Quaternions and Hilbert spaces

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Abstract

This is a compilation of quaternionic number systems, quaternionic function theory, quaternionic Hilbert spaces and Gelfand triples.

The difference between quaternionic differential calculus and Maxwell based differential calculus is explained.

Contents

1 Introduction

Hilbert spaces are a special kind of vector space. What makes this space special is the fact that its set of closed subspaces has the same relational structure as quantum logic has.

It is not generally known that separable Hilbert spaces can only handle number systems that form division rings. This was inescapably proven by Maria Pia Solèr.

Only three suitable division rings exist: the real numbers, the complex numbers and the quaternions. The first two are contained in the last one. Thus the most elaborate separable Hilbert space is a quaternionic Hilbert space.

See: “Division algebras and quantum theory” by John Baez.http://arxiv.org/abs/1101.5690

According to my experience hardly any scientist knows that quaternionic number systems, and continuous quaternionic functions exist in 16 versions that only differ in their discrete symmetry.

Also most scientist do not notice what separable stands for. It means that eigenspaces of operators can only contain a countable number of eigenvalues. For example operators whose eigenspaces contain all rational numbers may exist, but operators whose eigenspaces contain all (or a closed set of) real numbers can only exist in a non-separable Hilbert space, such as a Gelfand triple.

By the way, each infinite dimensional separable Hilbert space owns a Gelfand triple.

Great resemblance exists between Maxwell-like equations and quaternionic differential equations. However, also significant differences exist. This paper indicates what these differences are.

2 Quaternion geometry and arithmetic

Quaternions and quaternionic functions offer the advantage of a very compact notation of items that belong together.

Quaternions can be considered as the combination of a real scalar and a 3D vector that has real coefficients. This vector forms the imaginary part of the quaternion. Quaternionic number systems are division rings. Other division rings are real numbers and complex numbers. Octonions do not form a division ring.

Bi-quaternions exist whose parts exist of a complex scalar and a 3D vector that has complex coefficients. Bi-quaternions do not form division rings. This paper does not use them.

2.1 Notation

We indicate the real part of quaternion a by the suffix a_0.

We indicate the imaginary part of quaternion a by bold face $\boldsymbol{a}$.

$$a = a_0 + \boldsymbol{a} \tag{1}$$

$$a^* = a_0 - \boldsymbol{a} \tag{2}$$

2.2 Sum

$$c = c_0 + \boldsymbol{c} = a + b \tag{1}$$

$$c_0 = a_0 + b_0 \tag{2}$$

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b} \tag{3}$$

2.3 Product

$$f = f_0 + \boldsymbol{f} = d\ e \tag{1}$$

$$f_0 = d_0 e_0 - \langle \boldsymbol{d}, \boldsymbol{e} \rangle \tag{2}$$

$$\boldsymbol{f} = d_0 \boldsymbol{e} + e_0 \boldsymbol{d} \pm \boldsymbol{d} \times \boldsymbol{e} \tag{3}$$

The $\pm$ sign indicates the influence of right or left handedness of the number system[1].

$\langle \boldsymbol{d}, \boldsymbol{e} \rangle$ is the inner product of $\boldsymbol{d}$ and $\boldsymbol{e}$.

$\boldsymbol{d} \times \boldsymbol{e}$ is the outer product of $\boldsymbol{d}$ and $\boldsymbol{e}$.

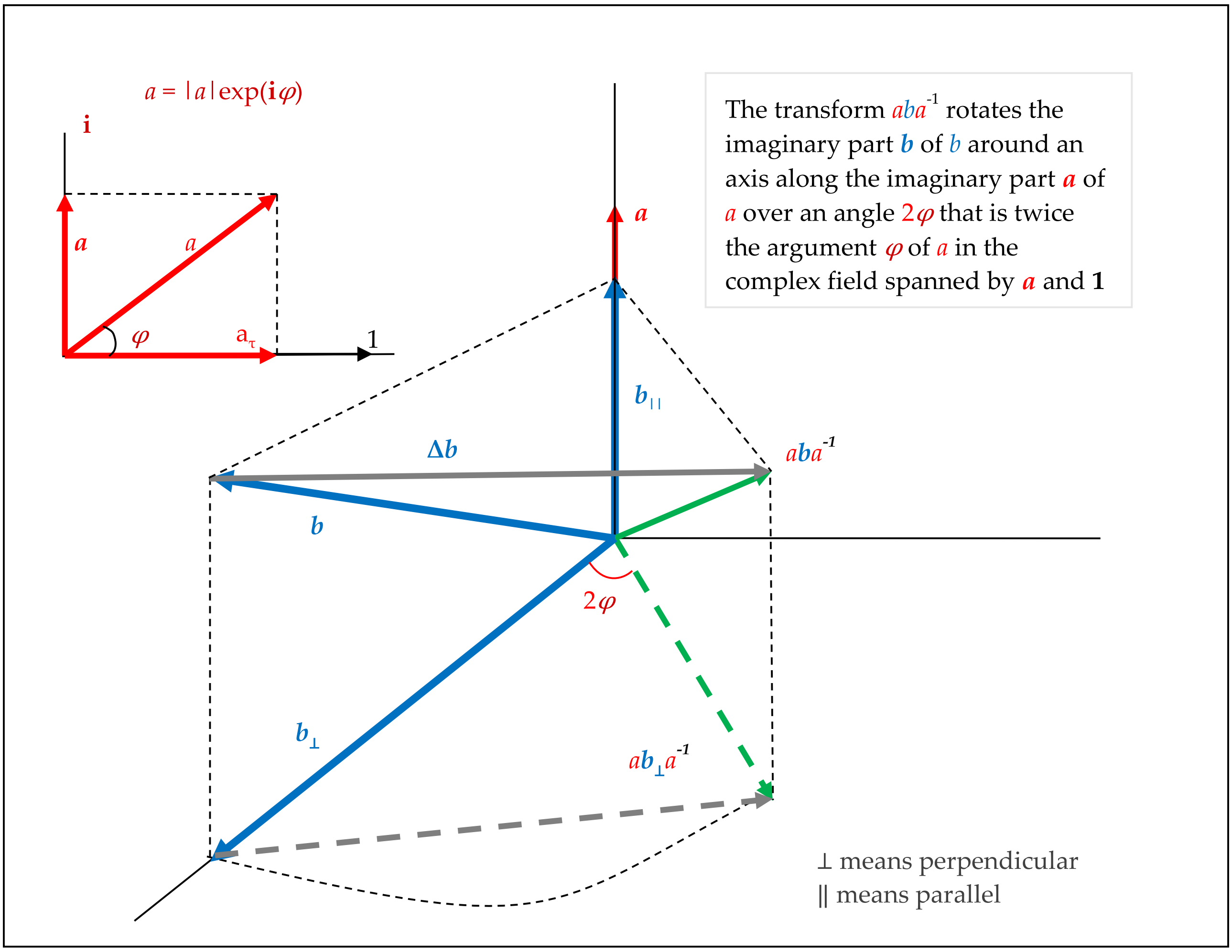
2.4 Norm

$$|a| = \sqrt{a_0 a_0 + \langle \boldsymbol{a}, \boldsymbol{a} \rangle} = \sqrt{a a^*} \tag{1}$$

2.5 Quaternionic rotation

In multiplication quaternions do not commute. Thus, in general $a\, b/a \neq b$. In this multiplication the imaginary part of b that is perpendicular to the imaginary part of a is rotated over an angle φ that is twice the complex phase of a.

[1] Quaternionic number systems exist in 16 symmetry flavors. Within a coherent set all elements belong to the same symmetry flavor.



This means that if $\varphi = \pi/4$, then the rotation $c = a\, b/a$ shifts $\boldsymbol{b}_\perp$ to another dimension. This fact puts quaternions that feature the same size of the real part as the size of the imaginary part is in a special category. They can switch states of tri-state systems.

2.6 Split

It is possible to split quaternions.

$$q = q_+ + q_- \tag{1}$$

$$q_\pm = \frac{q \pm \boldsymbol{i}q\boldsymbol{j}}{2} \tag{2}$$

For quaternionic functions:

$$f(x_+, x_-) = f_+(x_+, x_-) + f_-(x_+, x_-) \tag{1}$$

$$f_{\pm}(x_+, x_-) = \frac{f(x_+, x_-) \pm \boldsymbol{i}\, f(x_+, x_-)\, \boldsymbol{j}}{2} \tag{2}$$

3 Quaternionic Fourier transform

We use the quaternion split in the definition of the quaternionic Fourier transform.

$$\tilde{f}_{\pm}(\tilde{u}, \tilde{v}) = \int \exp\big(-\boldsymbol{i}\,(x\tilde{u} \pm y\tilde{v})\big)\, f_{\pm}(x, y)\, \mathrm{d}x\, dy \;= \int f_{\pm}(x) \exp\big(-\boldsymbol{j}\,(x\tilde{u} \pm y\tilde{v})\big)\, dx$$

$$\tilde{f}_{+}(\tilde{u}, \tilde{v}) = \int \exp\big(-\boldsymbol{i}\,(x\tilde{u} + y\tilde{v})\big)\, f_{+}(x, y)\, \mathrm{d}x\, dy \;=$$

$$½ \int \exp\big(-\boldsymbol{i}\,(x\tilde{u} + y\tilde{v})\big)\, f(x, y)\, \mathrm{d}x\, dy$$

$$+½ \int \exp\big(-\boldsymbol{i}\,(x\tilde{u} + y\tilde{v})\big)\, \boldsymbol{i}\, f(x, y) \boldsymbol{j}\, \mathrm{d}x\, dy$$

$$+$$

4 The separable Hilbert space

We will specify the characteristics of a generalized quaternionic infinite dimensional separable Hilbert space $\mathfrak{H}$. The adjective "quaternionic" indicates that the inner products of vectors and the eigenvalues of operators are taken from the number system of the quaternions. Separable Hilbert spaces can be using real numbers, complex numbers or quaternions. These three number systems are division rings. In fact the quaternionic number system comprises all division rings.

4.1 Notations and naming conventions

$\{f_x\}_x$ means ordered set of f_x . It is a way to define discrete functions.

The use of bras and kets differs slightly from the way Dirac uses them.

$|f\rangle$ is a ket vector.

$\langle f|$ is a bra vector.

A is an operator.

$A^{\dagger}$ is the adjoint operator of operator A.

| on its own, is a nil operator.

We will use capitals for operators and lower case Greek characters for quaternions and eigenvalues. We use Latin characters for ket vectors, bra vectors and eigenvectors. Imaginary and anti-Hermitian objects will be indicated in **bold** text. Real numbers get subscript $_0$.

Due to the non-commutative product of quaternions, special care must be paid to the ordering of factors inside products. In this paper a particular ordering is selected. It is one out of a lager set of possibilities.

4.2 Quaternionic Hilbert space

The Hilbert space $\mathfrak{H}$ is a **linear space**. That means for the elements $|f\rangle$, $|g\rangle$ and $|h\rangle$ of $\mathfrak{H}$ and quaternionic numbers α and β a linear space is defined. $|f\rangle$, $|g\rangle$ and $|h\rangle$ are ket vectors.

4.2.1 Ket vectors

For **ket** vectors hold

$$|f\rangle + |g\rangle = |g\rangle + |f\rangle = |g + f\rangle \tag{1}$$

$$(|f\rangle + |g\rangle) + |h\rangle = |f\rangle + (|g\rangle + |h\rangle) \tag{2}$$

$$|\alpha f\rangle = |f\rangle \alpha\,; \;\; |f\rangle = |\alpha f\rangle \alpha^{-1} \tag{3}$$

$$|(\alpha + \beta) f\rangle = |f\rangle \alpha + |f\rangle \beta \tag{4}$$

$$(|f\rangle + |g\rangle) \alpha = |f\rangle \alpha + |g\rangle \alpha \tag{5}$$

$$|f\rangle 0 = |0\rangle \tag{6}$$

$$|f\rangle 1 = |f\rangle \tag{7}$$

4.2.2 Bra vectors

The **bra** vectors form the dual Hilbert space $\mathfrak{H}^\dagger$ of $\mathfrak{H}$.

$$\langle f| + \langle g| = \langle g| + \langle f| = \langle f + g| \tag{1}$$

$$(\langle f| + \langle g|) + \langle h| = \langle f| + (\langle g| + \langle h|) \tag{2}$$

$$\langle \alpha f| = \alpha^* \langle f|\,; \;\; \langle f| = (\alpha^*)^{-1} \langle \alpha f| \tag{3}$$

$$\langle f\,(\alpha + \beta)| = \alpha^* \langle f| + \beta^* \langle f| \tag{4}$$

Notice the quaternionic conjugation that affects the coefficients of bra vectors.

$$(\langle f| + \langle g|)\alpha = \langle f|\, \alpha + \langle g|\, \alpha \tag{5}$$

$$0\, \langle f| = \langle 0| \tag{6}$$

$$1\, \langle f| = \langle f| \tag{7}$$

4.2.3 Scalar product

The scalar product couples Hilbert space $\mathfrak{H}^\dagger$ to its dual $\mathfrak{H}$.

$$\langle f|g\rangle = \langle g|f\rangle^* \tag{1}$$

$$\langle f + g|h\rangle = \langle f|h\rangle + \langle g|h\rangle \tag{2}$$

$$\langle \alpha f|g\rangle = \alpha^* \langle f|g\rangle = \alpha^* \langle g|f\rangle^* = \langle g|\alpha f\rangle^* \tag{5}$$

$$\langle f|\alpha g\rangle = \langle f|g\rangle\, \alpha = \langle g|f\rangle^*\, \alpha = \langle \alpha g|f\rangle^* \tag{6}$$

$\langle f|$ is a bra vector. $|g\rangle$ is a ket vector. α is a quaternion. $\langle f|g\rangle$ is quaternion valued.

If the Hilbert space represents both dual spaces, then the scalar product is also called an ***inner product***.

4.2.4 Separable

In mathematics a topological space is called separable if it contains a countable dense subset; that is, there exists a sequence $\{|x_n\rangle\}_{n=1}^{\infty}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence.

Every continuous function on the separable space $\mathfrak{H}$ is determined by its values on this countable dense subset.

4.2.5 Base vectors

The Hilbert space $\mathfrak{H}$ is **separable**. That means that a countable row of elements $\{|f_n\rangle\}$ exists that **spans** the whole space.

If $\langle f_n|f_m\rangle = \delta(m,n) = [1 \text{ when } n = m;\ 0 \text{ otherwise}]$ then $\{|f_n\rangle\}$ forms an **orthonormal base** of the Hilbert space.

A ket base $\{|k\rangle\}$ of $\mathfrak{H}$ is a minimal set of ket vectors $|k\rangle$ that together span the Hilbert space $\mathfrak{H}$.

Any ket vector $|f\rangle$ in $\mathfrak{H}$ can be written as a linear combination of elements of $\{|k\rangle\}$.

$$|f\rangle = \sum_k (|k\rangle \langle k|f\rangle) \tag{1}$$

A bra base $\{\langle b|\}$ of $\mathfrak{H}^\dagger$ is a minimal set of bra vectors $\langle b|$ that together span the Hilbert space $\mathfrak{H}^\dagger$.

Any bra vector $\langle f|$ in $\mathfrak{H}^\dagger$ can be written as a linear combination of elements of $\{\langle b|\}$.

$$\langle f| = \sum_k (\langle k|f\rangle \langle b|) \tag{2}$$

Usually base vectors are taken such that their norm equals 1. Such a base is called an orthonormal base.

4.2.6 Operators

Operators act on a subset of the elements of the Hilbert space.

4.2.6.1 Linear operators

An operator Q is linear when for all vectors $|f\rangle$ and $|g\rangle$ for which Q is defined and for all quaternionic numbers α and β:

$$\begin{aligned} |Q\ \alpha\ f\rangle + |Q\ \beta\ g\rangle = |Q\ f\rangle\alpha + |Q\ g\rangle\,\beta = \\ Q(|\alpha\ f\rangle + |\beta\ g\rangle) = Q(|f\rangle\alpha + |g\rangle\beta) \end{aligned} \tag{1}$$

Operator B is **colinear** when for all vectors $|f\rangle$ for which B is defined and for all quaternionic numbers α there exists a quaternionic number γ such that:

$$|\alpha\ B\ f\rangle = |B\ f\rangle\,\gamma\alpha\gamma^{-1} \equiv |B\ \gamma\alpha\gamma^{-1}\ f\rangle \tag{2}$$

If $|f\rangle$ is an eigenvector of operator A with quaternionic eigenvalue a,

$$A|f\rangle = \ |f\rangle a$$

then $|b\ f\rangle$ is an eigenvector of A with quaternionic eigenvalue $b^{-1}a\ b$.

$$A|b\ f\rangle = |A\ b\ f\rangle = \ |A\ f\rangle\ b = |f\rangle a\ b = \ |b\ f\rangle\ b^{-1}a\ b$$

$A^{\dagger}$ is the **adjoint** of the **normal** operator A.

$$\langle f\ |A\ g\rangle = \ \langle f\ A^{\dagger}|g\rangle = \langle g\ |A^{\dagger}\ f\rangle^{*} \tag{4}$$

$$A^{\dagger\,\dagger}\ = \ A \tag{5}$$

$$(A + B)^{\dagger}\ \ = \ B^{\dagger} + A^{\dagger} \tag{6}$$

$$(A \cdot B)^{\dagger}\ \ = \ B^{\dagger}A^{\dagger} \tag{7}$$

If $A = A^{\dagger}$, then A is a **self adjoint** operator.

$|$ is a nil operator.

4.2.6.2 Operator construction

The construct $|f\rangle\langle g|$ acts as a linear operator. $|g\rangle\langle f|$ is its adjoint operator.

The using an orthonormal base $\{|q_i\rangle\}$ that belong to quaternionic eigenvalues $\{q_i\}$ and a quaternionic function $f(q)$ a linear operator F can be defined such that for all vectors $|g\rangle$ and $|h\rangle$ holds:

$$\langle g|F\ h\rangle = \sum_i \{\langle g|q_i\rangle f(q_i)\langle q_i|h\rangle\} \tag{7}$$

$$F \equiv \sum_i \{|q_i\rangle f(q_i)\langle q_i|\} \tag{8}$$

For the orthonormal base $\{|q_i\rangle\}$ holds:

$$\langle q_j|q_k\rangle = \delta_{jk} \tag{9}$$

We will use

$$F \equiv |q_i\rangle f(q_i)\langle q_i| \tag{10}$$

as a shorthand for equations (7) and (8).

$$F^\dagger \equiv |q_i\rangle f(q_i)^*\langle q_i| \tag{11}$$

$$|q_i\rangle f(q_i)\langle q_i| = |q_i\, f(q_i)\rangle\langle q_i| = |q_i\rangle\langle f(q_i)^*\, q_i| \tag{12}$$

The eigenspace of reference operator $\mathcal{R}$ defined by

$$\mathcal{R} \equiv \sum_i \{|q_i\rangle q_i \langle q_i|\} \tag{13}$$

represents the countable parameter space of discrete function $f(q_i)$.

F and $\mathcal{R}$ are constructed operators.

If collection $\{q_i\}$ covers all rational members of a quaternionic number system then this definition specifies a reference operator for which the eigenspace represents the parameter space of all discrete functions that can be defined with this number system.

Quaternionic number systems exist in several versions that only differ in the way that the elements are ordered. We will identify these different versions with special superscripts. When relevant, this will also be done with the number systems, with the operators, with the eigenvectors and with the eigenvalues.

$$\mathcal{R}^{⓪} \equiv \sum_i \left\{|q_i^{⓪}\rangle\, q_i^{⓪}\, \langle q_i^{⓪}|\right\} \tag{14}$$

$\mathcal{R}^{⓪}$ is a member of a set of reference operators $\{\mathcal{R}^x\}$. The superscript ^{x} specifies the symmetry flavor of the number system $\{q^x\}$.

The superscript ^{x} can be ⓪, ①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩, ⑪, ⑫, ⑬, ⑭, or ⑮.

Often, we will use the same letter for identifying eigenvectors, eigenvalues and the corresponding operator.

Definition 8 specifies a normal operator. The set of eigenvectors of a normal operator form an orthonormal base of the Hilbert space.

A self adjoint operator has real numbers as eigenvalues.

4.2.6.3 Normal operators

The most common definition of continuous operators is:

A **continuous** operator is an operator that creates images such that the inverse images of open sets are open.

Similarly, a **continuous** operator creates images such that the inverse images of closed sets are closed.

If $|a\rangle$ is an eigenvector of normal operator A with eigenvalue a then

$$\langle a|A|a\rangle = \langle a|a|a\rangle = \langle a|a\rangle\, a \qquad (1)$$

indicates that the eigenvalues are taken from the same number system as the inner products.

A normal operator is a continuous linear operator.

A normal operator in $\mathfrak{H}$ creates an image of $\mathfrak{H}$ onto $\mathfrak{H}$**.** It transfers closed subspaces of $\mathfrak{H}$ into closed subspaces of $\mathfrak{H}$.

The normal operators N have the following property.

$$N: \mathfrak{H} \Rightarrow \mathfrak{H} \qquad (2)$$

Thus the normal operator N maps separable Hilbert space $\mathfrak{H}$ onto itself.

N commutes with its **(Hermitian) adjoint** $N^\dagger$:

$$NN^\dagger = N^\dagger N \qquad (2)$$

Normal operators are important because the spectral theorem holds for them.

Examples of normal operators are

- **unitary** operators: $U^\dagger = U^{-1}$, unitary operators are bounded;
- **Hermitian** operators (i.e., self-adjoint operators): $N^\dagger = N$;
- **Anti-Hermitian** or anti-self-adjoint operators: $N^\dagger = -N$;
- **Anti-unitary** operators: $U^\dagger = -U^{-1}$, anti-unitary operators are bounded;

- **positive operators**: $N = MM^\dagger$
- **orthogonal projection** operators: $P^\dagger = P = P^2$.

For normal operators hold:

$$AB = A_0B_0 - \langle \boldsymbol{A}, \boldsymbol{B} \rangle + A_0\boldsymbol{B} + \boldsymbol{A}B_0 \pm \boldsymbol{A} \times \boldsymbol{B} \qquad (3)$$

$$N_0 = ½(N+N^\dagger) \qquad (4)$$

$$\boldsymbol{N} = ½(N-N^\dagger) \qquad (5)$$

$$NN^\dagger = N_0N_0 + \langle \boldsymbol{N}, \boldsymbol{N} \rangle = N_0^2 - \boldsymbol{N}^2 \qquad (6)$$

4.2.6.4 Spectral theorem

For every compact self-adjoint operator T on a real, complex or quaternionic Hilbert space $\mathfrak{H}$, there exists an orthonormal basis of $\mathfrak{H}$ consisting of eigenvectors of T. More specifically, the orthogonal complement of the kernel (null space) of T admits, either a finite orthonormal basis of eigenvectors of T, or a countable infinite orthonormal basis of eigenvectors of T, with corresponding eigenvalues $\{\lambda_n\} \subset \mathbb{R}$, such that $\lambda_n \to 0$. Due to the fact that $\mathfrak{H}$ is separable the set of eigenvectors of T can be extended with a base of the kernel in order to form a complete orthonormal base of $\mathfrak{H}$.

If T is compact on an infinite dimensional Hilbert space $\mathfrak{H}$, then T is not invertible, hence $\sigma(T)$, the spectrum of T, always contains 0. The spectral theorem shows that $\sigma(T)$ consists of the eigenvalues $\{\lambda_n\}$ of T, and of 0 (if 0 is not already an eigenvalue). The set $\sigma(T)$ is a compact subset of the real line, and the eigenvalues are dense in $\sigma(T)$.

A normal operator has a set of eigenvectors that spans the whole Hilbert space $\mathfrak{H}$.

In quaternionic Hilbert space a normal operator has quaternions as eigenvalues.

The set of eigenvalues of a normal operator is NOT compact. This is due to the fact that $\mathfrak{H}$ is separable. Therefore the set of eigenvectors is countable. As a consequence the set of eigenvalues is countable. Further, in general the eigenspace of normal operators has no finite diameter.

A continuous bounded linear operator on $\mathfrak{H}$ has a compact eigenspace. The set of eigenvalues has a closure and it has a finite diameter.

4.2.6.5 Eigenspace

The set of eigenvalues $\{q\}$ of the operator Q form the eigenspace of Q.

4.2.6.6 Eigenvectors and eigenvalues

For the eigenvector $|q\rangle$ of normal operator Q holds

$$|Q\, q\rangle = |q\, q\rangle = |q\rangle q \qquad (1)$$

$$\langle q\, Q^{\dagger}| = \langle q\, q| = q^{*}\langle q| \tag{2}$$

$$\forall_{|f\rangle \in \mathfrak{H}} \left[\{\langle f|Q\, q\rangle\}_q = \{\langle f|q\rangle q\}_q = \left\{\langle q\, Q^{\dagger}|f\rangle^{*}\right\}_q = \{(q^{*}\langle q|f\rangle)^{*}\}_q \right] \tag{3}$$

The eigenvalues of 2^n-on normal operator are 2^n-ons. For Hilbert spaces the eigenvalues are restricted to elements of a division ring.

$$Q = \sum_{j=0}^{n-1} \mathrm{I}_j Q_i \tag{4}$$

The Q_j are self-adjoint operators.

4.2.6.7 Unitary operators

For unitary operators holds:

$$U^{\dagger} = U^{-1} \tag{1}$$

Thus

$$U U^{\dagger} = U^{\dagger} U = I \tag{2}$$

Suppose $U = I + C$ where U is unitary and C is compact. The equations *(2)* and $C = U - I$ show that C is normal. The spectrum of C contains 0, and possibly, a finite set or a sequence tending to 0. Since $U = I + C$, the spectrum of U is obtained by shifting the spectrum of C by 1.

The unitary transform can be expressed as:

$$U = exp(\tilde{\mathrm{I}}\, \Phi/\hbar) \tag{3}$$

$$\hbar = h/(2\,\pi) \tag{4}$$

Φ is Hermitian. The constant h refers to the granularity of the eigenspace.

Unitary operators have eigenvalues that are located in the unity sphere of the 2^n-ons field.

The eigenvalues have the form:

$$u = exp(\boldsymbol{i}\,\varphi/\hbar) \tag{5}$$

φ is real. $\boldsymbol{i}$ is a unit length imaginary number in 2^n-on space. It represents a direction.

u spans a sphere in 2^n-on space. For constant $\boldsymbol{i}$, u spans a circle in a complex subspace.

4.2.6.7.1 ***Polar decomposition***

Normal operators N can be split into a real operator A and a unitary operator U. U and A have the same set of eigenvectors as N.

$$N = \|N\|\, U = A\, U = U\, A = A\, exp\left(\tilde{\mathrm{I}}\frac{\Phi}{\hbar}\right) = exp\left(\Phi_r + \tilde{\mathrm{I}}\frac{\Phi}{\hbar}\right) \tag{1}$$

Φ_r is a positive normal operator.

4.2.6.8 Ladder operator

4.2.6.8.1 General formulation

Suppose that two operators X and N have the commutation relation:

$$[N, X] = c\,X \quad (1)$$

for some scalar c. If $|n\rangle$ is an eigenstate of N with eigenvalue equation,

$$|N\,n\rangle = |n\rangle\,n \quad (2)$$

then the operator X acts on $|n\rangle$ in such a way as to shift the eigenvalue by c:

$$|N\,X\,n\rangle = |(X\,N + [N,X])n\rangle = |(X\,N + c\,X)n\rangle \quad (3)$$
$$= |X\,N\,n\rangle + |X\,n\rangle\,c = |X\,n\rangle\,n + |X\,n\rangle\,c = |X\,n\rangle(n + c)$$

In other words, if $|n\rangle$ is an eigenstate of N with eigenvalue n then $|X\,n\rangle$ is an eigenstate of N with eigenvalue $n + c$.

The operator X is a *raising operator* for N if c is real and positive, and a *lowering operator* for N if c is real and negative.

If N is a Hermitian operator then c must be real and the Hermitian adjoint of X obeys the commutation relation:

$$[N,\, X^\dagger] = -\,\mathrm{c}\,\,X^\dagger \quad (4)$$

In particular, if X is a lowering operator for N then $X^\dagger$ is a raising operator for N and vice-versa.

4.2.7 Unit sphere of $\mathfrak{H}$

The ket vectors in $\mathfrak{H}$ that have their norm equal to one form together the **unit sphere** Θ of $\mathfrak{H}$.

The orthonormal base vectors are all member of the unit sphere.

4.2.8 Bra-ket in four dimensional space

The Bra-ket formulation can also be used in transformations of the four dimensional curved spaces.

The bra $\langle f|$ is then a covariant vector and the ket $|g\rangle$ is a contra-variant vector. The inner product acts as a metric.

$$s = \langle f|g\rangle \quad (1)$$

The effect of a linear transformation L is then given by

$$s_L = \langle f|Lg\rangle \quad (2)$$

The effect of a the transpose transformation $L^{\dagger}$ is then given by

$$\langle f L^{\dagger} \, | g \rangle = \langle f | L g \rangle \tag{3}$$

For a unitary transformation U holds:

$$\langle N f | N g \rangle = \langle f | N^{\dagger} N g \rangle = \langle f | N N^{\dagger} g \rangle = \langle N N^{\dagger} f | g \rangle = \langle N^{\dagger} N f | g \rangle \tag{4}$$

$$\langle U f | U g \rangle = \langle f | g \rangle \tag{5}$$

$$\langle \nabla f | \nabla g \rangle = \langle f | \nabla^{\dagger} \nabla \mathrm{g} \rangle = \langle f | \nabla \nabla^{\dagger} \mathrm{g} \rangle = \langle \nabla \nabla^{\dagger} f | \mathrm{g} \rangle = \langle \nabla^{\dagger} \nabla f | \mathrm{g} \rangle \tag{6}$$

Notice that

$$\nabla \nabla^{\dagger} = \nabla^{\dagger} \nabla = \nabla_0 \nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle = \nabla_0^2 - \boldsymbol{\nabla}^2 \tag{7}$$

4.2.9 Closure

The closure of $\mathfrak{H}$ means that converging rows of vectors converge to a vector of $\mathfrak{H}$.

In general converging rows of eigenvalues of Q do not converge to an eigenvalue of Q.

Thus, the set of eigenvalues of Q is open.

At best the density of the coverage of the set of eigenvalues is comparable with the set of 2^n-ons that have rational numbers as coordinate values.

With other words, compared to the set of real numbers the eigenvalue spectrum of Q has holes.

The set of eigenvalues of operator Q includes 0. This means that Q does not have an inverse.

The rigged Hilbert space $\mathcal{H}$ can offer a solution, but then the direct relation with quantum logic is lost.

4.2.10 Canonical conjugate operator P

The existence of a canonical conjugate represents a stronger requirement on the continuity of the eigenvalues of canonical eigenvalues.

Q has eigenvectors $\{|q\rangle\}_q$ and eigenvalues q_s.

P has eigenvectors $\{|p\rangle\}_p$ and eigenvalues p_s.

For each eigenvector $|q\rangle$ of Q we define an eigenvector $|p\rangle$ and eigenvalues p_s of P such that:

$$\langle q | p \rangle = \langle p | q \rangle^* = exp \, (\boldsymbol{i} \, p_s \, q_s / \hbar) \tag{1}$$

$\hbar = h/(2\pi)$ is a scaling factor. $\langle q|p\rangle$ is a quaternion. $\boldsymbol{i}$ is a unit length imaginary quaternion. q_s and p_s are quaternionic (eigen)values corresponding to $|q\rangle$ *and* $|p\rangle$.

4.2.11 Displacement generators

Variance of the scalar product gives:

$$\boldsymbol{i}\,\hbar\,\delta\langle q|p\rangle = -p_s\langle q|p\rangle\delta q \tag{1}$$

$$\boldsymbol{i}\,\hbar\,\delta\langle p|q\rangle = -q_s\langle p|q\rangle\delta p \tag{2}$$

In the rigged Hilbert space $\mathcal{H}$ the variance can be replaced by differentiation.

Partial differentiation of the function $\langle q|p\rangle$ gives:

$$\boldsymbol{i}\,\hbar\,\frac{\partial}{\partial q_s}\langle q|p\rangle = -p_s\langle q|p\rangle \tag{3}$$

$$\boldsymbol{i}\,\hbar\frac{\partial}{\partial p_s}\langle p|q\rangle = -q_s\langle p|q\rangle \tag{4}$$

4.3 Quaternionic L^2 space

The space of quaternionic measurable functions is a separable quaternionic Hilbert space. For example quaternionic probability density distributions are measurable.[2]

This space is spanned by an orthonormal basis of quaternionic measurable functions. The shared affine-like versions of the parameter space of these functions is called **Palestra**[3]. When the Palestra is non-curved, then this base has a canonical conjugate, which is the quaternionic Fourier transform of the original base.

As soon as curvature of the Palestra arises, this relation is disturbed.

With other words: “In advance the Palestra has a virgin state.”

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[2] http://en.wikipedia.org/wiki/Lp_space#Lp_spaces

[3] The name Palestra is suggested by Henning Dekant’s wife Sarah. It is a name from Greek antiquity. It is a public place for training or exercise in wrestling or athletics

5 Gelfand triple

The separable Hilbert space only supports countable orthonormal bases and countable eigenspaces. The rigged Hilbert space $\mathcal{H}$ that belongs to an infinite dimensional separable Hilbert space $\mathfrak{H}$ is a Gelfand triple. It supports non-countable orthonormal bases and continuum eigenspaces.

A rigged Hilbert space is a pair $(\mathfrak{H}, \Phi)$ with $\mathfrak{H}$ a Hilbert space, Φ a dense subspace, such that Φ is given a topological vector space structure for which the inclusion map i is continuous.

Identifying $\mathfrak{H}$ with its dual space $\mathfrak{H}^\dagger$, the adjoint to i is the map

$$i^*: \mathfrak{H} = \mathfrak{H}^\dagger \to \Phi^\dagger \tag{1}$$

The duality pairing between Φ and $\Phi^\dagger$ has to be compatible with the inner product on $\mathfrak{H}$, in the sense that:

$$\langle u, v \rangle_{\Phi \times \Phi^\dagger} = (u, v)_{\mathfrak{H}} \tag{2}$$

whenever $u \in \Phi \subset \mathfrak{H}$ and $v \in \mathfrak{H} = \mathfrak{H}^\dagger \subset \Phi^\dagger$.

The specific triple $(\Phi \subset \mathfrak{H} \subset \Phi^\dagger)$ is often named after the mathematician Israel Gelfand).

Note that even though Φ is isomorphic to $\Phi^\dagger$ if Φ is a Hilbert space in its own right, this isomorphism is *not* the same as the composition of the inclusion i with its adjoint $i^\dagger$

$$i^\dagger i: \Phi \subset \mathfrak{H} = \mathfrak{H}^\dagger \to \Phi^\dagger \tag{3}$$

5.1 Understanding the Gelfand triple

The Gelfand triple of a real separable Hilbert space can be understood via the enumeration model of the real separable Hilbert space. This enumeration is obtained by taking the set of eigenvectors of a normal operator that has rational numbers as its eigenvalues. Let the smallest enumeration value of the rational enumerators approach zero. Even when zero is reached, then still the set of enumerators is countable. Now add all limits of converging rows of rational enumerators to the enumeration set. After this operation the enumeration set has become a continuum and has the same cardinality as the set of the real numbers. This operation converts the Hilbert space $\mathfrak{H}$ into its Gelfand triple $\mathcal{H}$ and it converts the normal operator in a new operator that has the real numbers as its eigenspace. It means that the orthonormal base of the Gelfand triple that is formed by the eigenvectors of the new normal operator has the cardinality of the real numbers. It also means that linear operators in this Gelfand triple have eigenspaces that are continuums and have the cardinality of the real numbers[4]. The same reasoning holds for complex number based Hilbert spaces and quaternionic Hilbert spaces and their respective Gelfand triples.

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[4] This story also applies to the complex and the quaternionic Hilbert spaces and their Gelfand triples.

6 Categories of operators

6.1 Functions as Hilbert space operators

Paul Dirac introduced the bra-ket notation that eases the formulation of Hilbert space habits.

By using reverse bra-ket notation, operators that reside in the Hilbert space and correspond to continuous functions, can easily be defined by starting from an orthonormal base of vectors. In this base the vectors are normalized and are mutually orthogonal. The vectors span a subspace of the Hilbert space. This works both in separable Hilbert spaces as well as in non-separable Hilbert spaces.

Let $\{q_i\}$ be the set of rational quaternions in a selected quaternionic number system and let $\{|q_i\rangle\}$ be the set of corresponding base vectors. They are eigenvectors of a normal operator $\mathcal{R} = |q_i\rangle q_i \langle q_i|$. Here we enumerate the base vectors with index i.

$$\mathcal{R} = |q_i\rangle q_i \langle q_i| \tag{1}$$

$\mathcal{R}$ is the configuration parameter space operator.

$\mathcal{R}_0 = \left(\mathcal{R} + \mathcal{R}^\dagger\right)/2$ is a self-adjoint operator. Its eigenvalues can be used to order the eigenvectors. The ordered eigenvalues can be interpreted as ***progression values***.

$\boldsymbol{\mathcal{R}} = \left(\mathcal{R} - \mathcal{R}^\dagger\right)/2$ is an imaginary operator. Its eigenvalues can be used to order the eigenvectors. The eigenvalues can be interpreted as ***spatial values*** and can be ordered in several ways.

Let $f(q)$ be a quaternionic function.

$$F = |q_i\rangle f(q_i) \langle q_i| \tag{2}$$

f defines a new operator that is based on function $f(q)$. Here we suppose that the target values of f belong to the same version of the quaternionic number system as its parameter space does.

Operator f has a countable set of discrete quaternionic eigenvalues.

For this operator the reverse bra-ket notation is a shorthand for

$$\langle x|F\, y\rangle = \sum_i \langle x|q_i\rangle f(q_i) \langle q_i|y\rangle \tag{3}$$

This formula uses the Kronecker delta $\langle q_i|q_j\rangle = \delta_{ij}$. In a non-separable Hilbert space, such as the Gelfand triple, the continuous function $\mathcal{F}(q)$ can be used to define an operator, which features a continuum eigenspace.

$$\mathcal{F} = |q\rangle \mathcal{F}(q) \langle q| \tag{7}$$

Via the continuous quaternionic function $\mathcal{F}(q)$, the operator $\mathcal{F}$ defines a curved continuum $\mathcal{F}$. This operator and the continuum reside in the Gelfand triple, which is a non-separable Hilbert space.

$$\mathfrak{R} = |q\rangle q \langle q| \tag{8}$$

The function $\mathcal{F}(q)$ uses the eigenspace of the reference operator $\mathfrak{R}$ as a flat parameter space that is spanned by a quaternionic number system $\{q\}$. The continuum $\mathcal{F}$ represents the target space of function $\mathcal{F}(q)$.

Here we no longer enumerate the base vectors with index i. We just use the name of the parameter. If no conflict arises, then we will use the same symbol for the defining function, the defined operator and the continuum that is represented by the eigenspace.

For the shorthand of the reverse bra-ket notation of operator $\mathcal{F}$ the integral over q replaces the summation over q_i.

$$\langle x|\mathcal{F}\, y\rangle = \int_q \langle x|q\rangle \mathcal{F}(q)\langle q|y\rangle\ dq \tag{9}$$

This formula uses the Dirac delta function $\langle q|p\rangle = \delta(p-q)$.

Remember that quaternionic number systems exist in several versions, thus also the operators F and $\mathcal{F}$ exist in these versions. The same holds for the parameter space operators. When relevant, we will use superscripts in order to differentiate between these versions.

Thus, operator $f^x = |q_i^x\rangle f^x(q_i^x)\langle q_i^x|$ is a specific version of operator f. Function $f^x(q_i^x)$ uses parameter space $\mathcal{R}^x$.

Similarly, $\mathcal{F}^x = |q^x\rangle \mathcal{F}^x(q^x)\langle q^x|$ is a specific version of operator $\mathcal{F}$. Function $\mathcal{F}^x(q^x)$ and continuum $\mathcal{F}^x$ use parameter space $\mathfrak{R}^x$.

In general the dimension of a subspace loses its significance in the non-separable Hilbert space.

The continuums that appear as eigenspaces in the non-separable Hilbert space $\mathcal{H}$ can be considered as quaternionic functions that also have a representation in the corresponding infinite dimensional separable Hilbert space $\mathfrak{H}$. Both representations use a flat parameter space $\mathfrak{R}$ or $\mathcal{R}$ that is spanned by quaternions. $\mathcal{R}$ is spanned by rational quaternions.

The parameter space operators will be treated as reference operators. The rational quaternionic eigenvalues $\{q_i\}$ that occur as eigenvalues of the reference operator $\mathcal{R}$ in the separable Hilbert space map onto the rational quaternionic eigenvalues $\{q_i\}$ that occur as subset of the quaternionic eigenvalues $\{q\}$ of the reference operator $\mathfrak{R}$ in the Gelfand triple. In this way the reference operator $\mathcal{R}$ in the infinite dimensional separable Hilbert space $\mathfrak{H}$ relates directly to the reference operator $\mathfrak{R}$, which resides in the Gelfand triple $\mathcal{H}$.

The examples $\mathcal{R}, F, \mathfrak{R}$ and $\mathcal{F}$ are constructed operators.

6.1.1 Symmetry centers

Symmetry centers $\mathfrak{S}^x$ are anti-Hermitian reference operators. The symmetry flavor ^{x} of the symmetry center $\mathfrak{S}^x$, which is maintained by operator $\mathfrak{S}^x = |\mathfrak{s}_i^x\rangle \mathfrak{s}_i^x \langle \mathfrak{s}_i^x|$ is determined by the affine Cartesian preordering of its eigenspace. The superscript ^{x} can be ⓪, ①, ②, ③, ④, ⑤, ⑥, or ⑦. Apart from the affine Cartesian ordering the symmetry centers feature a spherical coordinate ordering that starts from the affine Cartesian ordering.

The eigenspace of a symmetry centers can float on the eigenspace of reference operator $\mathcal{R}^{⓪}$. As a consequence, many symmetry centers can coexist in an infinite dimensional separable Hilbert space.

The spherical ordering of the symmetry center defines a spin value and its relation to reference operator $\mathcal{R}^{⓪}$ defines a relative position. The Cartesian coordinate axes of $\mathfrak{S}^x$ and $\mathcal{R}^{⓪}$ are parallel.

6.1 Stochastic operators

Stochastic operators do not get their data from a continuous quaternionic function. Instead a stochastic process delivers the eigenvalues. Again for quaternionic stochastic operators these eigenvalues are quaternions and the real parts of these quaternions may be interpreted as progression values.

The mechanisms that control the stochastic processes do not belong to the Hilbert space. Stochastic operators only act in a step-wise fashion. Their eigenspace is countable. If the real parts of the eigenvalues are interpreted as progression, then some of these stochastic operators may be considered to act in a cyclic fashion.

These mechanisms can synchronize the progression values with the model wide progression that is set by a selected reference operator.

6.1.1 Density operators

The eigenspace of a stochastic operator may be characterized by a continuous density distribution. In that case the corresponding stochastic process must ensure that this continuous density distribution fits. The density distribution can be constructed afterwards or after each regeneration cycle. Constructing the density distribution involves a reordering of the imaginary parts of the produced eigenvalues. A different operator can then use the continuous density distribution in order to generate its functionality. The real parts of the eigenvalues may then reflect the reordering. The construction of the density distribution is a pure administrative action that is performed as an aftermath. The constructed density operator represents a continuous function and may reside both in the separable Hilbert space and in the Gelfand triple.

7 Change of base

In quaternionic Hilbert space a change of base can be achieved by:

$$\langle x|\tilde{\mathcal{F}}\, y\rangle = \int_{\tilde{q}} \langle x\,|\tilde{q}\rangle \left\{ \int_{q} \langle \tilde{q}|q\rangle \mathcal{F}(q) \langle q|\tilde{q}\rangle\, dq \right\} \langle \tilde{q}|y\rangle\, d\tilde{q} \tag{1}$$

$$= \int_{\tilde{q}} \langle x|\tilde{q}\rangle \tilde{\mathcal{F}}(\tilde{q}) \langle \tilde{q}|y\rangle\, d\tilde{q}$$

$$\tilde{\mathcal{F}}(\tilde{q}) = \int_{q} \langle \tilde{q}|q\rangle \mathcal{F}(q) \langle q|\tilde{q}\rangle\, dq \tag{2}$$

$$\tilde{\mathfrak{R}}(\tilde{q}) = \int_{q} \langle \tilde{q}|q\rangle q \langle q|\tilde{q}\rangle\, dq \tag{3}$$

$$\langle x|\tilde{\mathfrak{R}}\, y\rangle = \int_{\tilde{q}} \langle x|\tilde{q}\rangle \tilde{\mathfrak{R}}(\tilde{q}) \langle \tilde{q}|y\rangle\, d\tilde{q} \tag{4}$$

$$\tilde{\mathfrak{R}} = |\tilde{q}\rangle \tilde{q} \langle \tilde{q}| \tag{5}$$

However, as we see in the formulas this method merely achieves a rotation of parameter spaces and functions. In the complex number based Hilbert space it would achieve no change at all.

7.1 Quaternionic Fourier transform

A Fourier transform uses a different approach. It is not a direct transform between parameter spaces, but instead it is a transform between sets of mutually orthogonal functions, which are formed by inner products, which are related to different parameter spaces. The quaternionic Fourier transform exists in three versions. The first two versions have a reverse Fourier transform.

The left oriented Fourier transform is defined by:

$$\tilde{\mathcal{F}}_L(\tilde{q}_L) = \int_{q} \langle \tilde{q}_L|q\rangle\, \mathcal{F}(q)\, dq \tag{1}$$

Like the functions $\langle q|q'\rangle$ and $\langle \tilde{q}_L|\tilde{q}'_L\rangle$, the functions $\langle \tilde{q}_L|q\rangle$ and $\langle q|\tilde{q}_L\rangle$ form sets of mutually orthogonal functions, as will be clear from:

$$\langle q|q'\rangle = \delta(q - q') \tag{2}$$

$$\langle \tilde{q}_L|\tilde{q}'_L\rangle = \delta(\tilde{q}_L - \tilde{q}'_L) \tag{3}$$

$$\int_{\tilde{q}_L} \langle q'|\tilde{q}_L\rangle \langle \tilde{q}_L|q\rangle\, d\tilde{q}_L = \delta(q - q') \tag{4}$$

$$\int_{q} \langle \tilde{q}'_L|q\rangle \langle q|\tilde{q}_L\rangle\, dq = \delta(\tilde{q}_L - \tilde{q}'_L) \tag{5}$$

The reverse transform is:

(6)

$$\mathcal{F}(q) = \int_{\tilde{q}_L} \langle q|\tilde{q}_L\rangle \tilde{\mathcal{F}}_L(\tilde{q}_L)\, d\tilde{q}_L = \int_{\tilde{q}_L}\int_{q'} \langle q|\tilde{q}_L\rangle\langle\tilde{q}_L|q'\rangle \mathcal{F}(q')\, d\tilde{q}_L\, dq'$$

$$= \int_{q'} \left\{\int_{\tilde{q}_L} \langle q|\tilde{q}_L\rangle\langle\tilde{q}_L|q'\rangle\, d\tilde{q}_L\right\} \mathcal{F}(q')\, dq' = \int_{q'} \delta(q-q')\mathcal{F}(q')\, dq'$$

The reverse bra-ket form of the operator $\tilde{\mathcal{F}}_L$ equals:

$$\tilde{\mathcal{F}}_L = |\tilde{q}_L\rangle \tilde{\mathcal{F}}_L(\tilde{q}_L)\langle\tilde{q}_L| \tag{7}$$

Operator $\tilde{\mathfrak{R}}_L$ provides the parameter space for the left oriented Fourier transform $\tilde{\mathcal{F}}_L(\tilde{q}_L)$ of function $\mathcal{F}(q)$ in equations (1) and (6).

$$\tilde{\mathfrak{R}}_L = |\tilde{q}_L\rangle \tilde{q}_L \langle\tilde{q}_L| \tag{8}$$

Similarly the right oriented Fourier transform can be defined.

$$\tilde{\mathcal{F}}_R(\tilde{q}) = \int_q \mathcal{F}(q')\langle q'|\tilde{q}\rangle\; dq' \tag{9}$$

The reverse transform is:

$$\mathcal{F}(q) = \int_{\tilde{q}_R} \tilde{\mathcal{F}}_R(\tilde{q}_R)\langle q|\tilde{q}_R\rangle\, d\tilde{q}_R = \int_{\tilde{q}_R}\int_{q'} \mathcal{F}(q')\langle q'|\tilde{q}_R\rangle\langle\tilde{q}_R|q\rangle\, dq'\, d\tilde{q}_R \tag{10}$$

$$= \int_{q'} \mathcal{F}(q')\left\{\int_{\tilde{q}_R} \langle q'|\tilde{q}_R\rangle\langle\tilde{q}_R|q\rangle\;\; d\tilde{q}_R\right\} dq' = \int_{q'} \mathcal{F}(q')\,\delta(q-q')\, dq'$$

Also here the functions $\langle q|q'\rangle$, $\langle\tilde{q}_R|\tilde{q}_R'\rangle$, $\langle\tilde{q}_R|q\rangle$ and $\langle q|\tilde{q}_R\rangle$ form sets of mutually orthogonal functions.

The reverse bra-ket form of the operator $\tilde{\mathcal{F}}_R$ equals:

$$\tilde{\mathcal{F}}_R = |\tilde{q}_R\rangle \tilde{\mathcal{F}}_R(\tilde{q}_R)\langle\tilde{q}_R| \tag{11}$$

Operator $\tilde{\mathfrak{R}}_R$ provides the parameter space for the right oriented Fourier transform $\tilde{\mathcal{F}}_R(\tilde{q}_R)$ of function $\mathcal{F}(q)$ in equations (9) and (10).

$$\tilde{\mathfrak{R}}_R = |\tilde{q}_R\rangle \tilde{q}_R \langle\tilde{q}_R| \tag{12}$$

The third version of the Fourier transform is:

$$\tilde{\mathcal{F}}(\tilde{q}_L, \tilde{q}_R) = \frac{\tilde{\mathcal{F}}_L(\tilde{q}_L) + \tilde{\mathcal{F}}_R(\tilde{q}_R)}{2} = ½ \int_q \{\langle\tilde{q}_L|q\rangle\mathcal{F}(q) + \mathcal{F}(q)\langle q|\tilde{q}_R\rangle\}\, dq \tag{13}$$

In contrast to the right and left version, the third version has no reverse.

8 Symmetry flavor

Quaternionic number systems can be mapped to Cartesian coordinates along the orthonormal base vectors 1, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$; with $\boldsymbol{ij} = \boldsymbol{k}$

Due to the four dimensions of quaternions, quaternionic number systems exist in 16 well-ordered versions $\{q^x\}$ that differ only in their discrete Cartesian symmetry set. The quaternionic number systems $\{q^x\}$ correspond to 16 versions $\{q_i^x\}$ of rational quaternions.

Half of these versions are right handed and the other half are left handed. Thus the handedness is influenced by the symmetry flavor.

The superscript ^{x} can be ⓪, ①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩, ⑪, ⑫, ⑬, ⑭, or ⑮.

This superscript represents the ***symmetry flavor*** of the superscripted subject.

The reference operator $\mathcal{R}^{\textcircled{0}} = \left|q_i^{\textcircled{0}}\right\rangle q_i^{\textcircled{0}} \left\langle q_i^{\textcircled{0}}\right|$ in separable Hilbert space $\mathfrak{H}$ maps into the reference operator $\Re^{\textcircled{0}} = |q^{\textcircled{0}}\rangle q^{\textcircled{0}} \langle q^{\textcircled{0}}|$ in Gelfand triple $\mathcal{H}$.

The symmetry flavor of the symmetry center $\mathfrak{S}^x$, which is maintained by operator $\mathfrak{S}^x = |\mathfrak{s}_i^x\rangle \mathfrak{s}_i^x \langle \mathfrak{s}_i^x|$ is determined by its Cartesian ordering and then compared with the reference symmetry flavor, which is the symmetry flavor of the reference operator $\mathcal{R}^{\textcircled{0}}$. ↑↑↑↑

Now the symmetry related charge follows in three steps.

1. Count the difference of the spatial part of the symmetry flavor of $\mathfrak{S}^x$ with the spatial part of the symmetry flavor of reference operator $\mathcal{R}^{\textcircled{0}}$.
2. If the handedness changes from **R** to **L**, then switch the sign of the count.
3. Switch the sign of the result for anti-particles.

Electric charge equals symmetry related charge divided by 3.

Symmetry flavor					
Ordering x y z τ	Super script	Handedness Right/Left	Color charge	Electric charge * 3	Symmetry center type. Names are taken from the standard model
↑↑↑↑	⓪	R	N	+0	neutrino
↓↑↑↑	①	L	R	−1	down quark
↑↓↑↑	②	L	G	−1	down quark
↓↓↑↑	③	L	B	−1	down quark
↑↑↓↑	④	R	B	+2	up quark
↓↑↓↑	⑤	R	G	+2	up quark
↑↓↓↑	⑥	R	R	+2	up quark
↓↓↓↑	⑦	L	N	−3	electron
↑↑↑↓	⑧	R	N	+3	positron
↓↑↑↓	⑨	L	$\bar{R}$	−2	anti-up quark
↑↓↑↓	⑩	L	$\bar{G}$	−2	anti-up quark
↓↓↑↓	⑪	L	$\bar{B}$	−2	anti-up quark
↑↑↓↓	⑫	R	$\bar{B}$	+1	anti-down quark
↓↑↓↓	⑬	R	$\bar{R}$	+1	anti-down quark
↑↓↓↓	⑭	R	$\bar{G}$	+1	anti-down quark
↓↓↓↓	⑮	L	N	−0	anti-neutrino

Per definition, members of coherent sets $\{a_i^x\}$ of quaternions all feature the same symmetry flavor that is marked by superscript ^{x}.

Also continuous functions and continuums feature a symmetry flavor. Continuous quaternionic functions $\psi^x(q^x)$ and corresponding continuums do not switch to other symmetry flavors ^{y}.

The reference symmetry flavor $\psi^y(q^y)$ of a continuous function $\psi^x(q^y)$ is the symmetry flavor of the parameter space $\{q^y\}$.

The symmetry related charge conforms to the amount of reordering that is required when the symmetry center or one of its elements is mapped onto the reference space $\mathcal{R}^{⓪}$.

The concept of symmetry flavor sins against the cosmologic principle, which states that universe does not contain specific directions. It also claims that universe has no origin. Affine Cartesian ordering does not apply a selected spatial origin. That does not say that universe cannot have a unique spatial origin. That origin would be the spatial origin of reference operator $\mathcal{R}^{⓪}$. All symmetry centers own a unique spatial origin. That origin maps onto a dynamic location in $\mathcal{R}^{⓪}$.

8.1.1 Symmetry flavor conversion tools

Quaternionic conjugation

$$(\psi^x)^* = \psi^{(7-x)}; x = ⓪,①,②,③,④,⑤,⑥,⑦$$

Via quaternionic rotation, the following normalized quaternions ϱ^x can shift the indices of symmetry flavors of coordinate mapped quaternions and for quaternionic functions:

$$\varrho^{①} = \frac{1+\boldsymbol{i}}{\sqrt{2}};\ \varrho^{②} = \frac{1+\boldsymbol{j}}{\sqrt{2}};\varrho^{③} = \frac{1+\boldsymbol{k}}{\sqrt{2}};\ \varrho^{④} = \frac{1-\boldsymbol{k}}{\sqrt{2}};\ \varrho^{⑤} = \frac{1-\boldsymbol{j}}{\sqrt{2}};\ \varrho^{⑥} = \frac{1-\boldsymbol{i}}{\sqrt{2}}$$

$$\boldsymbol{ij} = \boldsymbol{k};\ \ \boldsymbol{jk} = \boldsymbol{i};\ \ \boldsymbol{ki} = \boldsymbol{j}$$

$$\varrho^{⑥} = \left(\varrho^{①}\right)^*$$

For example

$$\psi^{③} = \varrho^{①}\psi^{②}/\varrho^{①}$$

$$\psi^{③}\varrho^{①} = \varrho^{①}\psi^{②}$$

$$\psi^{⓪} = \varrho^x\psi^{⓪}/\varrho^x; \psi^{⑦} = \varrho^x\psi^{⑦}/\varrho^x$$

Also strings of symmetry flavor convertors change the index of symmetry flavor of the multiplied quaternion or quaternionic function. The convertors can act on each other.

For example:

$$\varrho^{①}\varrho^{②} = \varrho^{②}\varrho^{③} = \varrho^{③}\varrho^{①} = \frac{1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}}{2}$$

The result is an isotropic quaternion. This means:

$$\varrho^{①}\psi^{②}/\varrho^x = \varrho^{②}\psi^{③}/\varrho^x = \psi^{(x+1)}$$

Here $(x+1)$ means $\boldsymbol{i} \to \boldsymbol{j} \to \boldsymbol{k} \to \boldsymbol{i} \to \boldsymbol{j} \to \boldsymbol{k}$, or ①→②→③→①→②→③ and so on.

9 Quaternionic functions

9.1 Norm

Square-integrable functions are normalizable. The norm is defined by:

$$\|\psi\|^2 = \int_V |\psi|^2 \, dV \tag{1}$$

$$= \int_V \{|\psi_0|^2 + |\boldsymbol{\psi}|^2\} dV$$

$$= \|\psi_0\|^2 + \|\boldsymbol{\psi}\|^2$$

9.2 Differentiation

Under rather general conditions the change of a quaternionic function $f(q)$ can be described by:

$$df(q) = c^\tau \, dq_\tau + c^x \, dq_x + c^y \, dq_y + c^z \, dq_z = df_\nu(q) e^\nu = \sum_{\mu=0\ldots3} \frac{\partial f}{\partial q_\mu} dq^\mu = c_\mu(q) dq^\mu \tag{1}$$

Here the coefficients $c^\mu(q)$ are full quaternionic functions. dq_μ are real numbers. e^ν are quaternionic base vectors.

More violent conditions require the inclusion of higher order partial differential terms. For example:

$$df(q) = \sum_{\mu=0\ldots3} \left(\frac{\partial f}{\partial q_\mu} + \left(\sum_{\nu=0\ldots3} \frac{\partial^2 f}{\partial q_\mu \partial q_\nu} dq^\nu \right) \right) dq^\mu \tag{2}$$

Under more moderate and sufficiently short range conditions the function $f(q)$ behaves more linearly.

$$df(q) = c_0^\tau \, dq_\tau + c_0^x \, \boldsymbol{i} \, dq_x + c_0^y \, \boldsymbol{j} \, dq_y + c_0^z \, \boldsymbol{k} \, dq_z = c_0^\mu(q) \, e_\mu \, dq_\mu \tag{2}$$

Here the coefficients $c_0^\mu(q)$ are real functions.

Thus, in a rather flat continuum we can use the quaternionic nabla ∇.

$$\nabla = \left\{\frac{\partial}{\partial\tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} = \frac{\partial}{\partial\tau} + \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} = \nabla_0 + \boldsymbol{\nabla} \tag{3}$$

9.2.1 Moderate continuity conditions

If g is differentiable then the quaternionic nabla ∇g of g exists.

The quaternionic nabla ∇ is a shorthand for $\nabla_0 + \boldsymbol{\nabla}$

$$\nabla_0 = \frac{\partial}{\partial\tau} \tag{3}$$

$$\boldsymbol{\nabla} = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} \tag{4}$$

$$h = h_0 + \boldsymbol{h} = \nabla g \tag{4}$$

$$h_0 = \nabla_0 g_0 - \langle \boldsymbol{\nabla}, \boldsymbol{g} \rangle \tag{5}$$

$$\boldsymbol{h} = \nabla_0 \boldsymbol{g} + \boldsymbol{\nabla} g_0 \pm \boldsymbol{\nabla} \times \boldsymbol{g} \tag{6}$$

$$\phi = \nabla\psi \Rightarrow \phi^* = (\nabla\psi)^* \tag{7}$$

$$(\nabla\psi)^* = \nabla_0\psi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle - \nabla_0\boldsymbol{\psi} - \boldsymbol{\nabla}\psi_0 \mp \boldsymbol{\nabla} \times \boldsymbol{\psi} \tag{8}$$

$$\nabla^*\psi^* = \nabla_0\psi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle - \nabla_0\boldsymbol{\psi} - \boldsymbol{\nabla}\psi_0 \pm \boldsymbol{\nabla} \times \boldsymbol{\psi} \tag{9}$$

Similarity of these equations with Maxwell equations is not accidental. In Maxwell equations several terms in the above equations have been given special names and special symbols. Similar equations occur in other branches of physics. Apart from these differential equations also integral equations exist.

9.2.2 Gauge transformation

For a function χ that obeys the ***quaternionic wave equation***[5]

$$\nabla^*\nabla\chi = \nabla_0\nabla_0\chi + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\chi\rangle = 0 \quad (1)$$

the value of ϕ in

$$\phi = \nabla\psi \quad (2)$$

does not change after the gauge transformation[6]

$$\psi \rightarrow \psi + \xi = \psi + \nabla^*\chi \quad (3)$$

$$\nabla\xi = 0 \quad (4)$$

$$\chi = \chi_0 + \boldsymbol{\chi} \quad (5)$$

Thus in general:

$$\nabla^*\nabla\psi = \nabla_0\nabla_0\psi + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\psi\rangle = \rho \neq 0 \quad (6$$

ρ is a quaternionic function.

Its real part ρ_0 represents an object density distribution.

Its imaginary part $\boldsymbol{\rho} = \boldsymbol{v}\,\rho_0$ represents a current density distribution.

Equation (1) forms the basis of the generalized (quaternionic) Huygens principle[7].

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[5] Be aware, this is the quaternionic wave equation. This is not the common form of the wave equation, which is complex number based.

[6] The qualification gauge transformation is usually given to a transformation that leaves the Laplacian untouched. Here we use that qualification for transformations that leave the quaternionic differential untouched.

[7] The papers on Huygens principle use the complex number based wave equation, which differs from the quaternionic wave equation.

$$\nabla^*\nabla\chi_0 = 0 \tag{7}$$

Equation (7) has 3D isotropic wave fronts as its solution. χ_0 is a scalar function. By changing to polar coordinates it can be deduced that a general solution is given by:

$$\chi_0(r,\tau) = \frac{f_0(\boldsymbol{i}r - c\tau)}{r} \tag{8}$$

Where $c = \pm 1$ and $\boldsymbol{i}$ represents a base vector in radial direction. In fact the parameter $\boldsymbol{i}r - c\tau$ of f_0 can be considered as a complex number valued function.

$$\nabla^*\nabla\boldsymbol{\chi} = 0 \tag{9}$$

Here $\boldsymbol{\chi}$ is a vector function.

Equation (9) has one dimensional wave fronts as solutions:

$$\boldsymbol{\chi}(z,\tau) = \boldsymbol{f}(\boldsymbol{i}z - c\tau) \tag{10}$$

Again the parameter $\boldsymbol{i}z - c\tau$ of $\boldsymbol{f}$ can be interpreted as a complex number based function.

The imaginary $\boldsymbol{i}$ represents the base vector in the x, y plane. Its orientation θ may be a function of z.

That orientation determines the polarization of the wave front.

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\boldsymbol{f} = c\boldsymbol{f}' \\
&\frac{\partial^2\boldsymbol{f}}{\partial\tau^2} = c\frac{\partial}{\partial\tau}\boldsymbol{f}' = c^2\boldsymbol{f}'' \\
&\frac{\partial\boldsymbol{f}}{\partial z} = \boldsymbol{i}\boldsymbol{f}' \\
&\frac{\partial^2\boldsymbol{f}}{\partial z^2} = \boldsymbol{i}\frac{\partial}{\partial z}\boldsymbol{f}' = -\boldsymbol{f}'' \\
&\frac{\partial^2\boldsymbol{f}}{\partial\tau^2} + \frac{\partial^2\boldsymbol{f}}{\partial z^2} = (c^2 - 1)\boldsymbol{f}''
\end{aligned} \tag{11}$$

If $c = \pm 1$, then $\boldsymbol{f}$ is a solution of the quaternionic wave equation.

9.2.3 Non-homogeneous wave equation

The non-homogeneous wave equation runs:

$$\nabla^*\nabla\chi = \nabla_0\nabla_0\chi + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\chi \rangle = \xi \tag{1}$$

Depending on local conditions equation it restricts to the homogeneous wave equation:

$$\nabla^*\nabla\chi = \nabla_0\nabla_0\chi + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\chi \rangle = 0 \tag{2}$$

or to the (screened) Poisson equation:

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \chi - \lambda^2 \chi = \rho \tag{3}$$

The function ρ may represent a distribution of triggers.

$$\nabla_0\nabla_0\chi = \xi - \rho = -\lambda^2 \chi \tag{4}$$

The 3D solution of equation (3) is determined by the screened Green's function $G(r)$.

$$G(r) = \frac{\exp(-\lambda\, r)}{r} \tag{6}$$

This Green's function corresponds to the Yukawa potential. For $\lambda = 0$ it corresponds to the Coulomb potential. Green's functions represent solutions for point sources.

$$\chi = \iiint G(\boldsymbol{r} - \boldsymbol{r}')\, \rho(\boldsymbol{r}')\, d^3\boldsymbol{r}' \tag{7}$$

Equation (1) does not involve a Green's function. If $\lambda \neq 0$ then equation (4) has a solution

$$\chi = a(\boldsymbol{x}) \exp(\pm i\, \omega\, \tau);\ \lambda = \pm i\, \omega \tag{8}$$

ω represents a parameter space wide clock frequency.

The non-homogeneous wave equation can be split into continuity equations:

$$\nabla\chi = \phi\ ;\ \nabla^*\phi = \rho \tag{9}$$

And

$$\nabla^*\chi = \varphi\ ;\nabla\ \varphi = \rho \tag{10}$$

9.3 Displacement generator

The definition of the differential is

$$\Phi = \nabla\psi \tag{1}$$

In Fourier space the nabla becomes a displacement generator.

$$\widetilde{\Phi} = \mathcal{M}\widetilde{\psi} \tag{2}$$

$\mathcal{M}$ is the ***displacement generator***

A small displacement in configuration space becomes a multiplier in Fourier space.

In a paginated space-progression model the displacements are small and the displacement generators work incremental. The multipliers act as superposition coefficients.

9.4 The coupling equation

The coupling equation follows from peculiar properties of the differential equation. We start with two normalized functions ψ and φ and a normalizable function $\Phi = m\ \varphi$.

$$||\psi|| = ||\varphi|| = 1 \tag{1}$$

These normalized functions are supposed to be related by:

$$\Phi = \nabla\psi = m\ \varphi \tag{2}$$

$\Phi = \nabla\psi$ defines the differential equation. (3)

$\nabla\psi = \Phi$ formulates a continuity equation. (4)

$\nabla\psi = m\,\varphi$ formulates the coupling equation. (5)

It couples ψ to φ. m is the coupling factor.

$$\nabla\psi = m_1\,\varphi \tag{6}$$

$$\nabla^*\varphi = m_2\,\zeta \tag{7}$$

$$\nabla^*\nabla\psi = m_1\,\nabla^*\varphi = m_1 m_2 \zeta = \rho \tag{8}$$

Be aware,

$$(\nabla\psi)^* = \Phi^* \neq \nabla^*\psi^* = \Phi^* + 2\boldsymbol{\nabla}\times\boldsymbol{\psi} \tag{9}$$

Each double differentiable quaternionic function corresponds to a normalized density distribution.

9.4.1 In Fourier space

The Fourier transform of the coupling equation is:

$$\mathcal{M}\tilde{\psi} = m\tilde{\varphi} \tag{1}$$

$\mathcal{M}$ is the ***displacement generator***

9.5 Difference with Maxwell-based differential equations

9.5.1 Maxwell-like equations

Similarity of the quaternionic differential equations with Maxwell based differential equations is not accidental. In Maxwell equations several terms in the differential equations have been given special names and special symbols. Similar equations occur in other branches of physics.

In the quaternionic differential calculus holds:

$$\phi = \nabla\varphi \equiv (\nabla_0 + \boldsymbol{\nabla})(\varphi_0 + \boldsymbol{\varphi}) = \nabla_0\varphi_0 - \langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle + \boldsymbol{\nabla}\varphi_0 + \nabla_0\boldsymbol{\varphi} \pm \boldsymbol{\nabla}\times\boldsymbol{\varphi} \tag{1}$$

Now we can define special symbols for the terms.

$$\mathfrak{E} \equiv -\boldsymbol{\nabla}\varphi_0 - \nabla_\tau \boldsymbol{\varphi} \tag{2}$$

$$\boldsymbol{\mathcal{B}} \equiv \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{3}$$

With these definitions:

$$\boldsymbol{\phi} = -\mathfrak{E} \pm \boldsymbol{\mathcal{B}} \tag{4}$$

In addition hold:

$$\nabla_\tau \boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \times \nabla_\tau \boldsymbol{\varphi} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla}\varphi_0 - \boldsymbol{\nabla} \times \mathfrak{E} = -\boldsymbol{\nabla} \times \mathfrak{E} \tag{5}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\varphi}) = \boldsymbol{\nabla}\langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle - \langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\boldsymbol{\varphi} \tag{6}$$

$$\langle\boldsymbol{\nabla}, \boldsymbol{\mathcal{B}}\rangle = 0 \tag{7}$$

$$\nabla_\tau \mathfrak{E} = -\nabla_\tau \boldsymbol{\nabla}\varphi_0 - \nabla_\tau \nabla_\tau \boldsymbol{\varphi} \tag{8}$$

$$\langle\boldsymbol{\nabla}, \mathfrak{E}\rangle = -\langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\varphi_0 - \nabla_\tau\langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \tag{9}$$

The following equation is ***not*** a Maxwell-like equation. We use a control switch $\alpha = -1$ for quaternionic differential equations and $\alpha = +1$ for Maxwell based differential equations:

$$\phi_0 = -\alpha\, \nabla_\tau \varphi_0 - \langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \tag{10}$$

$$\boldsymbol{\nabla}\phi_0 = -\alpha\, \nabla_\tau \boldsymbol{\nabla}\varphi_0 - \boldsymbol{\nabla}\langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \tag{11}$$

$$\nabla_\tau \phi_0 = -\alpha\, \nabla_\tau \nabla_\tau \varphi_0 - \nabla_\tau\langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \tag{12}$$

$$\nabla_\tau \phi_0 - \langle\boldsymbol{\nabla}, \mathfrak{E}\rangle = -\alpha\, \nabla_\tau \nabla_\tau \varphi_0 - \nabla_\tau\langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle + \langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\varphi_0 + \nabla_\tau\langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \tag{13}$$

$$= -\alpha\, \nabla_\tau \nabla_\tau \varphi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0$$

$$-\boldsymbol{\nabla}\phi_0 - \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} + \alpha\, \nabla_\tau \mathfrak{E} \quad (14)$$

$$= +\alpha\, \nabla_\tau \boldsymbol{\nabla} \varphi_0 + \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle - \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi} - \alpha\, \nabla_\tau \boldsymbol{\nabla} \varphi_0 - \alpha\, \nabla_\tau \nabla_\tau \boldsymbol{\varphi}$$

$$= -\alpha\, \nabla_\tau \nabla_\tau \boldsymbol{\varphi} + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi}$$

With

$$\zeta_0 = \nabla_\tau \phi_0 - \langle \boldsymbol{\nabla}, \mathfrak{E} \rangle \quad (15)$$

and

$$\boldsymbol{\zeta} = -\boldsymbol{\nabla}\phi_0 - \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} + \alpha\, \nabla_\tau \mathfrak{E} \quad (16)$$

follow the non-homogeneous equations:

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 - \alpha\, \nabla_\tau \nabla_\tau \varphi_0 = \zeta_0 \quad (17)$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi} - \alpha\, \nabla_\tau \nabla_\tau \boldsymbol{\varphi} = \boldsymbol{\zeta} \quad (18)$$

$$(\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle - \alpha\, \nabla_\tau \nabla_\tau) \varphi = \zeta \quad (19)$$

For $\alpha = -1$, These equations have a Euclidean signature!

9.5.2 Maxwell based differential calculus

For Maxwell based differential calculus, instead of the equation for ϕ_0 an equivalent equation is used in order to define a gauge.

$$\varkappa \equiv \alpha\, \nabla_t \varphi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \quad (1)$$

In the gauge, the control factor α can be -1, 0 or 1. For the Lorentz gauge holds $\alpha = 1$

This means that in Maxwell based differential equations the equivalent of the real part of ϕ is ignored or is used as a gauge. Further, ∇_τ is replaced by ∇_t. We use another symbol $\mathcal{E}$ for $\mathfrak{E}$.

With the Lorentz gauge the Maxwell equations run as:

$$\mathcal{E} \equiv -\boldsymbol{\nabla}\varphi_0 - \nabla_t \boldsymbol{\varphi} \tag{3}$$

$$\mathcal{B} \equiv \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{4}$$

$$\nabla_t \varkappa = \alpha\, \nabla_t \nabla_t \varphi_0 + \nabla_t \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{5}$$

$$\boldsymbol{\nabla} \varkappa = \alpha\, \nabla_t \boldsymbol{\nabla} \varphi_0 + \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{6}$$

$$\nabla_t \varkappa + \langle \boldsymbol{\nabla}, \mathcal{E} \rangle = \alpha \nabla_t \nabla_t \varphi_0 + \nabla_t \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 - \nabla_t \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{7}$$

$$= \alpha \nabla_t \nabla_t \varphi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 = (\alpha \nabla_t \nabla_t - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle) \varphi_0$$

$$-\boldsymbol{\nabla} \varkappa - \alpha \nabla_t \mathcal{E} + \boldsymbol{\nabla} \times \mathcal{B} \tag{8}$$

$$= -\alpha \nabla_t \boldsymbol{\nabla} \varphi_0 - \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle + \alpha \nabla_t \boldsymbol{\nabla} \varphi_0 + \alpha \nabla_t \nabla_t \boldsymbol{\varphi} + \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi}$$

$$= -\alpha \nabla_t \boldsymbol{\nabla} \varphi_0 + \alpha \nabla_t \boldsymbol{\nabla} \varphi_0 + \nabla_t \nabla_t \boldsymbol{\varphi} - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi}$$

In quaternionic differential calculus $\alpha = -1$ and $\phi_0 = -\varkappa$. If in Maxwell equations the Lorentz gauge $\alpha = 1$ is applied, then:

$$(\alpha\, \nabla_t \nabla_t - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle) \varphi_0 = \rho_0 = \nabla_t \varkappa + \langle \boldsymbol{\nabla}, \mathcal{E} \rangle \tag{9}$$

$$\alpha \frac{\partial^2 \varphi_0}{\partial t^2} - \frac{\partial^2 \varphi_0}{\partial x^2} - \frac{\partial^2 \varphi_0}{\partial y^2} - \frac{\partial^2 \varphi_0}{\partial z^2} = \rho_0 \tag{10}$$

$$(\alpha\, \nabla_\tau \nabla_\tau - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle) \boldsymbol{\varphi} = \boldsymbol{J} = \boldsymbol{\nabla} \times \mathcal{B} - \alpha\, \nabla_\tau \mathcal{E} - \boldsymbol{\nabla} \varkappa \tag{11}$$

(12)

$$\alpha\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2}-\frac{\partial^2\boldsymbol{\varphi}}{\partial x^2}-\frac{\partial^2\boldsymbol{\varphi}}{\partial y^2}-\frac{\partial^2\boldsymbol{\varphi}}{\partial z^2}=\boldsymbol{J}$$

This corresponds to the Minkowski signature.

$$\{\rho_0,\boldsymbol{J}\}\Leftrightarrow\{\nabla_t\varkappa-\langle\boldsymbol{\nabla},\boldsymbol{\mathcal{E}}\rangle,-\boldsymbol{\nabla}\varkappa+\boldsymbol{\nabla}\times\boldsymbol{\mathcal{B}}-\alpha\,\nabla_\tau\boldsymbol{\mathcal{E}}\}\quad(13)$$

$$=\{\nabla_t\varkappa,-\boldsymbol{\nabla}\varkappa\}+\{\langle\boldsymbol{\nabla},\boldsymbol{\mathcal{E}}\rangle,\boldsymbol{\nabla}\times\boldsymbol{\mathcal{B}}-\alpha\,\nabla_\tau\boldsymbol{\mathcal{E}}\}$$

Adding equation (1) as an extra Maxwell equation would bring Maxwell equations more in conformance with the equations of quaternionic differential calculus.

Notice the difference of the Minkowski signature of these equations with the Euclidean signature of the wave function of quaternionic differential calculus. This difference is enforced by the selection of the value of α.

9.5.3 Difference with Maxwell based equations

The difference between the Maxwell-Minkowski based approach and the Hamilton-Euclidean based approach will become clear when the difference between the coordinate time t and the proper time τ is investigated. This becomes difficult when space is curved, but for infinitesimal steps space can be considered flat. In that situation holds:

Coordinate time step vector = proper time step vector + spatial step vector (1)

Or in quaternionic format ($\Delta\tau\equiv\ \Delta t_0$):

$$\Delta t=\Delta t_0+\boldsymbol{i}\,\Delta x+\boldsymbol{j}\,\Delta y+\boldsymbol{k}\,\Delta z\quad(2)$$

Or in Pythagoras format:

$$(\Delta t)^2\ =\ (\Delta\tau)^2+(\Delta x)^2+(\Delta y)^2+(\Delta z)^2\quad(3)$$

$$(\Delta\tau)^2\ =\ (\Delta t)^2-(\Delta x)^2-(\Delta y)^2-(\Delta z)^2\quad(4)$$

In quaternionic terms this means that t corresponds to quaternionic distance.

$$t = |x| = |\tau + \boldsymbol{x}| \tag{5}$$

This influence is easily recognizable in the corresponding wave equations:

In Maxell-Minkowski format the homogeneous wave equation uses coordinate time t. It runs as:

$$\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial z^2} = 0 \tag{6}$$

Papers on Huygens principle work with this formula or it uses the version with polar coordinates.

For 3D the general solution runs:

$$\psi\ = f(r - ct)/r, \text{ where } c = \pm 1; f \text{ is real} \tag{7}$$

For 1D the general solution runs:

$$\psi\ = f(x - ct), \text{ where } c = \pm 1; f \text{ is real} \tag{8}$$

In comparison the quaternionic differential calculus produces another homogeneous wave equation, which uses proper time τ. In this case we use the quaternionic nabla ∇:

$$\nabla = \left\{\frac{\partial}{\partial \tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} = \nabla_0 + \boldsymbol{\nabla}; \tag{9}$$

$$\nabla^* = \nabla_0 - \boldsymbol{\nabla} \tag{10}$$

$$\nabla\psi\ =\ \nabla_0\, \psi_0 - (\boldsymbol{\nabla}, \boldsymbol{\psi}) + \nabla_0\, \boldsymbol{\psi}\ +\ \boldsymbol{\nabla}\, \psi_0\ \pm\ \boldsymbol{\nabla}\ \times \boldsymbol{\psi} \tag{11}$$

The $\pm$ sign reflects the choice between right handed and left handed quaternions.

In this way the quaternionic format of the wave equation runs:

$$\nabla^*\nabla\psi\ =\ \nabla_0\nabla_0\psi\ + (\boldsymbol{\nabla}, \boldsymbol{\nabla})\psi\ = 0 \tag{12}$$

$$\frac{\partial^2\psi}{\partial\tau^2}+\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=0 \tag{13}$$

Where $\psi = \psi_0 + \boldsymbol{\psi}$

For the general solution holds: $f = f_0 + \boldsymbol{f}$

For the real part ψ_0 of ψ:

$$\psi_0 = f_0\,(\boldsymbol{i}\,r - c\,\tau)/r \tag{14}$$

where $c = \pm 1$ and $\boldsymbol{i}$ is an imaginary base vector in radial direction

For the imaginary part $\boldsymbol{\psi}$ of ψ:

$$\boldsymbol{\psi} = \boldsymbol{f}(\boldsymbol{i}\,z - c\,\tau) \tag{15}$$

where $c = \pm 1$ and $\boldsymbol{i} = \boldsymbol{i}(z)$ is an imaginary base vector in the x, y plane

The orientation $\theta(z)$ of $\boldsymbol{i}(z)$ in the x, y plane determines the polarization of the 1D wave front.

9.5.4 The screened Poisson equation

The screened Poisson equation runs:

$$\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\chi - \lambda^2\chi = \rho \tag{1}$$

In Maxwell based differential calculus this corresponds to:

$$\frac{\partial}{\partial t}\frac{\partial}{\partial t}\chi = \lambda^2\,\chi \tag{2}$$

A solution of this equation is

$$\chi = a(\boldsymbol{x})\,\exp(\pm\,\lambda\,t) \tag{3}$$

This differs significantly from the quaternionic differential calculus version.

10 Integral continuity equations

The integral equations that describe cosmology are:

$$\int_V \nabla \rho \, dV = \int_V s \, dV \tag{1}$$

$$\int_V \nabla_0 \rho_0 \, dV = \int_V \langle \boldsymbol{\nabla}, \boldsymbol{\rho} \rangle \, dV + \int_V s_0 \, dV \tag{2}$$

$$\int_V \nabla_0 \, \boldsymbol{\rho} \, dV = -\int_V \boldsymbol{\nabla} \rho_0 \, dV - \int_V \boldsymbol{\nabla} \times \boldsymbol{\rho} \, dV + \int_V \boldsymbol{s} \, dV \tag{3}$$

$$\frac{d}{d\tau} \int_V \rho \, dV + \oint_S \hat{\boldsymbol{n}} \rho \, dS = \int_V s \, dV \tag{4}$$

Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface *S*, $\boldsymbol{v}(\tau, \boldsymbol{q})$ is the velocity at which the charge density $\rho_0(\tau, \boldsymbol{q})$ enters volume *V* and s_0 is the source density inside *V*. If ρ_0 is stable then in the above formula ρ stands for

$$\rho = \rho_0 + \boldsymbol{\rho} = \rho_0 + \frac{\rho_0 \boldsymbol{v}}{c} \tag{4}$$

It is the flux (flow per unit of area and per unit of progression) of ρ_0 . τ stands for progression.

11 Metric

The differential of the sharp allocation function $\wp$ defines a kind of quaternionic metric.

$$ds(q) = ds^{\nu}(q) e_{\nu} = d\wp = \sum_{\mu=0\ldots3} \frac{\partial \wp}{\partial q_{\mu}} dq_{\mu} = c^{\mu}(q) dq_{\mu} \tag{1}$$

q is the quaternionic location.

ds is the metric.

c^{μ} is a quaternionic function.

Pythagoras:

$$c^2 dt^2 = ds\, ds^* = dq_0^2 + dq_1^2 + dq_2^2 + dq_3^2 \tag{2}$$

Minkowski:

$$dq_0^2 = d\tau^2 = c^2\, t^2 - dq_1^2 - dq_2^2 - dq_3^2 \tag{3}$$

In flat space:

$$\Delta s_{flat} = \Delta q_0 + \boldsymbol{i}\, \Delta q_1 + \boldsymbol{j}\, \Delta q_2 + \boldsymbol{k}\, \Delta q_3 \tag{4}$$

In curved space:

$$\Delta s_{\wp} = c^0\, \Delta q_0 + c^1\, \Delta q_1 + c^2\, \Delta q_2 + c^3\, \Delta q_3 \tag{5}$$

$d\wp$ is a quaternionic metric

It is a linear combination of 16 partial derivatives

12 Tri-state spaces

Quaternions not only fit in the representation of dynamic geometric data. They also match in representing three-fold states such as the RGB colors of quarks and the three generation flavors of fermions. In all these roles the real part of the quaternion plays the role of progression. Thus quaternions can also be used to model neutrino flavor mixing.

Say that a property is distributed over three mutually independent modes and these modes exist in a combination that superposes these three modes.

The property distribution is characterized by p_x, p_y, p_z

$$\cos^2(\theta_x) = \frac{p_x}{p_x + p_y + p_z}$$

$$\cos^2(\theta_x) + \cos^2(\theta_y) + \cos^2(\theta_z) = 1$$

The angles $\theta_x, \theta_y, \theta_z$ indicate a direction vector $\mathbf{n} = \{n_x, n_y, n_z\}$ in three dimensional state space.

$$|n_x|^2 = \frac{p_x}{p_x + p_y + p_z}$$

$$\cos(\theta_x) = n_x;\ |\boldsymbol{n}| = 1$$

If state mixing is a dynamic process, then the axis along direction vector $\boldsymbol{n}$ acts as the rotation axis. The concerned subsystem rotates smoothly as a function of progression. This is not a rotation in configuration space. Instead it is a rotation in tri-state space.

The fact that quaternions can rotate the imaginary part of other quaternions or of complete quaternionic functions also holds for tri-states. The quaternions that have equal real and imaginary size play a special role. They can shift an anisotropic property to another dimension. They can play a role in tri-state flavor switching.

13 Formula compendium

13.1 Vectors

Please notice that the vectors, which are treated here are 3D vectors and not imaginary parts of quaternions. However, all equations that are exposed here also hold for the imaginary parts of quaternions.

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \langle \boldsymbol{b}, \boldsymbol{a} \rangle = \delta_{ij} a_i b_j = |\boldsymbol{a}||\boldsymbol{b}| cos(\theta) \tag{1}$$

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} = \epsilon_{ijk} \widehat{\boldsymbol{x}}_{\boldsymbol{i}} a_j b_k \tag{2}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle^2 + \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{b} \rangle^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \tag{3}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \times \boldsymbol{c} \rangle = \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{c} \rangle \tag{4}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{0} \tag{5}$$

$$\langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{c} \times \boldsymbol{d} \rangle = \langle \boldsymbol{a}, \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d}) \rangle = \langle \boldsymbol{a}, \boldsymbol{c} \rangle \langle \boldsymbol{b}, \boldsymbol{d} \rangle - \langle \boldsymbol{a}, \boldsymbol{d} \rangle \langle \boldsymbol{b}, \boldsymbol{d} \rangle \tag{6}$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \langle \boldsymbol{a}, \boldsymbol{c} \rangle\, \boldsymbol{b} - \langle \boldsymbol{a}, \boldsymbol{b} \rangle \boldsymbol{c} \tag{7}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \langle \boldsymbol{a}, \boldsymbol{c} \rangle\, \boldsymbol{b} - \langle \boldsymbol{b}, \boldsymbol{c} \rangle \boldsymbol{a} \tag{8}$$

13.2 Nabla

13.3 Special

$$\boldsymbol{\nabla}\langle \boldsymbol{k}, \boldsymbol{x}\rangle = \boldsymbol{k} \tag{1}$$

$\boldsymbol{k}$ is constant.

$$\langle \boldsymbol{\nabla}, \mathbf{x}\rangle = \mathbf{3} \tag{2}$$

$$\boldsymbol{\nabla} \times \mathbf{x} = \mathbf{0} \tag{3}$$

$$\boldsymbol{\nabla}|\mathbf{x}| = \frac{\mathbf{x}}{|\mathbf{x}|} \tag{4}$$

$$\boldsymbol{\nabla}\frac{1}{|\mathbf{x}|} = -\frac{\mathbf{x}}{|\mathbf{x}|^3} \tag{5}$$

$$\langle \boldsymbol{\nabla}, \frac{\mathbf{k}}{|\mathbf{x}|}\rangle = -\frac{\langle \mathbf{k}, \mathbf{x}\rangle}{|\mathbf{x}|^3} \tag{6}$$

$$\langle \boldsymbol{\nabla}, \frac{\mathbf{x}}{|\mathbf{x}|^3}\rangle = \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\frac{1}{|\mathbf{x}|} = 4\pi\delta(\mathbf{x}) \tag{7}$$

$$\boldsymbol{\nabla} \times \left(\mathbf{k} \times \frac{\mathbf{x}}{|\mathbf{x}|^3}\right) = -\boldsymbol{\nabla} \times \left(\mathbf{k} \times \boldsymbol{\nabla}\frac{1}{|\mathbf{x}|}\right) = \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \frac{\mathbf{k}}{|\mathbf{x}|}\rangle \tag{8}$$

Under spherical conditions, the function $\frac{1}{|\mathbf{x}|}$ corresponds to the Green's function. In that case:

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) \tag{9}$$

13.4 Quaternionic nabla special

The following equations treat formulas that apply to the quaternionic nabla and parameter values.

$$x = x_0 + \boldsymbol{x}\,; x^* = x_0 - \boldsymbol{x}\,;\ \nabla = \nabla_0 + \boldsymbol{\nabla}\,; \nabla^* = \nabla_0 - \boldsymbol{\nabla} \tag{1}$$

$$\nabla x = 1 - 3\,;\ \nabla^* x = 1 + 3;\ \nabla x^* = 1 + 3 \tag{2}$$

$$\nabla(x^*x) = x$$

$$\nabla|x| = \nabla\sqrt{(x^*x)} = \frac{x}{|x|} \tag{3}$$

$$\nabla\frac{1}{|x-x'|} = -\frac{x-x'}{|x-x'|^3} \tag{4}$$

For the vector nabla case holds for spherical boundary conditions:

$$\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = 4\pi\,\delta(\boldsymbol{x}-\boldsymbol{x}') \qquad 8)$$

The next formula does not correspond to the vector nabla case.

$$\nabla^*\frac{x}{|x|^3} = \nabla\,\nabla^*\frac{1}{|x|} = (\nabla_0\nabla_0 + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\frac{1}{|x|} \tag{6}$$

$$= \frac{3\tau^2}{|x|^5} - \frac{1}{|x|^3} + \frac{3\tau^2}{|x|^5} = \frac{6\tau^2 - |x|^2}{|x|^5} = \frac{5\tau^2 - |\boldsymbol{x}|^2}{|x|^5}$$

$$\nabla_0\nabla_0\frac{1}{|x|} = -\nabla_0\frac{\tau}{|x|^3} = 3\frac{\tau^2}{|x|^5} - \frac{1}{|x|^3} \tag{7}$$

$$(\nabla_0\nabla_0 - \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\frac{1}{|x|} = -\frac{1}{|x|^3} \tag{8}$$

Thus, with spherical boundary conditions, $\frac{1}{4\pi\,|x-x'|}$ is suitable as the Green's function for the Poisson equation, but $\frac{1}{4\pi\,|x-x'|}$ does not represent a Green's function for the quaternionic operator $(\nabla_0\nabla_0 + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)$!

13.5 Functions

$$\boldsymbol{\nabla}a_0 = \hat{\boldsymbol{x}}_{\boldsymbol{i}}\partial_i a_0 \tag{1}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle = \partial_i a_i \tag{2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{a} = \epsilon_{ijk} \hat{\boldsymbol{x}}_{\boldsymbol{i}} \partial_j a_k \tag{3}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} a_0 \rangle = \boldsymbol{\nabla}^2 a_0 \tag{4}$$

$$\boldsymbol{\nabla}(a_0 b_0) = a_0 \boldsymbol{\nabla}(b_0) + b_0 \boldsymbol{\nabla}(a_0) \tag{5}$$

$$\langle \boldsymbol{\nabla}, a_0 \boldsymbol{a} \rangle = \langle \mathbf{a}, \boldsymbol{\nabla} a_0 \rangle + a_0 \langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle \tag{6}$$

$$\langle \boldsymbol{\nabla} a_0, \boldsymbol{\nabla} b_0 \rangle = \langle \boldsymbol{\nabla}, a_0 \boldsymbol{\nabla} b_0 \rangle - a_0 \boldsymbol{\nabla}^2 b_0 \tag{7}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{a} \times \boldsymbol{b} \rangle = \langle \mathbf{b}, \boldsymbol{\nabla} \times \boldsymbol{a} \rangle - \langle \mathbf{a}, \boldsymbol{\nabla} \times \boldsymbol{b} \rangle \tag{8}$$

$$\langle \boldsymbol{\nabla} a_0, \boldsymbol{\nabla} \times \boldsymbol{a} \rangle = -\langle \boldsymbol{\nabla}, \boldsymbol{a} \times \boldsymbol{\nabla} a_0 \rangle \tag{9}$$

$$\langle \boldsymbol{\nabla} \times \boldsymbol{a}, \boldsymbol{\nabla} \times \boldsymbol{b} \rangle = \langle \boldsymbol{b}, \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) \rangle - \langle \boldsymbol{\nabla}, (\boldsymbol{\nabla} \times \boldsymbol{a}) \times \boldsymbol{b} \rangle \tag{10}$$

$$\boldsymbol{\nabla} \times (a_0 \boldsymbol{a}) = a_0 \boldsymbol{\nabla} \times \boldsymbol{a} - \boldsymbol{a} \times \boldsymbol{\nabla} a_0 \tag{11}$$

$$\boldsymbol{\nabla} \times (a_0 \boldsymbol{\nabla} a_0) = (\boldsymbol{\nabla} a_0) \times \boldsymbol{\nabla} b_0 \tag{12}$$

$$\langle \mathbf{a}, \boldsymbol{\nabla} \times \boldsymbol{b} \rangle = \langle \mathbf{a} \times \boldsymbol{\nabla}, \boldsymbol{b} \rangle \tag{13}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{a} \rangle = 0 \tag{14}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} a_0 = \mathbf{0} \tag{15}$$

$$\langle \boldsymbol{\nabla} \times \boldsymbol{\nabla}, \boldsymbol{a} \rangle = 0 \tag{16}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) = \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{a}\rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle \boldsymbol{a} \tag{17}$$

14 Remarks

14.1 Non-homogeneous wave equation

$$\nabla^* \nabla \psi = \nabla_0 \nabla_0 \psi + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle \psi = \xi \tag{1}$$

The corresponding Poisson equation is

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle \psi = \rho \tag{2}$$

The non-homogeneous wave equation corresponds to two continuity equations.

$$m\,\varphi = \nabla \psi\,;\; \nabla^* \varphi = \frac{\xi}{m} \tag{3}$$

14.2 Green's function

The Green's function

$$G(\boldsymbol{x} - \boldsymbol{x}') = \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{1}$$

makes sense for the integration of imaginary parts of quaternionic functions, as follows from:

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = 4\pi\, \delta|\boldsymbol{x} - \boldsymbol{x}'| \tag{2}$$

The equivalent

$$F(x - x') = \frac{1}{|x - x'|}$$

does not act as a Green's function. This follows from:

$$\nabla_0 \nabla_0 \frac{1}{|x|} = -\nabla_0 \frac{\tau}{|x|^3} = \frac{3\tau^2}{|x|^5} - \frac{1}{|x|^3} \tag{2}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \frac{1}{|x|} = \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \frac{1}{|x|} \rangle = -\langle \boldsymbol{\nabla}, \frac{\boldsymbol{x}}{|x|^3} \rangle = \frac{3\tau^2}{|x|^5} \tag{3}$$

$$\nabla \nabla^* \frac{1}{|x - x'|} = (\nabla_0 \nabla_0 + \boldsymbol{\nabla}, \boldsymbol{\nabla}) \frac{1}{|x - x'|} = \frac{6\tau^2}{|x|^5} - \frac{1}{|x|^3} \tag{4}$$