is an increasing grasp of counting, measuring, and the infinite. There is a cesure between constructive S_5 and non-constructive S_6.

S_5 is constructive, uses three-valued logic to eliminate Cantorian nonsense, and uses to label ‘bijection by abstraction’ to capture the paradox that we can identify a bijection in potential infinity but lose identification when we create the actual infinity by the mental act of abstraction (and mapping onto the real continuum).

S_6 is the standard mathematical realm, is non-constructive, uses two-valued logic to support the Cantorian figments, and uses the loss of identification as an argument that there would be ‘different kinds of infinity’.

For S_5, ‘bijection by abstraction’ is just a term, and might as well be ‘no-bijection by abstraction’ (given the loss of identification). But the didactics of the situation is that (a) Cantor’s proofs have evaporated, (b) we want to grasp the paradox, (c) we want closure of the discussion, without the unsettling “this is too complex for you now”. Having $\mathbb{N} \sim \mathbb{R}$ or that there is only one kind of infinity, allows for simplicity, and creates room for the learning of the other elements in the discussion: (i) the construction of $\mathbb{R}$, (ii) properties such as 1 = 0.999..., (iii) the distinction between counting and measuring, (iv) the notion of bijection and ‘equality of sets’.

A mathematician who has a firm root in S_6 may be offended by S_5. I take the liberty to quote from an email and keep this anonymous (March 2013): “Your proposal is anti-scientific and thus anti-Occam’s-spirit. You want to obscure a distinction that actually is important. Occam says: why a complex explanation when there is a simple one ? You propose to no longer speak about a distinction, but this distinction explains all kinds of issues. You propose to close your eyes, so that you don’t see some phenomena: yes, then you don’t need an explanation ! (....) You change a definition in order to remove an imagined conflict with a metaphysical stance which you *want* to hold about mathematics, but the pay-off for mathematics is zero. It’s obscurantism. Newspeak. Big Brother. Abolish certain words from the dictionary and certain “problems” dissappear because they can’t be stated in words any more. In fact they were not problems at all, they were challenges, and they’ve been surmounted, and this has borne enormous fruit. (....) My impression is strengthened that you are building an elaborate construction in order to be able to tell lies to children. You’re a neo-Pythagorean: mathematical truth has to be bent to conform to your world-view (in this case, your view of the sociology of mathematicians). I just don't see the point. I don't see a problem. I do see a major misapprehension: you seem to think that modern non-constructivist mathematics is an abstract game. There you are wrong. It helps real people to effectively and constructively solve real problems, e.g. in modern statistics, in modern quantum engineering. (...) We don't solve real problems by redefining the concept "countable" so that the real numbers are no longer "uncountable". (...) But who wants to discuss d(C, C) with C = infinity? Only you, as far as I know."

The latter statements may have some weak points: (a) "modern non-constructivist mathematics" still allows "constructively solve real problems" - constructivists will challenge the "constructiveness" of non-constructivist methods, (b) there is a blindness on d(C, C) with C = infinity, which I propose should be lifted, (c) my 'sociology of mathematicians' would be that they confuse their abstract thoughts for reality: but this is not an axiom that I employ here but a result from empirical observation, see also Colignatus (2013).

The key point is that this paper has been written in the context of education. If S_6 teachers want to continue with the nonsense of the transfinites while there is no necessity for it - perhaps since they would be emotionally attached to 'no-bijection by abstraction' - then they are still free to do so, but they ought to inform their students that there is another way to look at it too, namely S_5.

The argument of openness of mind cuts two ways. In my educational ladder, S_5 is followed by S_6, for historical reasons, since it is useful to know what the illusions of Cantor have been, and how most mathematicians followed him. If there would be a key result for the real world that relies on the transfinites, I am interested to hear. Conversely, a teacher of non-constructivist denomination would be required to explain the approach of S_5. Students ought to have a chance to be inocculated against nonsense, instead of being lured into it by fallacies.

7. Conclusion

Apart from the more mundane conclusion that it indeed appears feasible to set up a highschool course on infinity without the need to refer to the transfinites, the following conclusions are possible.

7.1 A summary of the differences

Given the onslaught since 1874 (if not earlier with Zeno's paradoxes) it may be useful to put the different approaches in a table.

Topic	*Cantor*	*ALOE, EWS* and *COTP* (*Occam*)
Logic	two – valued	three – valued
Cantor' s Theorem	Accept	Reject, like with Russell' s paradox
Potential & actual infinity	Commits the fallacy of composition	Proper distinction
Diagonal	Assumption causes rejection	Is not defined in potential form
Mutation rule on diagonal	Creates a new number	Waiting rule
Bijection	Impossible to create	By abstraction
Cardinality	$\mathbb{N} < \mathbb{R}$	$\mathbb{N} \sim \mathbb{R}$

In the latter view the following statements mean precisely the same: (i) the shift in perspective from potential infinity to actual infinity (other than a mere name: thus the continuum), (ii) the imagination of the continuous interval of [0, 1], (iii) regarding this imagination as a constructive act (for geometry), (iv) accepting this to be what we mean by a 'bijection by abstraction' between ℕ and ℝ, (v) the specification in the steps above for the definition of 'bijection by abstraction'.

7.2 Conclusion on the continuum

As holds for evolutionary biology where we tend to forget what 'deep time' is, we may forget for the natural numbers what infinity really means. The googol is 10^100. Let *g*[*n*] = *n*^...^*n* with *n* times ^. For example *g*[2] = 2^(2^2) = 16. Try *g*[googol], or apply *g* a googol times to itself, as in *g*[... *g*[googol]...]. These are just small numbers compared to what is possible.

The unrepenting constructivist (S_3) has a strong position and might actually be right. On the other hand, we have not shown that S_6 is inconsistent, and the inclusion of Cantor's Theorem as a separate axiom might work. There might be theoretical advantages to assume a continuity with a higher cardinality than the set of natural numbers. The main reason to accept the diagonal argument and thus different cardinal numbers for ℕ and ℝ is rather not 'mathematical' but 'philosophical'. Instead of getting entangled in logical knots we might also use Occam's razor and assume the same cardinality. Above considerations on 'bijection by abstraction' would support the latter.

Kronecker's apparant suggestion to use the potential and actual infinities as the demarcation is not convincing. It is rather on how those are applied. The demarcation remains depending upon necessity. Attributed to Occam is the statement now known as his razor: "entia non sunt multiplicanda praeter necessitatem"

7.3 Conclusion on the foundations

A consequence of "A Logic of Exceptions" (ALOE, draft 1981, 2007, 2nd edition 2011) is that it refutes 'the' general proof of Cantor's Theorem (on the power set), so that it only holds for finite sets but not for 'any' set. The diagonal argument on the real numbers can be rejected as well (a new finding in 2011, explained in this paper). There is a 'bijection by abstraction' between ℕ and ℝ. If no contradiction turns up it would become feasible to use the notion of a 'set of all sets' 𝕊, as it would no longer be considered a contradiction that the power set of 𝕊 would be an element and subset of 𝕊 itself.

7.4 Conclusion on constructivism

The specification of the construction steps S_1, ..., S_6 worked well in identifying the various mathematical and philosophical aspects in the various arguments. S_3 would be the potential infinite and S_4 the actual infinite, and the latter would still be constructive but with some abstraction. The two concepts of infinity would be two faces of the same coin. The confusions about S_6, nonconstructivism and the transfinites, derive rather from logic than from infinity.

Appendix A: Rejection of Cantor's original proof

Taken from Hart (2011):

> Wenn eine nach irgendeinerm Gesetze gegebenen unendliche Reihe von einander verschiedener reeller Zahlgrößen
>
> $$\omega_1, \omega_2, \ldots, \omega_\nu, \ldots \qquad (4)$$
>
> vorliegt, so läßt sich in jedem vorgegebenen Intervalle $(\alpha \ldots \beta)$ eine Zahl η (und folglich unendlich viele solcher Zahlen) bestimmen, welche in der Reihe (4) nicht vorkommt; dies sol nun bewiesen werden. *G. Cantor* [1874]

Unfortunately Hart (2011) uses Dutch so we now use the text from Wikipedia March 6 2012 after checking that it fits with Hart (2011):

Next Cantor proves his second theorem: Given any sequence of real numbers x_1, x interval [a, b], one can determine a number in [a, b] that is not contained in the give

To find such a number, Cantor builds two sequences of real numbers as follows: Fi numbers of the given sequence x_1, x_2, x_3, ... that belong to the interior of the interva Designate the smaller of these two numbers by a_1, and the larger by b_1. Similarly, fi numbers of the given sequence belonging to the interior of the interval [a_1, b_1]. Desi by a_2 and the larger by b_2. Continuing this procedure generates a sequence of inte b_2], ... such that each interval in the sequence contains all succeeding intervals. Thi sequence a_1, a_2, a_3, ... is increasing, the sequence b_1, b_2, b_3, ... is decreasing, an the first sequence is smaller than every member of the second sequence.

Cantor now breaks the proof into two cases: Either the number of intervals generat infinite. If finite, let [a_N, b_N] be the last interval. Since at most one x_n can belong to th b_N], any number belonging to the interior besides x_n is not contained in the given se

If the number of intervals is infinite, let $a_\infty = \lim_{n\to\infty} a_n$.[13] At this point, Cantor coul noting that a_∞ is not contained in the given sequence since for every n, a_∞ belongs b_n] but x_n does not.[14]

Instead Cantor analyzes the situation further. He lets $b_\infty = \lim_{n\to\infty} b_n$,[15] and then br two cases: $a_\infty = b_\infty$ and $a_\infty < b_\infty$. In the first case, as mentioned above, a_∞ is not co sequence. In the second case, any real number in [a_∞, b_∞] is not contained in the gi Cantor observes that the sequence of real algebraic numbers falls into the first cas how his proof handles this particular sequence.[16]

CCPO - WIP (2011) also uses the term ‘bijection in the limit’ as equivalent to the term ‘bijection by abstraction’. This allows for a better link up to above arguments that uses the term ‘limit’ as well. But we can proceed with what we have available now.

Let us now redo this method of proof using the $\mathbb{R}[1]$, ... , $\mathbb{R}[d]$. As said the numbers are ranked up to 10^d. For clarity we can take the news $D[d] = \mathbb{R}[d] \setminus \mathbb{R}[d\text{-}1]$, and then rank the digits as $X[d] = D[1] \cup D[2] \cup ... \cup D[d] = \{x_0, x_1, ..., x_{10^d}\}$, where the union maintains order. Taking the interval from [a, b] generates [$a[d]$, $b[d]$]. For example, if we start on [0, 1] then [$a[1]$, $b[1]$] = [0.1, 0.2], then [0.11, 0.12], [0.111, 0.112] and so on. (Rather nicely we might think of the limit value of 1/9.)

We now take $\mathbb{R}[d]$ @ $\mathbb{R}$. Subsequently also $X[d]$ @ X. Clearly X is only a permutation of $\mathbb{R}$, and all numbers are represented. Let us denoted the final interval as $[\alpha, \beta]$.

The suggestion that there is an $\eta \in [\alpha, \beta]$ but $\eta \notin \mathbb{R}$ is erroneous since we see that all elements of $\mathbb{R}$ are represented in X.

Thus there is something crooked in this method of proof. Note that there is no finite number to find the final interval. Note that taking the interior of $[\alpha, \beta]$ is impossible if $\alpha = \beta$. Taking the interior of [$a[d]$, $b[d]$] is quite possible since the numbers are defined such that $a[d] \neq b[d]$. But the notion of an ‘interior’ apparently loses ‘grip’ when we take the step of abstraction.

(Regard for example the series with limits α = 0.9999... and β = 1.000.... It is common to conclude that $\alpha = \beta$ so that there is no η inbetween. Notably, with 1 / 3 = 0.3333... we want 1 = 3 * 1/ 3 = 3 * 0.3333...)

This completes the rejection of Cantor's original proof.

Discussion: Cantor assumes that he can define the various notions on limit and $\mathbb{R}$ independently, but they get only meaning in their mutual dependence, and then must be constructed in a dependent manner.

His proof *seems* to work since he assumes that $\mathbb{R}$ is built up in the manner of a potential infinite, but the proof uses that all elements have an actual infinity of digits. Instead, the proof can only use numbers up to a certain digital depth *d,* and create the interval only alongside the construction of $\mathbb{R}$ itself. The notion of an interior uses a distance measure that relies on actual infinites, and this apparently also conflicts with the construction of $\mathbb{R}$ from $\mathbb{R}[d]$.

Appendix B helps our understanding of this issue by looking at $\sqrt{2}$. The definition of $\sqrt{2}$ doesn't depend upon the construction of $\mathbb{R}$, while the definition of the diagonal does, and the notion of an interior apparently does too.

Appendix B: Comparison to the irrational numbers

The reasoning in the main body of the paper finds a parallel in the discussion on the irrational numbers, for example the square root of 2. Rather than calling such a number 'irrational' it is conceivable to say that it is 'rational by abstraction'. In this case, however, this is merely a play of words. In the case of 'bijection by abstraction' there is also a shift of perspection because this allows us to regard $\mathbb{N}$ and $\mathbb{R}$ as equally large and only ordered differently. Let us discuss this issue in more detail.

Let us first copy the ancient proof ascribed to Hippasus that $\sqrt{2}$ is irrational, i.e. cannot be expressed as a ratio of two integers. Thus there are no integer numerator n and denominator d such that $\sqrt{2} = n / d$. Take an isosceles right triangle, with sides 1, then the hypothenuse is $\sqrt{1 + 1} = \sqrt{2}$ indeed, so we indeed have such a length. Assume that n and d exist. Regard these as the simplest possible, e.g. 2 / 10 reduces to 1 / 5. Thus n and d cannot both be even numbers. Squaring gives $n^2 = 2\,d^2$ or n^2 is an even number. Note that the square of an uneven number will always be uneven again. Thus if n^2 is even, it follows that n is even, and hence d is uneven. If n is even then we get a new integer number $m = n / 2$. Hence, $n^2 = 2\,d^2$ gives $4\,m^2 = 2\,d^2$ or $2\,m^2 = d^2$ from which it follows that d^2 is even. From this it follows that d cannot be uneven. But we had already derived that d is uneven. Contradiction. Hence, there are no such numbers n and d such that $\sqrt{2} = n / d$.

Our $\mathbb{R}$ concerned the interval [0, 1] and hence we now consider $1/\sqrt{2}$. If $1/\sqrt{2}$ is regarded as a process towards a numerical value then it belongs to S_3 and if it is understood as a completed number then it belongs to S_4. For $\mathbb{R}[d]$ we can find a best approximation. Since @ has been used for sets, it may be wise to use @@ for numbers. Then:

(1) $x[d] = num[d] / den[d] \approx 1/\sqrt{2}$

(2) $(\mathbb{N}[d]$ @ $\mathbb{N}) \Rightarrow (x[d]$ @@ $1/\sqrt{2})$

(3) then $1/\sqrt{2}$ might be labelled as 'rational by abstraction'.

The phrase 'rational by abstraction' only is a change of words from 'irrational number', for there is no change in perspective. Changing the words does not add to anything. We still need to specify the numerator and denominator in the steps, and develop notions of convergence, for which Weierstraß is excellent.

Above proof that $\sqrt{2}$ is irrational uses that it is a completed number. If it is only in the process of being constructed then the proof collapses, since we cannot use yet that the outcome of squaring is 2 (because of the approximation). Thus, there is scope for a fallacy of composition, w.r.t. being completed or in construction. In S_3 we would have an argument in each $\mathbb{R}[d]$. Assume that there are numerator and denominator $num[d]$ / $den[d] = 1/\sqrt{2}$ (with a factor 10^d cancelling), etcetera, and deduce that it is 'irrational[d]', also with the meaning that it would not be present in the list (since we haven't made the step towards completed $\mathbb{R}$). In S_4 we construct the whole $\mathbb{R}$ and then it is present. It can still be called 'irrational'. There is no advantage in labelling it as 'rational by abstraction'. The term 'irrational' is somewhat quaint, in comparison to 'irrational people', but historically useful, because of the conceptual linkages of 'proportion, ratio, logos, calculation, reasoning'.

Thus we find some parallels in this issue on the S_3 / S_4 frontier with respect to $\sqrt{2}$ on 'rational by abstraction', on one hand, and the other issue on the S_5 / S_6 frontier on 'bijection by abstraction', on the other hand. The parallel is not only a phrase 'by abstraction' but also a scope for a fallacy of composition. The difference is however that first issue does not concern a change of perspective, so that it merely amounts to a different label for the same situation, without illumination, while the second issue does concern a change of perspective. It makes a difference to be able to hold that $\mathbb{N}$ and $\mathbb{R}$ are equally large and only ordered differently. The key consideration is of course that Cantor's proofs have collapsed, so that, bearing other proofs, it becomes a philosophical issue to regard $\mathbb{N}$ and $\mathbb{R}$ as different in cardinality.

The notion of 'completion' shows this choice with respect to the construction of the real numbers and their properties. Conventional reasoning is: (i) first construct $\mathbb{R}$, (ii) consider $\sqrt{2}$ as a completed number, (iii) then consider limiting processes around $\sqrt{2}$ and within $\mathbb{R}$ and its completed numbers. Alternatively, we can imagine a limiting process that occurs simultaneously while $\mathbb{R}$ is constructed (as is suggested as the proper approach in Appendix A). Alternatively, one might argue that it shouldn't matter (but then consider Appendix A).The distinction may lie in the point that the definition of $\sqrt{2}$ doesn't depend upon the construction of $\mathbb{R}$, while the definition of the diagonal does.

Literature

Colignatus is the name of Thomas Cool in science. He is an econometrician and teacher of mathematics in Scheveningen, Holland. See http://thomascool.eu.

Aczel, A.D. (2000), “The mystery of the Aleph. Mathematics, the Kabbalah, and the Search for Infinity”, Four Walls Eight Windows

Bochenski, I.M. (1956, 1970), “A history of formal logic”, 2nd edition, Chelsea

Bendegem, J.P. van (2012), “A Defense of Strict Finitism”, Constructivist Foundations, Vol 7, No 2, March, http://www.univie.ac.at/constructivism/journal/7/2/141.vanbendegem.pdf

Bradley, M. (2012), “Review ‘A Logic of Exceptions’”, European Mathematical Society website, http://www.euro-math-soc.eu/node/2494

Brady, R. and P. Rush (2008), “What is Wrong with Cantor’s Diagonal Argument?”, Logique et Analyse, issue 202:185-219

Cariani, P. (2012), “Infinity and the Observer: Radical Constructivism and the Foundations of Mathematics”, Constructivist Foundations, Vol 7, No 2, March, http://www.univie.ac.at/constructivism/journal/7/2/116.cariani.pdf

Colignatus (1981, 2007, 2011), “A logic of exceptions”, 2nd edition, Thomas Cool Consultancy & Econometrics, http://thomascool.eu/Papers/ALOE/Index.html

Colignatus (2009), “Elegance with Substance”, Dutch University Press, http://thomascool.eu/Papers/Math/Index.html

Colignatus (2011), “Conquest of the Plane”, 1st edition, Thomas Cool Consultancy & Econometrics, http://thomascool.eu/Papers/COTP/Index.html

Colignatus (2011a), “Definition & Reality in the General Theory of Political Economy”, 3rd edition, Thomas Cool Consultancy & Econometrics, http://thomascool.eu/Papers/Drgtpe/Index.html

Colignatus (2011b, 2012), “Contra Cantor Pro Occam - Work in Progress”, http://thomascool.eu/Papers/ALOE/2011-08-07-ContraCantorProOccam.pdf

Colignatus (2011e), “Reading Notes on “Conquest of the Plane” and “Contra Cantor Pro Occam””, http://thomascool.eu/Papers/COTP/2011-08-23-ReadingNotesForCOTP.pdf

Colignatus (2011f), “Neoclassical mathematics for the schools”, http://thomascool.eu/Papers/Math/2011-09-06-NeoclassicalMathematics.pdf

Colignatus (2011g), “Brain research and mathematics education: some comments”, http://thomascool.eu/Papers/Math/2011-07-11-COTP-Damasio.pdf

Colignatus (2013), “What a mathematician might wish to know about my work”, http://thomascool.eu/Papers/Math/2013-03-26-WAMMWTKAMW.pdf

Crowe, M. J. (1988), “Ten Misconceptions about Mathematics and Its History” in William Aspray and Philip Kitcher, “History and philosophy of modern mathematics. Volume XI: Minnesota Studies in the Philosophy of Science”, www.mcps.umn.edu/philosophy/11_11Crowe.pdf

Davis, P.J, and R. Hersh (1980, 1983), “The mathematical experience”, Penguin

edition
DeLong, H. (1971), “A profile of mathematical logic”, Addison Wesley
Dijksterhuis, E.J. (1990), “Clio’s stiefkind. Een keuze uit zijn werk door K. van Berkel”, Bert Bakker
Edwards, H. (1988), “Kronecker’s place in history”, in William Aspray and Philip Kitcher, eds. “History and philosophy of modern mathematics”, p139–144, Minnesota Stud. Philos. Sci., XI, Univ. Minnesota Press, http://www.mcps.umn.edu/assets/pdf/11_ModernMath.pdf
Edwards, H. (2008), “Kronecker’s Algorithmic Mathematics”, Presented at “Computability in Europe 2008,” Athens, June 19, http://math.nyu.edu/faculty/edwardsd/talks.htm
Gill, R.D. (2008), “Book review. Thomas Colignatus. A Logic of Exceptions: Using the Economics Pack Applications of *Mathematica* for Elementary Logic”, NAW 5/9 nr. 3 sept., http://www.nieuwarchief.nl/serie5/pdf/naw5-2008-09-3-217.pdf
Gill, R.D. (2012), “Book reviews. Thomas Colignatus. (1) Elegance with Substance, (2) Conquest of the Plane”, Nieuw Archief voor Wiskunde, March, p66-68, http://www.nieuwarchief.nl, http://www.math.leidenuniv.nl/~gill/reviewCOTP.html
Gamboa, J. M. (2011), “Review ‘Conquest of the Plane’”, European Mathematical Society website, http://www.euro-math-soc.eu/node/2081
Gowers, T. (2003), “What is so wrong with thinking of real numbers as infinite decimals?”, http://www.dpmms.cam.ac.uk/~wtg10/decimals.html
Hart, K.P. (2011), “Verzamelingenleer”, http://dutiaw37.twi.tudelft.nl/~kp/onderwijs/verzamelingenleer/dictaat/dictaat-A5.pdf
Hodges, W. (1998), “An editor recalls some hopeless papers”, Bulletin of Symbolic Logic 4 (1):1-16
Hiele, P.M. van (1973), “Begrip en Inzicht”, Muusses.
Kauffman, L.H. (2012), “The russell operator”, Constructivist Foundations, Vol 7, No 2, March, http://www.univie.ac.at/constructivism/journal/7/2/112.kauffman.pdf
Kerkhove, B. van & J.P. van Bendegem, “The Many Faces of Mathematical Constructivism”, Constructivist Foundations, Vol 7, No 2, March, http://www.univie.ac.at/constructivism/journal/7/2/097.vankerkhove.pdf
Kuiper, J.J.C. (2004), “Ideas and Explorations : Brouwer’s Road to Intuitionism”, Thesis University of Utrecht, http://igitur-archive.library.uu.nl/dissertations/2004-0303-084333/inhoud.htm
Riegler, A. (2005), “Editorial. The Constructivist Challenge”, Constructivist Foundations, vol 1, no 1, p1-8, http://www.univie.ac.at/constructivism/journal/coretexts/riegler2005editorial.pdf
Quale, A. (2012), “On the Role of Constructivism in Mathematical Epistemology”, Constructivist Foundations, Vol 7, No 2, March, http://www.univie.ac.at/constructivism/journal/7/2/104.quale.pdf
Wallace, D.F. (2003), “Everything and more. A compact history of ∞”, Norton

Wikipedia (2011), “http://en.wikipedia.org/wiki/Cantor’s_theorem” and “http://en.wikipedia.org/wiki/Controversy_over_Cantor's_theory” Retrieved from source on August 6 2011

Wikipedia (2012), “http://en.wikipedia.org/wiki/Cantor’s_first_uncountability_proof” Retrieved from source on March 6 2012