W.B.VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

ALGEBRAIC STRUCTURES ON THE FUZZY INTERVAL [0, 1)

Algebraic Structures on the Fuzzy Interval [0, 1)

W. B. Vasantha Kandasamy
Florentin Smarandache

Educational Publisher Inc.
Ohio
2014

This book can be ordered from:

Education Publisher Inc.
1313 Chesapeake Ave.
Columbus, Ohio 43212,
USA
Toll Free: 1-866-880-5373

Peer reviewers:

Dr. Christopher Dyer, University of New Mexico, Gallup, USA.
Prof. Jason Dyer, Tucson, Arizona, USA.
Scientist Liviu Ardeias, Schelklingen, Germany.

Many books can be downloaded from the following
Digital Library of Science:
http://www.gallup.unm.edu/eBooks-otherformats.htm

ISBN-13: 978-1-59973-259-6
EAN: 9781599732596

Printed in the United States of America

CONTENTS

PREFACE

In this book we introduce several algebraic structures on the special fuzzy interval [0, 1). This study is different from that of the algebraic structures using the interval [0, n) $n \neq 1$, as these structures on [0, 1) has no idempotents or zero divisors under $\times$. Further [0, 1) under product $\times$ is only a semigroup. However by defining min(or max) operation in [0, 1); [0, 1) is made into a semigroup.

The semigroup under $\times$ has no finite subsemigroups but under min or max we have subsemigroups of order one, two and so on. [0, 1) under + modulo 1 is a group and [0, 1) has finite subgroups.

We study [0, 1) with two binary operations min and max resulting in semiring of infinite order. This has no subsemirings which is both an ideal and a filter. However pseudo semiring under min and $\times$ has subsemirings which is both a filter and an ideal. Construction and study of pseudo rings on [0, 1) is interesting as distributive law is not true. Study of algebraic structures on the fuzzy interval [0, 1) is innovative and interesting.

We wish to acknowledge Dr. K Kandasamy for his sustained support and encouragement in the writing of this book.

W.B.VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

Chapter One

INTRODUCTION

In this book, we for the first time introduce a new type of special fuzzy set algebraic structures using [0,1). These new structures are of infinite cardinality and enjoy special features.

To make this book a self contained one we give the references of the books used in this study. For semigroups refer [6]. Further the notion of group semirings and semigroup semirings are used in later chapter of this book. Please refer [6, 7].

We use also the concept of special fuzzy set pseudo grouprings and special fuzzy set pseudo semigroup rings.

In this book $S = \{[0,1), \min, \max\}$ is a semiring of infinite order.

$R = \{[0, 1), +, \times\}$ is a pseudo special fuzzy ring of infinite order as $(a + b)\, c \neq ac + bc$ in general for $a, b, c \in [0, 1)$.

S is known (defined) as the special fuzzy set pseudo semiring if min and $\times$ operation is used and R will be known as the special fuzzy set pseudo ring.

However S is not a special integral domain for take

$$(0.7 + 0.3)\ 0.2 = (0.14 + 0.06) = 0.20.$$

Now $(0.7 + 0.3)\ 0.2 = 1 \times 0.2 = 0$. So $(a + b)\ c \neq ac + bc$ in general for $a, b, c \in [0, 1)$.

Study in this direction is carried out in this book. The study of algebraic structures using $[0, n)$, $n \neq 1$ and greater than one has been systematically developed in [8]. When $n = 1$we call the new algebraic structure as special fuzzy algebraic structures such as special fuzzy or special fuzzy groups under '+' and so on.

The special feature is these $S = \{[0, 1), +\ x\}$ is a special pseudo fuzzy set which has no zero divisors and these rings donot satisfy the distributive laws.

Chapter Two

SEMIGROUP AND GROUP STRUCTURES ON THE SPECIAL FUZZY INTERVAL [0,1)

In the first place we wish to state that the interval [0,1) contains all elements on the continuous interval 0 to 1 excluding 1. Thus [0, 1) has infinite number of elements in it. We give on [0, 1) two algebraic structures and then study them.

We call this fuzzy interval [0, 1) or any section of it as a special fuzzy interval.

We see [0, 1) under usual product is a semigroup. Further [0, 1) under product is never a group as no element x in [0, 1) is a multiplicative identity; hence cannot have the concept of x inverse for any element x in [0, 1) as $1 \notin [0, 1)$.

DEFINITION 2.1: *Let $M = \{[0, 1), \times\}$ be the semigroup. M is defined as the special fuzzy set semigroup under $\times$.*

Example 2.1: Let $M = \{[0, 1), \times\}$ be the special fuzzy set semigroup.

For $x = 0.5$ and $y = 0.2$ in M we see $x \times y = 0.10$ and for $x_1 = 0.9$ and $y_1 = 0.7$ in M we have $x_1 \times y_1 = 0.63$ and so on.

Take $P_2 = \{0, ½, 1/2^2, \ldots, 1/2^n ; n \to \infty\} \subseteq [0, 1)$.

P_2 under product $\times$ is a special fuzzy set subsemigroup of M.

Infact M has infinite number of special fuzzy set subsemigroups and each is of infinite order.

$P_m = \{0, 1/m, 1/m^2, \ldots, 1/m^n; n \to \infty\} \subseteq [0, 1)$ is a special fuzzy set subsemigroup for $m \in N \setminus \{0,1\}$.

This is not only the way special fuzzy set subsemigroups can be got.

We can have other ways of finding them.

Example 2.2: Let $M = \{[0, 1), \times\}$ be the special fuzzy set semigroup. Now take $P = \{[0, 0.07), \times\} \subseteq M$; P is again a special fuzzy set subsemigroup of M.

We can have any $P_i = \{[0\ j/i] \mid j < i$, i, j positive numbers $i \neq 0$ or 1, $j \neq 0\} \subseteq M$ leads to special fuzzy set subsemigroup.

Infact M has infinite number of such special fuzzy set subsemigroups and each P_i is of infinite order. Further if $i = 1$ we do not get P_1 as we cannot find a j with $j < i$, so only we avoid $i = 1$.

It is important to observe that the special fuzzy set subsemigroups in example 2.1 is different from those special fuzzy set subsemigroups given in example 2.2.

The differences are very obvious for in example 2.1 the elements are discrete as elements of a special fuzzy set subsemigroup and that of the subsemigroups in example 2.2 are continuous subintervals in [0, 1).

Example 2.3: Let M = {[0, 1), $\times$} be a special fuzzy set subset semigroup.

Let P = {0, 3/5, $(3/5)^2$, …, $(3/5)^n$ as $n \to \infty$} $\subseteq$ M be a special fuzzy set subsemigroup of M.

Take B = {0, 2/7, $(2/7)^2$, …, $(2/7)^n$ as $n \to \infty$} $\subseteq$ M to be a special fuzzy set subsemigroup of M.

Example 2.4: Let W = {[0, 1), $\times$} be the special set fuzzy semigroup.

T = {0, 2/11, $(2/11)^2$, $(2/11)^3$, …, $(2/11)^n$ as $n \to \infty$} $\subseteq$ W be the special fuzzy set subsemigroup of W.

V = {[0.03]} $\subseteq$ W is a fuzzy set special subsemigroup of W. V is continuous and not discrete like T.

Thus we have two types of special fuzzy set subsemigroups.

Can these special fuzzy set subsemigroups be special fuzzy set ideals of the special fuzzy set semigroup?

The answer is no in general.

The following examples prove our claim.

Example 2.5: Let M = {[0, 1), $\times$} be the special fuzzy set semigroup.

P = {0, 1/5, $1/5^2$, $1/5^3$, …} is the special fuzzy set subsemigroup of M.

We see P is not a special fuzzy set ideal of M.

For $3/7 \in M$, $3/7 \times 1/5 = 3/35 \notin P$, so P is not a special fuzzy set ideal only a subsemigroup of M.

Let $T = \{[0, 0.3]\} \subseteq M$ be the special fuzzy set subsemigroup of M. Clearly T is a special fuzzy set ideal of M.

For one of the largest elements in M say M = (9.999999) and y = 0.3 we have $x.y \in T$.

Thus some closed intervals can be taken as special fuzzy set subsemigroup which also happens to be a special fuzzy set ideal of M.

Example 2.6: Let $M = \{[0, 1), \times\}$ be a special fuzzy set semigroup.

P = (0.2, 0.4) is not a subset fuzzy set subsemigroup of M.

For $0.2 \times 0.2 = 0.04 \notin P$ so the very closure axiom is not true.

Hence the claim. $P_1 = (0.61, 0.9)$ is not a subset fuzzy set subsemigroup of M.

For 0.7 and $0.63 \in P_1$ we see $0.63 \times 0.7 = 0.441 \notin P_1$ so P_1 is not a fuzzy set subsemigroup of M.

Inview of this we have the following theorem.

THEOREM 2.1: *Let $M = \{[0, 1), \times\}$ be the special fuzzy set semigroup.*

(i) Only intervals of the form $P = \{[0, a) \mid a < 1\} \subseteq M$ are special fuzzy set subsemigroups.
(ii) An interval of the form $(a, b) \subseteq [0, 1)$; $0 < a$ can never be a special fuzzy set subsemigroup of M.

The proof is direct and hence left as an exercise to the reader.

Example 2.7: Let M = {[0, 1), $\times$} be a special fuzzy set semigroup.

P = {[0, 0.7), $\times$} $\subseteq$ M is a special set fuzzy subsemigroup of M. Clearly P is also a special fuzzy set ideal of M.

We see if T = {[0, 0.3), $\times$} $\subseteq$ M is again a special fuzzy set ideal of M then T $\cap$ P = T that is T is contained in P as ideal.

However if W = {0, 1/2, $1/2^2$, …, $1/2^n$, …, n $\to \infty$} and L = {0, 2/7, $2^2/7^7$, …, $(2/7)^n$, …} be the two special fuzzy set subsemigroups of M.

We see W $\cap$ L = {0} and W $\cup$ L is not a special fuzzy set subsemigroup of M. However W $\cup$ L can generate a special fuzzy set subsemigroup of M.

Example 2.8: Let M = {[0, 1), $\times$} be the special fuzzy set semigroup. We see P_i = {[0, p_i), $\times$} $\subseteq$ M are special fuzzy set subsemigroups of M, where $0 < p_1 < p_2 < p_3 < p_4 < \ldots < p_n \ldots < 1$.

Hence $\{0\} \subset P_1 \subseteq P_2 \subseteq P_3 \subseteq \ldots \subseteq P_n \subseteq \ldots \subset M$. So these collection forms an infinite chain of special fuzzy set ideals of M.

However all special fuzzy set subsemigroups of M do not form a chain of this type.

For take
$V_1 = \{0, 1/2, (1/2)^2, \ldots, (1/2)^n, \ldots, \infty\}$,
$V_2 = \{0, 1/3, (1/3)^2, \ldots, (1/3)^n, \ldots, \infty\}$,
$V_3 = \{0, 1/5, (1/5)^2, \ldots, (1/5)^n, \ldots, \infty\}$,
$V_4 = \{0, 3/4, (3/4)^2, \ldots, (3/4)^n, \ldots, \infty\}$ and so on be special fuzzy set subsemigroups of M.

We see these are not comparable in general. Further $V_i \cap V_j = \{0\}$ we do not even have any common elements.

However consider the special fuzzy set subsemigroup V_1; let $W_1 = \{0, (1/2)^2, (1/2)^4, (1/2)^6, \ldots, (1/2)^{2n}, \ldots\} \subseteq V_1$, W_1 is a special fuzzy set subsemigroup of V_1 and infact also a special fuzzy set subsemigroup of the special fuzzy set subsemigroup V_1.

Let us denote by

$W_2 = \{0, (1/2)^3, (1/2)^6, (1/2)^9, \ldots, (1/2)^{3n}, \ldots\}$,
$W_3 = \{0, (1/2)^4, (1/2)^8, (1/2)^{12}, \ldots, (1/2)^{4n}, \ldots\}$,
$W_4 = \{0, (1/2)^5, (1/2)^{10}, (1/2)^{15}, \ldots, (1/2)^{5n}, \ldots\}$;
$W_5 = \{0, (1/2)^6, (1/2)^{12}, (1/2)^{18}, \ldots, (1/2)^{6n}, \ldots\}$ and so on.

$W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$ however $W_1 \cap W_2 \neq \{0\}$.

We see these W_i's as special fuzzy set subsubsemigroups of W_1.

Also W_1 is a special fuzzy set subideal over the special fuzzy set sub subsemigroups W_i.

For we have if $x \in W_1$ and $y \in W_i$ then $xy \in W_1$ $(i \neq 1)$; however $xy \notin W_i$ $(i \neq 1)$ for all $x \in W_1$ and $y \in W_i$; $(i = 3, 5, 7, \ldots)$.

Similarly if $T_1 = \{0, (1/3), (1/3)^2, \ldots, (1/3)^n, \ldots\} \subseteq M$; T_1 is a special fuzzy set subsemigroup of M and not a special fuzzy set ideal of M, however T_1 is a special fuzzy set subideal of M over T_i $(i \neq 1)$.

$T_i = \{0, (1/3)^i, (1/3)^{2i}, \ldots\}$ $i \neq 1$, $i = 2, 3, \ldots$.

We see T_1 is a special fuzzy set subideal of M over infinite number of special fuzzy set subsemigroups of M.

Also each $T_i \subsetneq T_1$ $(i \neq 1)$. Thus though the discrete special fuzzy set subsemigroups are not special fuzzy set ideals of M still they are special fuzzy set subideals over appropriate special fuzzy set subsemigroups of M.

Let $B_1 = \{0, 1/19, (1/19)^2, \ldots, (1/19)^n, \ldots\} \subseteq M$. B_1 is only a special fuzzy set subsemigroup of M; however B_1 is not a special fuzzy set ideal of M.

$B_i = \{0, (1/19)^i, (1/19)^{2i}, (1/19)^{3i}, \ldots, (1/19)^{ni}, \ldots\} \subseteq B_1$ is a special fuzzy set subsemigroups of M. i = 2, 3, ….; B_1 is a special fuzzy set subideal of M over the special fuzzy set subsemigroups B_i of B_1.

$C_1 = \{0, 12/21, (12/21)^2, (12/21)^3, \ldots, (12/21)^n, \ldots\} \subseteq M$. C_1 is a special fuzzy set subsemigroup of M.

$C_2 = \{0, (12/21)^2, (12/21)^4, (12/21)^6, \ldots, (12/21)^{2n}, \ldots\} \subseteq C_1$. C_2 is a special fuzzy set subsemigroup of M.

C_2 is a special fuzzy set subsubsemigroup of $C_1 \subseteq M$. C_1 is a special fuzzy set subideal of $C_2 \subseteq M$.

Example 2.9: Let $M = \{[0, 1), \times\}$ be a special fuzzy set semigroup.

We see given any discrete special fuzzy set subsemigroup P_1 then P_1 is a special fuzzy set subideal over special fuzzy set sub subsemigroups of P_1.

Infact any P_n can be realized as a special fuzzy set subideal over a special fuzzy set sub subsemigroup of M.

Infact P_n is a special fuzzy set subideal over infinite number of special fuzzy set sub subsemigroups.

THEOREM 2.2: *The special fuzzy set semigroup $M = \{[0, 1), \times\}$ is never a monoid.*

Proof is direct as the multiplicative identity $1 \notin [0, 1)$.

THEOREM 2.3: *The special fuzzy set semigroup $M = \{[0, 1), \times\}$ has infinite number of special fuzzy set subsemigroups which are not special fuzzy set ideals.*

Can be proved by examples by the interested reader.

THEOREM 2.4: *The special fuzzy set subsemigroup $M = \{[0, 1), \times\}$ has infinite number of special fuzzy set subsemigroups which are special fuzzy set ideals.*

Left as an exercise for the reader to prove.

THEOREM 2.5: *The special fuzzy set semigroup $M = \{[0, 1), \times\}$ has infinite number of special fuzzy set subideals over infinite number of special fuzzy set sub subsemigroups of M.*

The proof is direct as evident from the examples given.

THEOREM 2.6: *The special fuzzy set semigroup $M = \{[0, 1), \times\}$ has no nontrivial special fuzzy set subsemigroup of finite order.*

Now having seen some of the properties of these substructures we proceed onto define the notion of special cyclic fuzzy set subsemigroups of $M = \{[0, 1), \times\}$.

DEFINITION 2.2: *Let $M = \{[0, 1), \times\}$ be a special fuzzy set semigroup. $P \subseteq M$ be a proper special fuzzy set subsemigroup of M. We say P is special fuzzy set cyclic subsemigroup; if P is generated by a single element.*

We define for $(p/q)^n \in M$ $(p < q)$;

if $n \to \infty$ $(p/q)^n \to 0$ and define this as fuzzy set special convergence in M.

We give examples of them.

Example 2.10: Let $M = \{[0, 1), \times\}$ be the special fuzzy set semigroup.

$P = \{0, 1/3, 1/3^2, 1/3^3, \ldots, 1/3^n, \ldots\} \subseteq M$ is a special fuzzy set subsemigroup of M. We see P is a special fuzzy set cyclic subsemigroup of M.

$W = \{[0, 0.3), \times\} \subseteq M$ is a special fuzzy set subsemigroup of M and W is not a special fuzzy set cyclic subsemigroup of M. Infact W is not generated by any collection of elements.

Take $V = \{0, 1/2, 1/2^2, 1/2^3, \ldots, 1/2^n, \ldots, 1/3, 1/3^2, 1/3^3, \ldots, 1/3^n, \ldots, 1/2^3, 1/2^23, 1/23^2, 1/2^23^2, \ldots, 1/2^m3^n \ldots\} \subseteq M$ is a special fuzzy set subsemigroup of M.

Infact V is not cyclic the special generators of V are {1/2, 1/3}.

Thus we see M has special fuzzy set subsemigroup which has two generators, three generators so on say n generators, $n \to \infty$.

Inview of all these we have the following theorem.

THEOREM 2.7: *Let M = {[0, 1), ×} be a special fuzzy set subsemigroup.*

(1) *M has special fuzzy set subsemigroups which are special cyclic.*
(2) *M has special fuzzy set subsemigroups which has n generators; $n \in N \setminus \{1\}$.*
(3) *M has special fuzzy set subsemigroups with countable number of special generators.*
(4) *M has special fuzzy set subsemigroups with uncountable number of generators.*

Proof: We will prove all the parts by illustrative examples.

Take
$B = \{0, 1/p, (1/p)^2, \ldots, (1/p)^n, \ldots n \to \infty\} \subseteq M$ ($p \in Z^+ \setminus \{1\}$). B is a cyclic special fuzzy set subsemigroup generated only by (1/p).

Hence part (1) is true.

Consider T = $\{0, 1/p_1, 1/p_2, \ldots, 1/p_n; (1/p_1)^2, \ldots, (1/p_1)^m, \ldots, (1/p_2)^m, \ldots, (1/p_n)^m, m \to \infty, \left(1\big/p_i^t p_j^s\right), \frac{1}{p_i^t p_j^s p_k^l}, \ldots, \frac{1}{p_1^{t_1} p_2^{t_2} \ldots p_n^{t_n}}, i \neq j, j \neq k$ and $k \neq i$, $t_r \to \infty$, $1 \leq r \leq n$, t, s, and $l \to \infty\}$ be the special fuzzy set subsemigroup of M.

We see T has n generators viz. $1/p_1, 1/p_2, \ldots, 1/p_n$. Hence (2) is true.

Proof of (3).

Let
W = $\{\langle 0, r_1/p_1, r_2/p_2, \ldots, r_n/p_n, n = \infty\rangle\}$ with $r_i < p_i$, $1 \leq i \leq n = \infty$ be the special fuzzy set subsemigroup of M. W has countably infinite number of generators.

Let S = $\{[0,a), \times \mid a < 0.97\} \subseteq M$; S is a special fuzzy set subsemigroup and S has infinite number of generators, that is uncountable number of generators.

Now having seen all these types of special fuzzy set subsemigroups we now give the following corollary.

Corollary 2.1: Let M = $\{[0,1), \times\}$ be the special fuzzy set semigroup (M as in theorem 2.1). Only special fuzzy set subsemigroups in (4) alone are a special fuzzy set ideals and the special fuzzy set subsemigroups in (1), (2) and (3) are not special fuzzy set ideals of M.

Proof is direct from the very structure of the special fuzzy set subsemigroups.

We see these special fuzzy set semigroups M = $\{[0,1), \times\}$ does not contain any zero divisors or units or idempotents.

The special fuzzy set subsemigroups are of infinite order. This special fuzzy set subsemigroup contains infinite number of special fuzzy set subsemigroups which are not special fuzzy set subideals of M.

Further the collection of all special fuzzy set subideals form an infinite chain which has no smallest special fuzzy subideal. The chain is of the form;

$$\{0\} \subseteq \ldots \subseteq \{[0, p_1)\} \subseteq \{[0, p_n)\} \subseteq \ldots. \{[0, 1)\}.$$

This chain is unique for it has no proper least ideal and no proper greatest ideal.

Infact there is infinitely many chains under the assumption if

$\{[0, p_1)\} \subseteq \{[0, p_2)\}$ we can insert infinitely many ideals in between them; as for instance $\left\{[0, \frac{p_1 + p_2}{2})\right\}$ and so on.

The study in this direction is not only innovative but very interesting and can lead to several results which cannot be done in Q or R as they were fields.

However if a special set fuzzy subsemigroup of the form $V_1 = \{0, 1/2, 1/2^2, \ldots, 1/2^n, \ldots\} \subseteq M = \{[0, 1), \times\}$ is taken we can have a chain of special set fuzzy subsemigroups of the form

$$V_{2^2} = \{0, 1/2^2, (1/2^2)^2, \ldots\},$$
$$V_{(2^2)^2} = \{0, (1/2^2)^2, \ldots\},$$
$$V_{(2^2)^4} = \{0, (1/2^2)^4, \ldots\},$$
$$V_{(2^2)^8} = \{0, (1/2^2)^8, \ldots\},$$
$$\vdots$$
$$V_{(2^2)^n} = \{0, (1/2^2)^n, \ldots\} \text{ and so on such that}$$
$$\{0\} \subseteq \ldots \subseteq V_{(2^2)^n} \subseteq \ldots \subseteq V_{(2^2)^8} \subseteq V_{(2^2)^4} \subseteq$$

$$V_{(2^2)^2} \subseteq V_{2^2} \subseteq V_1 \qquad \ldots \text{ I}$$

This has the greatest fuzzy set subsemigroup however no proper least fuzzy set subsubsemigroup.

However they are not fuzzy set subideals of V_1.

By replacing 2 by any odd prime or for that matter by any number we can get infinitely many such chains of fuzzy set subsubsemigroups.

Infact chain I of V_1 is not unique we can have many such chains using V_1.

However all the special fuzzy set subsemirings in the chain I are cyclic for they can be generated by a single element.

Thus the study of the special fuzzy set semigroup $M = \{[0,1), \times\}$ is itself a very innovative and an interesting research for this structure is very dense in giving special fuzzy set subsemigroups and special fuzzy set subideals.

Next we study the structure of $N = ([0,1), +)$ under addition.

It is interesting to record at this point this interval [0, 1) also has no crisp special fuzzy set or pure special fuzzy set so correspondingly we can call $M = \{[0,1), \times\}$ non crisp special fuzzy set semigroup or pure special fuzzy set semigroup.

Now we give on $N = \{[0,1), +\}$ a group structure. As planned this study is also interesting and important.

Let us consider the pure (crisp) fuzzy set [0,1); we define addition as follows . 0 is taken as the additive identity.

Take $x \in [0,1)$ we have a unique $y \in [0,1)$ such that $x + y = 1$ we define this as $x + y \equiv 0 \pmod 1$ so that for instance if $x = 0.002$ then $y = 0.998$ is such that $x + y = 0.002 + 0.998 = 1.000 = 0 \pmod 1$.

Thus we define addition modulo 1. Hence if $x, y \in [0,1)$ then

$$x + y = t \in [0,1) \text{ (if } t < 1)$$

$$x + y = 0 \text{ (if } t = 1)$$

$$x + y = (t-1) \text{ (if } t > 1).$$

This is the way operation + is performed on [0,1).

It is easily verified $M = \{[0,1), +\}$ is a special fuzzy set group which is commutative and is of infinite order.

Consider $P = \{0.2, 0.4, 0.6, 0.8, 0\} \subseteq M$. It is easily verified P is a special fuzzy set group under +, a subgroup of order five.

Let $T = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0\} \subseteq M$ be a special fuzzy set subgroup of order ten.

Let $X = \{0.01, 0.02, \ldots, 0.98, 0.99, 0\} \subseteq M$. X is a special fuzzy set subgroup of M and is of order 100.

$Y = \{0.5, 0\} \subseteq M$ is again a special fuzzy set subgroup of M of order two.

$W = \{0.05, 0.10, 0.15, \ldots, 0.95, 0\} \subseteq M$ is again special fuzzy set subgroup of order 20.

Let $V = \{0.005, 0.010, 0.015, 0.020, 0.025, 0.030, 0.035, \ldots, 0.996, 0\} \subseteq M$. V is a special fuzzy set subgroup of order 200 in M.

Let $S = \{0, 0.0005, 0.010, \ldots, 0.9995\} \subseteq M$ be a special fuzzy set subgroup of order 2,000 and so on.

Let $S_1 = \{0, 0.00005, 0.0010, \ldots, 0.99995\} \subseteq M$ be a special fuzzy set subgroup of order 20,000 and so on.

Let B = {0, 0.25, 0.50, 0.75} $\subseteq$ (M, +) is a special fuzzy set subgroup of order 4.

Thus we see all the special fuzzy set subgroups are of finite order.

C ={0, 0.125, 0.250, 0.375, 0.5, 0.625, 0.750, 0.875} $\subseteq$ M is a special fuzzy set subgroup of M of order 8.

It is interesting to note all special fuzzy set subsemigroup of M are of infinite order and that of the special fuzzy set subgroups are of finite order. Order of special fuzzy set subgroups are 2, 4, 5, 8, 10, 20 and so on.

It is left as an open conjecture; M = {[0,1), $\times$} has no special fuzzy set subgroups of infinite order.

All special fuzzy set subgroups are of finite order. However order of M is itself infinite.

All the special fuzzy set finite subgroups given above are all cyclic.

Interested reader can study the special features of these substructures.

Further as M is of infinite order; most of the classical theorems like Lagrange, Sylow or Cauchy theorem cannot be applied as they are theorems pertaining to groups of finite order.

Finally the notion of normal subgroups never find any meaning as M is a commutative group.

Now we see in M = {[0,1), +} the subsets

M = {[0, p_1), $p_1 < 1$, +} is not a group under +; infact N is not even closed under '+'.

L = {[0, 0.35), +} is only a subset and not special fuzzy set subgroup.

Now if $0.1 \in M = \{[0,1), +\}$ we see $\underbrace{0.1+0.1+...+0.1}_{10 \text{ times}}$ = 1 = 0 so 0.1 is of order 10; 0.5 is of order 2.

0.01 is of order 100 and so on. Every element x in M is such that nx = 1 (mod 1) ($n \in N$). So M is an additive abelian group of infinite order.

Every element in M is a torsion element.

This is a unique example of an additive infinite group in which every element is a torsion element.

For instance if $0.9998217 \in M$ then $10^7 \times 0.9998217 \equiv 0$ (mod 1).

Now this is the main and unique feature of this group.

We proceed on to discuss about other two types of semigroups on special fuzzy set semigroups.

Let $M_{max} = \{[0.1), max\}$ be the special set fuzzy semigroup. For a = 0.007 and b = 0.731 in M_{max};

max {a, b} = max {0.007, 0.731} = 0.731.

It is easily verified M_{max} is a semigroup defined as the special fuzzy set semigroup.

Further max{0,a} = a for all $a \neq 0 \in [0,1)$ and max{a,a} = a.

Thus the special feature enjoyed by this M_{max} semigroup is that every element in [0,1) is an idempotent. Clearly the order of M_{max} is infinite.

This semigroup is different from the M = $\{[0, 1), \times\}$.

This semigroup has several subsemigroups both of finite and infinite order. This is yet another interesting feature about this semigroup.

Take $X = \{0.23\} \subseteq M_{max}$ is a special fuzzy set subsemigroup of M_{max}.

Infact every singleton set is a special fuzzy set subsemigroup of M_{max}.

Consider $Y = \{0.246, 0.21007, 0.3\} \subseteq M_{max}$ is again a special fuzzy set subsemigroup of M_{max}.

Infact M_{max} has finite special fuzzy set subsemigroups of all orders including infinite.

Also all intervals $P_i = \{[0, p_i], \max\}$ are special fuzzy set subsemigroups of M_{max}.

Not only all these intervals of the form

$A = \{[a, b] \mid 0 \leq a \text{ and } b < 1, a, b \in [0, 1)\} \subseteq M_{max}$ are also special fuzzy set subsemigroups of S.

Thus by using the operation max on $M = \{[0, 1), \max\}$ we are in a position to have a very rich structure for the semigroup M_{max}.

Also intervals of the form

$T = \{[a, b] \mid 0 < a \text{ and } b < 1\} \subseteq [0,1)$ is a subsemigroup of $M = \{[0, 1), \max\}$.

For $0 < a < b < 1$ we have infinite number of subsemigroups with max operation on them.

Finally if on $M = \{[0, 1)\}$ we define the min operation, then also we get special fuzzy set semigroup.

We see every set of the form

$P = \{x_1, x_2, \ldots, x_n \mid x_i \in [0, 1), 1 \leq i \leq n\}$ under the min operation is a special fuzzy set subsemigroup of M.

Thus we have infinite number of finite special fuzzy set subsemigroups in M.

M = {[0,1), min} also have infinite number of infinite special fuzzy set subsemigroups of the form N = {[0, p) | p < 1} $\subseteq$ [0, 1).

As we have infinite number of p's, p > 0 (or p < 1) we get infinite number of infinite special fuzzy set subsemigroups of M = {[0, 1), min}.

Infact all intervals of the form

V = {[a, b] | 0 < a < b , 1} $\subseteq$ [0, 1) also pave way for an infinite number of special fuzzy set subsemigroups in {[0, 1), min}, the special fuzzy set semigroup under the operation min.

Further the fuzzy intervals

W = {[a, b) | 0 ≤ a < b < 1} $\subseteq$ [0, 1) also form an infinite collection of all special fuzzy set subsemigroups of infinite order.

Also intervals of the form

B = {(a, b] | 0 ≤ a < b < 1} $\subseteq$ [0, 1) form an infinite collection of special fuzzy set subsemigroups of infinite order in [0, 1).

Thus like M = {[0,1), max} special fuzzy set semigroups we see M = {[0,1), min} the special fuzzy set semigroup have infinite number of special fuzzy set subsemigroups of infinite as well as finite order.

All properties with max can be obtained for the special fuzzy set semigroup with min also.

Now we proceed onto build groups and semigroups using the interval [0,1) and describe them by examples.

Example 2.11: Let S = $\{(a_1, a_2, a_3) \mid a_i \in [0,1); 1 \leq i \leq 3, \times\}$ be the collection of all row matrices with entries from [0,1). Define the operation $\times$ on S, for if X = (a_1, a_2, a_3) and Y = (b_1, b_2, b_3) are in S then X $\times$ Y = $(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_1b_1, a_2b_2, a_3b_3) \in$ S.

Thus (S, $\times$) is the special fuzzy set row matrix semigroup. Clearly (S, $\times$) is of infinite order and is commutative.

Another special feature of this special fuzzy set row matrix semigroup is that S has infinite number of zero divisors.

Take X = (0.001, 0, 0.075) and Y = (0, 0.93107, 0) $\in$ S.

We see X $\times$ Y = (0, 0, 0) $\in$ S; thus S has infinite number of zero divisors.

We have in S a subset say $P_1 = \{(a_1, 0, 0) \mid a_1 \in [0, 1)\} \subseteq S$; which is a special fuzzy set row matrix subsemigroup of the special fuzzy set row matrix semigroup S.

Take $P_2 = \{(0, a_2,0) \mid a_2 \in [0, 1)\} \subseteq S$; P_2 is a special fuzzy set row matrix subsemigroup of S.

Similarly $P_3 = \{(0, 0, a_3) \mid a_3 \in [0, 1)\} \subseteq S$; P_3 is a special fuzzy set row matrix subsemigroup of S.

We see $P_i \cap P_j = \{(0, 0, 0)\}$ if $i \neq j$; $1 \leq i, j \leq 3$.

Further $S = P_1 + P_2 + P_3$; is a direct sum of special fuzzy set row matrix subsemigroups. S has also other types of special fuzzy set subsemigroups given by

$M_1 = \{(a_1, a_2, a_3) \mid a_i \in [0, 0.7); 1 \leq i \leq 3\} \subseteq S$ is again a special fuzzy set row matrix subsemigroup of S.

M_1 is of infinite order. M_1 has zero divisors and infact M_1 has special fuzzy set row matrix subsemigroups such that M_1 is the direct sum.

Let $N_1 = \{(a_1,0, 0); a_1 \in [0, 0.7)\} \subseteq M_1 \subseteq S$ is again a special fuzzy set row matrix subsemiring of M_1 as well as of S.

$N_2 = \{(0, a_2, 0) \mid a_2 \in [0, 0.7)\} \subseteq M_1 \subseteq S$ is a special fuzzy set row matrix subsemigroup of M_1 and $N_3 = \{(0, 0, a_3) \mid a_3 \in$

$[0, 0.7)\} \subseteq M_1 \subseteq S$ is a special fuzzy set row matrix subsemigroup of M_1 .

We see $N_i \cap N_j = \{(0, 0, 0)\}$ $i \neq j$, $1 \leq i, j \leq 3$ and $M_1 = N_1 + N_2 + N_3 \underset{\neq}{\subset} S$.

Let $M_2 = \{(a_1, a_2, a_3) \mid a_i \in [0, 0.3], 1 \leq i \leq 3\} \subseteq S$ be a special fuzzy set row matrix subsemigroup of S.

$V_1 = \{(a_1, 0, 0) \mid a_1 \in [0, 0.3]\} \subseteq M_2 \subseteq S$;
$V_2 = \{(0, a_2, 0) \mid a_2 \in [0, 0.3]\} \subseteq M_2 \subseteq S$ and
$V_3 = \{(0, 0, a_3) \mid a_3 \in [0, 0.3]\} \subseteq M_2 \subseteq S$ are three special fuzzy set row matrix subsemigroups of M_2 and also of S.

$V_i \cap V_j = \{(0, 0, 0\}$; $1 \leq i, j \leq 3$, $i \neq j$ and $M_2 = V_1 + V_2 + V_3$ as the direct sum of special fuzzy set subsemigroups of M_2 and sum is not equal to S.

Let $T_1 = \{(a_1, a_2, a_3) \mid a_i \in [0.2, 0.52], 1 \leq i \leq 3\} \subseteq S$ be a special fuzzy set subsemigroup of S.

T_1 is of infinite order and has infinite number of zero divisors.

Let
$B_1 = \{(a_1, 0, 0) \mid a_1 \in [0.2, 0.52]\} \subseteq T_1 \subseteq S$,
$B_2 = \{(0, a_2, 0) \mid a_2 \in [0.2, 0.52]\} \subseteq T_1 \subseteq S$ and
$B_3 = \{(0, 0, a_3) \mid a_3 \in [0.2, 0.52]\} \subseteq T_1 \subseteq S$ be special fuzzy set row matrix subsemigroups of T_1 and that of S.

As subsemigroups of T_1 we see $T_1 = B_1 + B_2 + B_3$ however $B_1 + B_2 + B_3 \neq S$.

Thus we see B_i's are the subdirect sum of special fuzzy set subsemigroups of a special fuzzy set subsemigroup.

Infact we have infinite number of infinite cardinality special fuzzy set subsemigroups of a special fuzzy set subsemigroup.

Now let $D_1 = \{(a_1, a_2, a_3) \mid a_i \in \{0, 1/2, 1/2^2, 1/2^3, \ldots, 1/2^n ; n \to \infty\}; 1 \leq i \leq 3\} \subseteq S$ be the special fuzzy set row matrix subsemigroup of S.

Clearly it is impossible to find special fuzzy set row matrix subsemigroups of S so that D_1 is a direct sum of subsemigroups of S; the special fuzzy set row matrix semigroup.

However D_1 is of infinite order D_1 has infinite number of zero divisors.

Let $E_1 = \{(a_1, 0, 0) \mid a_1 \in \{1/2, 0, 1/2^2, 1/2^3, \ldots, 1/2^n, \ldots\}\} \subseteq D_1$, $E_2 = \{(0, a_2, 0) \mid a_2 \in \{1/2, 0, 1/2^2, \ldots, 1/2^n, \ldots\}\} \subseteq D_1$ and $E_3 = \{(0, 0, a_3) \mid a_3 \in \{1/2, 0, 1/2^2, \ldots, 1/2^n, \ldots\}\} \subseteq D_1$ be special fuzzy set subsemigroups of $D_1 \subseteq S$.

Clearly $E_i \cap E_j = \{(0, 0, 0)\}$, $i \neq j$, $1 \leq i, j \leq 3$ and $D_1 = E_1 + E_2 + E_3$.

But D_1 cannot be any part of the direct sum of S; further $E_1 + E_2 + E_3 = D_1 \underset{\neq}{\subset} S$, but D_1 is the direct sum of special fuzzy set subsemigroups of D_1.

Infact S has infinite number of special fuzzy set subsemigroups such that none of them can be a part of the direct sum.

For take $P_n = \{(a_1, a_2, a_3) \mid a_i \in \{1/n, 0, 1/n^2, 1/n^3, \ldots, 1/n^m, \ldots\}, 1 \leq i \leq 3\} \subseteq S$; P_n is a special fuzzy set subsemigroup of S $(n \geq 2)$.

Infact $P_2 = \{(a_1, a_2, a_3) \mid a_i \in \{1/2, 0, 1/2^2, 1/2^3, \ldots, 1/2^n, \ldots\}, 1 \leq i \leq 3\} \subseteq S$, is a special fuzzy set subsemigroup of S for $n = 2$.

$P_3 = \{(a_1, a_2, a_3) \mid a_i \in \{0, 1/3, 1/3^2, 1/3^3, \ldots, 1/3^n, \ldots\}, 1 \leq i \leq 3\} \subseteq S$ is a special fuzzy set subsemigroup of S of infinite order.

Thus we have an infinite collection of infinite number of special fuzzy set subsemigroups of infinite order which can never form the part of any direct sum of S as special fuzzy set subsemigroups.

Next we describe by an example the special fuzzy set column matrix semigroup.

Example 2.12: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in \{[0, 1)\}, 1 \le i \le 6 \right.$$

under natural product $\times_n$ of column matrices} be the special fuzzy set column matrix semigroup.

Clearly S is of infinite order and S is a commutative special fuzzy set column matrix semigroup.

$$\text{Let } X = \begin{bmatrix} 0.3 \\ 0 \\ 0 \\ 0 \\ 0.4 \\ 0.6 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.7 \\ 0.52 \end{bmatrix} \in S$$

$$X \times_n Y = \begin{bmatrix} 0.3 \\ 0 \\ 0 \\ 0 \\ 0.4 \\ 0.6 \end{bmatrix} \times_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.7 \\ 0.52 \end{bmatrix}$$

$$= \begin{bmatrix} 0.3 \times 0 \\ 0 \times 0 \\ 0 \times 0 \\ 0 \times 0 \\ 0.4 \times 0.7 \\ 0.6 \times 0.52 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.28 \\ 0.312 \end{bmatrix} \in S.$$

This is the way the natural product $\times_n$ on S is defined.

S has zero divisors but has no idempotents and units.

Take $X = \begin{bmatrix} 0.8 \\ 0.1 \\ 0.4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.7 \\ 0.4 \\ 0.9 \end{bmatrix} \in S.$

$$X \times_n Y = \begin{bmatrix} 0.8 \\ 0.1 \\ 0.4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.7 \\ 0.4 \\ 0.9 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8\times 0 \\ 0.1\times 0 \\ 0.4\times 0 \\ 0\times 0.7 \\ 0\times 0.4 \\ 0\times 0.9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in S.$$

Thus S has zero divisors.

Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in \{[0, 0.4)\}, 1 \leq i \leq 6 \right\} \subseteq S$$

be again a special fuzzy set subsemigroup of S and is of infinite order.

This P too has infinite number of zero divisors and has no units and idempotents.

Consider

$$W_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_1, a_2 \in \{[0, 0.4)\} \subseteq P, \right.$$

$$W_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_3 \in \{[0, 0.4)\} \subseteq P, \right.$$

$$W_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \end{bmatrix} \middle| a_4 \in \{[0, 0.4)\} \subseteq P, \right.$$

$$W_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ 0 \end{bmatrix} \middle| a_5 \in \{[0, 0.4)\} \subseteq P \text{ and} \right.$$

$$W_5 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \end{bmatrix} \middle| a_6 \in \{[0, 0.4)\} \subseteq P \right.$$

are some five special fuzzy set subsemigroups of P.

$$\text{Infact } W_i \cap W_j = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} \text{ if } i \neq j, 1 \leq i \leq 5.$$

Further $P = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5$ is the direct sum of special fuzzy set subsemigroups of P.

However $P \underset{\neq}{\subset} S$ so is not a direct sum of S.

Let

$$V_1 = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_i \in \{[0, 1); 1 \leq i \leq 3\} \subseteq S\right.$$

and

$$V_2 = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ a_3 \\ a_1 \\ a_2 \end{bmatrix} \middle| a_i \in \{[0, 1); 1 \leq i \leq 3\} \subseteq S\right.$$

be special fuzzy set subsemigroups of S.

Clearly

$$V_1 \cap V_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ and } V_1 \oplus V_2 = S;$$

that is V is the direct sum of special fuzzy set subsemigroups of S.

We see $V_1^{\perp} = V_2$ and $V_2^{\perp} = V_1$. They are orthogonal as subsemigroups.

Let

$$X = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in \{0, 1/5, 1/5^2, 1/5^3, \ldots, 1/5^n, \ldots\}\ 1 \leq i \leq 6 \right\}$$

be the special fuzzy set subsemigroup of S.

Clearly S cannot be written as a direct sum using X.

For no collection of subspaces with X make it as a direct sum of S.

However X can be written as a direct sum of special fuzzy set subsemigroups of X.

For take

$$Z_1 = \left\{\left.\begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right| a_1 \in \{0, 1/5, 1/5^2, \ldots, 1/5^n, \ldots\}\right\} \subseteq S,$$

$$Z_2 = \left\{\left.\begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right| a_2 \in \{0, 1/5, 1/5^2, \ldots, 1/5^n, \ldots\}\right\} \subseteq S,$$

$$Z_3 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ a_3 \\ a_4 \\ 0 \\ 0 \end{bmatrix}\right| a_3, a_4 \in \{0, 1/5, 1/5^2, \ldots, 1/5^n, \ldots\}\right\} \subseteq S$$

and

$$Z_4 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ a_6 \end{bmatrix}\right| a_5, a_6 \in \{0, 1/5, 1/5^2, \ldots, 1/5^n, \ldots\}\right\} \subseteq S$$

be special fuzzy set subsemigroups of the special fuzzy set subsemigroup X of S.

We see $X = Z_1 + Z_2 + Z_3 + Z_4$ with $Z_i \cap Z_j = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$, $i \neq j$,

$1 \leq i, j \leq 4$ and we call this as the subdirect sum of special fuzzy set subsemigroups of the subsemigroup X.

Example 2.13: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \middle| a_i \in \{[0, 1); 1 \leq i \leq 12, \times_n\} \right.$$

be the special fuzzy set semigroup of infinite order.

Every special fuzzy set subsemigroup of S is also of infinite order and is commutative.

S has infinite number of ideals and each of them is of infinite order.

However S has special fuzzy set subsemigroups which are not ideals.

Let

$$M_1 = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| \, a_1 \in [0, 1) \right\} \subseteq S$$

be the special fuzzy set subsemigroup of S. Clearly M_1 is also a special fuzzy set ideal of S.

Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| \, a_1, a_2 \in \{0, 1/2\ 1/2^2, 1/2^3, \ldots, 1/2^n, \ldots\} \right.$$

$$\subseteq [0, 1)\} \subseteq S$$

be the special fuzzy subset subsemigroup of S.

Clearly P is not a special fuzzy set ideal of S for if

$$A = \begin{bmatrix} 1/2 & 1/2^7 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \in P \text{ and}$$

$$T = \begin{bmatrix} 0.7 & 0.3 \\ 0 & 0 \\ 0.9 & 0.8 \\ 0.4 & 0.81 \\ 0.9 & 0.1 \\ 0 & 0.4 \end{bmatrix} \in S.$$

We find $T \times_n A = \begin{bmatrix} 0.7 & 0.3 \\ 0 & 0 \\ 0.9 & 0.8 \\ 0.4 & 0.81 \\ 0.9 & 0.1 \\ 0 & 0.4 \end{bmatrix} \times_n \begin{bmatrix} 1/2 & 1/2^7 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0.(7/2) & 0.(3/2^7) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \notin P.$$

Hence P is not a special fuzzy set ideal of S.

Thus all special fuzzy set subsemigroups of S are not in general ideals of S.

We find zero divisors in S.

For take $A = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ a_2 & 0 \\ 0 & a_3 \\ 0 & 0 \\ 0 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & a_1 \\ a_2 & a_3 \\ 0 & 0 \\ 0 & 0 \\ a_4 & a_5 \\ a_6 & 0 \end{bmatrix} \in S$.

We see $A \times_n B = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ a_2 & 0 \\ 0 & a_3 \\ 0 & 0 \\ 0 & a_4 \end{bmatrix} \times_n \begin{bmatrix} 0 & a_1 \\ a_2 & a_3 \\ 0 & 0 \\ 0 & 0 \\ a_4 & a_5 \\ a_6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

However S has no idempotents.

Infact S has infinite number of zero divisors.

Further S has infinite number of subsemigroups which are not ideals and S has also infinite number of ideals.

Also we can write S as a direct sum of special fuzzy set subsemigroups (ideals).

For if $M_1 = \left\{ \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in [0, 1) \right\} \subseteq S$ is an ideal of S.

$$M_2 = \left\{\begin{bmatrix} 0 & 0 \\ a_1 & 0 \\ a_2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| a_1, a_2 \in [0, 1)\right\} \subseteq S$$

is an ideal of S (M_2 is a special fuzzy set subsemigroup of S).

$$M_3 = \left\{\begin{bmatrix} 0 & 0 \\ 0 & a_1 \\ 0 & a_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| a_1, a_2 \in [0, 1)\right\} \subseteq S$$

is a special fuzzy set subsemigroup of S as well as an ideal of S.

$$M_4 = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ a_1 & a_2 \\ a_3 & 0 \\ 0 & 0 \end{bmatrix} \middle| a_1, a_2, a_3 \in [0, 1)\right\} \subseteq S$$

is a special fuzzy set subsemigroup which is an ideal of S.

$$M_5 = \left\{\left.\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & a_1 \\ a_2 & a_3 \end{bmatrix}\right| a_1, a_2, a_3 \in [0, 1)\right\} \subseteq S$$

is a special fuzzy set subsemigroup which is also an ideal of S.

$$\text{We see } M_i \cap M_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ if } i \neq j \text{ and further}$$

$S = M_1 + M_2 + M_3 + M_4 + M_5$ is a direct sum of special fuzzy set ideals (subsemigroups) of S.

Example 2.14: Let

$$S = \left\{\left.\left[\begin{array}{c} a_1 \\ \hline a_2 \\ a_3 \\ \hline a_4 \\ a_5 \\ a_6 \\ \hline a_7 \\ a_8 \\ \hline a_9 \end{array}\right]\right| a_i \in [0, 1), 1 \leq i \leq 9\right\}$$

be the special fuzzy set semigroup under $\times_n$, the natural product of matrices.

S has infinite number of zero divisors, has no idempotents or units.

S has infinite number of special fuzzy set subsemigroups which are not ideals of S.

S has also infinite number of special fuzzy set subsemigroups which are ideals of S.

Every ideal and every subsemigroup of S is of infinite order.

This study is interesting as every special fuzzy set semigroup can also be realized as a fuzzy subsemigroup of any appropriate semigroup.

So this study can only a generalized study of fuzzy semigroups.

Example 2.15: Let S = {Collection of all super matrices

$$M = \left\{ \left[\begin{array}{c|ccc|c} a_1 & a_2 & a_3 & a_4 & a_5 \\ \hline a_6 & \cdots & \cdots & \cdots & a_{10} \\ a_{11} & \cdots & \cdots & \cdots & a_{15} \\ a_{16} & \cdots & \cdots & \cdots & a_{20} \\ a_{21} & \cdots & \cdots & \cdots & a_{25} \\ \hline a_{26} & \cdots & \cdots & \cdots & a_{30} \\ a_{31} & \cdots & \cdots & \cdots & a_{35} \end{array} \right] \right| \left. a_i \in [0, 1), 1 \leq i \leq 35 \right\}$$

be a special fuzzy set semigroup of infinite order.

This super matrix special fuzzy set semigroup has infinite number of special fuzzy set subsemigroups and special fuzzy set ideals all of which are infinite order.

Infact S has infinite cyclic special fuzzy set subsemigroups.

For instance take

$$B = \left\{ \left[\begin{array}{c|ccc|c} a_1 & 0 & 0 & 0 & 0 \\ \hline 0 & ... & ... & ... & 0 \\ 0 & ... & ... & ... & 0 \\ 0 & ... & ... & ... & 0 \\ 0 & ... & ... & ... & 0 \\ \hline 0 & ... & ... & ... & 0 \\ 0 & ... & ... & ... & 0 \end{array}\right] \middle| \; a_1 \in \{0, 1/2, 1/2^2, 1/2^3, \ldots, 1/2^n, \ldots n \to \infty\} \right\} \subseteq S .$$

B is a special fuzzy set subsemigroup of infinite order.

B is cyclic for B is generated by

$$\left\langle \left[\begin{array}{c|ccc|c} 1/2 & 0 & 0 & 0 & 0 \\ \hline 0 & ... & ... & ... & 0 \\ 0 & ... & ... & ... & 0 \\ 0 & ... & ... & ... & 0 \\ 0 & ... & ... & ... & 0 \\ \hline 0 & ... & ... & ... & 0 \\ 0 & ... & ... & ... & 0 \end{array}\right] \right\rangle .$$

S has infinite number of special fuzzy set subsemigroups which are cyclic and are of infinite order.

Further none of these cyclic special fuzzy set subsemigroups are ideals.

Now we just give examples of special fuzzy set semigroups using 'min' operation (The 'max' operation is considered as a matter of routine work).

Example 2.16: Let S = $\{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 1); 1 \leq i \leq 4\}$ be a special fuzzy set row matrix semigroup under min operation.

Clearly S gives an idempotent special fuzzy set semigroup of infinite order.

S has infinite number of special fuzzy set subsemigroups both of finite order as well as of infinite order.

We have special fuzzy set subsemigroups of order one. For take

$M_1 = \{(0.3, 0.7, 0)\} \subseteq S$, M_1 under min operation is an idempotent semigroup of order one.

Let $M_2 = \{(0, 0.5, 0), (0, 0, 0), (0.7, 0, 0.371)\} \subseteq S$; M_2 is a special fuzzy set subsemigroup of order three.

Clearly M_2 is an idempotent special fuzzy set subsemigroup such that min $(x, y) = (0, 0, 0)$ if $x \neq y$, $(x, y \in M_2)$
min $(x, x) = x$ and for $x \in M_2$.

Let $M_3 = \{(0.1, 0.3, 0.5), (0.8, 0.4, 0.7)\} \subseteq S$ be again a special fuzzy set subsemigroup of order two as
min $\{(0.1, 0.3, 0.5), (0.8, 0.4, 0.7)\} = \{(0.1, 0.3, 0.5)\} \in M_3$.

Now several interesting observations are made.

(1) Can any two elements of S be a special fuzzy set subsemigroup? (under min operation).
(2) Can every singleton set of S be a special fuzzy set subsemigroup under min operation?
(3) Can every n-elements of S be a special fuzzy set subsemigroup under min operation?

We answer all the three questions in the following.

Let $M_2 = \{(0.8, 0.4, 0.2), (0.6, 0.7, 0.315)\} \subseteq S$.

Is M_2 a special fuzzy set subsemigroup?

Consider
min {(0.8, 0.4, 0.2), (0.6, 0.7, 0.315)}
= (min {0.8, 0.6}, min {0.4, 0.7}, min {0.2, 0.315},
= $(0.6, 0.4, 0.2) \notin M_2$.

So M_2 is not a special fuzzy set subsemigroup. So the answer in general for question (1) is that any two elements of S is not always a special fuzzy set subsemigroup of S.

Now will every singleton set of S be a special fuzzy set subsemigroup. The answer is yes as every element in S under the min operation is an idempotent.

Now can we have conditions on the two element set of S so that S is a special fuzzy set subsemigroup under the min operation.

To this end we define a crude 'min' ordering on elements of S which is under the min operation.
Let $A = (x_1, x_2, x_3)$ and $B = (y_1\ y_2\ y_3) \in S$.

We define $A \leq_{\min} B$ if $\min (x_i, y_i) = x_i$ for i = 1, 2, 3 hence crude min (A, B) = A.

So we define crude minimum for the elements $A, B \in S$ if min (A, B) = A or B.

It is pertinent to keep on record that in general min (A, B) = C where $C \neq A$ or $C \neq B$.

So if every pair of elements in S which are crude min orderable, then certainly that set forms a special fuzzy set subsemigroup of S.

Further 0 = (0, 0, 0), A = $(x_1, x_2, x_3) \in S$ we have min {0, A} = {0} so all pairs of the form $M_2 = \{(0, 0, 0), (x_1, x_2, x_3)\} \subseteq S$ is always a special fuzzy set subsemigroup of S under min operation.

Likewise $M_2 = \{X = (x_1\ x_2\ x_3), Y = (y_1, y_2, y_3)\} \subseteq S$ is a special fuzzy set such that min $\{(x_1, x_2, x_3), (y_1, y_2, y_3)\} = (z_1, z_2, z_3) \neq X$ or Y. So in this case M_2 will not be a special fuzzy set subsemigroup.

For take M_2 = {(0.2, 0.7, 0.3) = X, Y = (0.8, 0.5, 0.4)} $\subseteq$ S; M_2 is not a special fuzzy set subsemigroup of S.

For min (X, Y) = min {(0.2, 0.7, 0.3), (0.8, 0.5, 0.4)}

= {(min {0.2, 0.8}, min {0.7, 0.5}, min {0.3, 0.4})} = {(0.2, 0.5, 0.3)} $\not\in M_2$; hence the claim.

Now $M_2 \cup$ {(0.2, 0.5, 0.3)} = M_2 is a special fuzzy set subsemigroup of S.

Suppose M_3 = {X = (0.4, 0.2, 0), Y = (0.2, 0.7, 0.5), Z = (0.1, 0.5, 0.6)} $\subseteq$ S.

Is M_3 a special fuzzy set subsemigroup?

We find min {X, Y}, min {X, Z} and min {Z, Y}.

min {X, Y} = min {(0.4, 0.2, 0), (0.2, 0.7, 0.5)} = {(min {0.4, 0.2}, min {0.2, 0.7}, min {0, 0.5})}
= {(0.2, 0.2, 0)} … I

min {Y, Z} = min {(0.2, 0.7, 0.5), (0.1, 0.5, 0.6)}
= {(min {0.2, 0.1}, min {0.7, 0.5}, min {0.5, 0.6})}
= {(0.1, 0.5, 0.5)} … II

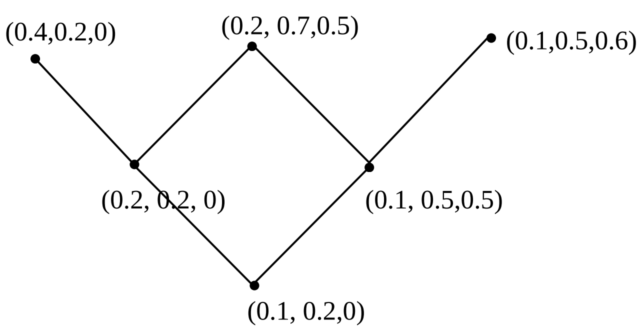
min (X, Z) = min {(0.4, 0.2, 0), (0.1, 0.5, 0.6)}
= {(min {0.4, 0.1}, min {0.2, 0.5}, min {0, 0.6})}
= {(0.1, 0.2, 0)} … III

Clearly I, II and III are distinct and are not in M_3.

Now we find min {I, II}
= min {(0.2, 0.2, 0), (0.1, 0.5, 0.5)}
= {(min {0.2, 0.1}, {min {0.2, 0.5}, min {0, 0.5})}
= {(0.1, 0.2, 0)} which is III.

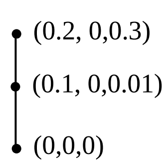
Thus {(0.2, 0.2, 0), (0.1, 0.2, 0), (0.1, 0.5, 0.3)} $\cup$ M_3 is a special fuzzy set subsemigroup of S.

The semilattice associated with
$M_3 \cup$ {(0.2, 0.2, 0), (0.1, 0.2, 0), (0.1, 0.5, 0.5)} is as follows:



M_4 = {(0, 0, 0), (0.2, 0, 0.3), (0.1, 0, 0.01))} $\subseteq$ S.

Clearly M_4 is a special fuzzy set subsemigroup (semilattice under min operation)



Consider

M_5 = {(0, 0, 0, 0), (0.3, 0, 0.2), (0, 0.2, 0), (0.4, 0.2, 0.2)} $\subseteq$ S, M_5 is a special fuzzy set subsemigroup.

For

min {(0, 0, 0), (0.3, 0, 0.2)}
= {(0, 0, 0)}

min {(0, 0, 0), (0, 0.2, 0)}
= {(0, 0, 0)}

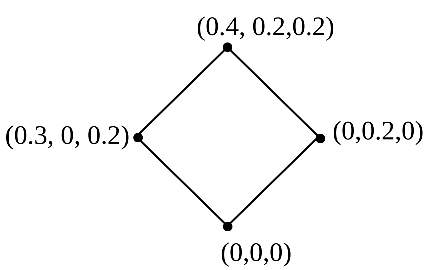
min {(0, 0, 0), (0.4, 0.2, 0.2)}
= {(0, 0, 0)}

min {(0.3, 0, 0.2), (0, 0.2, 0)}
= {(0, 0, 0)}

min {(0.3, 0, 0.3), (0.4, 0.2, 0.2)}
= {(0, 0.2, 0)}

min {(0, 0.2, 0.3), (0.4, 0.2, 0.2)}
= {(0, 0.2, 0)}.

Thus the diagram is the semilattice which is as follows:



It is important to keep on record we have infinite number of special fuzzy set subsemigroups of order two, order three, order four and so on.

Infact every singleton element of S is a special fuzzy set subsemigroup of S.

Now none of these special fuzzy set subsemigroups are ideals of S.

We give now infinite special fuzzy set subsemigroups of S.

Take $M_1 = \{(a_1, a_2, a_3) \mid a_i \in [0, 0.3), 1 \leq i \leq 3\} \subseteq S$ to be the special fuzzy set subsemiring of S. Clearly M_1 is an ideal of S under min operation. Clearly order of M_1 is infinite.

Let $M_2 = \{(a_1, a_2, a_3) \mid a_i \in [0, 0.652), 1 \leq i \leq 3\} \subseteq S$ be the special fuzzy set subsemigroup of S of infinite order. M_2 is also an ideal of S under min operation.

Thus we have infinite number of ideals in S under min operation given by $M_n = \{(a_1, a_2, a_3) \mid a_i \in [0, 0.n) \mid 0 < n < 1, 1 \leq i \leq 3\} \subseteq S$ is of infinite order.

Consider $P_1 = \{(a_1, a_2, a_3) \mid a_i \in [0.3, 0.8) \subseteq [0, 1)\} \subseteq S$ be the special fuzzy set subsemigroup of S.

Clearly P_1 is not a special fuzzy set ideal of S as if (0.1, 0.2, 0) $\in$ S and (0.3, 0.4, 0.5) $\in P_1$ we see

$$\min \{(0.1, 0.2, 0), (0.3, 0.4, 0.5)\}$$
$$= \{(0.1, 0.2, 0)\} \notin P_1 \text{ hence the claim.}$$

It is pertinent to keep on record that we have infinite number of special fuzzy set subsemigroups of S such that they are not ideals of S.

Take $M = \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in [c, d]; 0 < c < d < 1\} \subseteq S$ are all (for varying c and d) only special fuzzy set subsemigroups of S and not ideals of S.

Finally consider $N = \{(a_1, a_2, a_3) \mid a_i \in [t, 1), 1 \leq i \leq 3\} \subseteq S$; N is only a special fuzzy set subsemigroup and is not an ideal.

For if $X = (b_1, b_2, b_3) \in S$; $b_i < t$ $(1 \leq i \leq 3)$ take $A = (a_1, a_2, a_3) \in N$ se see $\min(X, A) = \{(b_1, b_2, b_3), (a_1, a_2, a_3)\}$

$$= \{(b_1, b_2, b_3)\} \notin N.$$

So N is not a special fuzzy set ideal of S.

We have infinite number of special fuzzy set subsemigroups which are not ideals of S.

Example 2.17: Let

$$S = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{array} \right] \middle| a_i \in [0, 1), 1 \leq i \leq 6 \right\}$$

be the special fuzzy set semigroup under min operation. This has infinite number of special fuzzy set subsemigroups some of which are ideals and some of them are not ideals.

Let

$$M_1 = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right] \middle| a_1, a_2 \in [0, 1) \right\} \subseteq S$$

be a special fuzzy set subsemigroup; M_1 is a special fuzzy set subsemigroup which is also an ideal.

Take

$$A = \begin{bmatrix} 0.2 \\ 0.39 \\ 0.418 \\ 0.61 \\ 0 \\ 0.79 \end{bmatrix} \in S \text{ and } B = \begin{bmatrix} 0.01 \\ 0.04 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in M_1.$$

We find

$$\min\{A, B\} = \min \left\{ \begin{bmatrix} 0.2 \\ 0.39 \\ 0.418 \\ 0.61 \\ 0 \\ 0.79 \end{bmatrix} \begin{bmatrix} 0.01 \\ 0.04 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} \min\{0.2, 0.01\} \\ \min\{0.39, 0.04\} \\ \min\{0.418, 0\} \\ \min\{0.61, 0\} \\ \min\{0, 0\} \\ \min\{0.79, 0\} \end{bmatrix}$$

$$= \begin{bmatrix} 0.01 \\ 0.04 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in M_1.$$

Clearly M_1 is a special fuzzy set ideal of S.

Now let

$$M_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle| \, a_1, a_2 \in [0,1) \right\} \subseteq S,$$

M_2 is a special subset subsemigroup of S and M_2 is also an ideal of S.

Further

$$M_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a \\ b \end{bmatrix} \middle| \, a, b \in [0,1) \right\} \subseteq S$$

is a special subset subsemigroup of S under min operation.

We see M_3 is also an ideal of S.

$$M_i \cap M_j = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ if } i \neq j,\ 1 \leq i, j \leq 3 \text{ and}$$

$S = M_1 + M_2 + M_3$; that is S is a direct sum of subsemigroups.

Let

$$N_1 = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right| a_1, a_2 \in [0,1)\right\} \subseteq S,$$

$$N_2 = \left\{\left.\begin{bmatrix} 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right| a_1, a_2 \in [0,1)\right\} \subseteq S,$$

$$N_3 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \right| a_1, a_2 \in [0,1)\right\} \subseteq S,$$

$$N_4 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \end{bmatrix} \right| a_1, a_2 \in [0,1)\right\} \subseteq S$$

and

$$N_5 = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} \middle| a_1, a_2 \in [0,1)\right\} \subseteq S$$

are five special fuzzy set subsemigroups which are also ideals of S.

$$\text{We see } N_i \cap N_j \neq \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} \text{ if } i \neq j,\ 1 \leq i, j \leq 5.$$

Also $S \subseteq N_1 + N_2 + N_3 + N_4 + N_5$ so S is not a direct sum of special fuzzy set subsemigroups (or ideals). All these special fuzzy set subsemigroups (ideals) are of infinite order.

Let

$$W_1 = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in [0.7, 1), 1 \leq i \leq 6\right\}$$

be a special fuzzy set subsemigroup of S.

W_1 is not an ideal for if

$$A = \begin{bmatrix} 0 \\ 0.8 \\ 0.1 \\ 0.45 \\ 0.031 \\ 0.110 \end{bmatrix} \in S \text{ and } B = \begin{bmatrix} 0.5 \\ 0.6 \\ 0.7 \\ 0.8 \\ 0.91 \\ 0.875 \end{bmatrix} \in W_1.$$

We find min {A, B}

$$= \min \left\{ \begin{bmatrix} 0 \\ 0.8 \\ 0.1 \\ 0.45 \\ 0.031 \\ 0.110 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.6 \\ 0.7 \\ 0.8 \\ 0.91 \\ 0.875 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 0 \\ 0.6 \\ 0.1 \\ 0.45 \\ 0.031 \\ 0.110 \end{bmatrix} \notin W_1;$$

so W_1 is not an ideal of S; only a special fuzzy set subsemiring.

We see order of W_1 is of infinite and infact S has infinite number of special fuzzy set subsemigroups which are not ideals of S.

Let

$$V_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \, a_i \in [a, b], 0 < a < b < 1; 1 \leq i \leq 6 \right\} \subseteq S$$

be a special fuzzy set subsemigroup of S.

Clearly V_1 is not an ideal of S.

For if $A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$ where $0 < x_i < a$, $1 \leq i \leq 6$, and

$$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} \in V,$$

then $\min \{A, B\} = A \not\in V_1$ so V_1 is not an ideal of S.

Thus we have infinite number of special fuzzy set subsemigroups which are not ideals of S.

Let

$$B_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in \{0, 1/2, 1/2^2, 1/2^3, \ldots, 1/2^n \text{ as } n \to \infty\} \subseteq [0,1]; \; 1 \leq i \leq 6 \right\} \subseteq S$$

be the special fuzzy set subsemigroup of S.

B_1 is not an ideal of S but B_1 is of infinite order.

$$D = \left\{ \begin{bmatrix} 0.3 \\ 0.1 \\ 0.2 \\ 0.4 \\ 0.5 \\ 0.7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.1 \\ 0.2 \\ 0.4 \\ 0.5 \\ 0.7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0.2 \\ 0.4 \\ 0.5 \\ 0.7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.4 \\ 0.5 \\ 0.7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0.7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \subseteq S$$

is a special fuzzy set subsemigroup of S.

D is not an ideal of S. D is of finite order and o(D) = 7.

We have infinite number of special fuzzy set subsemigroups of finite order.

$$R_1 = \left\{ \begin{bmatrix} 0.3 \\ 0.4 \\ 0.7 \\ 0 \\ 0.91 \\ 0.72 \end{bmatrix} \right\} \subseteq S$$

is a special fuzzy set subsemigroup of order one of S under the min operation.

$$R_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right\} \text{ where } a_1, a_2, a_3, a_4, a_5, a_6 \in [0,1)$$

are fixed so that $o(R_2) = 2$ is always a special fuzzy set subsemigroup of order two and is not an ideal of S.

$$M_2' = \left\{ \begin{bmatrix} 0.2 \\ 0.1 \\ 0.7 \\ 0.8 \\ 0.9 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 0.2 \\ 0.01 \\ 0.3 \\ 0.7 \\ 0.31 \\ 0.014 \end{bmatrix} \right\} \subseteq S$$

is a special fuzzy set subsemigroup of S.

$$N_2 = \left\{ \begin{bmatrix} 0.4 \\ 0.2 \\ 0.31 \\ 0.0.41 \\ 0.07 \\ 0.521 \end{bmatrix}, \begin{bmatrix} 0.7 \\ 0.4 \\ 0.13 \\ 0.025 \\ 0.06 \\ 0.6 \end{bmatrix} \right\} \subseteq S$$

is only a subset and is not a special fuzzy set subsemigroup of S as under the min operation; N_2 is not closed.

$$D_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 0.31 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.2 \\ 0.7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.521 \\ 0.7301 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \subseteq S$$

is a special fuzzy set subsemigroup of S of order 4 under the 'min' operation.

Infact D_1 is not an ideal of S.

However S has infinite number of special fuzzy set subsemigroups of order four.

$$\text{Let } E = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ where } a_1, a_2 \in [0,1)\}$$

be the special fuzzy set subsemigroup of S of order three.

Infact S has infinite number of special fuzzy subset subsemigroups of order three which are not ideals.

We see none of the finite special fuzzy set subsemigroups of S are ideals thus all ideals of S are of infinite order.

Let

$$F = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in [0.5, 1), 1 \leq i \leq 6 \right\} \subseteq S$$

be the special fuzzy set subsemigroup of infinite order under the min operation.

Clearly F is not an ideal of S. Thus we have an infinite collection of special fuzzy set subsemigroups which are not ideals and are of infinite order.

Example 2.18: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 12 \right\}$$

be the special fuzzy set semigroup under the min operation.

Take

$$R_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| a_i \in [0, 1) \right\} \subseteq S$$

to be a special fuzzy set subsemigroup of S and is of infinite order.

Clearly R_1 is also an ideal of S of infinite order under min operation.

$$R_2 = \left\{\begin{bmatrix} 0 & a_1 & a_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| a_1, a_2 \in [0, 1)\right\} \subseteq S$$

is a special fuzzy set subsemigroup of infinite order and is also an ideal of S.

Take

$$R_3 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| a_1, a_2, a_3 \in [0, 1)\right\} \subseteq S,$$

R_3 is a special fuzzy set subsemigroup as well as an ideal of S.

Now

$$R_4 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & 0 & 0 \\ a_2 & a_3 & 0 \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 3\right\} \subseteq S$$

is again a special fuzzy set subsemigroup as well as an ideal of S.

Finally

$$R_5 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & 0 & a_3 \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 3\right\} \subseteq S$$

is a special fuzzy set subsemigroup under as well as an ideal of S.

Further

$$R_i \cap R_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ if } i \neq j, 1 \leq i, j \leq 5 \text{ and}$$

$$S = R_1 + R_2 + R_3 + R_4 + R_5$$

a direct sum of special fuzzy set subsemigroups.

Suppose we take

$$B_1 = \left\{\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| a_i \in \{0, 1/3, 1/3^2, \ldots, 1/3^n,\right.$$

$$\left. n \to \infty\} \subseteq [0,1)\right\} \subseteq S.$$

B_1 is only a special fuzzy set subsemigroup and is not an ideal of S.

Let

$$B_2 = \left\{ \begin{bmatrix} 0 & a_1 & a_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 1/3, 1/3^2, \ldots, 1/3^n, \ldots\} \subseteq [0,1) \right\} \subseteq S.$$

B_2 is a only a special fuzzy set subsemigroup and is not an ideal of S.

Let

$$B_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_i \in \{0, 1/3, 1/3^2, \ldots, 1/3^n, \ldots \} \subseteq [0,1), 1 \leq i \leq 3 \right\} \subseteq S.$$

B_3 is a special fuzzy set subsemigroup and is not an ideal of S.

$$B_4 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & 0 & 0 \\ a_2 & a_3 & 0 \end{bmatrix} \middle| \; a_i \in \{0, 1/3, 1/3^2, \ldots, 1/3^n, \ldots \} \subseteq [0,1)\} \; 1 \leq i \leq 3 \right\} \subseteq S.$$

B_4 is only a special fuzzy set subsemigroup and is not an ideal of S.

Now

$$B_5 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & 0 & a_3 \end{bmatrix}\right| a_i \in \{0, 1/3, 1/3^2, \ldots, 1/3^n, \ldots\} \subseteq [0,1)\}\ 1 \leq i \leq 3\right\} \subseteq S$$

is only a special fuzzy set subsemigroup and is not an ideal of S.

Finally

$$B_i \cap B_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ if } i \neq j,\ 1 \leq i, j \leq 5 \text{ and}$$

$T = B_1 + B_2 + B_3 + B_4 + B_5$ that is

$$T = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix}\right| a_i \in \{0, 1/3, 1/3^2, \ldots, 1/3^n, \ldots \infty\} \subseteq [0,1)\}\ 1 \leq i \leq 12\right\} \subseteq S$$

and T is only a special fuzzy set subsemigroup and not an ideal.

This form of direct sum will be known as subdirect subsum of special fuzzy set subsemigroups of S.

We have infinite number of special fuzzy set subsemigroups which can be written as a subdirect subsum of special fuzzy set subsemigroups.

We also have infinite number of finite order special fuzzy set subsemigroup which can be written as a subdirect subsum of special fuzzy set subsemigroups.

Example 2.19: Let

$$S = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}\right| a_i \in [0, 1), 1 \leq i \leq 16\right\}$$

be the special fuzzy set semigroup clearly o(S) = ∞.

S has infinite number of finite special fuzzy set subsemigroups none of them are ideals of S.

Take

$$M_1 = \left\{\begin{bmatrix} 0.7 & 0 & 0 & 0.51 \\ 0.6 & 0.3 & 0.1 & 0.84 \\ 0.9 & 0.4 & 0.5 & 0.75 \\ 0.2 & 0.1 & 0.9 & 0.03 \end{bmatrix}\right\} \subseteq S,$$

M_1 is a special fuzzy set subsemigroup of order one and M_1 is not an ideal of S.

Take

$$M_2 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}\right| a_i \in [0, 1), 1 \leq i \leq 16\right\}$$

such that $o(M_2) = 2$.

M_2 is always a special fuzzy set subsemigroup of order two and is not an ideal of S.

Take

$$M_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} \subseteq S$$

is a special fuzzy subset subsemigroup of order three.

Clearly M_3 is not an ideal in fact we have infinite number of special fuzzy set subsemigroups of order three and none of them are ideals of S.

Consider

M_4 =

$$\left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

$\subseteq$ S;

M_1 is a special fuzzy set subsemigroup of order four.

Infact we have infinite number of special fuzzy set subsemigroups of order four which are not ideals of S.

Consider

$$M_5 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0.9 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.11 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 & 0 & 0 & 0.72 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.63 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0.59 \\ 0 & 0 & 0.74 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} \subseteq S,$$

M_5 is a special fuzzy set subsemigroup which is not a special fuzzy set ideal of S.

Infact of special fuzzy set subsemigroups of order five none of them are ideals of S.

Now let

$$M_6 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.19 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.41 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.31 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.11 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.15 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.19 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \end{bmatrix},$$

$$\left.\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7 \end{bmatrix}\right\} \subseteq S$$

be a special fuzzy set subsemigroup of order 17. Infact S has infinitely many special fuzzy set subsemigroups of order 17 none of which are ideals of S.

Inview of all these we have the following theorem.

THEOREM 2.8: *Let S = {Collection of all $m \times n$ matrices with entries from [0, 1)} be the special fuzzy set semigroup ($m \geq 1$, $n \geq 1$; $m < \infty$ and $n < \infty$) under min operation.*

(i) S has infinite number of special fuzzy set subsemigroups which are not ideals of infinite order.

(ii) *S has infinite number special fuzzy set subsemigroups which are also ideals of S; all of them are of infinite order.*

(iii) *S has infinite number of special fuzzy set subsemigroups none of which are ideals of S.*

Proof of (i) Take M_1 = {Collection of all $m \times n$ matrices with entries from [a, 1) $0 < a < 1$}, M_2 = {Collection of all $m \times n$ matrices with entries from [a, b], $0 < a < b < 1$} and M_3 = {Collection of all $m \times n$ matrices with entries form $(a, b) \subset [0, 1)$, $0 < a < b < 1$} be the three types of special fuzzy set subsemigroups of S.

It is easily verified that none of them are ideals for if $A = (a_{ij})$ and atleast one $a_{ij} < a$ or $a_{ij} = 0$ then min {A, P} for any P in M_i, $1 \leq i \leq 3$ is not in M_i, hence M_i's are not ideals for i = 1, 2, 3. Further all the three special fuzzy set semigroups are of infinite order.

Proof of (ii) Let N_t = {Collection of all $m \times n$ matrices with entries from [0, t) or (0, t), $0 < t < 1$} $\subseteq$ S. N_t is a special fuzzy set subsemigroup which is also an ideal of S. Clearly N_t is of infinite order and t can take an infinite number of values in [0,1). Hence the claim.

Proof of (iii) Let P_1 = {$m \times n$ matrices with only one non zero entry and the rest zeros} $\subseteq$ S with $o(P_1) = 1$. P_1 is a special fuzzy set subsemigroup of order one and P_1 is not an ideal.

Let P_2 = {(0), M_1 = {Only one $m \times n$ matrix with only one non zero entry say at a place}, M_2 = {Only one $m \times n$ matrix with a non zero entry different from M_1}} $\subseteq$ S be special fuzzy set subsemigroups which are not an ideals of S.

Hence the result.

Now we can, on matrices with entries from [0,1) define the operation 'max' so that the structure is a special fuzzy set semigroup.

We now proceed onto give examples to this end.

Example 2.20: Let
$S = \{(a_1, a_2, a_3, a_4)$ where $a_i \in [0, 1), 1 \leq i \leq 4\}$ be a special fuzzy set semigroup under the max operation.

That is if $X = \{(0.8, 0.2, 0.9, 0.7)\}$ and
$Y = (0.2, 0.7, 0.5, 0.8)$ are in S.

max {X, Y}
= max {(0.8, 0.2, 0.9, 0.7), (0.2, 0.7, 0.5, 0.8)}
= (max {0.8, 0.2}, max {0.2, 0.7}, max {0.9, 0.5}, max {0.7, 0.8})
= $(0.8, 0.7, 0.9, 0.8) \in S$.

This is the way operations are performed on S.

Let $X = (0.7, 0.5, 0.3, 0.9)$
and $Y = (0.2, 0.3, 0.1, 0.7) \in S$.

We see max (X, Y) = X and this is a special case for in general max $(X, Y) \neq X$ or Y it may be a $A \in S$ where $A \neq X$ and $A \neq Y$.

Let $M = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 0.6), 1 \leq i \leq 4\} \subseteq S$, M is a special fuzzy set subsemigroup.

Clearly M is not an ideal of S for if $P = (0.7, 0.8, 0.3, 0) \in S$ and $X = (0.5, 0.2, 0.5, 0.2) \in M$.

max (P, X)
= max {(0.7, 0.8, 0.3, 0), (0.5, 0.2, 0.5, 0.2)}
= {(max {0.7, 0.5}, max {0.8, 0.2}, max {0.3, 0.5} max {0, 0.2})
= $(0.7, 0.8, 0.5, 0.2) \notin M$ as 0.7 and $0.8 \notin [0, 0.6)$.

Thus M is not a special fuzzy set ideal of S.

Let N = $\{(a_1, a_2, a_3, a_4) \mid a \in [0.3, 0.7), 1 \leq i \leq 4\} \subseteq S$ be a special fuzzy set subsemigroup of S.

Clearly N is not an ideal of S. For take X = (0.2, 0.1, 0.9, 0.01) $\in$ S and B = (0.3, 0.4, 0.5, 0.6) $\in$ N we find

max {X, B}

= max {(0.2, 0.1, 0.9, 0.01), (0.3, 0.4, 0.5, 0.6)}

= (max {0.2, 0.3}, max {0.1, 0.4}, max {0.9, 0.5}
max {0.01, 0.6}))

= (0.3, 0.4, 0.9, 0.6) $\notin$ N so N is not an ideal of S.

Consider B = $\{(a_1, a_2, a_3, a_4) \mid a_i \in \{0, 1/7, 1/7^2, 1/7^3, \ldots 1/7^n, \ldots; n \to \infty, 1 \leq i \leq 4\} \subseteq S$ be the special subset subsemigroup of S.

Clearly B is not an ideal of S, for X = (1/3, 1/2, 1/5, 1/6) $\in$ S and A = $(1/7, 0, 1/7^2, 1/7^3) \in B$

We have max {X, A}= X; so B is not an ideal of S.

Consider V = $\{(a_1, a_2, a_3, a_4) \mid a_i \in [0.7, 1), 1 \leq i \leq 4\} \subseteq S$; V is a special fuzzy set subsemigroup of S.

Clearly V is also an ideal of S.

We see we can have infinite number of ideals of infinite order, for $V_i = \{(a_1, a_2, a_3, a_4) \mid a_j \in [i, 1); 0 < i < 1, 1 \leq j \leq 4\}\} \subseteq S$ are special fuzzy set subsemigroups which are also ideals as i can be infinite in number and are of infinite order.

The statement of theorem 2.1 is also true in case of special fuzzy set semigroups under max operation.

Example 2.21: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle| \; a_i \in [0,1),\ 1 \leq i \leq 7 \right\}$$

be the special fuzzy set semigroup under the max operation.

$$P = \left\{ \begin{bmatrix} 0.3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.7 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \subseteq S$$

is a special fuzzy set subsemigroup of S and is not an ideal of S.

Clearly o(P) = 2.

$$M = \left\{ \begin{bmatrix} 0.1 \\ 0.5 \\ 0.8 \\ 0.148 \\ 0.012 \\ 0.9 \\ 0.075 \end{bmatrix} \right\} \subseteq S$$

is a special fuzzy set subsemigroup of S of order one and is not an ideal of S.

Infact S has infinite number of special fuzzy set subsemigroups of order one none of them is an ideal barring the zero ideal.

$$W = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.3 \\ 0.7 \\ 0.001 \\ 0.025 \\ 0.512 \\ 0.301 \\ 0.709 \end{bmatrix} \right\} \subseteq S$$

is a special fuzzy set subsemigroup of order two and is not in ideal of S.

Infact S has infinite number of special fuzzy set subsemigroups none of them are special fuzzy set ideals.

Let

$$M = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a \in [0,1) \right\} \subseteq S;$$

M is a special fuzzy set subsemigroup of S which is not a special fuzzy set ideal of S of infinite order.

Let

$$D_2 = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right| a_1, a_2 \in [0,1)\ \right\} \subseteq S$$

be a special fuzzy set ideal of S of infinite order and is not an ideal of S.

$$\text{Let } C_3 = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right| a_1, a_2\ a_3 \in [0,1)\right\} \subseteq S$$

is only a special fuzzy set subsemigroup of S which is not an ideal of S of infinite order.

Let

$$T_2 = \left\{\left.\begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right| a_1, a_2\ \in [0,1)\right\} \subseteq S$$

be the special fuzzy set subsemigroup of S of infinite order.

Clearly T_2 is not a special fuzzy set ideal of S.

Let

$$C_4 = \left\{\left.\begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \\ a_4 \end{bmatrix}\right| a_1, a_2, a_3, a_4 \in [0,1)\right\} \subseteq S$$

be a special fuzzy set subsemigroup of S; clearly C_4 is of infinite order.

But C_4 is not an ideal of S.

Let

$$M_5 = \left\{\left.\begin{bmatrix} 0 \\ a_1 \\ 0 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}\right| a_i \in [0,1);\ 1 \le i \le 5\right\} \subseteq S$$

is only a special fuzzy set subsemigroup of S and is not an ideal of S.

$$M_6 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in [0,1);\ 1 \le i \le 6 \right\} \subseteq S$$

is only a special fuzzy set subsemigroup and is not an ideal of S.

Let

$$P_6 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ 0 \end{bmatrix} \middle| \; a_i \in [0,1),\ 1 \le i \le 6 \right\} \subseteq S$$

be a special fuzzy set subsemigroup of S and is not an ideal of S.

Let

$$T_4 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ a_3 \\ a_4 \\ 0 \end{bmatrix} \middle| \; a_i \in [0,1),\ 1 \le i \le 4 \right\} \subseteq S$$

be a special fuzzy set subsemigroup of S and is not an ideal of S.

Thus we have seen infinite number of special fuzzy set subsemigroups which are not ideals of S.

Let

$$R = \left\{ \begin{bmatrix} 0.3 \\ 0.4 \\ 0 \\ 0 \\ 0 \\ 0.8 \\ 0.7 \end{bmatrix}, \begin{bmatrix} 0.7 \\ 0.8 \\ 0.9 \\ 0.1 \\ 0.8 \\ 0.8 \\ 0.9 \end{bmatrix} \right\} \subseteq S$$

is a finite special fuzzy set subsemigroup of S of order two which is not an ideal of S.

Let

$$L = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ 0 \\ 0 \\ a_3 \end{bmatrix} \middle| \; a_i \in [0,0.4),\ 1 \leq i \leq 3 \right\} \subseteq S$$

be a special fuzzy set subsemigroup of S, clearly L is not a special fuzzy set ideal of S.

$$D = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ 0 \\ 0 \\ a_3 \end{bmatrix} \middle| \; a_i \in [0.4,1), 1 \leq i \leq 3 \right\} \subseteq S$$

is a special fuzzy set subsemigroup of S of infinite order. Clearly D is not a special fuzzy set ideal of S of infinite order.

Let

$$V_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_i \in [0.7,1) \right\} \subseteq S$$

be a special fuzzy set subsemigroup of S and is not an ideal of S.

Now

$$E = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ 0 \end{bmatrix} \middle| \; a_i \in [0.3, 1), 1 \leq i \leq 7 \right\} \subseteq S$$

is a special fuzzy set subsemigroup of S and is also a special fuzzy set ideal of S.

It is pertinent to observe that

$$W = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \\ a_4 \end{array}\right] \middle| \; a_i \in [0.6, 1), 1 \leq i \leq 4 \right\} \subseteq S$$

is a special fuzzy set subsemigroup of S and not an ideal of S.

For any

$$A = \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{array}\right] \in S \text{ with even one of } a_4 \text{ or } a_5 \text{ or } a_6 \neq 0;$$

is such that max {A, w} with $w \in W$ is not in W so W is only a special fuzzy set subsemigroup of S; and is not an ideal of S.

So we can say all special fuzzy set subsemigroups which has in the column matrices atleast one zero cannot be an ideal of S; only a subsemigroup of S.

Let

$$P_1 = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \\ 0 \end{bmatrix} \right| a_i \in [0, 1); 1 \leq i \leq 6 \right\} \subseteq S;$$

P_1 is a special fuzzy set subsemigroup and not an ideal of S.

Likewise

$$P_2 = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ a_3 \\ a_4 \\ 0 \\ a_5 \end{bmatrix} \right| a_i \in [0, 1), 1 \leq i \leq 5 \right\} \subseteq S$$

is only a special fuzzy set subsemigroup and is not an ideal of S.

$$P_3 = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right| a_i \in [0, 1), 1 \leq i \leq 6 \right\} \subseteq S$$

is only a special fuzzy set subsemigroup of S and is not an ideal of S.

Let

$$P_4 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ 0 \\ a_3 \\ 0 \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 3 \right\} \subseteq S;$$

be the special fuzzy set subsemigroup of S and is not an ideal of S.

Now

$$P_5 = \left\{ \begin{bmatrix} a_1 \\ a_4 \\ 0 \\ 0 \\ 0 \\ a_2 \\ a_3 \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 4 \right\} \subseteq S$$

is a special fuzzy set subsemigroup of S is not an ideal of S.

$$P_6 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 5 \right\} \subseteq S$$

is also a special fuzzy set subsemigroup of S and is not an ideal of S under max operation.

Example 2.22: Let S = {Collection of all matrices

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 15 \right\}\}$$

be the special fuzzy set semigroup under the operation max.

S is of infinite order.

S has infinite number of finite special fuzzy set subsemigroups and also infinite number of special fuzzy set subsemigroups of infinite order.

$$P_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \middle| a_1 \in [0, 1) \right\} \subseteq S$$

is an infinite special fuzzy set subsemigroup.

Clearly P_1 is not an ideal.

Let

$$P_2 = \left\{\begin{bmatrix} 0 & a_2 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix} \middle| a_2 \in [0, 1)\right\} \subseteq S$$

be an infinite special fuzzy set subsemigroup.

$$P_3 = \left\{\begin{bmatrix} 0 & 0 & a_3 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix} \middle| a_3 \in [0, 1)\right\} \subseteq S$$

is an infinite special fuzzy set subsemigroup and so on.

$$P_5 = \left\{\begin{bmatrix} 0 & 0 & 0 & ... & a_5 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix} \middle| a_5 \in [0, 1)\right\} \subseteq S$$

is an infinite special fuzzy set subsemigroup of S.

$$P_6 = \left\{\begin{bmatrix} 0 & 0 & 0 & ... & 0 \\ a_6 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix} \middle| a_6 \in [0, 1)\right\} \subseteq S$$

is an infinite special fuzzy set subsemigroup and so on.

$$P_{14} = \left\{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{14} & 0 \end{bmatrix} \middle| a_{14} \in [0, 1)\right\} \subseteq S$$

and

$$P_{15} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{15} \end{bmatrix}\right| a_{15} \in [0, 1)\right\} \subseteq S$$

are the collection of 15 special fuzzy set subsemigroups of S of infinite order and none of them are ideals of S.

Likewise

$$P_{12} = \left\{\left.\begin{bmatrix} a_1 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right| a_1, a_2 \in [0, 1)\right\} \subseteq S$$

is a special fuzzy set subsemiring which is not an ideal of S of infinite order likewise

$$P_{1,15} = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{15} \end{bmatrix}\right| a_1, a_{15} \in [0, 1)\right\} \subseteq S$$

is a special fuzzy set subsemigroup of S of infinite order which is not an ideal of S.

We can develop this to

$$P_{1,2,3} = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right| a_1, a_2, a_3 \in [0, 1)\right\} \subseteq S$$

$P_{1,2,3}$ is a special fuzzy set subsemigroup of S and is not an ideal of S.

$$P_{6,10,14} = \left\{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ a_6 & 0 & 0 & 0 & a_{10} \\ 0 & 0 & 0 & a_{14} & 0 \end{bmatrix} \middle| a_6, a_{10}, a_{14} \in [0, 1)\right\} \subseteq S$$

is a special fuzzy set subsemigroup of S and is not an ideal of S.

Let

$$P_{4,9,11,13} = \left\{\begin{bmatrix} 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & a_9 & 0 \\ 0 & a_{11} & 0 & a_{13} & 0 \end{bmatrix} \middle| a_4, a_9, a_{11}, a_{13} \in [0, 1)\right\} \subseteq S$$

be a special fuzzy set subsemigroup of S and is not an ideal.

Likewise

$$P_{2,5,8,10,15} = \left\{\begin{bmatrix} 0 & a_2 & 0 & 0 & a_5 \\ 0 & 0 & a_8 & 0 & a_{10} \\ 0 & 0 & 0 & 0 & a_{15} \end{bmatrix} \middle| a_2, a_5, a_8, a_{10}, a_{15} \in [0, 1)\right\} \subseteq S$$

is again a special fuzzy set subsemigroup of S and is not an ideal.

As an extreme case we can have

$$P_{1,2,\ldots,15} = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & 0 & a_4 \\ 0 & a_5 & 0 & a_6 & a_{10} \\ a_7 & 0 & a_8 & a_{11} & a_9 \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 11\right\} \subseteq S;$$

is a special fuzzy set subsemigroup of S and is not an ideal of S.

Now we do have other types of special fuzzy set subsemigroups which are not ideals of infinite order.

Let

$$B_t = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \end{bmatrix} \middle| \; a_i \in [0, t), 0 < t < 1, 1 \leq i \leq 15 \right\} \subseteq S$$

be the special fuzzy set subsemigroup of S.

Clearly $o(B_t) = \infty$ by varying t we get infinite number of subsemigroups all of infinite order and none of them is an ideal of S.

Let

$$C_{a,b} = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \end{bmatrix} \middle| \; a_i \in [a, b), 0 < a < b < 1, \right.$$

$$\left. 1 \leq i \leq 15 \right\} \subseteq S$$

be the special fuzzy set subsemigroup of S.

Clearly $C_{a,b}$ is not an ideal and $o(C_{a,b}) = \infty$. We have infinite number of pairs a, b $\in$ [0, 1) such that $C_{a,b}$ is a subsemigroup none of them is an ideal and each is of infinite order.

Let

$$E_s = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \end{bmatrix} \middle| \; a_i \in [a, 1), 0 < a < 1, 1 \leq i \leq 15 \right\} \subseteq S$$

be the special fuzzy set subsemigroup. $o(E_s) = \infty$ and E_s is a special fuzzy set ideal of S.

We have infact infinite number of such special fuzzy set subsemigroups which are ideals of S.

If we take $M_{1/3} = \left\{\begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \end{bmatrix} \middle| \; a_i \in \{0, 1/3, 1/3^2, 1/3^3,\right.$

$\left. \dots, 1/3^n, \dots, \; n \to \infty\}, 1 \leq i \leq 15\right\} \subseteq S$ be a special fuzzy set subsemigroup of S but $M_{1/3}$ is not a special fuzzy set ideal of S; $o(M_{1/3}) = \infty$.

Likewise

$$M_{1/n} = \left\{\begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \end{bmatrix} \middle| \; a_i \in \{0, 1/n, 1/n^2, 1/n^3, \dots, \right.$$

$$\left. 1/n^m; m \to \infty\}, 1 \leq i \leq 15\right\} \subseteq S$$

is a special fuzzy set subsemigroup of S.

We have such large collection of special fuzzy set subsemigroups which are not ideals and are of infinite order.

Example 2.23: Let

$$S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 0.9), 1 \leq i \leq 9\right\}$$

be a special fuzzy set semigroup of infinite order under max operation .

We find the properties enjoyed by S. S has infinite number of special fuzzy set subsemirings of finite order which are not ideals of S.

For take

$$P_1' = \left\{ \begin{bmatrix} 0.8 & 0.13 & 0.7 \\ 0.14 & 0.6 & 0.23 \\ 0.4 & 0.01 & 0 \end{bmatrix} \right\} \subseteq S$$

is a special fuzzy set subsemigroup of finite order in particular one.

$$P_2' = \left\{ \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \subseteq S$$

is again a special fuzzy set subsemigroup of finite order in fact of order 1.

$$P_3' = \left\{ \begin{bmatrix} 0 & 0.7 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0 & 0 \end{bmatrix} \right\} \subseteq S$$

is again a special fuzzy set subsemigroup of order one which is not an ideal of S.

Infact S has infinite number of special fuzzy set subsemigroups and none of them can be ideals of S.

Now consider

$$R_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0.3 & 0.4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \subseteq S,$$

R_1 is also a special fuzzy set subsemigroup and not an ideal of S. Infact S has infinite number of special fuzzy set subsemigroups of order two and none of them are ideals of S.

Consider

$$P_1 = \left\{\begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.7 & 0.5 \end{bmatrix}, \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0.9 & 0.6 \end{bmatrix}\right\} \subseteq S$$

P_1 is a special fuzzy set subsemigroup of finite order; $o(P_1) = 2$ and P_1 is not an ideal of S.

Finally

$$P_2 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.7 & 0.5 \end{bmatrix}, \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0.9 & 0.6 \end{bmatrix}\right\} \subseteq S$$

is again a special fuzzy set subsemigroup of order three and none of them are ideals of S.

We can have finite special fuzzy set subsemigroups of S none of them are ideals.

We have every order of such special fuzzy set subsemigroups from the set N = 1, 2, 3, …, $n < \infty$.

Let

$$W = \left\{\begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \middle| \; a_i \in [a, b), 0 < a < b < 1, 1 \leq i \leq 3\right\} \subseteq S$$

be a special fuzzy set subsemigroups of S. $o(W) = \infty$ but W is not an ideal of S.

Infact S has infinite number of such special fuzzy set subsemigroups under max operation which are not ideals of S.

Now consider

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & 0 & 0 \\ a_4 & 0 & 0 \end{bmatrix} \middle| \; a_i \in [0,0.5), 1 \leq i \leq 4 \right\} \subseteq S.$$

Clearly T is only a special fuzzy set subsemigroup and is not an ideal of S under max.

Take

$$U = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 1), 1 \leq i \leq 9 \right\} \subseteq S;$$

clearly U is a special fuzzy set subsemigroup and also an ideal of S under max operation.

Thus S has both ideals and special fuzzy set subsemigroups which are not ideals. For the time being we will denote by S_{max} all special fuzzy set semigroups under max operation and S_{min} all special fuzzy set semigroups under min operation.

We have the following theorem.

THEOREM 2.9: *Let S_{max} and S_{min} be special fuzzy set semigroup.*

(1) If $I \subseteq S$ and I_{max} is an ideal in S_{max} then on the set of $I \subseteq S$ with the min operation given then I_{min} is only special fuzzy set subsemigroup and is not an ideal.

(2) If $I \subseteq S$ and I_{min} is an ideal of S_{min} then I with max operation is not an ideal; only a special fuzzy set subsemigroup.

Proof follows from observation. However to this end we give an example or two.

Example 2.24: Let S = {Collection of all 1×7 matrices $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ where $a_i \in [0, 1)$, $1 \leq i \leq 7$}. S under max operation denoted by S_{max} is a special fuzzy set semigroup.

Now S under min operation denoted by S_{min} is a special fuzzy set semigroup.

Let P = $\{(a_1, a_2, \ldots., a_7) \mid a_i \in [0, 0.3), 1 \leq i \leq 7\} \subseteq S$. P under the min operation denoted by P_{min} is a special fuzzy set subsemigroup which is also an ideal of S_{min}.

Now the same P under max operation denoted by P_{max} is only a special fuzzy set subsemigroup and is not an ideal of S_{max}. Hence one part of theorem.

Further we have such P's to be infinite in number. For we take $P_a = \{(a_1, \ldots, a_7) \mid a_i \in [0, a); 1 \leq i \leq 7\} \subseteq S$; $0 < a < 1$; clearly a can take infinite number of values between 0 and 1.

Consider B = $\{(a_1, a_2, \ldots, a_7) \mid a_i \in [a, 1), 1 \leq i \leq 7\} \subseteq S$; if on B we give the max operation then B_{max} is a special fuzzy set subsemigroup as well as an ideal of S_{max}.

Now B_{min} is only a special fuzzy set subsemigroup and is not an ideal of S_{min}. Hence the other part of the theorem.

Example 2.25: Let

$$S = \left\{\text{Collection of all matrices } \left.\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \right| a_i \in [0, 1), \right.$$

$$1 \leq i \leq 12\}$$

be a special fuzzy set semigroup under max or min operation.

Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in [0, 0.7), 1 \leq i \leq 12 \right\} \subseteq S;$$

P under max operation is only a special fuzzy set subsemigroup and is not an ideal of S_{max}.

However P under the min operation is a special fuzzy set subsemigroup which is an ideal of S_{min}.

Now consider the set

$$R = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in [0.5, 1); 1 \leq i \leq 12 \right\} \subseteq S.$$

Clearly R under the max operation is a special fuzzy set subsemigroup and R_{max} is also an ideal of S_{max}.

However R_{min} is only a special fuzzy set subsemigroup of S_{min} and is not an ideal of S_{min}.

Thus we see R is a S_{max} ideal and the same R is not a S_{min} ideal.

Consider

$$T = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \middle| a_i \in [a, b); 0 < a < b < 1, 1 \leq i \leq 12\right\} \subseteq S,$$

T is a special fuzzy set subsemigroup of S_{max} as well as S_{min}, however T is not an ideal of S_{max} and S_{min}.

Let

$$W = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \middle| a_i \in \{0, 1/11, 1/11^2, \ldots, 1/11^n \text{ as}\right.$$

$$\left. n \to \infty\} \; 1 \leq i \leq 12\right\} \subseteq S,$$

T be a special fuzzy set subsemigroup of both S_{min} and S_{max}.

Clearly S, W is not an ideal of S_{max} or S_{min}.

Thus we see S has special fuzzy set subsemigroups which are not ideals with respect to both max or min operation.

Let

$$V_1 = \left\{ \begin{bmatrix} 0.3 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \subseteq S.$$

V_1 is both a special fuzzy set subsemiring with respect to max and min operation.

Clearly V_1 is not an ideal with respect to both min and max operation.

Example 2.26: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & \cdots & a_7 \\ a_8 & a_9 & \cdots & a_{14} \\ a_{15} & a_{16} & \cdots & a_{21} \\ a_{22} & a_{23} & \cdots & a_{28} \end{bmatrix} \middle| a_i \in [0, 1); 1 \leq i \leq 28 \right\}$$

be a special fuzzy set under min or max operation.

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & \cdots & a_7 \\ a_8 & a_9 & \cdots & a_{14} \\ a_{15} & a_{16} & \cdots & a_{21} \\ a_{22} & a_{23} & \cdots & a_{28} \end{bmatrix} \middle| a_i \in [0, 0.4); 1 \leq i \leq 28 \right\} \subseteq S$$

is a special fuzzy set subsemigroup as well as an ideal of S under the min operation.

However P under max operation is only a special fuzzy set subsemigroup and not an ideal under max operation.

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \\ a_{15} & a_{16} & \ldots & a_{21} \\ a_{22} & a_{23} & \ldots & a_{28} \end{bmatrix} \middle| a_i \in [0.6, 1); 1 \leq i \leq 28 \right\} \subseteq S$$

is a special fuzzy set subsemigroup under min operation but is not an ideal under min operation.

However W is a special fuzzy set subsemigroup as well as ideal of S under the max operation.

Let

$$Q = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \\ a_{15} & a_{16} & \ldots & a_{21} \\ a_{22} & a_{23} & \ldots & a_{28} \end{bmatrix} \middle| a_i \in [a, b); 1 \leq i \leq 28 \right\} \subseteq S$$

where $0 < a < b < 1$ be a special fuzzy set subsemigroup under the max operation as well as under the min operation.

However Q is not an ideal with respect to max as well as min operation.

Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \\ a_{15} & a_{16} & \ldots & a_{21} \\ a_{22} & a_{23} & \ldots & a_{28} \end{bmatrix} \middle| a_i \in \{0, 1/5, 1/5^2, \ldots, \right.$$

$$\left. 1/5^n \ldots, n \to \infty\} \subseteq [0, 1), 1 \leq i \leq 28 \right\} \subseteq S$$

be the special fuzzy set subsemigroup under the operation max as well as min and under both the operation max as well as min M is not an ideal.

$$T = \left\{ \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} \middle| \, a \in [0, 1) \right\} \subseteq S$$

is a special fuzzy set subsemigroup under both the operations max as well as min.

However T is an ideal with respect to the min operation but T is not an ideal with respect to the max operation.

Let

$$L = \left\{ \begin{bmatrix} 0.3 & 0.4 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0.9 & \dots & 0.7 \end{bmatrix} \right\} \subseteq S$$

be a special fuzzy set subsemigroup under the operation max (or min) of S.

Clearly L is not an ideal with respect to max or min operation on S.

Now having seen examples of special fuzzy set matrix semigroups we now proceed onto describe special fuzzy set polynomial semigroups.

Example 2.27: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 1), \text{‘+’ modulo } 1 \right\}$$

be a group.

For if $a = 0.3x^i \in S$ there is a unique $b = 0.7x^i \in S$ is such that $a + b = 0$ that is a is the additive inverse of b and vise versa.

Let $p(x) = 0.3 + 0.7x^3 + 0.4x^7$
and $q(x) = 0.8 + 0.5x^2 + 0.4x^3 + 0.2x^5 + 0.9x^7 \in S$.

To find p(x) + q(x);
$= 0.3 + 0.7x^3 + 0.4x^7 + 0.8 + 0.5x^2 + 0.4x^3 + 0.2x^5 + 0.9x^7$
$= (0.3 + 0.8) + 0.5x^2 + (0.7x^3 + 0.4x^3) + 0.2x^5 + (0.4x^7 + 0.9x^7)$
$= 0.1 + 0.5x^2 + 0.1x^3 + 0.2x^5 + 0.3x^7 \in S$.

This is the way '+' operation is performed on S. Further $0 \in S$ is such that $a(x) + 0 = 0 + a(x) = a(x)$.

We see every element p(x) in S has a unique q(x) such that $p(x) + q(x) = 0$.

Thus (S, +) is the special fuzzy set polynomial group under '+'.

We see (S, +) has special fuzzy set polynomial subgroups also; for take

$$T = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\} \subseteq [0, 1) \right\} \subseteq S.$$

Clearly (T, +) is an infinite special fuzzy set polynomial subgroup of S.

Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 0.01, 0.02,, 0.03, \ldots, 0.09, 0.10, 0.11, 0.12, \ldots, 0.19, \ldots, 0.99\} \subseteq S. \right.$$

Clearly (P, +) is again a special fuzzy set polynomial subgroup of S.

Now having seen example of special fuzzy set polynomial group we now proceed onto describe special fuzzy set polynomial semigroup under '×'.

Example 2.28: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 1), \times \right\}$$

be the special fuzzy set polynomial semigroup under ×.

The operation × is $S_\times$ is defined as follows:

Let $p(x) = 0.3 + 0.5x^3$
and $q(x) = 0.9 + 0.2x + 0.3x^2 \in S_\times$.

$p(x) \times q(x)$
$= (0.3 + 0.5x^3) \times (0.9 + 0.2x + 0.3x^2)$
$= (0.3 + 0.5x^3) \times 0.9 + (0.3 + 0.5x^3) \times 0.3x^2$
$= 0.27 + 0.45x^3 + 0.06x + 0.10x^4 + 0.09x^2 + 0.15x^5$.

Let 0.3x + 0.2x is 0.3x + 0.2x only we do not write it as 0.5x while performing the × operation on $S_\times$.

Let

$$P_{1/2} = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 1/2, 1/2^2, 1/2^3, \ldots\} \subseteq S \right.$$

is a special fuzzy set polynomial subsemigroup of $S_\times$.

Clearly this is not an ideal of $S_\times$.

Let

$$P_{1/3} = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 1/3, 1/3^2, 1/3^3, \ldots n \to \infty\} \right. \subseteq S_\times.$$

$P_{1/3}$ is the special fuzzy set subsemigroup of $S_\times$ and is not an ideal of $S_\times$.

Infact $S_\times$ has infinite number of special fuzzy set subsemigroups which are not ideals of $S_\times$.

Let

$$M_\times = P_{1/2} = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 0.2] \subseteq [0, 1)\right\} \subseteq S$$

be a special fuzzy set subsemigroup of $S_\times$. Clearly $M_\times$ is an ideal of $S_\times$. Infact $S_\times$ also has a collection of ideals which are of infinite cardinality and infinite in number.

Now we can define S_{min} and S_{max} special fuzzy set polynomial semigroups.

For instance let $S = P_{1/2} = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0,1)\right\};$

now if $p(x) = 0.3x^3 + 0.7$ and $q(x) = 0.5 + 0.2x^7 + 0.4x^9$ are in S then;

$\min (p(x), q(x)) = \min (0.3x^3 + 0.7, 0.5 + 0.2x^7 + 0.4x^9)$

$= (\min (0.7, 0.5) + \min (0.3x^3, 0) + \min (0, 0.2x^7) + \min (0, 0.4x^9))$

$= 0.5.$

$\max \{p(x), q(x)\} =$

$\max \{0.7 + 0.3x^3, 0.5 + 0.2x^7 + 0.4x^9\}$

$= \{\max \{0.7, 0.5\} + \max \{0.3x3, 0\} + \max \{0.2x^7, 0\} + \max \{0.4x^9, 0\}\}$

$= 0.7 + 0.3x^3 + 0.2x^7 + 0.4x^9$ is in S_{max}.

Thus S_{min} and S_{max} lead to special fuzzy set polynomial semigroups of infinite order. We can have substructures using them; which will be described by an example or two.

Example 2.29: Let $S_{min} = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 1)\right\}$ be the special fuzzy set semigroup under the min operation.

We see S_{min} has several special fuzzy set subsemigroups of finite order and infinite order which are not ideals.

Also S_{min} has special fuzzy set subsemigroups which are ideals but all of them are of infinite order.

$P_1 = \{0.051 + 0.73x^2 + 0.91x^3 + 0.031x^7\}$ is a special fuzzy set subsemigroup of finite order say of order one.

$P_2 = \{0.531x + 0.21, 0.73x + 0.023\} \subseteq S_{min}$ is a special fuzzy set subsemigroup of order two and clearly. P_2 is not an ideal.

Likewise we can say S_{min} has infinite number of special fuzzy set subsemigroups of order one, order two, order three, and so on and of order n; $n < \infty$.

Let

$$M = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 1/2, 1/2^2, \ldots 1/2^n, \ldots\}\right\} \subseteq S_{min}.$$

M is a special fuzzy set subsemigroup of S_{min} and is not an ideal of S_{min}.

$$N = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 1/7, 1/7^2, \ldots, 1/7^{10}\} \text{ and } \{0, 1/5, 1/5^2, \ldots, 1/5^{21}\}\right\} \subseteq S_{min}$$

is also a special fuzzy set subsemigroup of S_{min} and is not an ideal of S_{min}. Infact N is of infinite order.

$$T = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 0.02, 0.003, 0.031, 0.00015, \right.$$

$$\left. 0.01072, 0.007, 0.19, 0.7\}\right\} \subseteq S_{min}$$

is a special fuzzy set subsemigroup under the min operation and is not an ideal. T is of infinite order.

Let

$$T_1 = \left\{ \sum_{i=0}^{9} a_i x^i \,\middle|\, a_i \in \{0, 0.7, 0.714, 0.0014, 0.05, \right.$$

$$\left. 0.0703, 0.1, 0.156, 0.1030.7\right\} \subseteq S_{min}.$$

T_1 is a special fuzzy set subsemigroup of finite order under min operation and is not an ideal of S_{min}.

Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 0.3) \right\} \subseteq S_{min}.$$

V is a special fuzzy set subsemigroup as well as the ideal of S_{min}.

Thus we see we have ideals of infinite order.

If we take

$$W = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 0.8) \right\} \subseteq S_{min},$$

W is a special fuzzy set subsemigroup as well as an ideal of S_{min} under the min operation.

Suppose

$$B = \left\{\sum_{i=0}^{12} a_i x^i \,\middle|\, a_i \in [0, 0.8)\right\} \subseteq S_{min}$$

be the special fuzzy set subsemigroup of S_{min}.

Clearly B is an ideal. Thus S_{min} has infinite number of ideals of infinite order but of finite degree polynomials.

This is the special property enjoyed by S_{min} special fuzzy set polynomial semigroups.

Now we proceed onto study the new notion of special fuzzy set polynomial semigroups under max operation.

This concept will be illustrated by the following example.

Example 2.30: Let

$$S = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 1)\right\}$$

be the special fuzzy set semigroup under the max operation.

Let

$$P_1 = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0.5, 1)\right\} \subseteq S_{max},$$

P_1 is a special fuzzy set subsemigroup under the max operation.

P_1 is also a special fuzzy set ideal of S_{max}.

Let

$$M_1 = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0.3, 0.7)\right\} \subseteq S_{max};$$

M_1 is a special fuzzy set subsemigroup of S_{max} but M_1 is not an ideal of S_{max}.

Let

$$M_2 = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 0.6)\right\} \subseteq S_{max};$$

M_2 is a special fuzzy set subsemigroup of S_{max} but M_2 is not an ideal of S_{max}.

Let

$$B_2 = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 1/2, 1/2^2, 1/2^3, \ldots, 1/2^{12}\}\right\} \subseteq S_{max};$$

B_2 is a special fuzzy set subsemigroup of S_{max}. B_2 is not an ideal of S_{max}.

Now

$$W_2 = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0.3, 0.4, 0.72, 0.591, 0.032, 0.0676\}\right\} \subseteq S_{max};$$

W_2 is only a special fuzzy set subsemigroup of S_{max} and is not an ideal of S_{max}.

We have no finite special fuzzy set subsemigroups which are ideals under the max operation. However in contrast we have seen S_{min} has special fuzzy set ideals which are not special fuzzy set ideal in S_{max} under max operation.

For

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 0.7) \right\} \subseteq S_{max}$$

is not an ideal of S_{max}, however M is an ideal under the min operation.

Likewise

$$N = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0.8, 1) \right\} \subseteq S_{max}$$

is a special fuzzy set subsemigroup which is also an ideal of S_{max} but N is only a special fuzzy set subsemigroup under min operation which is not an ideal of S_{min}.

This is the way we get the ideals. We see S_{max} also has finite special fuzzy set subsemigroups which are not ideals.

Now we will proceed onto enumerate that we have three special fuzzy set polynomial semigroups under max or min or $\times$. However under '+' these special fuzzy set polynomial semigroups are groups.

We have seen ideals of special fuzzy set semigroups.

It is pertinent to keep on record that

$$S_{\times} = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 1) \right\}$$

under product operation has ideals of infinite order.

We further see $S_{\times}$ has several special fuzzy set subsemigroups which are not ideals. This study is interesting and innovative.

Chapter Three

SEMIRINGS AND PSEUDO RINGS USING [0,1)

In this chapter we for the first time proceed onto define, develop and describe algebraic structures using the special fuzzy interval [0,1).

We first define the operations min and max on [0,1).

We see under the min and max operation [0,1) becomes a semiring. This semiring will be denoted by S(min, max) = {[0,1), min, max}. Clearly S(min, max) is of infinite order and is both a commutative and an associative semiring.

One is interested in finding substructures of S(min, max).

In the first place we find the subsemiring of S(min, max).

We also call the semiring; S(min, max) as special fuzzy set semiring.

Take X = {0,73} and Y = {0.9102} in S(max, min);
max {X, Y} = max {{0.73} {0.9102}}
= 0.9102

and min {X, Y} = min {0.73, 0.9102}
= {0.73}.

Thus we see P = {X, Y} = {0.73, 0.9102} is itself a subsemiring of S(min, max).

Thus subsemirings of S(min, max) can be of order two.

Infact every singleton element in S(min, max) is a subsemiring of S(min, max).

For take P = {0.0023} $\in$ [0,1). min {0.0023, 0.0023} = {0.0023} and

max {0.0023, 0.0023}
= {0.0023}.

Every singleton element is a subsemiring of S(min, max).

Similarly every pair of elements in S(min, max) is again a special fuzzy set subsemiring.

Also every triple element x, y, z $\in$ [0,1) is again a special fuzzy set subsemiring.

For take P = {0.003, 0.42, 0.9215} $\subseteq$ S(min, max). P is a special fuzzy set subsemiring of S(min, max).

Likewise W = {0.9, 0.735, 0.128, 0.618} $\subseteq$ S (min, max) is again a special fuzzy set subsemiring of S(min, max).

Infact S(min, max) contains special fuzzy set subsemirings of every order from 1 to ∞.

Take P = {[0, 0.36)} $\subseteq$ {[0, 1)}. P is a special fuzzy set subsemiring of infinite order.

Now M = {[0, p); p < 1} $\subseteq$ {[0, 1)}; M is a special fuzzy set subsemiring under max and min operation.

Suppose N = {[a, b] | 0 < a and b < 1} $\subseteq$ [0, 1) be a special fuzzy set subsemiring under the max and min operation.

For instance take N = {[0.003, 0.07]} $\subseteq$ [0, 1).

Clearly N is a special fuzzy set subsemiring under max and min operations.

Also T = {[0.6, 1)} $\subseteq$ [0, 1) is again a special fuzzy set subsemiring under max and min operation of S(min, max).

W = {(0.5, 0.7))} $\subseteq$ [0, 1) is also a special fuzzy set subsemiring of S(min, max).

B = {[0.2, 0.7)} $\subseteq$ {[0, 1)} is a special fuzzy set subsemiring of S(min, max).

V = {(0.007, 0.7) $\subseteq$ [0, 1) is a special fuzzy set subsemiring of S(min, max).

We have infinite number of special fuzzy set subsemirings of S(min, max) which are mainly subintervals of the interval [0, 1).

If x = {0.0073} and y = {0.62} $\in$ V.

Then max {x, y} = {0.0073, 0.62} = {0.62} and min {x, y} = {0.0073, 0.62} = 0.0073 and so on.

This is the way operations are performed on V.

It is interesting to note we have special fuzzy set subsemirings of all possible order from 1 to ∞ and also countable infinity and uncountable infinity.

However this special fuzzy set semiring has infinite number of idempotents in it. Cleary S(min, max) has no zero divisors or units.

Now we have to study whether S(min, max) has ideals.

Recall S(min, max) has ideals.

We say $P \subseteq S$ (min, max) is an fuzzy special set ideal of S(min, max) if

(i) If $x, y \in P$, $\max\{x, y\} \in P$.
(ii) For every $x \in S$ and $y \in P$; $\min\{x, y\} \in P$.

We see singleton sets are not special fuzzy set ideals.

For if $D = \{0.007\} \in S(\max, \min)$ and if $0.00001 \in S$ then $\min = \{0.007, 0.00001\} = \{0.00001\} \notin D$.

So D is not a special fuzzy set ideal only a special fuzzy set subsemiring of S(min, max). We see all special fuzzy set subsemirings of any finite cardinality in general is not a special fuzzy set ideal of S(min, max).

In view of all these we have the following theorems.

THEOREM 3.1: *S(min, max) has infinite number of special fuzzy set subsemirings of order uncountable infinity.*

Proof: Consider subsets of the form $\{(a, b) \mid 0 \leq a \text{ or } b < 1\}$ or $\{[0, p) \mid p < 1\}$ or $\{(a, 1) \mid 0 < a\}$ or $\{[a, 1) \mid 0 < a\}$ or $\{[a, b] \mid 0 \leq a \text{ and } b < 1\}$.

All these subsets are special fuzzy set subsemirings of S(max, min) of infinite order and infact of uncountable infinity.

THEOREM 3.2: *Let S(max, min) be a special fuzzy set semiring. S(max, min) has infinite number of special fuzzy subsemirings of finite order.*

Proof: Consider $P = \{x_1, x_2, \ldots, x_n\} \subseteq S(\max, \min)$; $n = 1, 2, 3, \ldots, n$ and $n < \infty$.

Clearly each P is a special fuzzy set subsemiring we have infinite number of such special fuzzy set subsemirings all of them are of finite order.

THEOREM 3.3: *Let S(min, max) be a special fuzzy set semiring. None of the special fuzzy set subsemirings are of finite order is a special fuzzy set ideal of S(max, min).*

Proof: Let P = $\{x_1, x_2, \ldots, x_n\}$; $n < \infty$ where each $x_i \in [0, 1)$; $y \in [0,1) \setminus P$ such that $y < x_i$ for atleast one x_i.

In such case we see min $\{y, x_i\} = y$ and $y \notin P$. Hence the claim of the theorem.

THEOREM 3.4: *Let S(min, max) be the special fuzzy set semiring; S(min, max) has special fuzzy set ideals all of which are uncountably infinite.*

Proof: Let S(min, max) be the special fuzzy set semiring. Consider P = $\{[0, p); p < 1\} \subseteq$ S (min, max); P is clearly a special fuzzy set subsemiring.

However P is also a special fuzzy set ideal of S(min, max). For any $x \in P$ and $y \in$ S(min, max) \ P we see min (x, y) = x for all $y \in$ S(min, max) \ P. Thus P is a special fuzzy set ideal of S(min, max).

Thus S(min, max) has infinite number of special fuzzy set ideals of the form P for varying $p \in [0,1)$.

THEOREM 3.5: *Let S(min, max) be a special fuzzy set semiring. S(min, max) has infinite number of special fuzzy set subsemiring of uncountably infinite cardinality which are not special fuzzy set ideals.*

Proof: Let S(min, max) be the special fuzzy set semiring. Take B = $\{(a, b] \mid 0 < a \text{ and } b < 1\} \subseteq$ S(min, max). B is a special fuzzy set subsemiring of S(min, max).

Clearly B is not a special fuzzy set ideal of S(min, max) as for any x $\in$ { S(min, max) \ B}; x < a we see for any y $\in$ B min{x, y} = x and x $\notin$ B. So B is only a special fuzzy set subsemiring which is not a special fuzzy set ideal of S(min, max).

B is of uncountable infinity so we have infinite number of uncountably infinitely many special fuzzy set subsemirings which are not ideals.

We illustrate all these theorems by some examples.

Example 3.1: Let S(min, max) be the special fuzzy set semiring. M = {0.03, 0.135, 0.79, 0.092, 0.31, 0.08432} $\subseteq$ S(min, max). M is a special fuzzy set subsemiring. Clearly M is not a special fuzzy set ideal of S(min, max).

For take x = 0.00000732 $\in$ S(max, min) and any y $\in$ M. We see min {x, y} = {0.00000732, y}

= {0.00000732} $\notin$ M so M is not a special fuzzy set ideal of S(min, max); only special fuzzy set subsemiring of S(min, max).

Example 3.2: Let S(min, max) be a special fuzzy set semiring; P = {(0.007, 0.059)} $\subseteq$ S(min, max) be the special fuzzy set subsemiring of S(min, max). P is only a special fuzzy set subsemiring and not a special fuzzy set ideal of S(min, max).

Take 0.0000015 $\in$ S(min, max) and for any y $\in$ P; we see min {0.0000015, y} = {0.0000015} $\notin$ P; hence P is only a special fuzzy set subsemiring and not a special fuzzy set ideal of S(min, max).

Example 3.3: Let S(min, max) be a special fuzzy set semiring. Take T = {[0.02, 0.5]} $\subseteq$ [0,1); T is a special fuzzy set subsemiring of S(min, max).

Clearly T is not a special fuzzy set ideal of S(min, max) for; if x = 0.0001 $\in$ S(min, max), take any y $\in$ T we see min (x, y) = x = 0.0001 $\notin$ T. Thus T is only a special fuzzy set subsemiring and is not an ideal of S(min, max).

Example 3.4: Let S(min, max) be the special fuzzy set semiring. M = {[a, 1) | 0 < a} $\subseteq$ S(min, max) be the special fuzzy set subsemiring of S(min, max).

M is not a special fuzzy set ideal of S(min, max) for if x < a and y $\in$ M then min {x, y} = x $\notin$ M; thus M is only special fuzzy set subsemiring and not an ideal of S(min, max).

We now proceed onto define and describe the notion of filter in special fuzzy set semiring; S(min, max).

Let P $\subseteq$ S(min, max); we see P to be a filter (special fuzzy set filter) of S(min, max) if the following conditions are satisfied.

(i) min {x, y} $\in$ P for all x, y $\in$ P.
(ii) For every x $\in$ S and y $\in$ P we have max {x, y} $\in$ P.

We will first give examples of special fuzzy set filters of a special fuzzy set semiring S(min, max).

Example 3.5: Let S(min, max) be the special fuzzy set semiring.

M = {0.007, 0.08, 0.015, 0.0017, 0.042, 0.089, 0.97, 0.019} $\in$ S(min, max) be the special fuzzy set subsemiring.

Clearly M is not special fuzzy set filter of S(min, max) for if x = 0.9 $\in$ S(min, max) and y $\in$ M we say max {x, y} = {0.9, y} = 0.9 $\notin$ M; thus M is not a special fuzzy set filter of S(min, max).

Example 3.6: Let S(min, max) be the special fuzzy set semiring. Let M = {[0.06, 1)} $\subseteq$ S(min, max).

M is a special fuzzy set subsemiring of S(min, max). M is special fuzzy set filter of S(min, max).

For any $x \in$ S(min, max) or in particular for $x \in [0,1) \setminus [0.06, 1)$ we see max $\{x, y\}$ for every $y \in [0.06, 1)$ is in $[0.06, 1)$, hence the claim.

Example 3.7: Let S(min, max) be the special fuzzy set semiring.

Take P = $\{[0, 0.09]\} \subseteq$ S(min, max) to be the special fuzzy set subsemiring of S(min, max).

For any $x = 0.12 \in$ S(min, max) and for any $y \in P$ we see max $\{x, y\} = 0.12$ so P is not a special fuzzy set filter of S(min, max).

Example 3.8: Let S(min, max) be the special fuzzy set semiring. Take R = $\{[0.2, 0.7]\} \subseteq [0, 1)$; R is a special fuzzy set subsemiring of S(min, max).

R is not a special fuzzy set filter of S for if $0.9 \in$ S(min, max) and any $y \in R = [0.2, 0.7]$.

We see max $\{y, 0.9\} = 0.9$ and clearly 0.9 is not an element in R. Hence the claim.

Inview of all these we have the following theorems.

THEOREM 3.6: *Let S(min, max) be the special fuzzy set semiring and $P = \{[0, p) \mid p < 1\} \subseteq [0, 1)$ be the special fuzzy set subsemiring of S(min, max). P is not a special fuzzy set filter of S(min, max).*

Proof: Given P, choose a $p_1 > p$ and $p_1 \in [0,1)$ then max $\{p_1, x\} = p_1$ for every $x \in P$; as $p_1 \notin P$; we see P is not a special fuzzy set filter of S(max, min).

Corollary 3.1: Let S(max, min) be the special fuzzy set subsemiring. If T = {[a, b]}; 0 < a and b < 1 be a special fuzzy set subsemiring then T is not a special fuzzy set filter of S(min, max).

THEOREM 3.7: *Let S(min, max) be the special fuzzy set semiring. Special fuzzy set subsemirings of S (min, max) of finite order are not special fuzzy set filters of S.*

Proof: Given P = $\{x_1, x_2, \ldots, x_n\}$ ($n < \infty$) and P $\in$ [0,1) is a special fuzzy set subsemiring of finite order. Let y $\in$ [0,1) such that $y > x_i$, for $1 \leq i \leq n$. We see max $\{y, x_i\}$ = y for all $x_i \in$ P so y $\notin$ P; hence P is not a special fuzzy set filter of S(min, max).

Thus no special fuzzy set subsemiring of finite order can be a special fuzzy set filter of S(min, max).

This is possible for any given finite set P in [0,1); we have an element y $\in$ [0,1) which is greater than every other element in P. Hence max (y, x_i) for every $x_i \in$ P is only y and y $\notin$ P; hence the claim.

THEOREM 3.8: *Let S(min, max) be special fuzzy set semiring. All special fuzzy set subsemirings of the form P = {[a, 1) | 0 < a} $\subseteq$ [0,1) are special fuzzy set filters of S(min, max).*

Proof: Given P = {[a, 1) | 0 < a} = {[0, 1)} is a subset of S(min, max) and P is a special fuzzy set subsemiring of S(min, max).

Clearly P is a special fuzzy set filter of S as for any x $\in$ S(min, max) and p $\in$ P; we have max {x, p} = p as x < p. Hence the claim.

Now S(min, max) has infinite number of filters and also S(min, max) has infinite number of special fuzzy set subsemirings which are not filters.

THEOREM 3.9: *Let S(min, max) be the special fuzzy set semiring.*

Consider P = {[a, b] | 0 < a and b < 1} $\subseteq$ S(min, max). P is only a special fuzzy set subsemiring and not a special fuzzy set filter.

Proof: Given P = {[a, b] | 0 < a and b < 1} $\subseteq$ S(min, max). P is a special fuzzy set subsemiring of S(min, max). We see P is not a special fuzzy set filter. For if x $\in$ P and y $\in$ S with y > b then max {y, x} = y $\notin$ P, hence the claim of the theorem.

We give few examples before we construct semirings using [0,1).

Example 3.9: Let S(min, max) be a special fuzzy set semiring.

Consider M = {[0.05, 0.6]} $\subseteq$ {[0, 1)}, M is a special fuzzy set subsemiring. Further if 0.92 $\in$ S and 0.5731 $\in$ M we see max {0.5731, 0.92} = 0.92 $\notin$ M; hence M is not a special fuzzy set filter of S.

Example 3.10: Let S(min, max) be the special fuzzy set semiring. Take T = {(0.09, 0.9)} $\subseteq$ {[0, 1)} $\subseteq$ S(min, max).

It is easily verified T is a special fuzzy set subsemiring and is not a special fuzzy set ideal of S for if 0.9 $\in$ T and 0.9092 $\in$ S then max {0.9, 0.9092} = 0.9092 $\notin$ T; hence T is not a special fuzzy set filter of S only a special fuzzy set subsemiring of S.

Example 3.11: Let S(min, max) be a special fuzzy set semiring. Let M = {0, 1/2, $1/2^2$, …, $1/2^{90}$} $\subseteq$ S(min, max): M is a special fuzzy set subsemiring of S(min, max).

However if 0.92 $\in$ S and $1/2^n$; n $\leq$90 $\in$ M we see max {0.92, $1/2^n$} = 0.92 $\notin$ M; so M is not a special fuzzy set filter only a special fuzzy set subsemiring of S.

Example 3.12: Let S(min, max) be a special fuzzy set semiring.

Consider P = {[0, 0.092)} $\subseteq$ {[0, 1)}; P is a special fuzzy set subsemiring of S(min, max).

We see if x = 0.923 $\in$ S and y = 0.09 $\in$ P then max {x, y} = max {0.092, 0.923} = 0.923 $\notin$ P; so P is only a special fuzzy set subsemiring and not a special fuzzy set filter of S.

Example 3.13: Let S(min, max) be a special fuzzy set semiring. Let P = {[0.37, 1)} $\subseteq$ S(min, max) be the special fuzzy set subsemiring of S(min, max).

We see if x = 0.363996 $\in$ S and y = 0.371108 $\in$ P then max {x, y} = max {0.363996, 0.371108} = 0.371108 $\in$ P.

Thus P is a special fuzzy set filter of S(min, max).

Now we give algebraic structures on [0,1), we will first illustrate this situation by some examples.

Example 3.14: Let S(min, max) be the special fuzzy set semiring. M = {(a_1, a_2, …, a_6) | a_i $\in$ [0, 1), $1 \leq i \leq 6$}, we see M under (max, min) operation is a semiring known as the special fuzzy set row matrix semiring.

For if X = (0.3, 0.1, 0, 0.312, 0.10, 0.1135) and Y = (0, 0.71, 0.2, 0, 0.41703, 0.1132) $\in$ M then max (x, y) = max {(0.3, 0.1, 0, 0.312, 0.10, 0.1135), (0, 0.71, 0.2, 0, 0.41703, 0.1132)} = (max {0.3, 0}, max {0.1, 0.71}, max {0, 0.2}, max {0.312, 0}, max {0.10, 0.41703}, max {0.1135, 0.1132}).

= (0.3, 0.71, 0.2, 0.312, 0.41703, 0.1135).

Similarly min (X, Y) is calculated as min (X, Y) = (0, 0.1, 0, 0, 0.1, 0.1132) are in M.

It is easily verified M is special fuzzy set semiring under {min, max}.

Example 3.15: Let

$$P = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 6\right\}$$

be the special fuzzy set semiring under the {max, min} operation on it.

$$A = \begin{bmatrix} 0.3 \\ 0.13 \\ 0.415 \\ 0.031 \\ 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0.7 \\ 0.2 \\ 0.415 \\ 0 \\ 0.102 \end{bmatrix} \in P.$$

$$\max\{A, B\} = \max\left\{\begin{bmatrix} 0.3 \\ 0.13 \\ 0.415 \\ 0.031 \\ 0 \end{bmatrix}\begin{bmatrix} 0.7 \\ 0.2 \\ 0.415 \\ 0 \\ 0.102 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \max\{0.3, 0.7\} \\ \max\{0.13, 0.2\} \\ \max\{0.415, 0.415\} \\ \max\{0.031, 0\} \\ \max\{0, 0.102\} \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.2 \\ 0.415 \\ 0.031 \\ 0.102 \end{bmatrix} \text{ and}$$

$$\min\{A, B\} = \min\left\{\begin{bmatrix} 0.3 \\ 0.13 \\ 0.415 \\ 0.031 \\ 0 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.2 \\ 0.415 \\ 0 \\ 0.102 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \min\{0.3, 0.7\} \\ \min\{0.13, 0.2\} \\ \min\{0.415, 0.415\} \\ \min\{0.031, 0\} \\ \min\{0, 0.102\} \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.13 \\ 0.415 \\ 0 \\ 0 \end{bmatrix} \in P.$$

Thus $\{P_{\max, \min}\}$ is a special fuzzy set column matrix semiring.

Example 3.16: Let

$$S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 9\right\}$$

be the special fuzzy set square matrix semiring.

Take

$$A = \begin{bmatrix} 0 & 0.31 & 0.51 \\ 0.6 & 0.92 & 0.29 \\ 0.19 & 0.09 & 0.219 \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} 0.79 & 0.89 & 0.9 \\ 0.91 & 0.009 & 0.29 \\ 0.999 & 0.189 & 0.279 \end{bmatrix} \in S.$$

max {A, B}

$$= \begin{bmatrix} \max\{0, 0.79\} & \max\{0.31, 0.89\} & \max\{0.51, 0.9\} \\ \max\{0.6, 0.91\} & \max\{0.92, 009\} & \max\{0.29, 0.29\} \\ \max\{0.19, 0.999\} & \max\{0.09, 0.189\} & \max\{0.219, 0.279\} \end{bmatrix}$$

$$= \begin{bmatrix} 0.79 & 0.89 & 0.51 \\ 0.91 & 0.92 & 0.29 \\ 0.999 & 0.189 & 0.279 \end{bmatrix} \in S.$$

Now

$$\min\{A, B\} = \begin{bmatrix} 0 & 0.31 & 0.51 \\ 0.6 & 0.009 & 0.29 \\ 0.19 & 0.09 & 0.219 \end{bmatrix} \text{ are in S}$$

(min operation performed on similar lines).

Example 3.17: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 30 \right\}$$

be the special fuzzy set semiring of matrices under {min, max} operation.

Now all these special fuzzy set matrix semirings under min and max operation is a commutative semiring of infinite order.

Infact we see in these semirings every element is an idempotent under max (and) min operations as max {a, a} = a and min {a, a} = a for all a $\in$ [0, 1).

Now we proceed onto describe some substructures in these special fuzzy set matrix semirings by examples.

Example 3.18: Let S = $\{(a_1, a_2, a_3, a_4) \mid a_i \in [0,1), 1 \leq i \leq 4\}$ be the special fuzzy set row matrix semiring under the min max operation.

P $=\{(a_1, a_2, 0, 0) \mid a_i \in [0, 1), 1 \leq i \leq 2\} \subseteq S$ is a special fuzzy set matrix subsemiring of S.

This is one type of special fuzzy set matrix subsemiring. We can have atleast 4 + 6 + 4 = 14 such special fuzzy set row matrix subsemirings.

B = $\{(0\ 0\ 0\ a_1) \mid a_1 \in [0,1)\} \subseteq S$ is again a special fuzzy set row matrix subsemiring.

D = $\{(a_1, a_2, 0, a_3) \mid a_1 \in [0,1), 1 \leq i \leq 3\} \subseteq S$ is again a special fuzzy set row matrix subsemiring of S.

All these subsemirings are of infinite order. We have also other types of special fuzzy set subsemirings which are of finite order.

Let V = $\{(a_1, a_2, a_3, a_4) \mid a_i \in \{1/2, 1/3, 1/4, 1/2^3, 1/6, 1/7, 1/40\}, 1 \leq i \leq 4\} \subseteq S$.

We see V is a special fuzzy set subsemiring of finite order.

We can infact get infinite number of special fuzzy set subsemirings of finite order.

Apart from these special fuzzy set subsemirings of infinite order; we can get some more number of special fuzzy set subsemirings of infinite order given by the following.

Let W = $\{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 0.7), 1 \leq i \leq 4\} \subseteq S$; W is a special fuzzy set subsemiring of S and of infinite order.

We have such infinite collection under the max and min operations.

This semiring has infinite number of special fuzzy set subsemirings of infinite order.

Let $P_1 = \{(0, 0, 0, 0), (0.7, 0.2, 0.5, 0.6)\} \subseteq S$; P_1 is a special fuzzy set row matrix subsemiring of order two.

Clearly P_1 is not an ideal. $P_2 = \{(0, 0, 0, 0), (0.2, 0.3, 0.7, 0.5), (0.4, 0.5, 0.7, 0.8)\} \subseteq S$ is also a special fuzzy set row matrix subsemiring which is not an ideal of S.

Thus S has infinite number of special fuzzy set row matrix subsemirings none of them are ideals.

Example 3.19: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in \{0, 1/2, 1/2^2, \ldots, 1/2^{28}\} \subseteq [0, 1), 1 \leq i \leq 6 \right\}$$

be the special fuzzy set semiring under the max and min operation. $o(S) < \infty$.

Let

$$M_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_1, a_2 \in \{0, 1/2, 1/2^2, \ldots, 1/2^{28}\} \subseteq [0, 1)\right\} \subseteq S$$

be a special fuzzy set subsemiring of S and M_1 is not an ideal of S.

$$\text{Take } A = \begin{bmatrix} 0 \\ 1/2 \\ 1/2^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1/2^3 \\ 0 \\ 0 \\ 1/2 \\ 1/2^2 \\ 0 \end{bmatrix} \in S.$$

We see min {A, B} =

$$\min \left\{ \begin{bmatrix} 0 \\ 1/2 \\ 1/2^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1/2^3 \\ 0 \\ 0 \\ 1/2 \\ 1/2^2 \\ 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} \min\{0,1/2^3\} \\ \min\{1/2,0\} \\ \min\{1/2^2,0\} \\ \min\{0,1/2\} \\ \min\{0,1/2^2\} \\ \min\{0,0\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in S.$$

This we call as fuzzy zero divisors of the special fuzzy set semiring.

Now clearly

$$\max\{A, B\} = \begin{bmatrix} 1/2^3 \\ 1/2 \\ 1/2^2 \\ 1/2 \\ 1/2^2 \\ 0 \end{bmatrix}.$$

Let

$$M_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_1 \\ 0 \\ a_2 \\ 0 \end{bmatrix} \middle| a_1, a_2 \in \{0, 1/2, 1/2^2, \ldots, 1/2^{28}\} \right\} \subseteq S$$

be a special fuzzy set subsemiring of S.

Now

$$M_3 = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ 0 \\ a_2 \end{bmatrix} \middle| a_1, a_2 \in \{0, 1/2, 1/2^2, \ldots, 1/2^{28}\}\right\} \subseteq S$$

is a special fuzzy set subsemiring of S.

Further

$$\min \{M_i \cap M_j\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ if } i \neq j,\ 1 \leq i, j \leq 3$$

and max $\{M_1, M_2, M_3\} = S$.

This we call as the special direct sum of special fuzzy set subsemirings of S.

We see every element in S is an idempotent both with respect to max operation and min operation.

We can for these semirings with max and min operations have the concept of both ideals and filters.

Example 3.20: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in [0, 1), 1 \leq i \leq 12 \right\}$$

be the special fuzzy set semiring under min and max operation.

Clearly S is of infinite order. (P, max, min) be the subsemiring of S.

If for all $s \in S$ and $p \in P$; min $\{s, p\} \in P$ we define P to be an ideal of the semiring S.

Take

$$P_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1 \in [0, 1) \right\} \subseteq S; \{P_1, \max, \min\};$$

P_1 is a subsemiring of S.

We see for every $s \in S$ and $p \in P_1$ we have min $\{s_1, p\} \in P_1$. Hence P_1 is an ideal of S.

Take

$$P_2 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in [0, 1) \right\} \subseteq S; \{P_2, \max, \min\}$$

is a special fuzzy set subsemiring. P_2 is an ideal of S.

Consider

$$P_3 = \left\{\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in [0, 1)\right\} \subseteq S;$$

$\{P_3, \max, \min\}$ is a special fuzzy set subsemiring.

P_3 is an ideal of S.

We see

$$P_4 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ a_1 & 0 & 0 \\ 0 & 0 & 0 \\ a_2 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in [0, 1)\right\} \subseteq S$$

is also an ideal of S.

Thus we can have several ideals of the semiring $\{S, \max, \min\}$.

All subsemirings need not in general be ideals.

For let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in [0.5, 1), 1 \leq i \leq 12\right\} \subseteq S.$$

Clearly M is a special fuzzy set subsemiring under max and min operation but M is not an ideal of S.

Thus we have special fuzzy set subsemirings which are not in general ideals of S.

Let

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & 0 \\ a_6 & 0 & 0 \end{bmatrix} \middle| \; a_i \in [0.7, 1), 1 \leq i \leq 6 \right\} \subseteq S;$$

clearly M_1 is only a special fuzzy set subsemiring under the max min operation; but is not an ideal.

$$M_2 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \middle| \; a_i \in [0.3, 1), 1 \leq i \leq 3 \right\} \subseteq S;$$

clearly M_2 is only a special fuzzy set subsemiring under max operation but is not an ideal.

Let

$$N_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & a_4 & 0 \\ a_3 & a_5 & a_6 \end{bmatrix} \middle| \; a_i \in [0, 0.3), 1 \leq i \leq 6 \right\} \subseteq S;$$

N_1 is a special fuzzy set subsemiring of S which is also an ideal of S.

$$N_2 = \left\{ \begin{bmatrix} 0 & a_1 & a_2 \\ a_4 & 0 & a_3 \\ a_5 & a_6 & 0 \end{bmatrix} \middle| \; a_i \in [0, 0.7), 1 \leq i \leq 6 \right\} \subseteq S;$$

N_2 is a special fuzzy set subsemiring of S. We see N_2 is also an ideal of S.

Let

$$N_3 = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_i \in [0, 0.25), 1 \leq i \leq 3\right\} \subseteq S$$

be the special fuzzy set subsemiring of S. N_3 is also a special fuzzy set ideal of S.

We see S has infinite number of special fuzzy set subsemirings which are ideals and infinite subsemirings which are not ideals of S.

We wish to give the filters of these special fuzzy set semirings under the min and max operations.

Consider

$$P_1 = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0.3, 1), 1 \leq i \leq 3\right\} \subseteq S;$$

clearly P_1 is a special fuzzy set subsemiring of S under the operations min and max.

Further we see for all

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \in S \text{ and } B = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} \in P_1.$$

max (A, B) $\in P_1$.

Hence P_1 is a filter of S; however P_1 is not an ideal of S for

if $\begin{bmatrix} 0.1 & 0 & 0.8 \\ 0 & 0.5 & 0.1 \\ 0 & 0 & 0.003 \end{bmatrix} = C \in S$ then min (C, A) for any A in S is

not in P_1 hence P_1 is not an ideal of S.

We have infinite number of special fuzzy set subsemirings which are not ideals but are only filters.

So the natural question would be can an ideal in general be a filter?

Take

$$T_1 = \left\{ \begin{bmatrix} a_1 & 0 & a_2 \\ 0 & a_3 & 0 \\ a_4 & 0 & a_5 \end{bmatrix} \middle| \; a_i \in [0.3, 0.5), 1 \leq i \leq 5 \right\} \subseteq S$$

be a special fuzzy set subsemiring. Clearly T_1 is not an ideal of S and T_1 is also not a filter of S. Thus we have special fuzzy set subsemrings which are not ideals or filters of S.

Now we give examples of ideals of S which are not filters.

Take

$$B_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & 0 \\ a_6 & 0 & 0 \end{bmatrix} \middle| \; a_i \in [0, 0.34), 1 \leq i \leq 6 \right\} \subseteq S,$$

B_1 is a special fuzzy set subsemiring which is also an ideal of S.

However B_1 is not a filter for if T = $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$ where

$a_i \in (0.34, 1]$, $1 \leq i \leq 9$; then for any $A \in B_1$ we see
max (T, A) = T $\notin B_1$ so B_1 is not a filter only an ideal of S.

Inview of all these we have the following theorem.

THEOREM 3.10: *Let*
S = {m × n matrix with entries from [0, 1)} be the special fuzzy set semiring under min and max operations.

(1) S has special fuzzy set subsemirings which are not filters or ideals.
(2) S has special fuzzy set subsemirings which are filters and not ideals.
(3) *S has special fuzzy set subsemirings which are ideals and not filters.*

Proof : Take any
P = {m × n matrices with entries from [0, 0.7)} ⊆ S; clearly P is special fuzzy set subsemiring which is also an ideal of S.

Consider
M = {m × n matrices with entries from [0.3, 0.8] ⊆ [0,1)} ⊆ S; M is a special fuzzy set subsemiring which is not a ideal or a filter of S.

Take
N = {m × n matrices with entries from (0.6, 1) ⊆ [0,1)} ⊆ S; N is a special fuzzy set subsemiring which is a filter need not an ideal of S.

Hence the claim of the theorem.

Example 3.21: Let

$$S = \left\{ \left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right| a_i \in [0, 1), 1 \leq i \leq 16 \right\}$$

be the special fuzzy set semiring under min and max operations. We see S has special fuzzy set subsemiring which are neither filter nor ideal of S.

For

$$M = \left\{ \left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right| a_i \in [0.25, 0.714), 1 \leq i \leq 16 \right\} \subseteq S$$

is a special fuzzy set subsemiring of S which is not a filter; also M is not an ideal of S.

For if

$$A = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \\ \vdots & \vdots \\ 0.15 & 0 \\ 0 & 0 \end{bmatrix} \in S \text{ (all elements in the interval } [0, 0.2])$$

$$B = \begin{bmatrix} 0.3 & 0.4 \\ 0.5 & 0.6 \\ \vdots & \vdots \\ 0.7 & 0.42 \end{bmatrix} \in M, \text{ then } \min(A, B) = A \text{ and}$$

$A \notin S$ hence the claim. Thus M is not an ideal of S.

Now consider for any $B \in M$.

$$D = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \text{ (where } a_i \in [0.8, 1), 1 \leq i \leq 16)$$

in S we see max (B, D) = D for all $B \in M$.

Thus M is not a filter of S. Hence the claim.

Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in [0, 0.7), 1 \leq i \leq 16 \right\} \subseteq S$$

be the special fuzzy set subsemiring of S.

T is an ideal of S, however T is not a filter of S for if

$$D = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \in S \text{ (where } a_i \in [0.8, 1), 1 \leq i \leq 6).$$

max (D, A) = D for every $A \in T$.

Hence T is not a filter of S. T is only an ideal of S.

Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in [0.532, 1), 1 \leq i \leq 16 \right\} \subseteq S$$

be the special fuzzy set subsemiring of S.

Clearly V is not an ideal for

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} (a_i \in [0, 0.531), 1 \leq i \leq 16\})$$

is in S then; min{A, W} = A for every W $\in$ V.

Thus V is not an ideal of S.

However V is a filter of S as for every W $\in$ V and every L $\in$ S we have max {W, L} $\in$ V; hence V is a filter of S.

Next we consider the following example.

Example 3.22: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{18} \end{bmatrix} \middle| \; a_i \in [0, 9), 1 \leq i \leq 18 \right\}$$

be the special fuzzy set semiring under the max and min operation.

Now

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix} \middle| \; a_i \in [0, 9), 1 \leq i \leq 6 \right\} \subseteq S$$

is a only a special fuzzy set subsemiring of S.

Further T is an ideal of S however T is not a filter for if

$$A = \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{18} \end{bmatrix} \text{ where } a_i \in [0.4, 0.8], 1 \leq i \leq 8$$

then max (A, B) $\notin$ T for any B $\in$ T.

It is important to note that if in any special fuzzy set matrix subsemiring one or more entries of the matrix is zero as in the above example;

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \\ 0 & 0 & \ldots & 0 \end{bmatrix} \text{ where } a_i \in [0, 1), 1 \leq i \leq 12 \right\} \subseteq S$$

that is the last row is zero we see V is a special fuzzy set matrix subsemiring which is an ideal but not a filter.

So every special fuzzy set subsemiring in which one or more entries is always zero cannot be a filter. It can be a ideal in appropriate cases.

For take

$$N = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix} \middle| \; a_i \in \{0, 1/2, 1/2^2, \ldots, 1/2^{10}; \; 1 \leq i \leq 6\} \right. \subseteq S$$

N is a special fuzzy set subsemiring which is not an ideal of S.

For if

$$A = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{18} \end{bmatrix} \middle| \; a_i = 0.2, 0.4, 0.6 \text{ and } 0.8; \; 1 \leq i \leq 18 \right\}$$

A is not a filter or ideal of S only a special fuzzy set subsemiring of S.

Let

$$D = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{18} \end{bmatrix} \text{ where } a_i \in \{0, 0.7, 0.32, 0.004, \right.$$

$$0.3, 0.71, 0.0014, 0.4, 0.012, 0.00352, 0.00179\}, \; 1 \leq i \leq 18\}$$

be the special fuzzy set subsemiring.

Clearly D is not an ideal or a filter of S.

We can have infinite number of special fuzzy subset subsemirings all of which are of finite order and none of them are ideals or filters of S.

Take for instance

$$R = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_6 \\ a_7 & a_8 & \dots & a_{12} \\ a_{13} & a_{14} & \dots & a_{18} \end{bmatrix} \right. \text{where } a_i \in \{0, 0.31, 0.031, 0.0031,$$

0.00031, 0.000031, 0.0000031, 0.21, 0.2101, 0 .4, 0.5, 0.71, 0.071, 0.00712, 0.000031325}, $1 \leq i \leq 18\} \subseteq S$,

R is a special fuzzy set subsemiring of finite order. Clearly R is not an ideal or a filter of S.

In view of all these we have the following theorem.

THEOREM 3.11: *Let S = {m × n matrix with entries from [0,1)} be the special fuzzy set semiring under min, max operation. S has infinite number of finite special fuzzy set subsemirings none of them are ideals or filters.*

Proof is direct and left as an exercise to the reader.

Now we can have special fuzzy set semirings using (min, ×) and under (max, ×) operation S = [0,1) is not a semiring as under max, as S has no identity.

Let S = {[0,1), min, ×} be the semiring as {[0,1), min} contain 0 such that min {0, a} = 0 for every a ∈ [0, 1). S is a special quasi fuzzy set semiring of infinite order. S = {[0,1), min, ×} has infinite number of special fuzzy set subsemirings.

For let M = {[0,0.5), min, ×} ⊆ S be a special quasi fuzzy set subsemiring which is an ideal for any; m ∈ M and s ∈ S; ms ∈ M.

M is also a filter as min {m, s} ∈ M for all m ∈ M and s ∈ S. This is an interesting feature enjoyed by S = {[0,1), min, ×}

special fuzzy set semiring. Even if the term quasi is omitted still we know from the operation S = {[0, 1), min, $\times$}, it is a special quasi semiring.

A fuzzy set special subsemiring can be both a filter and a ideal of S = {[0, 1), min, $\times$}.

Take N = {[0, 0.7], min, $\times$} $\subseteq$ S; N is a special fuzzy set subsemiring and is an ideal and a filter.

Let P = {0, 1/2, $1/2^2$, …, $1/2^n$; n $\to \infty$} $\subseteq$ {[0, 1), min, $\times$} $\subseteq$ S is a special fuzzy set subsemiring and is not an ideal or a filter of S.

S = {[0,1), min, $\times$} has infinite number of special fuzzy set subsemirings which are not ideals or filters of S all of them of infinite order.

Infact S = {[0, 1), min, $\times$} has no finite special fuzzy set subsemiring all special fuzzy set subsemirings are of infinite order.

Example 3.23: Let S = {(a_1, a_2, a_3, a_4) | $a_i \in$ [0, 1), $1 \leq i \leq 4$, min, $\times$} be a special quasi fuzzy set semiring.

Let

A = (0.3, 0.2, 0.4, 0)
B = (0.2, 0.07, 0.21, 0.35) and
C = (0, 0.7, 0.21, 0.12) be in S.

To show for the above A, B, C in S;

C $\times$ min {A, B} = min{C $\times$ A, C $\times$ B}

C $\times$ (min {A, B})
= (0,0.07, 0.21, 0.12) $\times$ (min {(0.3, 0.2, 0.4, 0), (0.2, 0.07, 0.21, 0.35)}
= (0, 0.7, 0.21, 0.12) $\times$ (0.2, 0.07, 0.21, 0)
= (0, 0.049, 0.0441, 0) … I

Now min $\{C \times A, C \times B\}$

$= \min \{(0, 0.7, 0.21, 0.12) \times (0.3, 0.2, 0.4, 0),$
$(0, 0.7, 0.21, 0.12) \times (0.2, 0.007, 0.21, 0.35)\}$
$= \min \{(0, 0.14, 0.84, 0), (0, 0.049, 0.0441, 0.420)\}.$
$= (0, 0.049, 0.0441, 0)$ … II

I and II are identical

This is the way operations are performed on S.

Let $M = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 0.3), 1 \leq i \leq 4\} \subseteq S$, M is a special fuzzy set subsemiring. M is also an ideal of S. M is of infinite order.

M is also a filter of S.

Thus M is both an ideal and filter of S.

$N = \{(a_1, 0, a_2, 0) \mid a_i \in [0, 1), 1 \leq i \leq 2\} \subseteq S$; $(N, \min, \times) \subseteq S$ is a special fuzzy set subsemiring of S.

Infact N is an ideal of S as well as a filter of S.

Let $T = \{(a_1, 0, 0, 0) \mid a_1 \in [0,1)\} \subseteq S$, T is also a special fuzzy set subsemiring as well as an ideal and filter of S.

The representation of S as a direct sum is meaning less as the operation on S is min and $\times$.

Let $W = \{(a_1, 0, 0, 0) \mid a_i \in [0, 0.2]\} \subseteq T \subseteq S$; W is a special fuzzy set subsemiring as well as ideal and filter of S.

Infact we have an infinite collection of chain of special fuzzy set subsemirings given by

$W_1 = \{(a_1, 0, 0, 0) \mid a_1 \in [0, 0.0t_1]\} \subseteq W_2 = \{(a_1, 0, 0, 0) \mid a_1 \in [0, 0.0t_2]\} \subseteq W_3 = \{(a_1, 0, 0, 0) \mid a_1 \in [0, 0.0t_3]\} \subseteq \ldots \subseteq W_n = \{(a_1, 0, 0, 0) \mid a_1 \in [0, 0.0t_n]\} \subseteq \ldots \subseteq S$.

$0.0t_1 < 0.0t_2 < \ldots < 0.0t_n$. All these W_i's are both ideals and filters.

Now Consider B_1 $\{(a_1, a_2, 0, 0) \mid a_1, a_2\ [0, t_1] \subseteq [0.1) \subseteq S$.

B_1 is also a special fuzzy set subsemiring of S. B_1 is both an ideal and filter of S.

$B_2 = \{(a_1, a_2, 0, 0) \mid a_1, a_2 \in [0, t_2]0, 1)\ (t_1 < t_2)\} \subseteq S$ is again a special fuzzy set subsemiring which is both an ideal and filter of S. Infact we can get an infinite chain of filters (ideals) of say

$$B_1 \underset{\neq}{\subset} B_2 \underset{\neq}{\subset} B_3 \underset{\neq}{\subset} \ldots \underset{\neq}{\subset} B_n \underset{\neq}{\subset} \ldots \underset{\neq}{\subset} S.$$

This chain need not be a countable one. The major difference between (S, min, max) and (S, min, $\times$) is that in the later set we see the special fuzzy set semiring is such that it contains special fuzzy set subsemirings which are both ideals and filters simultaneously.

To the best of our knowledge this is not possible in case of the special fuzzy set semiring (S, min, max).

Example 3.24: Let

$$S = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 6\right\}$$

be a special fuzzy set semiring under the operation min and $\times$.

S is a semiring of infinite order. We see S has no special fuzzy set subsemirings of finite order.

Every subsemiring of S is of infinite order but some of them can be countable infinite.

Let

$$P_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \, a_1 \in \{0, 1/2, 1/2^2, \ldots, 1/2^n \text{ as } n \to \infty\} \subseteq [0, 1)\right\} \subseteq S.$$

Clearly P_1 is a special fuzzy set subsemiring of infinite order; the order being a countable infinity.

Further P_1 is not a filter and P_1 is not an ideal of S.

$$P_2 = \left\{ \begin{bmatrix} 0 \\ a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \, a_1 \in \{0, 1/5, 1/5^2, \ldots, 1/5^n \text{ as } n \to \infty\} \subseteq [0, 1)\right\} \subseteq S$$

is again a special fuzzy set subsemiring of infinite order which is not an ideal or filter of S.

We can also have

$$P_3 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix} \middle| \, a_i \in \{0, 1/3, 1/3^2, \ldots, 1/3^n; n \to \infty; 1 \leq i \leq 6\}\right\} \subseteq S,$$

P_3 is a special fuzzy set subsemiring of infinite order and is not an ideal or a filter of S.

Thus we can have infinite number of special fuzzy set subsemirings which are not ideals or filters of S.

Consider

$$M_1 = \left\{\left[\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_6 \end{array}\right] \middle| \; a_i \in [0, 0.314), 1 \leq i \leq 6\right\} \subseteq S;$$

M_1 is a special fuzzy set subsemiring as well as special fuzzy ideal and filter of S. Clearly $o(M_1) = \infty$.

Consider

$$M_2 = \left\{\left[\begin{array}{c} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \middle| \; a_1, a_2 \in [0, 0.2)\right\} \subseteq S;$$

M_2 is a special fuzzy set subsemiring of S which is both a filter and ideal of S.

Thus S has infinite number of special fuzzy set subsemirings which are filters and ideals of S.

Example 3.25: Let

$$S = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ \hline a_{13} & a_{14} & a_{15} \\ \hline a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \end{array}\right] \right| a_i \in [0, 1), 1 \leq i \leq 21 \right\}$$

be the special fuzzy set matrix semigroup of infinite order under (min, ×) operations.

This S has filters and ideals. Also S contains special fuzzy set subsemirings which are not ideals or filters.

Take

$$T_1 = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline a_4 & a_5 & a_6 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \right| a_i \in [0, 1), 1 \leq i \leq 6 \right\} \subseteq S;$$

T_1 is a special fuzzy set subsemiring which is both an ideal and filter of S.

Let

$$M = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline a_7 & a_8 & a_9 \\ \hline 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} \end{array}\right] \middle| \; a_i \in \{0, 1/2, 1/2^2, \ldots, 1/2^n, \ldots, n \to \infty), 1 \le i \le 12 \right\} \subseteq S$$

be the special fuzzy set subsemiring of S.

Clearly M is not an ideal or filter of S. We see M is of infinite order.

Infact we have an infinite collection of such special fuzzy set subsemirings which are not ideals or filters of S.

Now we proceed onto give a few more examples about zero divisors and idempotents in S = {[0, 1), min, ×}.

Example 3.26: Let

$$M = \left\{ \left[\begin{array}{c|ccc|c} a_1 & a_2 & a_3 & a_4 & a_5 \\ \hline a_6 & \ldots & \ldots & \ldots & a_{10} \\ a_{11} & \ldots & \ldots & \ldots & a_{15} \\ a_{16} & \ldots & \ldots & \ldots & a_{20} \\ \hline a_{21} & \ldots & \ldots & \ldots & a_{25} \\ a_{26} & \ldots & \ldots & \ldots & a_{30} \end{array}\right] \middle| \; a_i \in [0, 1), 1 \le i \le 30 \right\}$$

be the special fuzzy set semiring of super matrices.

M has special fuzzy set subsemirings which are not filters or ideals. Further M also has special fuzzy set subsemirings which are filters and ideals of S.

Take

$$P = \left\{ \left[\begin{array}{c|ccc|c} a_1 & a_2 & a_3 & a_4 & a_5 \\ \hline (0) & & (0) & & (0) \\ \hline (0) & & (0) & & (0) \end{array}\right] \middle| a_i \in [0, 1), 1 \leq i \leq 5\right\} \subseteq M,$$

P is a special fuzzy set subsemiring which is also a filter and an ideal of S.

Take

$$R = \left\{ \left[\begin{array}{c|ccc|c} a_1 & a_2 & a_3 & a_4 & a_5 \\ \hline a_6 & \ldots & \ldots & \ldots & a_{10} \\ a_{11} & \ldots & \ldots & \ldots & a_{15} \\ a_{16} & \ldots & \ldots & \ldots & a_{20} \\ \hline a_{21} & \ldots & \ldots & \ldots & a_{25} \\ a_{26} & \ldots & \ldots & \ldots & a_{30} \end{array}\right] \middle| a_i \in 2/11, (2/11)^2, \ldots (2/11)^n \right.$$

$$\text{as } n \to \infty\}, 1 \leq i \leq 30\} \subseteq S$$

to be the special fuzzy set subsemiring of S.

Clearly R is not an ideal and not a filter of S. Infact S has infinite number of special fuzzy set subsemirings which are not filters or ideals of S.

Example 3.27: Let S = $\{(a_1, a_2, a_3) \mid a_i \in [0, 1), 1 \leq i \leq 3\}$ be the special fuzzy set semiring under $\{\min, \times\}$. S has zero divisors given by A = (0, 0.31, 0.0041) and B = (0.743, 0, 0) $\in$ S; A $\times$ B = (0, 0, 0), so S has zero divisors.

Infact S has infinite number of zero divisors.

Every element in S under min operation is an idempotent. Infact every element A = $\{(a_1, a_2, a_3)\}$ under (min, $\times$) operation generates a subsemiring of infinite order.

For instance let A = $\{\langle(0, 1/2, 4/7)\rangle\} \in$ S.

We see A = $\{(0, 1/2, 4/7), (0, (1/2)^2, (4/7)^2), \ldots, (0, (1/2)^n, (4/7)^n)$, as $n \to \infty$, $(0, 0, 0)\} \subseteq$ S is a special fuzzy set subsemirings under (min, $\times$) operation.

Clearly A is generated by a single element, however A is not an ideal or filter of S.

We call such special fuzzy set subsemirings as specially principal or cyclic subsemirings. This is the special feature enjoyed by semirings under only the (min, $\times$) operation.

B = $\{\langle(0.01, 0.003, 0.0056)\rangle\}$ is again a special cyclic special fuzzy set subsemiring which is not an ideal or filter of S.

Infact S has infinite number of such special fuzzy cyclic subsemirings each of infinite order none of them are ideals or filters.

We can have special fuzzy set subsemirings generated by two elements say a and b in S where min {a, b} = a or b.

In this way we get the special fuzzy set subsemiring to be of infinite order. Clearly it is not an ideal or a filter.

Take M = $\{\langle x = (0.3, 0.4, 0.01), y = (0.7, 0.8, 0.5)\rangle\} \subseteq$ S, M is a special fuzzy set subsemiring and is of infinite order, generated by (x, y) $\in$ S and they are not ideals or filters of S.

We can on similar lines have special fuzzy set subsemirings generated by three elements, four elements and so on.

All of them are of infinite order and none of them is a filter or an ideal.

Let T = {⟨(0.31, 0, 0), (0, 0.25, 0), (0, 0, 0.03)⟩} $\subseteq$ S, T is an infinite special fuzzy set subsemiring and it is not an ideal or filter but it has only three generators.

Infact S has infinite number of special fuzzy set subsemirings generated by three elements of S.

x, y, z such that min{x, y} = x or y or z min{y, z} = x or y or z and min{z, x} = x or y or z; x, y, z $\in$ S. This can be extend to any finite number n.

Example 3.28: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 1), 1 \leq i \leq 9 \right\}$$

be the special fuzzy set semiring, clearly S is of infinite order.

Consider

$$P = \left\{ \left\langle \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\rangle \right\} \subseteq S,$$

P is a special fuzzy set subsemiring of infinite order which is a cyclic special fuzzy set subsemiring and P is not an ideal or a filter of S.

Take

$$M = \left\{\left\langle\begin{bmatrix} 0.31 & 0.015 & 0 \\ 0 & 0 & 0.6 \\ 0 & 0 & 0.27 \end{bmatrix}\right\rangle\right\} \subseteq S,$$

M is a special fuzzy set subsemiring of infinite order which is not a filter or an ideal of S.

Now

$$N = \left\{\left\langle\begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0.2 \\ 0 & 0 & 0.6 \end{bmatrix}, \begin{bmatrix} 0.2 & 0.2 & 0 \\ 0 & 0.3 & 0.4 \\ 0.5 & 0.7 & 0.8 \end{bmatrix}\right\rangle\right\} \subseteq S$$

is such that N is generated by two elements and N is a special fuzzy set subsemiring of S.

Clearly

$$\min\left\{\begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0.2 \\ 0 & 0 & 0.6 \end{bmatrix}, \begin{bmatrix} 0.2 & 0.2 & 0 \\ 0 & 0.3 & 0.4 \\ 0.5 & 0.7 & 0.8 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0.2 \\ 0 & 0 & 0.6 \end{bmatrix} \in N.$$

This is the way N is generated by these two elements.

N is not an ideal or filter of S.

We can have special fuzzy set subsemirings generated by any n elements ($n < \infty$). These are only subsemirings and are not ideals or filters.

Further they are infinite in number and all of them are of infinite order.

We see S has also special fuzzy set subsemirings which are not finitely generated.

Consider

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in \{0, 1/7, 1/7^2, 1/7^3, \ldots, \right.$$

$$\left. 1/7^n \text{ as } n \to \infty\} \subseteq [0, 1); 1 \leq i \leq 9 \right\} \subseteq S$$

be the special fuzzy set subsemiring of infinite order.

P is not an ideal or filter and is not finitely generated.

Let

$$L = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 0.7], 1 \leq i \leq 9 \right\} \subseteq S$$

be the special fuzzy set subsemiring of S of infinite order. L is infinitely generated. L is a filter as well as an ideal of S.

S has infinitely many such special fuzzy set subsemirings which are infinitely generated and is both an ideal and filter of S.

Infact S has infinite number of zero divisors and every element in S under the min operation is an idempotent of S. S contains special fuzzy subset subsemirings such that their product is zero.

For take

$$Z_1 = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_2 \end{bmatrix}\right| a_1, a_2 \in [0, 1)\right\} \subseteq S,$$

$$Z_2 = \left\{\left.\begin{bmatrix} 0 & a_1 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a_1, a_2 \in [0, 1)\right\} \subseteq S,$$

$$Z_3 = \left\{\left.\begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & 0 \\ a_2 & 0 & 0 \end{bmatrix}\right| a_1, a_2 \in [0, 1)\right\} \subseteq S,$$

$$Z_4 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & a_2 & 0 \end{bmatrix}\right| a_1, a_2 \in [0, 1)\right\} \subseteq S \text{ and}$$

$$Z_5 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_1 \\ 0 & 0 & 0 \end{bmatrix}\right| a_1 \in [0, 1)\right\} \subseteq S$$

are not special fuzzy set subsemirings which are ideals as well as filters of S.

Infact

$$Z_i \cap Z_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{if } i \neq j,\ 1 \leq i, j \leq 5.$$

Both min {A, B} = $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for every $A \in Z_i$ and

$B \in Z_j$ $(i \neq j)$, $1 \leq i, j \leq 5$.

Also

$$A \times B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ for every } A \in Z_i$$

for every $B \in Z_j$ $(i \neq j)$, $1 \leq i, j \leq 5$.

This is the unique feature enjoyed by these special fuzzy set subsemirings.

Let

$$y_1 = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a_1, a_2 \in [0, 0.7)\right\} \subseteq S,$$

$$y_2 = \left\{\left.\begin{bmatrix} 0 & a_1 & a_2 \\ 0 & 0 & 0 \\ 0 & 0 & a_3 \end{bmatrix}\right| a_1, a_2, a_3 \in [0, 0.5)\right\} \subseteq S,$$

$$y_3 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & 0 & 0 \end{bmatrix}\right| a_1, a_2 \in [0, 0.91)\right\} \subseteq S$$

and

$$y_4 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & 0 \end{bmatrix}\right| a_1, a_2 \in [0, 0.8501)\right\} \subseteq S$$

be the special fuzzy set subsemirings of S. All of them are of infinite order and each one of them is both a filter and an ideal of S.

Also

$$y_i \times y_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ if } i \neq j,\ 1 \leq i, j \leq 4 \text{ and}$$

$$\min\{y_i, y_j\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ if } i \neq j,\ 1 \leq i, j \leq 4.$$

Also if $A \in y_j$ and $B \in y_i$ and $i \neq j$; then

$$\min\{A, B\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$A \times B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we see these y_i's are different from Z_j for Z_j's are built on whole of [0, 1) where as y_j's are built on a section of the interval.

However still the results hold good for if

$$B_1 = \left\{\begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 1/7, (1/7)^2, \ldots, 1/7^n \text{ and } n \to \infty\} \subseteq [0, 1)\right\} \subseteq S,$$

$$B_2 = \left\{\begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 1/8, (1/8)^2, \ldots, 1/8^n \text{ and } n \to \infty\} \subseteq [0, 1)\right\} \subseteq S,$$

$$B_3 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 1/11, (1/11)^2, \ldots, 1/11^n \text{ and } n \to \infty\} \subseteq [0, 1)\right\} \subseteq S,$$

and
$$B_4 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in \{0, 1/6, 1/6^2, 1/6^3, \ldots, 1/6^n \text{ and } n \to \infty\} \subseteq [0, 1)\right\} \subseteq S$$

are all special fuzzy set subsemirings and none of them is an ideal or a filter.

However

$$B_i \times B_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ if } i \neq j, \; 1 \leq i, j \leq 4$$

and

$$\min\{B_i, B_j\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, i \neq j; j \leq 4.$$

Further for every $X \in B_i$ and $Y \in B_j$ ($i \neq j$) we have

$$\min\{X, Y\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$X \times Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, 1 \leq i, j \leq 4.$$

This is the special feature enjoyed by special fuzzy set subsemirings of infinite order which are not ideals or filters of S.

Finally if

$$A = \begin{bmatrix} 0 & 0 & 0.7 \\ 0.8 & 0.9 & 0 \\ 0 & 0.1 & 0.52 \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} 0.9 & 0.11 & 0 \\ 0 & 0 & 0.321 \\ 0.74 & 0 & 0 \end{bmatrix} \in S$$

we have

$$A \times B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ however}$$

$$\min(A, B) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In view of these results we have the following theorem.

THEOREM 3.12: *Let*
S = {Collection of all m × n matrices with entries from [0, 1)} be the special fuzzy set semiring with {min, ×} as the operations on S. Let A, B ∈ S. Min {A, B} = {(0)}, zero matrix if and only if A × B = {(0)}.

Proof follows from the very definition of min and × operation on S.

Example 3.29: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 6 \right\}$$

be the special fuzzy set semiring.

$$\text{Let } X = \begin{bmatrix} 0 \\ 0 \\ 0.71 \\ 0 \\ 0.69 \\ 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0.8 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 0 \end{bmatrix} \in S.$$

We see min (X, Y) = $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and

$$X \times Y = \begin{bmatrix} 0 \\ 0 \\ 0.71 \\ 0 \\ 0.69 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0.8 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let v = $\begin{bmatrix} 0.3 \\ 0.4 \\ 0.51 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and u = $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.01 \\ 0.003 \\ 0.0005 \end{bmatrix} \in S$;

we have $v \times u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and min $\{u, v\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$; hence the claim.

Infact S has infinite number of zero divisors.

Let

$$A = \begin{bmatrix} 0.001 \\ 0.2 \\ 0.14 \\ 0.03 \\ 0.004 \\ 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \end{bmatrix} \in S.$$

$$\min\{A, B\} = \min\left\{\begin{bmatrix} 0.001 \\ 0.2 \\ 0.14 \\ 0.03 \\ 0.004 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 0.001 \\ 0.2 \\ 0.14 \\ 0.03 \\ 0.004 \\ 0 \end{bmatrix} \in S.$$

$$A \times B = \begin{bmatrix} 0.001 \\ 0.2 \\ 0.14 \\ 0.03 \\ 0.004 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0001 \\ 0.04 \\ 0.042 \\ 0.012 \\ 0.002 \\ 0 \end{bmatrix} \in S.$$

We make the following observation for any A, B $\in$ S; min{A $\times$ B, A or B} = A $\times$ B; if A and B are not comparable.

We have the following theorem.

THEOREM 3.13: *Let S = {Collection of all n $\times$ m matrices with entries from [0, 1)} be the special fuzzy set semiring.*

If A, B $\in$ S; (min (A, B) $\neq$ A or B) then min{A $\times$ B, B (or A)} = A $\times$ B.

Proof is direct and hence left as an exercise to the reader.

Example 3.30: Let

$$S = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \right| a_i \in [0, 1), 1 \leq i \leq 18 \right\}$$

be the special fuzzy set semiring under (min, ×). S has zero divisors, special fuzzy set subsemirings which are not ideals. Also S has special fuzzy set subset semirings which are not filters.

However S contains special fuzzy set subset semirings which are filters as well as ideals.

All special fuzzy set subsemirings are of infinite order. Infact S has no special fuzzy set subsemiring which is of finite order.

Example 3.31: Let S = {Collection of all matrices

$$\left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \right| a_i \in [0, 1), 1 \leq i \leq 8 \right\}$$

be the special fuzzy set semiring under {min, ×}.

$$\text{Let } A = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.2 & 0.1 \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} 0.11 & 0.3 & 0.5 & 0.7 \\ 0.6 & 0.01 & 0.8 & 0.16 \end{bmatrix} \in S.$$

Now

$$\min\{A, B\} = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.4 & 0.01 & 0.2 & 0.1 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.2 & 0.1 \end{bmatrix} \times \begin{bmatrix} 0.11 & 0.3 & 0.5 & 0.7 \\ 0.6 & 0.01 & 0.8 & 0.16 \end{bmatrix}$$

$$= \begin{bmatrix} 0.011 & 0.06 & 0.15 & 0.28 \\ 0.24 & 0.003 & 0.16 & 0.016 \end{bmatrix}.$$

min {A × B, A (or B)} = A × B.

$$\text{Let } A = \begin{bmatrix} 0.1 & 0.2 & 0 & 0.4 \\ 0.5 & 0 & 0.7 & 0 \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} 0.4 & 0.3 & 0.2 & 0.5 \\ 0.6 & 0.1 & 0.9 & 0.2 \end{bmatrix} \in S.$$

$$A \times B = \begin{bmatrix} 0.04 & 0.06 & 0 & 0.2 \\ 0.3 & 0 & 0.63 & 0 \end{bmatrix}$$

$$\min\{A, B\} = \begin{bmatrix} 0.1 & 0.2 & 0 & 0.4 \\ 0.5 & 0 & 0.7 & 0 \end{bmatrix} = A.$$

min {A × B, A (or B) = A × B.

However min {A, B} = A still min {A × B, A (or B)} = A × B only.

Infact if A, B ∈ S are such that

$$A \times B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ then}$$

$$\min \{A \times B, A \text{ or } B\} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have studied about zero divisors, idempotents in special fuzzy set semiring with (min, ×).

Incase of special fuzzy set semirings S with (min, max) we see every element in S is an idempotent both with respect to min as well as max and those special fuzzy set semirings with (min and max) operation has finite special fuzzy set subsemirings however there exist no finite special fuzzy set subsemirings in case S has (min, ×) as the operation on it.

Also to the best of our knowledge we do not have special fuzzy set subsemirings which are both ideals and filters in case of (max, min) operation but if S has (min, ×) as the operation then S has special fuzzy set subsemirings which are simultaneously ideals and filters.

Now we proceed onto study special fuzzy set semiring polynomials under (min, ×) operations.

Example 3.32: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 1) \right\}$$

be the special fuzzy set semiring under (min, ×) operation.

Let $p(x) = 0.3 + 0.25\,x + 0.312x^2$
and $q(x) = 0.4 + 0.2x^3 + 0.32x$;
min (p(x), q(x)) = {0.3 + 0.25x} for min {p(x), q(x)} = {0.3 + 0.25x + 0.312x^2, 0.4 + 0.2x^3 + 0.32x}
= {min {0.3, 0.4} + min {0.25x, 0.32x} + {0.312x^2, 0.0x^2} + {0, 0.2x^3}}
= {0.3 + 0.25x + 0.x^2 + 0.x^3}

$= \{0.3 + 0.25x\}$.

This is the way min operation is performed on S.

Now

$p(x) \times q(x) = (0.3 + 0.25\,x + 0.312x^2) \times (0.4 + 0.2x^3 + 0.32x\,)$
$= \{0.12 + 0.1x + 0.1248x^2 + 0.06x^3 + 0.05x^4 + 0.0624x^5 + 0.096x + 0.0080x^2 + 0.09984x^3\}$

It is important to keep on record 0.3 x + 0.025x is not 0.325x for such operation on S is meaningless as we have defined only the two operations min and ×.

This is the way operations are performed on S.
Let $p(x) = 0.00251x$ and $q(x) = 0.1523x^3 \in S$
We see min $\{p(x), q(x)\} = 0$ and

$p(x) \times q(x) = 0.00251\,x \times 0.1523x^3$
$= (0.00251 \times 0.1523)x^4$.

If $p(x) = 0.13x^3$ and $q(x) = 0.3x^7$ then min $\{p(x), q(x)\} = 0$ and $p(x) \times q(x) = 0.13x^3 \times 0.3x^7 = 0.039x^{10}$.

To find $p(x) \times \min(q(x), r(x))$ where $p(x) = 0.3x$, $q(x) = 0.2x^2 + 0.1$ and $r(x) = 0.3 + 0.2x \in S$.

$p(x) \times \min(q(x), r(x))$
$= p(x) \times 0.1$
$= 0.3x \times 0.1$
$= 0.03x \quad \ldots \quad \text{I}$

$\min\{p(x), q(x), p(x).\, r(x)\}$
$= \min\{0.3x \times 0.1 + 0.2x^2, 0.3x \times 0.3 + 0.2x\}$
$= \min\{0.03x + 0.06x^3, 0.09x + 0.06x^2\}$
$= 0.03x \quad \ldots \quad \text{II}$

I and II are identical hence this is the way operations are performed on S.

Thus S is a special fuzzy set polynomial semiring under (min, $\times$) operations.

Take

$$P = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 0.5)\right\} \subseteq S.$$

P is a special fuzzy set subsemiring which is both an ideal and filter of S.

Now S has infinite number of special fuzzy set subsemirings which is both an ideal and filter of S.

Consider

$$M = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 1/2, 1/2^2, 1/2^3, \ldots, 1/2n, n \to \infty\}\right\} \subseteq S$$

to be a special fuzzy set ideal or a filter of S.

Infact M is of infinite order. We have infinite number of special fuzzy set subsemirings which are not ideals.

Infact S has idempotents under the min operation but does not contain zero divisors. We can have cyclic or principal special fuzzy set subsemirings.

Take B = $\langle\{0.3x\}\rangle$. 0.3x generates a special cyclic fuzzy set subsemiring which is not an ideal.

B = $\{0, 0.3x, (0.3x)^2, (0.3x)^3, \ldots, (0.3x)^n, \ldots$ as $n \to \infty\}$; clearly B is not a filter B is of infinite order.

It is nice to keep on record S has infinite number of special fuzzy set subsemirings of infinite order which are not ideals or filters of S but are special cyclic fuzzy set subsemirings.

Consider $D = \langle\{0.004x^7 + 0.2x + 0.01\}\}\rangle \subseteq S$. D is a special fuzzy set cyclic subsemiring.

$D = \{0, (0.1 + 0.2x + 0.004x^7), (0.1 + 0.2x + 0.004x^7)^2, \ldots \infty\} \subseteq S$ is of infinite order.

We can have special fuzzy set subsemirings generated by two elements also.

$$\text{Take } B_1 = \langle\{0.3x^2 + 0.4x + 0.31 = p(x) \quad q(x) = 0.03x^2 + 0.04x + 0.031\}\rangle;$$

B_1 is a special fuzzy set subsemiring of S of infinite order and B_1 is not an ideal or a filter of S.

However B_1 is generated only by these two elements and

$B_1 = \{0, p(x), q(x), (p(x))^2, (q(x))^2, (p(x))^3, (q(x))^3, p(x), q(x), (p(x))^2\, q(x), \ldots \infty\} \subseteq S$

$D_1 = \langle\{p(x) = 0.315x^3 + 0.21x + 0.001, q(x) = 0.0315x^3 + 0.021x + 0.0001, r(x) = 0.00315x^3 + 0.0021x + 0.00001\}\rangle \subseteq S$ is a special fuzzy set subsemiring of infinite order which is not an ideal or a filter of S.

Thus S can have special fuzzy set polynomial subsemirings generated by a finite number of elements say n.

It is pertinent to keep on record that none of these are ideals or filters of S and all of them are of infinite order.

Let

$$V = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 0.05)\right\} \subseteq S$$

is a special fuzzy set subsemiring of S which is an ideal as well as a filter of S.

Suppose

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 1/7, 1/7^2, 1/7^3, \ldots, \infty\} \subseteq S, \right.$$

M is a special fuzzy set subsemiring of S which is not an ideal as well as not a filter of S.

Thus S has infinite number of special fuzzy set subsemirings which are not ideals or filters of S.

Now we will introduce the notion of special fuzzy set pseudo rings.

Let

$$S = \{[0,1), +, \times\} \text{ where } a + b = \begin{cases} a + b \text{ if } a + b < 1 \\ (a+b) \text{ (mod 1) if } a + b > 1 \end{cases}$$

for a, b $\in$ [0,1) and $\times$ is the usual product.

(S, +) is an additive abelian group modulo 1.

For if x = 0.91 and y = 0.302 then

$$\begin{aligned} x+y &= 0.91 + 0.302 \text{ (mod 1)} \\ &= 1.212 \text{ (mod 1)} \\ &= 0.212 \in S. \end{aligned}$$

This is the way '+' operation is performed on S. 0 is the additive identity. We see the distributive law in general is not true.

For if 0.031 $\in$ S then 0.969 $\in$ S is the additive inverse of S for 0.031 + 0.969 = 1.000 = 0 (mod 1).

Let a = 0.02, b = 0.791 and c = 0.902 $\in$ S.

We find $a \times (b+c)$ and $a \times b + a \times c$;

$$\begin{aligned} a \times (b+c) &= 0.02 \times (0.791 + 0.902) \pmod 1 \\ &= 0.02 \times (1.693) \pmod 1 \\ &= 0.03386 \qquad \ldots \quad \text{I} \end{aligned}$$

Now consider $a \times b + a \times c$

$$\begin{aligned} &= 0.02 \times 0.791 + 0.02 \times 0.902 \\ &= 0.01582 + 0.01804 \\ &= 0.03386 \qquad \ldots \quad \text{II} \end{aligned}$$

I and II are identical.

Let $a = 0.005$, $b = 0.31$ and $c = 0.999 \in S$

$$\begin{aligned} a \times (b + c) &= 0.005 \times (0.031 + 0.999) \\ &= 0.005 \times (1.030) \\ &= 0.005150 \qquad \ldots \quad \text{I} \end{aligned}$$

$$\begin{aligned} a \times b + a \times c &= 0.005 \times 0.031 + 0.005 \times 0.999 \\ &= 0.000155 + 0.004995 \\ &= 0.005150 \qquad \ldots \quad \text{II} \end{aligned}$$

I and II are identical hence S is a special fuzzy set pseudo ring of infinite order. But in general distributive law is not true.

We have very many difficulties in finding special fuzzy set pseudo subring of $S = \{[0, 1), +, \times\}$ as it is a very dense interval under addition modulo 1.

Now using the operations of + and × we build several special fuzzy set pseudo rings which will be illustrated by examples and their properties will be explained.

Example 3.33: Let
$S = \{(a_1, a_2, a_3)$ where $a_i \in [0, 1), 1 \leq i \leq 3, +, \times\}$ be the special fuzzy set pseudo row matrix ring of infinite order.

We see $P_1 = \{(a_1, 0, 0) \mid a_1 \in [0, 1), +, \times\} \subseteq S$ is a special fuzzy set pseudo row matrix subring as well as ideal of S.

$P_2 = \{(0, a_1, 0) \mid a_1 \in [0, 1), +, \times\} \subseteq S$ is a special fuzzy set pseudo row matrix subring as well as ideal of S.

$P_3 = \{(0, 0, a_1) \mid a_1 \in [0, 1), +, \times\} \subseteq S$ is again a special fuzzy set pseudo subring of S.

Thus these special fuzzy set pseudo subring are of infinite order and $P_i \cap P_j = \{(0, 0, 0)\}$ of $i \neq j$ with $S = P_1 + P_2 + P_3$, $1 \leq i, j \leq 3$. Thus S is the direct sum of special fuzzy set pseudo subrings.

All special fuzzy set pseudo subrings of S may not in general lead to special fuzzy set pseudo ring direct sum.

For if we take $M_1 = \{(a_1, a_2, 0) \mid a_1\, a_2 \in [0, 1), +, \times\} \subseteq S$ and $M_2 = \{(0, a_1, a_2) \mid a_1\, a_2 \in [0, 1), +, \times\} \subseteq S$ both M_1 and M_2 are special fuzzy set pseudo subrings however $M_1 \cap M_2 \neq \{(0, 0, 0)\}$ and $S \subseteq M_1 + M_2$ so M_1 and M_2 cannot contribute to the direct sum of S as subrings.

Clearly S has zero divisors this S has no idempotents. Infact S has infinite number of zero divisors.

Let $P = \langle\{(0.2, 0, 0)\}\rangle \subseteq S$ generate a special fuzzy set pseudo subring and it is not the principal ideal of S.

Likewise $R = \langle\{(0, 0.03, 0)\}\rangle$ generates a special fuzzy set pseudo subring which is also not a principal ideal of S.

Infact S has infinite number of subrings which are principal ideals.

Let $T = \langle\{(0, 0.5, 0.2)\}\rangle$ be the special fuzzy set pseudo subring; clearly T is an ideal of S. We see $o(T) = \infty$.

However as S is a pseudo ring the question of filters does not arise.

Infact S = {[0,1), +, ×} has ideals which are principal. For if M = {⟨0.02⟩} $\subseteq$ S is a special fuzzy set pseudo subring which is not a principal ideal of S.

S has infinite number pseudo subrings which are not principal ideals and all of them are of infinite order.

We cannot find zero divisors in S = {[0,1), +, ×}.

Example 3.34: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 5 \right\}$$

be the special fuzzy set column matrix pseudo ring under (+, $\times_n$). S is commutative and is of infinite order.

S has zero divisors. S has infinite number of special fuzzy set column matrix pseudo subrings of infinite order.

Take

$$P_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_1, a_2 \in [0, 1) \right\} \subseteq S$$

P_1 is a special fuzzy set pseudo subring of S which is also an ideal of S of infinite order.

$$P_2 = \left\{ \begin{bmatrix} 0 \\ a_1 \\ 0 \\ a_2 \\ 0 \end{bmatrix} \middle| a_1, a_2 \in [0, 1) \right\} \subseteq S$$

is again a special fuzzy set pseudo subring of infinite order.

We see

$$P_1 \cap P_2 = \left\{ \begin{bmatrix} 0 \\ x \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| x \in [0, 1) \right\} \subseteq S$$

is again a special fuzzy set pseudo subring as well as ideal of S of infinite order.

$$P_3 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in \{0, 0.2, (0.2)^2, \ldots, (0.2)^n, \ldots\} \subseteq [0, 1), 1 \leq i \leq 5 \right\}$$

is a special fuzzy set pseudo subring of S of infinite order.

Clearly S is not an ideal of S. We have special fuzzy pseudo set subrings which are not ideals of S.

Now consider

$$N = \left\{ \left\langle \begin{bmatrix} 0.7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\rangle \right\} \subseteq S$$

be the special fuzzy set pseudo subring of S which is not an ideal of S.

It is left as an open problem will

$$N = \left\{ \left\langle \begin{bmatrix} 0.7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\rangle \right\} \cong M_1 = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \, a \in [0, 1) \right\} \subseteq S?$$

It is important to mention that isomorphism or homomorphism of special fuzzy set pseudo rings is the same as that of usual rings in these special fuzzy set pseudo rings.

S has zero divisors and no units or idempotents.

Infact every element in S is torsion free for if $x \in [0,1)$ then $x^n \to 0$ as $n \to \infty$, so $x^n \neq 1$ for any x or any finite or infinite number n. Hence we can say all elements are nilpotent of infinite order.

Example 3.35: Let

$$W = \left\{ \left[\begin{array}{c} a_1 \\ \hline a_2 \\ a_3 \\ a_4 \\ \hline a_5 \\ a_6 \\ \hline a_7 \end{array} \right] \,\middle|\, a_i \in [0, 1), 1 \leq i \leq 7 \right\}$$

be the special fuzzy set pseudo super matrix ring under + and $\times_n$ ($\times_n$ the natural product).

Let

$$A = \left[\begin{array}{c} a_1 \\ \hline a_2 \\ a_3 \\ a_4 \\ \hline a_5 \\ a_6 \\ \hline a_7 \end{array} \right] \text{ and } B = \left[\begin{array}{c} b_1 \\ \hline b_2 \\ b_3 \\ b_4 \\ \hline b_5 \\ b_6 \\ \hline b_7 \end{array} \right] \in W,$$

$$\text{then } A \times_n B = \left[\begin{array}{c} a_1 \\ \hline a_2 \\ a_3 \\ a_4 \\ \hline a_5 \\ a_6 \\ \hline a_7 \end{array} \right] \times_n \left[\begin{array}{c} b_1 \\ \hline b_2 \\ b_3 \\ b_4 \\ \hline b_5 \\ b_6 \\ \hline b_7 \end{array} \right] = \left[\begin{array}{c} a_1b_1 \\ \hline a_2b_2 \\ a_3b_3 \\ a_4b_4 \\ \hline a_5b_5 \\ a_6b_6 \\ \hline a_7b_7 \end{array} \right] \in W.$$

This is the way the operation $\times_n$ (natural product) is performed on W. W has infinite number of zero divisors no units and no idempotents.

W has infinite number of special fuzzy set pseudo subrings which are not ideals. However W has special fuzzy set pseudo subrings which are ideals.

It is left as an open problem in S = {m × n matrix with entries from [0,1)} and (S, +, $\times_n$) is a special fuzzy set pseudo ring.

Can S have infinite number of ideals or only a finite number of ideals?

However S has infinite number of special fuzzy set pseudo subrings which are not ideals of S. Further we have also special fuzzy set pseudo matrix rings which are non commutative.

Example 3.36: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 1), 1 \leq i \leq 9 \right\}$$

be the special fuzzy set matrix ring of infinite order where in S the product is the usual product × and not the natural product $\times_n$.

S is non commutative under usual product ×. For take

$$A = \begin{bmatrix} 0.3 & 0.1 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0.9 & 0 & 0 \\ 0.2 & 0 & 0.2 \\ 0 & 0.3 & 0.4 \end{bmatrix} \text{ in S.}$$

$$A \times B = \begin{bmatrix} 0.3 & 0.1 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} \times \begin{bmatrix} 0.9 & 0 & 0 \\ 0.2 & 0 & 0.2 \\ 0 & 0.3 & 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.27+0.02 & 0.06 & 0.02 \\ 0.04 & 0 & 0.04 \\ 0 & 0.21 & 0.28 \end{bmatrix}$$

$$= \begin{bmatrix} 0.29 & 0.06 & 0.02 \\ 0.04 & 0 & 0.04 \\ 0 & 0.21 & 0.28 \end{bmatrix} \qquad \ldots \text{ I}$$

Now

$$B \times A = \begin{bmatrix} 0.9 & 0 & 0 \\ 0.2 & 0 & 0.2 \\ 0 & 0.3 & 0.4 \end{bmatrix} \times \begin{bmatrix} 0.3 & 0.1 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 0.27 & 0.09+0.04 & 0 \\ 0.06 & 0.02 & 0.14 \\ 0 & 0.06 & 0.28 \end{bmatrix} = \begin{bmatrix} 0.27 & 0.13 & 0 \\ 0.06 & 0.02 & 0.14 \\ 0 & 0.06 & 0.28 \end{bmatrix} \qquad \ldots \text{ II}$$

Clearly I and II are distinct hence $A \times B \neq B \times A$ for $A, B \in S$ thus $(S, +, \times)$ is a non commutative special fuzzy set pseudo ring of infinite order.

If on the other hand we have $(S, +, \times_n)$ then it is a commutative special fuzzy set pseudo ring. Having seen examples of some non commutative special fuzzy set pseudo matrix rings we now give more properties.

Example 3.37: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 4 \right\}$$

be the special fuzzy set pseudo matrix ring under (+, ×) usual matrix product. S is a non commutative ring.

$$\text{Let } A = \begin{bmatrix} 0.3 & 0 \\ 0.4 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0.2 & 0.7 \end{bmatrix} \in S.$$

$$\text{Now } A \times B = \begin{bmatrix} 0.3 & 0 \\ 0.4 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ 0.2 & 0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \ldots \quad \text{I}$$

$$\text{Consider } B \times A = \begin{bmatrix} 0 & 0 \\ 0.2 & 0.7 \end{bmatrix} \times \begin{bmatrix} 0.3 & 0 \\ 0.4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0.48 & 0 \end{bmatrix} \quad \ldots \quad \text{II}$$

Clearly

$$A \times B \neq B \times A \text{ for } A \times B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and}$$

$$B \times A \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Infact (S, +, ×) has infinite number of right zero divisors which are not left zero divisors and vice versa.

However S has no idempotents. S has no units as $1 \notin S$.

If $(S, +, \times_n)$ is taken we see S is a commutative pseudo ring and has zero divisors.

$$\text{Take } A = \begin{bmatrix} 0.3 & 0 \\ 0.5 & 0.6 \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} 0 & 0.7 \\ 0 & 0 \end{bmatrix} \in S; \text{ we have } A \times_n B = B \times_n A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\text{However } A \times B = \begin{bmatrix} 0.3 & 0 \\ 0.5 & 0.6 \end{bmatrix} \times \begin{bmatrix} 0 & 0.7 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.21 \\ 0 & 0.5 \end{bmatrix} \qquad \ldots \text{ I}$$

$$B \times A = \begin{bmatrix} 0 & 0.7 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0.3 & 0 \\ 0.5 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.35 & 0.42 \\ 0 & 0 \end{bmatrix} \qquad \ldots \text{ II}$$

I and II are distinct and none of them are zero divisors only under natural product they are zero divisors.

Further

$$A = \begin{bmatrix} 0.3 & 0 \\ 0.4 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0.2 & 0.7 \end{bmatrix}$$

under natural product does not contribute to zero divisors for

$$A \times B = \begin{bmatrix} 0.3 & 0 \\ 0.4 & 0 \end{bmatrix} \times_n \begin{bmatrix} 0 & 0 \\ 0.2 & 0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0.08 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Can we have matrices A, B such that A × B =

$$A \times_n B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}?$$

$$\text{Let } A = \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.1 \end{bmatrix} \in S \text{ we find}$$

$$A \times A = \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.1 \end{bmatrix} \times \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.04 + 0.48 & 0.12 + 0.06 \\ 0.16 + 0.08 & 0.48 + 0.01 \end{bmatrix}$$

$$= \begin{bmatrix} 0.52 & 0.18 \\ 0.245 & 0.49 \end{bmatrix}.$$

Now

$$A \times_n A = \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.1 \end{bmatrix} \times \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.04 & 0.36 \\ 0.64 & 0.01 \end{bmatrix}.$$

We see $A \times A \neq A \times_n A$ in general.

We say two matrices; A and B are orderable if $A = (a_{ij})$ and $B = (b_{ij})$ then either $a_{ij} < b_{ij}$ or $a_{ij} > b_{ij}$ for every a_{ij} and b_{ij}.

$$\text{Take } A = \begin{bmatrix} 0.7 & 0.8 \\ 0.1 & 0.3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0.6 & 0.8 \\ 0.5 & 0.9 \end{bmatrix}$$

Clearly A and B are not orderable.

$$\text{Let } A = \begin{bmatrix} 0.3 & 0.4 \\ 0.1 & 0.8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0.4 & 0.6 \\ 0.41 & 0.92 \end{bmatrix} \in S.$$

We say A and B are specially orderable and $A < B$.
If $A \in (S, +, \times_n)$ then we have

$$A \geq A^2 \geq A^3 \geq \ldots \geq A^n \to \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ as } n \to \infty.$$

This is one of the special and striking features of special fuzzy set pseudo matrix ring under the natural product $\times_n$.

$$\text{Let } A = \begin{bmatrix} 0.3 & 0.4 \\ 0.1 & 0.8 \end{bmatrix} \in S$$

$$A \times A = \begin{bmatrix} 0.3 & 0.4 \\ 0.1 & 0.8 \end{bmatrix} \times \begin{bmatrix} 0.3 & 0.4 \\ 0.1 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.09+0.04 & 0.12+0.32 \\ 0.03+0.08 & 0.04+0.64 \end{bmatrix}$$

$$= \begin{bmatrix} 0.13 & 0.44 \\ 0.11 & 0.68 \end{bmatrix}$$

we see A and A × A we not comparable (that is orderable)

However

$$A \times_n A = \begin{bmatrix} 0.3 & 0.4 \\ 0.1 & 0.8 \end{bmatrix} \times_n \begin{bmatrix} 0.3 & 0.4 \\ 0.1 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.09 & 0.16 \\ 0.01 & 0.64 \end{bmatrix} \text{ and } A > A^2$$

$A \times_n A \times_n A = A^2 \times_n A$

$$= \begin{bmatrix} 0.09 & 0.16 \\ 0.01 & 0.64 \end{bmatrix} \times \begin{bmatrix} 0.3 & 0.4 \\ 0.1 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.027 & 0.064 \\ 0.001 & 0.512 \end{bmatrix} = A^3$$

We see $A > A^2 > A^3 >$ and so on.

$$\text{Let } A = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix} \in S.$$

Now

$$A \times_n A = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.04 & 0 \\ 0 & 0 \end{bmatrix}.$$

$A^3 = A^2 \times_n A$

$$= \begin{bmatrix} 0.04 & 0 \\ 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.008 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$A^4 = A^3 \times_n A = \begin{bmatrix} 0.008 & 0 \\ 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0016 & 0 \\ 0 & 0 \end{bmatrix} \text{ and so on.}$$

$A > A^2 > A^3 > A^4 > \ldots$

Also $A \times A = A \times_n A$ so
$A > A^2 > A^3 \ldots$ in $(S, +, \times)$ also for this particular A.

However this is not in general true for all A.

$$\text{For take } A = \begin{bmatrix} 0 & 0.9 \\ 0 & 0 \end{bmatrix} \in S.$$

$$A \times A = \begin{bmatrix} 0 & 0.9 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0.9 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\text{But } A \times_n A = \begin{bmatrix} 0 & 0.9 \\ 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 0 & 0.9 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.81 \\ 0 & 0 \end{bmatrix}.$$

$$A \times_n A \times_n A = \begin{bmatrix} 0 & 0.81 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0.9 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.729 \\ 0 & 0 \end{bmatrix} \text{ and so on.}$$

$A > A^2 > A^3 \ldots$ under natural product.

However if the matrix used is a column matrix or a rectangular matrix then the usual product cannot be defined so we only use the natural product $\times_n$ on S.

For if

$$S = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right| a_i \in [0, 1), 1 \leq i \leq 4 \right\}$$

be a special fuzzy set pseudo matrix ring under the natural product $\times_n$ and if

$$A = \begin{bmatrix} 0.3 \\ 0.1 \\ 0.4 \\ 0.5 \end{bmatrix} \in S; \text{ we have}$$

$$A \times_n A = \begin{bmatrix} 0.3 \\ 0.1 \\ 0.4 \\ 0.5 \end{bmatrix} \times_n \begin{bmatrix} 0.3 \\ 0.1 \\ 0.4 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.09 \\ 0.01 \\ 0.16 \\ 0.25 \end{bmatrix} \text{ (and } A > A \times_n A).$$

Consider

$$A \times_n A \times_n A = \begin{bmatrix} 0.09 \\ 0.01 \\ 0.16 \\ 0.25 \end{bmatrix} \times_n \begin{bmatrix} 0.3 \\ 0.1 \\ 0.4 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.027 \\ 0.001 \\ 0.064 \\ 0.125 \end{bmatrix} \text{ and } A > A^2 > A^3 \text{ and so on.}$$

$$\text{Let } A = \begin{bmatrix} 0.3 & 0.4 & 0.5 & 0.6 \\ 0 & 0 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0 & 0 \\ 0.4 & 0.1 & 0 & 0.7 \end{bmatrix} \in (S, \times_n, +)$$

Now

$$A \times_n A = \begin{bmatrix} 0.09 & 0.16 & 0.25 & 0.36 \\ 0 & 0 & 0.04 & 0.01 \\ 0.01 & 0.09 & 0 & 0 \\ 0.16 & 0.01 & 0 & 0.49 \end{bmatrix} \text{ we see } A > A \times_n A$$

and so on.

Example 3.38: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| a_i \in [0, 1), 1 \leq i \leq 10 \right\}$$

be the special fuzzy set pseudo ring under (+, $\times_n$). S has special fuzzy set pseudo subrings which are not ideals as well as special fuzzy set pseudo subrings which are ideals. Infact S has infinite number of zero divisors.

Every A in S is such that A > A $\times_n$ A > A $\times_n$ A $\times_n$ A > … and so on. This structure is commutative. S has no idempotents and units.

Clearly no A $\in$ S (A $\neq$ (0)) is such that $A^n = \underbrace{A \times_n A \times_n A \times_n \ldots \times_n A}_{n-\text{times}}$ is zero for any finite n. This study is interesting.

Now we build non commutative special fuzzy set pseudo ring which is non commutative.

Let S = {[0,1), +, ×} be the ring S has no unit. All n × n matrices under the usual matrix product will give non commutative special fuzzy set pseudo ring.

We can also have the notion of special fuzzy set pseudo polynomial ring which is as follows.

Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0,1) \right\}$$

be the special fuzzy set pseudo polynomial ring.

Let $p(x) = 0.3x^2 + 0.72x + 0.9$
and $q(x) = 0.9x^3 + 0.78x^2 + 0.4x + 0.321$ be in S.

$p(x) + q(x) = (0.3x^2 + 0.72x + 0.9) + 0.9x^3 + 0.78x^2 + 0.4x + 0.321$

$= 0.9x^3 + (0.3 + 0.78)x^2 + (0.72 + 0.4)x + (0.9 + 0.321)$

$= 0.9x^3 + 1.08 \pmod 1 x^2 + (1.12) \pmod 1 x + 1.221 \pmod 1$

$= 0.9x^3 + 0.08x^2 + 0.12x + 0.221 \in S.$

This is the way polynomial addition is made.

Let $a(x) = 0.03x^2 + 0.2x + 0.3$

and $b(x) = 0.7x + 0.1 \in s$

$a(x) \times b(x) = (0.03x^2 + 0.2x + 0.3) \times (0.7x + 0.1)$

$= 0.021x^3 + 0.14x^2 + 0.21x + 0.003x^2 + 0.02x + 0.03$

$= 0.021x^3 + 0.143x^2 + 0.23x + 0.03 \in S$.

This is the way $\times$ operation is performed on S.

Clearly S[x] has no zero divisors. We can have special fuzzy set pseudo subrings and ideals in S[x].

Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^{2i} \,\middle|\, a_i \in [0, 1) \right\} \subseteq S$$

be the special fuzzy set pseudo subring of S.

Clearly M is not an ideal for if $0.3 \in S[x]$ and $0.2x^2 + 0.4 \in M$ then $0.3\,x \times 0.2x^2 + 0.4 \in M$ then $0.3x \times 0.2x^2 + 0.4 = 0.06x^3 + 0.12x \in S[x]$ but $0.06x^3 + 0.12x \notin M$ as M contains only polynomials of even degree with coefficients from [0,1).

Can S[x] have ideals? This is left as an open problem. However S[x] have no zero divisors or idempotents or units.

We see S[x] has no polynomial where coefficient is 1. For no polynomial in S[x] is monoic.

Let $p(x) = 0.06x^2 + 0.19x + 0.08$.

This can be factored as $(0.2x + 0.1) \times (0.3x + 0.8)$

Thus $0.2x + 0.1 = 0$ and $0.3x + 0.8x = 0$

$0.2x = 0.9$ and $0.3x = 0.2$.

Now as coefficient of x can never be 1 in S[x] we see 0.2x = 0.9 and 0.3x = 0.2 are the values 0.2x and 0.3x can take. So solving has no meaning in this context.

We can however compare the roots by multiplying;

$$0.2\,x = 0.9 \text{ by } 0.3$$
$$0.3\,x = 0.2 \text{ by } 0.2$$
$$\text{So that } 0.06x = 0.18 \text{ or}$$
$$0.06x = 0.04$$

which ever solution is a feasible one we accept them.

Thus if $p(x) = 0.006x^3 + 0.035x^2 + 0.053x + 0.014 \in S$ then $p(x) = (0.2x + 0.7)\,(0.3x + 0.1)\,(0.1x + 0.2)$

$$\text{So } 0.2x + 0.7 = 0$$
$$0.3x + 0.1 = 0 \text{ and}$$
$$0.1x + 0.2 = 0.$$

This is turn implies

$$0.2\,x = 0.3$$
$$0.3\,x = 0.9 \text{ and}$$
$$0.1x = 0.8$$
$$\text{so that } 0.06x = 0.09$$
$$0.06x = 0.18 \text{ and}$$
$$0.06x = 0.48.$$

One can choose any of the needed value for 0.06x takes there three values.

The method of factorizing into linear factors is a difficult job. By studying the coefficients of the highest power of x and that of the constant term one can guess and write the product in linear terms and work out to see how we get the solution.

It is an open problem to study the situation.

Can the $p(x) \in S[x]$ have several sets of roots?

Now we can infact differentiate the polynomials in $p(x) \in S[x]$ and we see under (modulo 1) $p'(x) \in S[x]$.

For if $p(x) = 0.3x^7 + 0.2x^6 + 0.5x^3 + 0.3$ is in S[x] then $\frac{dp(x)}{dx} = p'(x) = 7 \times 0.3x^6 + 6 \times 0.2x^5 + 3 \times 0.5x^2$

$= 0.1x^6 + 0.2x^5 + 0.5x^2 \in S[x]$ thus is the way the differentiation of polynomials in S[x] are performed.

As we cannot find multiplicative inverse for any element in [0,1), we see it is not easy to define integration for we have to think of natural means to define division. It may be possible in due course of time.

Now we see special fuzzy set pseudo polynomial ring S[x] is a ring. We have subrings for take the special fuzzy set pseudo subring generated by the polynomial $p(x) = 0.2x^3 + 0.2x + 0.02$ then $P = \langle p(x) \rangle$ generates a special fuzzy set pseudo polynomial subring of infinite order. Infact p(x) is not an ideal.

Study in this direction is also open for such study is interesting and innovative for which p(x) may be ideals? This question is let as an open conjecture.

Finally we suggest that

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 1), \times, + \right\}$$

happens to be a very new study for we do not include 1 so no polynomial can be monic.

We call elements of S[x] to be special fuzzy polynomials.

We leave open the following questions.

Can S[x] have principal ideals?

Is S[x] a Euclidean domain?

Is S[x] a Unique Factorization Domain?

Can the theorem of fundamental algebra; every polynomial of degree n has only n and n roots true? No doubt. We can have n roots but will we have sets of n roots which are different? Study is this direction is very rich and may lead to may exciting results.

For S = {[0,1), +, ×} is a ring without unit.

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in [0, 1), +, \times \right\}$$

is also a ring without unit which has forced us to several open problems.

Also the concept of $\frac{dp(x)}{dx} = 0$ if and only if p(x) is a constant is not true for if $p(x) = 0.5x^2 + 0.7$ then $\frac{dp(x)}{dx} = 0$. Likewise $0.4x^5 + 0.2x^4 + 0.921 = q(x) \in S[x]$ is such that $\frac{dp(x)}{dx}$ = 0 even though p(x) and q(x) are not constant polynomials.

Now we proceed onto study other types of non commutative semirings and rings which are constructed using S = {[0,1), +, ×}.

Example 3.39: Let M = {SG where G = S_3 and S = {[0, 1), +, ×} be the special fuzzy set pseudo group ring. Elements of SG are $\sum s_i g_i$ where $g_i \in S_3$ we make $s_i e = s_i$ where $e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ the identity of S_3.

$$\text{Here } p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \text{ and}$$

$$p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \text{ are in } S_3.$$

If $a = 0.3p_1 + 0.2p_5 + 0.15$ and $b = 0.2p_4 + 0.2$ are in SG

then $a + b = \{0.3p_1 + 0.2p_5 + 0.15\} + \{0.2p_4 + 0.2\}$

$= \{0.3p_1 + 0.2p_5 + 0.2p_4 + 0.35\}$.

$a \times b = \{0.3p_1 + 0.2p_5 + 0.15\} \times \{0.2p_4 + 0.2\}$
$= 0.06p_1p_4 + 0.04p_5p_4 + 0.03p_4 + 0.06p_1 + 0.04p_5 + 0.03$
$= 0.06p_3 + 0.4 + 0.03p_4 + 0.06p_1 + 0.04p_5 + 0.03$
$= 0.06p_3 + 0.06p_1 + 0.03p_4 + 0.04p_5 + 0.43 \in S(G)$.

It is important and interesting to that $G \not\subseteq SG$ as $1 \notin S$ however $S \subseteq SG$ as $e \in G$.

Further we see $p_1^2 = e$ so $x = 0.1 + 0.1\, p_1 \in SG$ and

$$\begin{aligned} x^2 &= (0.1 + 0.1p_1)^2 \\ &= 0.01 + 0.01p_2 + 0.02p_1 \\ &= 0.02 + 0.02\, p_1. \end{aligned}$$

Of course SG has special fuzzy set pseudo subrings and ideals.

For take $A = \{SP_1$ where $P_1 = \{e, p_1\} \subseteq S_3\}$; A is a special fuzzy set pseudo subring and is not an ideal of SG.

Similarly $B_2 = \{SP_2$ where $P_2 = \{e, p_2\} \subseteq S_3\}$ is a special fuzzy set pseudo subring of SG and is not an ideal of SG.

Clearly SG is a non commutative fuzzy set pseudo ring of infinite order.

Let $B_3 = \{SP_3$ where $P_3 = \{e, p_3\} \subseteq S_3\}$ be the special fuzzy set pseudo subring of SG.

We see all the three special fuzzy set pseudo subrings of SG are commutative and is of infinite order.

$B_4 = \{SP_4$ where $P_4 = \{e, p_4, p_5\} \subseteq S_3\} \subseteq SG$ is also a commutative special fuzzy set pseudo subring of infinite order.

Clearly SG has no units; zero divisors and idempotents even though the group is of finite order.

Example 3.40: Let SG be the special fuzzy set pseudo group ring of the group $G = D_{2,9} = \{a, b \mid a^2 = 1 = b^9; bab = a\}$ over S, the special fuzzy set pseudo ring.

SG is of infinite order. SG has subrings which are commutative though SG is non commutative. However the subrings are also of infinite order.

Take $M_1 = \{SH$ where $H = \{e = 1, a \mid a^2 = 1\} \subseteq D_{2,9}\} \subseteq SG$ is a special fuzzy set pseudo subring of SG which is commutative but M_1 is not an ideal of SG.

Let $M_2 = \{SP$ where $P = \{1, b, b^2, \ldots, b^8$ with $b^9 = 1 = e\} \subseteq D_{2,9}\} \subseteq SG$ is again a special fuzzy set pseudo subring of infinite order which is commutative and is not an ideal of SG.

Also $M_3 = \{SR$ where $R = \{1, b^3, b^6\} \subseteq D_{2,9}\} \subseteq SG$ is also a special fuzzy set pseudo subring of infinite order which is commutative and not an ideal of SG.

However SG has no units, zero divisors or idempotents.

Example 3.41: Let SG be the special fuzzy set group ring of the group $G = S_4$ over the special fuzzy set pseudo ring $S = \{[0, 1), +, \times\}$.

SG is non commutative and is of infinite order. SG has both commutative and non commutative special fuzzy set pseudo subrings.

For take

N_1 = {SA_4 | A_4 is the alternative subgroup of S_4} $\subseteq$ SG; N_1 is a special fuzzy set pseudo subring which is non commutative.

Take

$$N_2 = \{SP \text{ where } P = \left\{\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\right\} \subseteq S_4\} \subseteq SG$$

be the special fuzzy set pseudo subring of SG. Clearly N_2 is commutative of infinite order but is not an ideal of SG.

This ring has no units, no zero divisors, and no idempotents. So we define for this new structure special fuzzy pseudo units and special fuzzy pseudo idempotents in the following.

DEFINITION 3.1: *Let SG be the special fuzzy set pseudo group ring of a group G over the special fuzzy set pseudo ring S. We say an element x of SG is a special fuzzy pseudo unit if $x^n \in S$. ($n \geq 2$); $x \in SG$ is a special fuzzy pseudo idempotent if $x^2 = sx$ where $s \in S$ (That is if $x = s_1g_1 + s_2g_2 + \ldots + s_ng_n$ then $x^2 = p_1g_1 + p_2g_2 + \ldots + p_ng_n$, $p_i, s_i \in S$, $1 \leq i \leq n$). We say for $x \in SG$ and a $y \in SG$ with $x \times y = s \in S$; we call y to be the special fuzzy pseudo inverse of x.*

We will illustrate all these situations by some examples.

Example 3.42: Let SG be the special fuzzy pseudo set group ring of the group G = $D_{2,7}$ over the special fuzzy set pseudo ring S = {[0,1), +, ×}. Let G = {a, b | $a^2 = b^7 = 1$, bab = a}.

Take $0.2 + 0.3a = x \in SG$. $x^2 = (0.2 + 0.3a)(0.2 + 0.3a) = 0.04 + 0.06a + 0.06a + 0.09 = (0.13 + 0.12a)$.

We call x to be a special fuzzy pseudo idempotent of S.

Let $x = 0.4b^3 \in SG$ we have $y = 0.2b^4$ in SG such that

$$\begin{aligned} x \times y &= 0.4b^3 \times 0.2b^4 \\ &= 0.08b^7 \\ &= 0.08 \in S. \end{aligned}$$

Thus y is a special pseudo fuzzy set inverse of x and vice versa.

Let $x = 0.07a \in SG$ we see $x^2 = (0.07a)^2$
$= 0.0049$ so x is a special pseudo fuzzy unit of SG.

Consider $x = 0.02b$; we see $x^7 = (0.02)^7$ is a special pseudo fuzzy unit of SG.

Thus SG has special pseudo fuzzy unit, special pseudo fuzzy idempotent and special pseudo fuzzy inverse.

Thus we find conditions for a special fuzzy set pseudo group ring to possess these properties.

Example 3.43: Let SG be the special fuzzy set pseudo group ring of the group A_4 over the special fuzzy set pseudo ring $S = \{[0, 1), \times, +\}$.

Consider $x = 0.04\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + 0.7 \in SG$.

$$\text{We see } x^2 = [(0.04\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + 0.7]^2$$

$$= [0.04\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}^2 + 2\,[0.04\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}] \times 0.7 + (0.7)^2$$

$$= 0.0016 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} +$$

$$0.056 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + 0.49$$

$$= 0.0016 + 0.49 + 0.056 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$= 0.4916 + 0.056 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \in SG,$$

so x is a special pseudo fuzzy set idempotent of SG.

Take

$$x = 0.1 + 0.2 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} + 0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \in SG$$

We see $x^2 = \ (0.1)^2 + [0.2 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}]^2 +$

$$(0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix})^2 + 2 \times 0.1 \times 0.2 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} + 2 \times 0.1$$

$$\times 0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} + 2 \times 0.2 \times 0.3 \times \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

$$= 0.01 + 0.04\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 2 & 4\end{pmatrix} + 0.09\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 1 & 4\end{pmatrix} +$$

$$0.08\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 1 & 4\end{pmatrix} + 0.06\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 2 & 4\end{pmatrix} + 0.12\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix}$$

$$= 0.13 + 0.1\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 2 & 4\end{pmatrix} + 0.17\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 1 & 4\end{pmatrix} \text{ is in SG.}$$

Thus x is a special pseudo fuzzy set idempotent of SG.

$$\text{Let } x = 0.1 + 0.5\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} \in SG$$

$$x^2 = [0.1 + 0.5\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}]^2$$

$$= (0.1)^2 + [0.5\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}]^2 + 2 \times 0.1 \times 0.5\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}$$

$$= 0.01 + 0.25 + 0.1\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}$$

$$= 0.26 + 0.1\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} \in SG.$$

Hence x is a special pseudo fuzzy set idempotent of SG.

Let

$$x = 0.3 + 0.2\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} + 0.6\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} +$$

$$0.4 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \in SG.$$

We find x^2 ;

$$x^2 = (0.3)^2 + [0.2 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}]^2 + [0.6 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}]^2 +$$

$$0.4 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}]^2 + 2 \times 0.6 \times 0.4 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} + 2 \times 0.2 \times 0.4 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} +$$

$$2 \times 0.2 \times 0.6 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} + 2 \times 0.3 \times$$

$$0.2 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + 2 \times 0.3 \times 0.6 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} + 2 \times$$

$$0.4 \times 0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$= 0.09 + 0.04 + 0.36 + 0.48 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} +$$

$$0.16 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} + 0.24 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} +$$

$$0.12 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + 0.36 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} +$$

$$0.24 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$= 0.49 + 0.60 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + 0.52 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} +$$

$$0.48 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \in SG.$$

Thus x is a special pseudo fuzzy set idempotent of S.

$$\text{Let } x = 0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \in SG$$

$$x^4 = [0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}]^4$$

$$= [0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}] [0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}]$$

$$[0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}]^2$$

$$= 0.09 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \times 0.09 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$= 0.0081 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$= 0.0081.$$

Thus $x^4 = s \in S$ and x is a special fuzzy pseudo unit.

$$\text{Let } x = 0.01 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \text{ and}$$

$$y = 0.9 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \in SG$$

We find

$$x \times y = 0.01 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \times 0.9 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

$$= 0.009 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$= 0.009 \in S.$$

Thus x is the special pseudo fuzzy inverse of y and vice versa in SG.

SG has infinite number of special pseudo fuzzy units and idempotents. Also SG has several elements which are fuzzy pseudo invertible in SG.

Example 3.44: Let SG be the special fuzzy set pseudo group ring of the group $G = \langle g \mid g^{24} = 1\rangle$ and $S = \{[0,1), \times, +\}$ be the special fuzzy set pseudo ring. SG is commutative. SG contains elements such that $x^n = s \in S$ and special fuzzy pseudo idempotents and inverses.

Let $x = 0.07g$ we see $x^{24} = (0.07)^{24} \in S$. Take $y = 0.2 + 0.7g^{12} \in SG$.

Now we find $y^2 = (0.2 + 0.7g^{12})^2$
$= (0.2)^2 + (0.7g^{12})^2 + 2 \times 0.2 \times 0.7g$
$= 0.04 + 0.49 + 0.28\, g^{12}$
$= 0.53 + 0.28g^{12} \in SG;$

so x is a special pseudo fuzzy idempotent of SG.

Let $y = 0.1 + 0.2g^6 + 0.3g^{12} + 0.5g^{18} \in SG$.

We find $y^2 = (0.1 + 0.2g^6 + 0.3g^{12} + 0.5g^{18})^2$

$= (0.1)^2 + (0.2g^6)^2 + (0.3g^{12})^2 + (0.5g^{18})^2 + 2 \times 0.1 \times 0.2g^6 + 2 \times 0.1 \times 0.3g^{12} + 2 \times 0.1 \times 0.5g^{18} + 2 \times 0.2g^6 \times 0.3g^{12} + 2 \times 0.2 \times 0.5g^6 \times g^{18} + 2 \times 0.3 \times 0.5g^{12} \times g^{18}$

$= 0.01 + 0.04g^{12} + 0.09 + 0.25g^{12} + 0.1g^{18} + 0.12g^{18} + 0.04g^6 + 0.06g^{12} + 0.2 + 0.3g^6$

$= (0.01 + 0.09 + 0.2) + (0.04g^6 + 0.3g^6) + (0.4g^{12} + 0.25g^{12} + 0.06g^{12}) + (0.1g^{18} + 0.12g^{18})$

$= 0.3 + 0.34g^6 + 0.35g^{12} + 0.22g^{18} \in SG$.

Thus x is a special pseudo fuzzy idempotent of SG.

Infact it is easily verified $v = 0.2 + 0.3x^3 + 0.6x^6 + 0.7x^9 + 0.11x^{12} + 0.01x^{15} + 0.42x^{18} + 0.02x^{21} \in SG$ where $g = x \in G = \{g \mid g^{24} = 1\}$ is a special pseudo fuzzy idempotent of SG.

Consider $t = 072x^{15} \in SG$ we have a $y = 0.1x^9 \in SG$ such that

$t \times y = 0.72x^{15} \times 0.1x^9 = 0.072x^{24} = 0.072 \in S$.

Thus t is the special pseudo fuzzy inverse of y and vice versa.

Let $t = 0.021x^{17} \in SG$ then $y = 0.015\, x^7 \in SG$ is such that

$t \times y = (0.021 \times 0.015) \in S$, so t is the special pseudo fuzzy inverse of y and vice versa.

We see SG has several pseudo fuzzy inverse.

It is pertinent to keep on record for a given x = ag, a $\in$ S and g $\in$ G we have infinitely many special fuzzy pseudo inverses y and vice versa, that is why the authors choose to call them as special pseudo fuzzy inverse elements.

Likewise if $x = a_1 + a_2g + a_3g^2 + \ldots + a_n g^{n-1}$ ($g^i \in G$ and $a_j \in S$; $0 \leq j \leq n$, $1 \leq i \leq n-1$) in SG is a special fuzzy pseudo idempotent if $x^2 = s \in S$ we can change infinitely many such x by retaining the g_i and only $a_i \in [0, 1)$ varies for that x.

Now we give some more examples.

Example 3.45: Let SG be the special fuzzy set pseudo group ring where $G = Q^+$. We see SG has no special pseudo fuzzy units for if x = sg; where s $\in$ S and g $\in$ G then $x^n = t$; t $\in$ S is impossible.

SG has no special fuzzy pseudo idempotents but SG has special fuzzy pseudo inverses.

For if $x = 0.37 \times 7/13$ then $y = 0.1 \times 13/7 \in SG$ is such that $x \times y = 0.37 \times 7/13 \times 0.1 \times 13/7 = 0.037 \in S$.

Hence y is the special pseudo fuzzy inverse of x and vice versa.

Example 3.46: Let SG be the special fuzzy set pseudo group ring where $G = S_3 \times D_{2,5} \times A_4$. SG has infinite number of special fuzzy pseudo units, special fuzzy pseudo idempotents and every element SG \ S has a special fuzzy pseudo inverse.

Inview of all these we have the following theorem.

THEOREM 3.14: *Let SG be the special fuzzy set pseudo group ring of a group G over the special fuzzy set pseudo ring $S = \{[0, 1), +, \times\}$.*

(i) SG has special pseudo fuzzy units if and only if $o(G) < \infty$ or G has elements of finite order.

(ii) *SG has special pseudo fuzzy idempotents if and only if $o(G) < \infty$ or G has finite subgroups.*
(iii) *SG has special pseudo fuzzy inverses.*

Proof is direct and hence left as an exercise to the reader.

Example 3.47: Let SG be the special fuzzy set pseudo group ring where

$$G = \left\{\left.\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| ad - bc \neq 0.\ a, b, c, d \in Z_{19}\right\}$$

be the group under product. SG is non commutative and has special pseudo fuzzy units, idempotents and inverses.

Now we proceed onto study special fuzzy set pseudo semigroup rings.

Example 3.48: Let SM be the special fuzzy set pseudo semigroup ring of the semigroup M = S(3) over the special fuzzy set pseudo ring S = {[0,1), +, ×}.

SM has special pseudo fuzzy units, idempotents and inverses and SM is non commutative.

Example 3.49: Let SZ_6 be the special fuzzy set pseudo semigroup ring of the semigroup $\{Z_6, \times\}$ over the special fuzzy set pseudo ring S = {[0,1), +, ×}.

Now x = 0.8(3) is a special pseudo fuzzy idempotents of SZ_6 as $x^2 = (0.8.3)^2 = 0.64.3$ (since $3^2 = 3 \bmod 6$).

Also $x = (0.2)4 \in SZ_6$ is a special pseudo fuzzy idempotent of SZ_6 as $x^2 = ((0.2)4)^2 = 0.04.4$.

Also $y = (0.18)3 + (0.5)4 \in SZ_6$ is a special pseudo fuzzy idempotent of SZ_6 as

$$y^2 = (0.18(3) + 0.5(4))^2$$
$$= (0.18)^2 \,.\, 3^2 + (0.5)^2\,(4)^2 + (0.18)\,(0.5)\,(3.4)$$

(3.4 $\equiv$ 0 mod 6 and
$3^2 \equiv 3$ mod 6 and
$4^2 = 4$ mod 6)

we get $y^2 = (0.18)^2 3 + (0.25)4$ is a special fuzzy pseudo idempotent of SZ_6.

Infact SZ_6 has infinite number of special fuzzy pseudo idemptotents.

Let $x = (0.1)\,5 \in SZ_6$
$x^2 = (0.1)^2\,(5)^2$ (therefore $5^2 \equiv 1 \pmod 6$)
$= 0.015$ is a special pseudo fuzzy unit of SZ_6.

Example 3.50: Let SZ_{15} be the special fuzzy set pseudo semigroup ring of the semigroup $(Z_{15}, \times)$ over the special fuzzy set pseudo ring $S = \{[0, 1), \times, +\}$.

SZ_{15} has special pseudo fuzzy units, idempotents and inverses for $x = (0.3)^2$ and $y = (0.5)^8$ in SZ_{15} such
$x \times y = (0.3)^2 \times (0.5)^8 = 0.15$ as $2.8 \equiv 1 \pmod{15}$ so x is the special pseudo fuzzy inverse of y and vice versa.

Take $x = (0.3).6 \in SZ_{15}$ is such that $y = (0.3)5$ given

$$x \times y = (0.3)6 \times (0.3)5$$
$$= (0.09)0$$
$$= 0 \text{ as } 6 \times 5 \equiv 0 \pmod{15}.$$

This is defined as special pseudo fuzzy zero divisors or just a zero divisor of the ring SZ_{15}. We see it is a zero divisors.

Suppose $x = (0.7)3$ and $y = (0.02)10 \in SZ_{15}$; then
$x \times y = (0.7)3 \times (0.02)10$

$= (0.014)0 = 0$ as $3 \times 10 = 30 \equiv 0 \pmod{15}$.

Take x = (0.2)6 + (0.03) (3)
and y = (0.72)10 + (0.8)5 $\in$ SZ_{15}

We find x $\times$ y = ((0.2) 6 + (0.03)3) $\times$ ((0.72)10 + (0.8)5)

= (0.2) $\times$ (0.72) 6 $\times$ 10 + (0.2) $\times$ (0.8) (6 $\times$ 5) +
(0.03) $\times$ 0.72) (3 $\times$ 10) + (0.8) [3 $\times$ 5]
= 0

(6 $\times$ 10 $\equiv$ 0 mod 15
5 $\times$ 6 $\equiv$ 0 mod 15
3 $\times$ 10 $\equiv$ 0 mod 15 and
3 $\times$ 56 $\equiv$ 0 mod 15).

Thus x is a special fuzzy pseudo zero divisor or just a zero divisor of y in SZ_{15}. It is important to keep on record that in case of special fuzzy set group ring SG of a group G over the special fuzzy set pseudo ring S = {[0, 1), +, $\times$} we do not have the notion of special pseudo fuzzy zero divisors.

Infact SG is either a special fuzzy set pseudo interval commutative or a ring according as G is commutative or non commutative respectively.

Example 3.51: Let SS(4) be the special fuzzy set pseudo semigroup ring of the symmetric semigroup S(4) over the special fuzzy set pseudo ring S = {[0,1), +, $\times$}.

SS(4) is a non commutative special fuzzy set pseudo ring with no zero divisors, that is SS(4) is a special fuzzy set pseudo division ring of infinite order which has special fuzzy pseudo idempotents, units and inverses but has no zero divisors.

Example 3.52: Let SZ_5 be the special fuzzy set pseudo semigroup ring of the semigroup (Z_5, $\times$) over the special fuzzy set pseudo ring S = {[0, 1), +, $\times$}. SZ_5 has no zero divisors.

Further SZ_5 is commutative hence SG is a special fuzzy set pseudo interval ring of infinite order.

Example 3.53: Let SZ_{45} be the special fuzzy set pseudo semigroup ring of the semigroup $(Z_{45}, \times)$ over the special fuzzy set pseudo ring $S = \{[0, 1), +, \times\}$.

SZ_{45} has zero divisors for take x = (0.5)9 and y = (0.2)5 in SZ_{45}

$$x \times y = (0.5) \times (0.2)\ (9 \times 5)$$
$$= 0 \text{ as } (9 \times 5 = 0 \bmod 45)).$$

Let x = (0.08)3 and y = (0.072)15 $\in SZ_{45}$ we see $x \times y = (0.08 \times (0.072) \times (3 \times 15) = 0 \pmod{45}$.

Thus x is a zero divisor.

Let x = (0.1112)9 and y = (0.25)10 $\in SZ_{45}$ we get
$$x \times y = (0.1112)\ 9 \times (0.25)10 = 0 \pmod{45}$$
$$(\text{and } 9 \times 10 = 90 = 0 \pmod{45})).$$

Thus x is a zero divisor in SZ_{45}. SZ_{45} has special pseudo fuzzy idempotents, units and inverses. So SZ_{45} is not a special fuzzy set interval pseudo ring.

Inview of this we have the following theorem.

THEOREM 3.15: *Let SG be the special fuzzy set pseudo group ring of a group G over the special fuzzy set pseudo ring S = {[0, 1), +, ×}. SG is a special fuzzy set integral domain or a special fuzzy set division ring according as G is a commutative group or a non commutative group respectively.*

Proof is direct and hence left as an exercise to the reader.

THEOREM 3.16: *Let SM be a special fuzzy set pseudo semigroup ring of the semigroup M over the special fuzzy set pseudo ring S = {[0, 1), +, ×}. SM is a special fuzzy set integral domain (or a division ring) according as the semigroup M is commutative or not.*

Or equivalently we can describe the above theorem as follows; SM has zero divisors if and only if the semigroup M has zero divisors or yet SM is a special fuzzy set integral domain or a special fuzzy set division ring if and only if M has no zero divisors.

The proof is direct hence left as exercise to the reader.

Example 3.54: Let SM be the special fuzzy set pseudo semigroup ring of the semigroup M = $\{C(Z_{12}), \times\}$. SM is not a special fuzzy set pseudo interval ring.

Example 3.55: Let SM be the special fuzzy set pseudo semigroup ring of the semigroup M = $\{Z_7$ (g, g_1), $\times$) where $g_1^2 = g_1$ and $g^2 = 0$, $g_1g = gg_1 = 0\}$ and S = $\{[0, 1), +, \times\}$ be the pseudo ring. S is not a special fuzzy set pseudo interval ring.

THEOREM 3.17: *Let SS(n) be the special fuzzy set semigroup ring of the symmetric semigroup S(n) over the ring S = {[0, 1), +, ×}. SS(n) is a special fuzzy set division ring.*

Proof is direct and hence left as an exercise to the reader.

THEOREM 3.18: *Let SZ_n be the fuzzy set pseudo semigroup ring of the semigroup (Z_n, ×) over the special fuzzy set ring S = {[0, 1), +, ×}.*

(i) SZ_n is not a special fuzzy set integral domain if and only if n is a composite number.
(ii) SZ_n is a special fuzzy set integral domain if and only if n is not a composite number.

Proof of the theorem is left as an exercise to the reader.

Next we give examples of non commutative special fuzzy set group semiring and special fuzzy set semigroup semiring.

Example 3.56: Let PG be the special fuzzy set group semiring of the group G = S_7 over the semiring P = $\{[0, 1), \min, \max\}$.

The operation on PG is as follows. '+' will denote max and '×' will denote the min.

Let $p(x) = 0.3g_1 + 0.7g_2 + 0.5$
and $q(x) = 0.4g_2 + 0.5g_3 + 0.7$ be in SG

max $(p(x), q(x)) = 0.3g_1 + 0.7g_2 + 0.7 + 0.5g_3$
and min $(p(x).\ q(x)) = 0.3g_1g_2 + 0.4g_2^2 + 0.4g_2 + 0.3g_1g_3 + 0.5g_2g_3 + 0.5g_3 + 0.3g_1 + 0.7g_2 + 0.5$
$= 0.3g_1g_2 + 0.4g_2^2 + 0.7g_2 + 0.3g_1g_3 + 0.5g_2g_3 + 0.5g_3 + 0.3g_1 + 0.5.$

This is the way operations are performed on PG.

It is pertinent to keep on record we denote max $\{0.7g_3, 0.8g_4\}$ by $0.7g_3 + 0.8g_4$ and max $\{0.5g_3, 0.8g_3\}$ by $0.8g_3$ and min of $\{0.7g_3, 0.8g_4\} = 0.7g_3g_4$ and min of $\{0.5g_3, 0.8g_3\} = 0.5\ g_3^2$ and so on.

Clearly we do not have zero divisors. We do not have special fuzzy set idempotents from singletons from elements in SG \ S.

However we have special pseudo fuzzy special idempotents and special pseudo fuzzy inverses and units.

They will be illustrated in the following examples.

Example 3.57: Let PA_4 be the special fuzzy set group semiring of the group A_4 over the special fuzzy set semiring P = {[0,1), min, max}.

Let

$$A = 0.3 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} +$$

$$0.2\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} + 0.8\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix} \text{ and}$$

$$B = 0.7 + 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} \in PA_4.$$

$$\max(A, B) = \{0.7 + 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} +$$

$$0.7\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} + 0.8\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix}\}$$

$$\min(A, B) = \{0.3 + 0.7\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} +$$

$$0.2\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} + 0.7\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix} + 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}$$

$$+ 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} +$$

$$0.2\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} +$$

$$0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}\}$$

$$= 0.3 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} + 0.2 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} +$$

$$0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} + 0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} + 0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$+ 0.2 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} + 0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

$$= (0.3 + 0.2) + (0.7) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} +$$

$$0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} + 0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} +$$

$$0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$0.3 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} +0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} +$$

$$0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + 0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} +$$

$$0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \in PA_4.$$

We see min $\{A, B\} \neq 0$ unless $A = 0$ or $B = 0$ likewise max $\{A, B\} \neq 0$ whatever be A and B even one them is not zero.

Thus PA_4 has no zero divisors.

$$\text{Let } x = 0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \text{ and}$$

$$y = 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \in PA_4.$$

$$\min(x, y) = 0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$= 0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$= 0.3 \in P.$$

Thus x is the special fuzzy pseudo inverse of y in SA_4.

$$\max(x, y) = 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

$$\text{Let } x = 0.8 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \text{ and}$$

$$y = 0.1 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \in PA_4.$$

Consider

$$\min(x, y) = 0.1 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

$$= 0.1 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$= 0.1 \in P.$$

Thus x is the special pseudo fuzzy inverse of y in PA_4 and vice versa.

$$\max \{x, y\} = 0.8 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} + 0.1 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}.$$

$$\text{Let } x = 0.3 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \in PA_4.$$

$$\max (x, x) = \{0.3 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

$$0.3 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\}$$

$$= \{0.3 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\}$$

$$= 0.3 + 0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$= 0.3 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \text{ as max is '+'}$$

max (x, x) = x so we can also remove the term special fuzzy pseudo idempotents instead say idempotents

$$\min \{x, x\} = \min \{0.3 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

$$0.3 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\}$$

$$= 0.3 + 0.3 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\}$$

$$\neq x.$$

Let

$$x = 0.5 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} + 0.6 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \in PA_4$$

$$\min (x, x) = \{0.5 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}^2 + 0.6 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}^2 +$$

$$0.5 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} + 0.5 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} + 0.6 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} + 0.5 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} + 0.5 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

$$= 0.6 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} + 0.6 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}.$$

So min (x, x) gives a special pseudo fuzzy set idempotent.

max (x, x)

$$= 0.5 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} + 0.6 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} +$$

$$0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}^2 +$$

$$0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} + 0.6 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} +$$

$$0.6 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}^2$$

$$= 0.7 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

Thus x is only a special pseudo fuzzy idempotent of PA_4.

This is the marked difference between the special fuzzy set groupring and special fuzzy set group semiring as the later can have both idempotents and pseudo special fuzzy idempotents.

Consider

$$x = 0.7 + 0.5 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \in PA_4$$

$$\min\{x, x\} = 0.7 + 0.5 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}^2 +$$

$$0.5 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$= 0.7 + 0.5 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$= 0.7 + 0.5 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

$$= x.$$

Thus x under min is the usual idempotent of PA_4

$$\max\{x, x\} = \{0.7 + 0.5 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}^2 +$$

$$0.7\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}\}$$

$$= 0.7 + 0.7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

Clearly x is not the usual idempotent only a special fuzzy pseudo idempotent of PA_4.

Let $x = 0.9 + 0.9 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \in PA_4$

$$\max\{x, x\} = 0.9 + 0.9 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} +$$

$$0.9 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}^2$$

$$= 0.9 + 0.9 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = x.$$

Thus under the operations of max, x is an idempotent of PA_4.

$$\min\{x, x\} = 0.9 + 0.9\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} + 0.9\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}^2$$

$$= 0.9 + 0.9\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} = x.$$

Thus x under the min operation is also an idempotent.

It is pertinent to keep on record $x \in PA_4$ may be a usual idempotent under one of the operations min or max and the same x may be a special pseudo fuzzy idempotent under one of the operations max or min.

$$\text{Let } x = 0.4 + 0.8\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} \in PA_4;$$

$$\min\{x, x\} = 0.4 + 0.4\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} + 0.8\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix}^2$$

$$= 0.4 + 0.4\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} + 0.8$$

$$\left(\because \begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix}^2 = \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix} = e\right)$$

$$= 0.8 + 0.4\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} \neq x;$$

so x of PA_4 is only a special fuzzy pseudo idempotent of PA_4.

$$\max\{x, x\} = 0.4 + 0.8\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} + 0.8\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix}$$

$$+ 0.8\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix}$$

$$= 0.8 + 0.8\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} \neq x$$

so x of PA_4 is not a usual idempotent under max and only a special pseudo fuzzy idempotent of PA_4.

$$\text{Let } x = 0.3 + 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} + 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} +$$

$$0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} \in PA_4;$$

$$\max\{x, x\} = 0.3 + 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} + 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} +$$

$$0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} + 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}^2 + 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix}^2$$

$$+ 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}^2 + 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} +$$

$$0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} +$$

$$0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} +$$

$$0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} + 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} + 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} +$$

$$0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} +$$

$$0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} +$$

$$0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}$$

$$= 0.3 + 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} + 0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} +$$

$$0.3\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} = x.$$

Thus max $\{x, x\} = x$ is the usual idempotent of PA_4.

min $\{x, x\} = x$. (easily verified).

Thus min $\{x, x\}$ also gives only a usual idempotent.

In view of this we propose the conditions under which the special fuzzy set semigroup rings can have usual idempotents and special pseudo fuzzy idempotents.

Example 3.58: Let PG be the special fuzzy set pseudo group semiring of the group $G = \{g \mid g^6 = 1\}$ and $P = \{[0, 1), +, \times\}$ be the special fuzzy set pseudo semiring.

Take $x = 0.3 + 0.3g^3 \in PG$

$$\min \{x, x\} = 0.3 + 0.3g^3 + 0.3g^3 + 0.3g^3g^3$$
$$= 0.3 + 0.3g^3 \text{ (as } g^3g^3 = 1 \text{ and max is '+')}$$
$$= x.$$

Hence x is the usual idempotent of PG under min. operation.

Consider max {x, x}
$$= 0.3 + 0.3g^3 + 0.3g^3 + 0.3g^3$$
$$= 0.3 + 0.3g^3 = x.$$

Thus even under max; x is the usual idempotent.

Let $x = 0.5 + 0.7g^3 \in PG$.
$$\min \{x, x\} = 0.5 + 0.7(g^3)^2 + 0.5g^3 + 0.5g^3$$
$$= 0.5 + 0.7 + 0.5g^3 \quad (\text{as } g^6 = (g^3)^2 = 1)$$
$$= 0.7 + 0.5g^3 \neq x.$$

So x is only a special fuzzy pseudo idempotent of PG.

Now consider
$$\max \{x, x\} = 0.5 + 0.7g^3 + 0.7g^3 + 0.7g^3$$
$$= 0.5 + 0.7g^3 = x.$$
So x is the usual idempotent of x.

Let $x = 0.9 + 0.2g^3 \in PG$

$$\text{Now } \max \{x, x\} = 0.9 + 0.9g^3 + 0.9g^3 + 0.2g^3$$
$$= 0.9 + 0.9g^3.$$

So x is not the usual idempotent only a special pseudo fuzzy idempotent of PG.

$\min \{x, x\} = 0.9 + 0.2g^3 + 0.2g^3 + 0.2(g^3)^2 = 0.2 + 0.2g^3$.
So x is not the usual idempotent of PG only a special pseudo fuzzy idempotent of PG.

Let $y = 0.4 + 0.4g^2 + 0.4g^4 \in PG$.

We now find

$$\begin{aligned}\min \{y, y\} &= 0.4 + 0.4g^4 + 0.4g^2 + 0.4g^2 + 0.4g^4 + 0.4g^2g^4 \\ &\quad + 0.4g^2g^4 + 0.4g^2 + 0.4g^4 \\ &= 0.4 + 0.4g^2 + 0.4g^4 \\ &= x \in PG.\end{aligned}$$

Thus x is the usual idempotent of PG and not a special fuzzy pseudo idempotent of PG.

$$\begin{aligned}\max \{y, y\} &= 0.4 + 0.4g^2 + 0.4g^4 + 0.4g^2 + 0.4g^4 + \\ &\quad 0.4 + 0.4g^2 + 0.4g^4 + 0.4 \\ &= 0.4 + 0.4g^2 + 0.4g^4\end{aligned}$$

(as max is + and max $\{0.4, 0.4g\} = 0.4 + 0.4g$
max $\{0.2g, 0.4g\} = 0.4g$ and so on.

However min $\{0.2g, 0.4g\} = 0.2g^2$ and
min$\{0.4, 0.2g\} = \{0.2g\}$.

In view of all these we have the following theorem.

THEOREM 3.19: *Let PG be the special fuzzy set group semiring of the group G over the special fuzzy set semiring P = {[0, 1), +, ×}. If G has finite subgroup $H = \{e = 1, h_1, h_2, \ldots, h_{t-1}\}$; $o(H) = t$ then*

(i) *$x = a + ah_1 + \ldots + ah_{t-1}$, $a \in [0, 1)$ is a usual idempotent under both max and min.*

(ii) *If $x = a_0 + a_1h_1 + a_2h_2 + \ldots + a_{t-1} h_{t-1}$, $a_i \in [0, 1)$ and a_i's are distinct $0 \leq i \leq t-1$, then in general x is only a special pseudo fuzzy idempotent under both max and min.*

Proof is direct and hence left as an exercise to the reader.

However it is important at this juncture to keep on record that PG where G is a group and P = {[0, 1), min, max}; the special fuzzy set semiring and PG the special fuzzy set group semiring has no zero divisors.

Now PG has special pseudo fuzzy units and inverses.

This will be illustrated by an example or two.

Example 3.59: Let PG be the special fuzzy set group semiring of the group G = $D_{2,7}$ over the semiring P = {[0, 1), min, max} where $D_{2,7} = \{a, b \mid a^2 = b^7 = 1 \text{ bab} = a\}$.

Let $x = 0.7b^5$ and $y = 0.2b^2 \in$ PG we see min {x, y} = 0.2 so x is the special fuzzy pseudo inverse of y and vice versa.

$$\max \{x, y\} = 0.7b^5 + 0.2b^2$$

Now let x = 0.9a $\in$ PG, x is a unit for min {0.9a, 0.9a} = 0.9 so x is a special pseudo fuzzy unit of PG.

Take $0.7b^2$ = x in PG the special pseudo fuzzy inverse of x is $y = 0.tb^5$ $0.t \in [0, 1)$, as $\min \{x, y\} = \{0.7b^2, 0.tb^5\} = 0.7$ if $0.7 < t$ and min {x, y} = 0.t if 0.7 > t.

However $0.t \neq 0$.

Example 3.60: Let PG be the special fuzzy set group semiring of the group G = S_3 over the special fuzzy set semiring P = {[0, 1), min, max}.

Take $x = 0.3 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \in$ PG; the special fuzzy pseudo inverse of x is $y = 0.7 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ hence

$$\min \{x, y\} = \min \left\{0.3\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, 0.7 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right\}$$

$$= 0.3 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$= 0.3 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = 0.3.$$

Hence the claim.

Consider $x = 0.8 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \in PG$; we see

$$\min \{x, x\} = \min \left\{0.8 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, 0.8 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\right\}$$

$$= 0.8 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$= 0.8 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = 0.8 \in P.$$

Thus x is the special fuzzy pseudo unit of PG.

Next we proceed onto describe special fuzzy set semigroup semirings and their properties.

Example 3.61: Let PS be the special fuzzy set semigroup semiring of the semigroup $S = \{Z_{12}, \times\}$ and $P = \{[0, 1), \min, \max\}$ be the special fuzzy set semiring. We see PS has zero divisors.

Take $x = 0.7(6)$ and $y = 0.8\,(4) \in PS$.

$\min \{x, y\} = 0.7\,(6 \times 4) = 0.7 \times 0 = 0$ as $(6 \times 4 \equiv 0 \bmod 12)$. Hence x is a zero divisor in PS.

PS has infinitely many zero divisors.

Take x = (0.1)3 and y = (0.7)4 ∈ PS we see min (x, y) = 0. Thus x is a zero divisor in PS. Let y = (0.7)4 ∈ PG, y is an idempotent for min {(0. 7)4, (0.7)4} = (0.7) (4) as 4 × 4 ≡ 4 (mod 12).

Now max {(0.7)4, (0.7)4} = (0.7)4 is again an idempotent of PS.

PS has idempotents which are not in general special fuzzy pseudo idempotents of PS.

Take x = (0.3)4 + (0.7)8 + 0.4 ∈ PS.

We find max {x, x} = x but
min {x, x} = (0.3)4 + (0.7)4 + 0.4 + (0.3)4 + (0.4)8 + (0.3)4 + (0.4)8 + (0.3)8 + (0.3)8

= 0.4 + (0.7)4 + (0.3)8 so under min; x is a special pseudo fuzzy idempotent but under max; x is an idempotent.

Thus PS has zero divisors, idempotents and special fuzzy pseudo idempotents.

So under min, x is a special pseudo fuzzy idempotent but under max; x is an idempotent. Thus PS has zero divisors, idempotents and special fuzzy pseudo idempotents.

Further take x = (0.7)11 ∈ PS; min {x, x} = 0.7.1 ($11^2 \equiv 1$ mod 12) = 0.7 ∈ P so x is a special pseudo fuzzy unit of SP. But max {x, x} = x.

Take x = (0.1251)5 ∈ PS, min {x, x} = 0.1251 ∈ P so x is again a special pseudo fuzzy unit of SP.

Take x = (0.3)5 and y = (0.721)3 ∈ SP;
min {x, y} = {(0.3) (5 × 3)}
= (0.3) 3
and max {x, y} = (0.3)5 + (0.721)3

Example 3.62: Let PS be the special fuzzy set semigroup semiring of the semigroup S = $\{Z_{15}, \times\}$ and P = {[0, 1), min, max} the special fuzzy set semiring. PS has zero divisors, idempotents, special pseudo fuzzy inverse, special fuzzy pseudo units and idempotents.

Let $x = (0.7)\ 10 \in PS$;

min $\{x, x\} = (0.7)\ (10^2) = (0.7)10$ (as $10^2 \equiv 10 \pmod{15}$)
= x thus x is an idempotent in PS under min.

max $\{x, x\} = (0.7)10 = x$, then x is also an idempotent under max operation.

Let $x = 0.3 + (0.7)5 + (.2)10 \in PS$.

We find max $\{x, x\} = x$ and min $\{x, x\} = 0.3 + (0.7)5^2 + (0.2)10^2 + (0.3)5 + (0.2)10 + (0.2)10 + (0.2)10 + (0.3)5 + (0.2)10$

$= 0.3 + (0.7)10 + 0.2\ 10 + (0.3)5 +(0.2)10$
$= 0.3 + (0.7)\ 10 + (0.3)5$
$\neq x$ (as + is max operation on PS).

So under min; x is only a special fuzzy pseudo idempotent of PS.

Let $y = \{(0.112)3\}$ and $x = (0.792)5 \in PS$ we see
max $\{x, y\} = (0.112)3 + (0.792)5$ and

min $\{x, y\} = (0.112)\ 3.5$
$= 0$ (as $3.5 \equiv 0 \pmod{15}$).

Hence x is a zero divisor in PS, infact we have infinite number of $y = t.3$ ($t \in [0, 0.3)$; $t \neq 0$) such that min $\{x, y\} = 0$.

Let $x = (0.4107)4 \in PS$ we find min $\{x, x\} = 0.4107 \in P$, so x is a special pseudo fuzzy unit of PS.

Let x = (0.2)6 ∈ PS. We see min {x, x} = (0.2)6 as
$6^2 = 6 \pmod{15}$ is an idempotent of PS.

Thus PS has idempotents, special pseudo fuzzy idempotents, zero divisors, special fuzzy pseudo units and special fuzzy pseudo inverses.

However we have special fuzzy set semigroup rings which has all the above special elements.

Example 3.63: Let PS be the special fuzzy set semigroup semiring where P = {[0, 1), min, max} be the special fuzzy set semiring and S = $\{(a_1, a_2, a_3, a_4) \mid a_i \in Z_{20}, 1 \leq i \leq 4, \times\}$ be the semigroup.

PS has idempotents, zero divisors, special fuzzy pseudo units, inverses and idempotents.

Example 3.64: Let PS be the special fuzzy set semigroup semiring where P = {[0, 1), min, max} and

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in Z_{24}; 1 \leq i \leq 9 \right.$$

under the natural product $\times_n$} be the semigroup.

Consider

$$x = 0.072 \begin{bmatrix} 3 & 0 & 2 \\ 1 & 4 & 5 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } y = 0.043 \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 0 \\ 6 & 8 & 6 \end{bmatrix} \in PS.$$

$$\min (x, y) = 0.043 \left(\begin{bmatrix} 3 & 0 & 2 \\ 1 & 4 & 5 \\ 0 & 0 & 4 \end{bmatrix} \times_n \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 0 \\ 6 & 8 & 6 \end{bmatrix} \right)$$

$$= 0.043 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (0).$$

So PS has zero divisors.

$$\text{Let } x = 0.215 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \in \text{PS}.$$

We see min {x, x} = x so x is an idempotent of PS.

$$\text{Let } x = 0.312 \begin{bmatrix} 5 & 7 & 1 \\ 23 & 11 & 11 \\ 11 & 7 & 5 \end{bmatrix} \in \text{PS}.$$

We see

$$\min \{x, x\} = 0.312 \begin{bmatrix} 5 & 7 & 1 \\ 23 & 11 & 11 \\ 11 & 7 & 5 \end{bmatrix} \times_n \begin{bmatrix} 5 & 7 & 1 \\ 23 & 11 & 11 \\ 11 & 7 & 5 \end{bmatrix}$$

$$= 0.312 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(as $5^2 \equiv 1 \pmod{24}$
$7^2 \equiv 1 \pmod{24}$
$11^2 \equiv 1 \pmod{24}$
$23^2 \equiv 1 \pmod{24}$).

So PS has special pseudo fuzzy units. Likewise S has also special pseudo fuzzy inverses.

Example 3.65: Let PS be the special fuzzy set semigroup ring where P = {[0, 1), min, max} be the special fuzzy set semiring and

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \middle| \; a_i \in Z_{10}; 1 \leq i \leq 7, \times_n \right\}$$

be the semigroup. We see PS has zero divisors and idempotents.

However PS has also special fuzzy set idempotents, units and inverses.

$$\text{Let } x = 0.3 \begin{bmatrix} 5 \\ 0 \\ 1 \\ 6 \\ 0 \\ 6 \\ 5 \end{bmatrix} \in \text{PS}.$$

max {x, x} = x

$$\min \{x, x\} = 0.3 \begin{bmatrix} 5 \\ 0 \\ 1 \\ 6 \\ 0 \\ 6 \\ 5 \end{bmatrix} \times_n \begin{bmatrix} 5 \\ 0 \\ 1 \\ 6 \\ 0 \\ 6 \\ 5 \end{bmatrix}$$

$$= 0.3 \begin{bmatrix} 5 \\ 0 \\ 1 \\ 6 \\ 0 \\ 6 \\ 5 \end{bmatrix}; 5^2 \equiv 5 \pmod{10},\ 6^2 \equiv 5 \pmod{10}$$

Thus x is an idempotent both under max and min.

$$\text{Let } x = 0.2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 0.5 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in PS$$

$$\min\{x, x\} = 0.2\begin{bmatrix}0\\1\\0\\0\\1\\1\\1\end{bmatrix} + 0.5\begin{bmatrix}1\\0\\1\\1\\0\\0\\0\end{bmatrix}.$$

$$\max\{x, x\} = x.$$

Thus x is a usual idempotent.

$$\text{Let } x = 0.75\begin{bmatrix}0\\7\\0\\5\\3\\7\\0\end{bmatrix} \in PS \text{ and } y = 0.21\begin{bmatrix}5\\0\\8\\0\\0\\0\\4\end{bmatrix} \in PS$$

is such that

$$\min\{x, y\} = 0.21\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix} \text{ so x is a zero divisor.}$$

But

$$\max\{x, y\} = 0.75\begin{bmatrix}0\\7\\0\\5\\3\\7\\0\end{bmatrix} + 0.21\begin{bmatrix}5\\0\\8\\0\\0\\0\\4\end{bmatrix}.$$

We can also find other special pseudo fuzzy idempotents and special pseudo fuzzy units.

$$\text{Take } x = 0.034\begin{bmatrix}1\\9\\1\\9\\9\\1\\9\end{bmatrix} \in PS;$$

$$\text{we see } \min\{x, x\} = 0.034\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\end{bmatrix}; \max\{x, x\} = x.$$

Thus x is a special fuzzy pseudo unit of PS under min x.

Example 3.66: Let PS be the special fuzzy set semigroup semiring of the semiring P = {[0, 1), min, max} and

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_7 \\ a_8 & a_9 & a_{10} & \dots & a_{14} \end{bmatrix} \middle| \; a_i \in Z_{20}; 1 \leq i \leq 4, \times_n \right\}$$

be the semigroup under $\times_n$. PS has special fuzzy pseudo units, idempotents and inverses.

This is the way properties are analyzed in PS.

Example 3.67: Let PS be the special fuzzy semigroup semiring of the semiring P = {[0, 1), min, max} and

$S = \{(a_1, a_2, \dots, a_9) \mid a_i \in Z, ; 1 \leq i \leq 9\}$ be the semigroup. PS has special fuzzy pseudo idempotents.

The question of units and inverses is a challenge as the identity of S is (1, 1, 1, 1, 1, 1, 1, 1, 1) = I is not in S as $1 \notin [0, 1)$.

Example 3.68: Let PS be the special fuzzy semigroup semiring where P = {[0, 1), min, max} be the special fuzzy set semiring and

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in Q^+ \cup \{0\}; 1 \leq i \leq 6, \times_n \right\}$$

be the semigroup.

PS has special pseudo fuzzy inverses, no unit but has idempotents in a very special form.

$$\text{Let } x = 0.31 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 0.25 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in \text{PS}.$$

We see max {x, x} = x.

$$\text{Now min } \{x, x\} = 0.31 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 0.25 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = x.$$

Thus x is an idempotent of PS.

$$\text{Let } x = 0.24 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 0.35 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in \text{PS}.$$

$$\min(x, x) = 0.24 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 0.35 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 0.24 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Now we are in a problem.
min (x, x) is not an idempotent or special fuzzy pseudo idempotent.

Let min (x, x) = y we now find

$$\min (y, y) = 0.24 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 0.35 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 0.24 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = y$$

is an idempotent we define such x as special pseudo fuzzy second stage idempotent or just to be more appropriate second stage idempotent.
Interested readers can construct second stage idempotents.

Chapter Four

SUGGESTED PROBLEMS

In this chapter we suggest some problems for the reader. Some problems are easy and others are difficult and can be treated as research problems.

1. Obtain some special features enjoyed by special fuzzy set semigroups on [0, 1).

2. Can
$S = \{(a_1, a_2, a_3) \mid a_i \in \{0, 1/3, 1/3^2, \ldots, 1/3^{25}\}, 1 \leq i \leq 3\}$ be a special fuzzy set semigroup under max?

    (i) Find o(S).
    (ii) Find all special fuzzy set subsemigroups of S.
    (iii) Can S have ideals?
    (iv) Find the number of special fuzzy set subsemigroups which are not ideals of S.
    (v) Can S have infinite subsemigroups which are not ideals of S?

3. Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in \{0, 1/2, 1/2^2, \ldots, 1/2^n, n \to \infty, 1 \leq i \leq 5\} \right.$$

be the special fuzzy set semigroup under the natural product $\times_n$.

(i) Find special fuzzy set subsemigroups which are not ideals.
(ii) Prove S has infinite number of zero divisors.
(iii) Find special fuzzy set ideals of S.

4. Let S = {Collection of all 8 × 3 matrices $\begin{bmatrix} a_1 & a_9 & a_{17} \\ a_2 & a_{10} & a_{18} \\ a_3 & a_{11} & a_{19} \\ a_4 & a_{12} & a_{20} \\ a_5 & a_{13} & a_{21} \\ a_6 & a_{14} & a_{22} \\ a_7 & a_{15} & a_{23} \\ a_8 & a_{16} & a_{24} \end{bmatrix}$

where $a_i \in [0,0.6]$, $1 \leq i \leq 24$} be the special fuzzy set semigroup under the natural product $\times_n$.

Study questions (i) to (iii) of problem 3 for this S.

5. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \end{bmatrix} \middle| \; a_i \in [0, 0.5), 1 \leq i \leq 40 \right\}$

under the natural product $\times_n$ be a special fuzzy set semigroup of infinite order.

Study questions (i) to (iii) of problem 3 for this S.

6. Let $S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in [0, 1), 1 \leq i \leq 16 \right\}$

be a special fuzzy set semigroup under the natural product $\times_n$.

Study questions (i) to (iii) of problem 3 for this S.

7. Let $S = \left\{ \left[\begin{array}{c} a_1 \\ \hline a_2 \\ a_3 \\ a_4 \\ a_5 \\ \hline a_6 \\ a_7 \\ \hline a_8 \end{array} \right] \middle| \; a_i \in [0, 0.75), 1 \leq i \leq 8 \right\}$ be the special

fuzzy set super matrix semigroup under natural product $\times_n$.

(i) Study questions (i) to (iii) of problem 3 for this S.

(ii) Can S have a finite subsemigroup?

8. Obtain some special and distinct features enjoyed by special fuzzy set semigroups under $\times$.

9. Compare special fuzzy set semigroups under $\times$ the operations and min.

10. Compare special fuzzy set semigroups under the operations $\times$ and max.

11. Which of the special fuzzy set semigroups enjoys more distinct features?

12. Does there exist a special fuzzy set semigroup without ideals?

13. Let $M = \{(a_1, a_2, \ldots, a_9) \mid a_i \in [0, 1), 1 \leq i \leq 9, \times\}$ be the special fuzzy set semigroup.

(i) Prove M has zero divisors.
(ii) Can M have idempotents?
(iii) Find subsemigroups which are not ideals.
(iv) Find ideals if any in M.
(v) Prove M has subsemigroups A, B with $A \times B = (0\ 0 \ldots 0)$.

14. Let $M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| a_i \in [0, 1),\right.$

$1 \leq i \leq 15\}$ be the special fuzzy set semigroup under $\times_n$.

Study questions (i) to (v) of problem 13 for this M.

15. Let $M_2 = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \\ \vdots & \vdots & & \vdots \\ a_{21} & a_{22} & \ldots & a_{25} \end{bmatrix} \right|$ $a_i \in [0, 1), 1 \leq i \leq 25,$

max} be the special fuzzy set semigroup under natural product $\times_n$.

Study questions (i) to (v) of problem 13 for this M_2.

16. Let N = $\left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in [0, 1)$, min} be the special fuzzy set semigroup.

(i) Study questions (i) to (v) of problem 13 for this N.
(ii) Can N have finite special fuzzy set subsemigroups?
(iii) Prove N cannot have ideals of finite order.

17. Let T = $\left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in [0, 1)$, max} be the special fuzzy set semigroup.

(i) Study questions (i) to (v) of problem 13 for this T.
(ii) Compare N in problem 16 with T in problem 17.

18. Let P = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right|$ $a_i \in [0, 1), 1 \leq i \leq 9\}$ be the special fuzzy set semigroup under $\times_n$.

(i) Study questions (i) to (v) of problem 18 for this P.
(ii) Prove P has infinite number of zero divisors.
(iii) Can P have idempotents?
(iv) Can P have finite subsemigroups?
(v) Find subsemigroups generated by a single element.
(vi) Find subsemigroups generated by 5 elements.

19. Let M = $\left\{ \left[\begin{array}{c|cccc|c} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \hline a_7 & \dots & \dots & \dots & \dots & a_{12} \\ a_{13} & \dots & \dots & \dots & \dots & a_{18} \\ \hline a_{19} & \dots & \dots & \dots & \dots & a_{27} \\ a_{25} & \dots & \dots & \dots & \dots & a_{30} \end{array}\right] \right|$ $a_i \in [0, 1)$, $1 \leq i \leq 30\}$ be the special fuzzy set semigroup under the product $\times_n$.

Study questions (i) to (vi) of problem 18 for this P.

20. If in problem 19 the operation $\times_n$ is replaced by min study M.

21. Study M in problem 19 when $\times_n$ operation is replaced by max.

22. Let M = $\left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ \hline a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ \hline a_{19} & a_{20} & a_{21} \end{array}\right] \right|$ $a_i \in [0, 1)$, $1 \leq i \leq 21\}$ be the special fuzzy set semigroup under $\times_n$.

Study question (i) to (vi) of problem 18 for this S.

23. Let $S_1 = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_9 \\ 0 & a_{10} & \dots & a_{17} \\ 0 & 0 & \dots & a_{24} \\ \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & a_{45} \end{bmatrix} \right| a_i \in [0, 1), 1 \leq i \leq 8\}$ be the special fuzzy set semigroup.

Study question (i) to (vi) of problem 18 for this S.

24. Study for S_1 questions (i) to (vi) of problem 18 if the matrices are just diagonal.

25. Can a matrix special fuzzy set ring have right zero divisors which are not left zero divisors?

26. Can $M = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 1), 1 \leq i \leq 4\}$; the special fuzzy set ring under $(+, \times)$ have infinite number of ideals?

27. Obtain some very special features enjoyed by the special fuzzy set ring.

Characterize those special fuzzy set square matrix rings in which $A \times B = A \times_n B = (0)$.

28. Let $S = \{[0, 1), +\}$ be the group under addition modulo 1.

(i) Study the special features enjoyed by S.
(ii) Can S have finite subgroups?
(iii) Can S have infinite ordered subgroups?

29. Let $M = \{(a_1, a_2, a_3, a_4) \mid a_i \in S = \{[0, 1), +\}\}$ be the group under addition.

Study question (i) to (iii) of problem 28 for this S.

30. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix} \middle| \; a_i \in S = [0, 1), +,\right.$

$1 \leq i \leq 30\}$ be the special fuzzy set group under +.

Study question (i) to (iii) of problem 28 for this S.

31. Let $N = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \middle| \; a_i \in S = \{[0, 1), +\}, \; 1 \leq i \leq 12\right\}$ be the special fuzzy set group under +.

Study question (i) to (iii) of problem 28 for this S.

32. Let $S_1 = \left\{ \sum_{i=0}^{\infty} a_i x^i \middle| \; a_i \in [0, 1), +\right\}$ be a group.

Study question (i) to (iii) of problem 28 for this S.

33. Let $S = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \\ a_{41} & a_{42} & \ldots & a_{50} \end{bmatrix} \middle| \; a_i \in S = [0, 1), +,\right.$

$1 \leq i \leq 50\}$ be the special fuzzy set group under +.

(i) Study question (i) to (iii) of problem 28 for this S.

(ii) Prove S have over n = 50 + $50C_2$ + … + $50C_{49}$ number of subgroups all of which are of infinite order.

(iii) Can S have n such subgroups of finite order?

34. Suppose in problem 33 if we replace + by min (or max or ×).

Study the question (1) to (iii) of problem 33.

35. Define for two special fuzzy set semigroups the notion of semigroup homomorphism.

36. Let $S_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \, a_i \in S = [0, 1), +, \; 1 \leq i \leq 5, \times \right\}$ and

$$S_2 = \left\{ \begin{bmatrix} a_1 & a_2 & ... & a_5 \\ a_6 & a_7 & ... & a_{10} \\ a_{11} & a_{12} & ... & a_{15} \end{bmatrix} \middle| \, a_i \in [0, 1), \times, \; 1 \leq i \leq 15 \right\}$$

be two special fuzzy set semigroups.

Find two homomorphisms ϕ_1 and ϕ_2 from S_1 to S_2 such that ker $\phi_1 \neq$ ker ϕ_2.

37. Let $M_1 = \{(a_1, a_2, \ldots, a_9) \mid a_i \in [0, 1); 1 \leq i \leq 9$ under max operation} and $M_2 = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \middle| \, a_i \in [0, 1), 1 \leq i \leq 12,\right.$ under min operation} be any two special fuzzy set semigroups.

(i) Does there exist a homomorphism $\phi : M_1 \to M_2$ such that $\ker \phi \neq (0)$?

(ii) Find the algebraic structure enjoyed by $\ker \phi$.

38. Obtain some special features enjoyed by semigroup homomorphism of special fuzzy set semigroups under $\times$ or min or max.

39. Let $S_1 = \left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{15} \\ a_{16} & a_{17} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{25} \end{bmatrix} \middle| \, a_i \in S = [0, 1), +,\right.$ $1 \leq i \leq 25\}$ and

$S_2 = \left\{\begin{pmatrix} a_1 & a_2 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{24} \end{pmatrix} \middle| \, a_i \in [0, 1),\right.$

$1 \leq i \leq 24, +\}$ be two special fuzzy set groups.

(i) Find a homomorphism ϕ from S_1 to S_2 so that $\ker \phi \neq (0)$.
(ii) What is the algebraic structure enjoyed by $\ker \phi$?
(iii) Find two homomorphism ϕ_1 and ϕ_2 from S_1 to S_2 so that $\ker \phi_1 \neq \ker \phi_2$.

40. Find some special features enjoyed by group homomorphism of special fuzzy set groups using [0, 1).

41. Let $S_1 = \{(a_1, a_2, a_3) \mid a_i \in [0, 1), 1 \leq i \leq 3, \max, \min\}$ be a special fuzzy set semiring.

(i) Find subsemirings of S_1 of infinite order.
(ii) Find zero divisors in S_1.
(iii) Prove S_1 has infinite number of finite subsemirings.
(iv) Can S_1 has finite ideals?
(v) Find those subsemirings which are ideals of S_1.
(vi) Can S_1 have units?
(vii) Prove every element in S_1 is an idempotent.

42. Let $S_2 = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix}\right| a_i \in [0, 1), 1 \leq i \leq 9, \max, \min\right\}$ be the special fuzzy semiring.

Study questions (i) to (vii) of problem 41 for this S_2.

43. Let $M_2 = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}\right| a_i \in [0, 1), 1 \leq i \leq 12, \max, \min\right\}$ be the special fuzzy set semiring.

Study questions (i) to (vii) of problem 41 for this M_2.

44. Compare special fuzzy set pseudo semirings with operation {min, max} and that of {min, ×}.

45. Let S = $\left\{\left.\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{19} & a_{20} \end{bmatrix}\right| a_i \in [0, 1), 1 \leq i \leq 20\right\}$ be the special fuzzy set semiring.

Study questions (i) to (vii) of problem 41 for this S.

46. Can semirings built using {min, max} operation using [0, 1) be free from zero divisors?

47. Characterize those special fuzzy set semirings which have no ideals.

48. Characterize those special fuzzy set semirings M in which every subsemiring is an ideal!

Does there exist one such?

49. Let S = $\left\{\left.\sum_{i=0}^{\infty} a_i x^i \right| a_i \in [0, 1)\right\}$ be the special fuzzy set polynomial pseudo semiring.

(i) Study questions (i) to (vii) of problem 41 for this S.
(ii) Can $p(x) = 0$; $p(x) \in S$ have solution?
(iii) Can S in this problem have principal ideals?
(iv) Can S in this problem have finite special fuzzy set pseudo subsemirings? Justify your claim.

50. Obtain any of the stricking features about special fuzzy set pseudo polynomial semirings.

51. Let R = {[0, 1), +, ×} be the special fuzzy set pseudo ring.

(i) Obtain some special features associated with R.
(ii) Compare R with special fuzzy set pseudosemirings; S = {[0, 1), min, ×} and S′ = {[0, 1), max, min}.

52. Let R = $\{(a_1, a_2, a_3) \mid a_i \in \{[0, 1), +, \times\}, 1 \leq i \leq 3\}$ be the special fuzzy set pseudo ring.

(i) Find zero divisors of R.
(ii) Prove R can be written as a direct sum.
(iii) Find 3 special fuzzy set pseudo subrings which are not ideals.
(iv) Find 5 ideals of R.
(v) Can R have finite subrings?
(vi) Find subrings of R so that R is not a direct sum.

53. Let R = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle|\, a_i \in \{[0, 1), +, \times; 1 \leq i \leq 10\}\right.$ be the special fuzzy set pseudo ring.

Study questions (i) to (vi) of problem 52 for this R.

54. Let $S = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_7 \\ a_8 & a_9 & \dots & a_{14} \\ a_{15} & a_{16} & \dots & a_{21} \\ a_{22} & a_{23} & \dots & a_{28} \end{bmatrix} \middle| \right.$ $a_i \in \{[0, 1), +, \times\};$

$1 \leq i \leq 28\}$ be the special fuzzy set pseudo ring.

Study questions (i) to (vi) of problem 52 for this R.

55. Let $S = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \middle| \right.$ $a_i \in \{[0, 1), +, \times\};$

$1 \leq i \leq 25\}$ be the special fuzzy set pseudo ring.

Study questions (i) to (vi) of problem 52 for this R.

56. Let $S = \left\{ \sum_{i=0}^{\infty} a_i x^i \middle| \right.$ $a_i \in [0, 1), +, \times\}$ be the special fuzzy set pseudo polynomial ring.

Study questions (i) to (vi) of problem 52 for this R.

57. Let $S = \{RG$ where $R = \{[0, 1), +, \times\}$ and $G = D_{2,12}\}$ be the special fuzzy set pseudo group ring.

(i) Prove S is non commutative.
(ii) Find the special pseudo fuzzy idempotents, units and inverses in S.
(iii) Can S have ideals?
(iv) Is it possible for S to have finite subrings?

58. Let S = {RG where R = {[0, 1), ×, +} and G = S_4} be the special fuzzy set pseudo group ring of G over R.

    Study questions (i) to (iv) of problem 57 for this R.

59. Let
    S = {RG where R = {[0, 1), +, ×} and G = {g | g^{15} = 1}} be the special fuzzy set pseudo group ring of G over R.

    Study questions (i) to (iv) of problem 57 for this S.

60. Let
    S = {RG where R = {[0, 1), +, ×} and G = {$D_{2,7} \times S_5$}} be the special fuzzy set pseudo group ring of G over R.

    Study questions (i) to (iv) of problem 57 for this S.

61. Let S = {RG where R = {[0, 1), +, ×} and G = $D_{2,11}$} be the special fuzzy set pseudo ring of G over R.

    (i) Can S have special fuzzy pseudo idempotents?
    (ii) Can S have idempotents?
    (iii) Can S have zero divisors?
    (iv) Can S have special fuzzy pseudo inverses?
    (v) Can S have special fuzzy pseudo units?
    (vi) Can S have subrings which are not ideals?
    (vii) Find those subrings which are ideals.

62. Let S = {RG where R = {[0, 1), +, ×} and G = S_8} be the special fuzzy set pseudo ring.

    Study questions (i) to (vii) of problem 61 for this S.

63. Find any of the special features enjoyed by special fuzzy set pseudo rings.

64. Let S = {RG where R = {[0, 1), +, ×} and G = S(7)} be the special fuzzy set pseudo semigroup ring.

Study questions (i) to (vii) of problem 61 for this S.

65. Let S = {RG where R = {[0, 1), +, ×} be the special fuzzy set pseudo ring and G = {Z_{12}, ×} be the semigroup} be the special fuzzy set pseudo semigroup ring of the semigroup G over R.

Study questions (i) to (vii) of problem 61 for this S.

66. Let S = {RG where R = {[0, 1), +, ×} be the special fuzzy set ring and G = {(a_1, a_2, a_3) | $a_i \in Z_5$, ×; $1 \leq i \leq 3$}} be the special fuzzy set pseudo semigroup ring of G over R.

Study questions (i) to (vii) of problem 61 for this S.

67. Let S = {RG where R = {[0, 1), +, ×} be the special fuzzy set pseudo ring and G = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \middle| \; a_i \in Z_9;\right.$ $1 \leq i \leq 8$, $\times_n$}} be the special fuzzy set pseudo semigroup ring of G over R.

Study questions (i) to (vii) of problem 61 for this S.

68. Let S = {RG where R = {[0, 1), +, ×} be the special fuzzy set pseudo ring and G = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in Z_{15};\right.$ $1 \leq i \leq 12$}} be the semigroup under natural product $\times_n$} be the special fuzzy set pseudo semigroup ring of G over R.

Study questions (i) to (vii) of problem 61 for this S.

69. Let S = {RG where R = {[0, 1), +, ×} be the special fuzzy set pseudo semigroup ring of the semigroup G over the special fuzzy set pseudo ring R.

Study questions (i) to (vii) of problem 61 for this S.

70. Let S = {RG where G = S(4) × Z_{18}, (Z_{18}, ×) is a semigroup} be the semigroup ring of G over the special fuzzy set pseudo ring R = {[0, 1), ×, +}.

Study questions (i) to (vii) of problem 61 for this S.

71. Let
$S = \{(a_1, a_2, \ldots, a_{10}) \mid a_i \in ([0, 1), \min, \max); 1 \leq i \leq 10\}$
be the special fuzzy set semiring.

(i) Find zero divisors if any in S.
(ii) Find those subsemirings which are not ideals.
(iii) Find a few ideals in S.
(iv) Can S have finite subsemirings?
(v) Can S have finite ideals?
(vi) Find idempotents of S.

72. Let $S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \middle| \; a_i \in \{[0, 1), \min, \max; 1 \leq i \leq 12\} \right.$ be the special fuzzy set semiring.

Study questions (i) to (vi) of problem 71 for this S.

73. Let S = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix}\right|$ $a_i \in$ {[0, 1), min, max}; $1 \leq i \leq 30$} be the special fuzzy set semiring.

Study questions (i) to (vi) of problem 71 for this S.

74. Let S = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{15} \\ a_{16} & a_{17} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{25} \end{bmatrix}\right|$ $a_i \in$ {[0, 1), min, max}; $1 \leq i \leq 25$} be the special fuzzy set semiring.

Study questions (i) to (vi) of problem 71 for this S.

75. Let S = {PG | P = {[0, 1), min, max} G = {S_5} be the group} be the special fuzzy set group semiring.

Study questions (i) to (vi) of problem 71 for this S.

76. Let S = {RG / R = {[0, 1) min, max} and G = {$D_{2,9}$}} be the special fuzzy set group semiring.

Study questions (i) to (vi) of problem 71 for this S.

77. Let S = {RG / R = {[0, 1) min, max} and G = $S_4 \times D_{2,7}$} be the special fuzzy set group semiring.

Study questions (i) to (vi) of problem 71 for this S.

78. Let S = $\left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \end{bmatrix} \right|$ $a_i \in$ {[0, 1), min, max};

$1 \leq i \leq 40$} be the special fuzzy set semiring.

Study questions (i) to (vi) of problem 71 for this S.

79. Let S = {RG / R = {[0, 1), min, max}, G = S(5) be the semigroup} be the special fuzzy set semigroup semiring of the symmetric semigroup S(5) over the semiring R.

Study questions (i) to (vi) of problem 71 for this S.

80. Let S = {RG where R = {[0, 1), min, max} and G = (Z_{20}, $\times$), be the semigroup} be the special fuzzy set semigroup semiring of the semigroup G over the semiring R.

Study questions (i) to (vi) of problem 71 for this S.

81. Let S = {RG where R = {[0, 1), min, max} and G = {Z_{16}, $\times$} be the semigroup} be the special fuzzy set semigroup semiring of G over R.

(i) Does S contain special pseudo fuzzy set idempotents?
(ii) Can S have usual idempotents?
(iii) Can S have special pseudo fuzzy units?
(iv) Can S have special pseudo fuzzy inverses?
(v) Can S have zero divisors?
(vi) Can S have filters which are not ideals?
(vii) Can S have ideals which are not filters?
(viii) Can S have subsemirings which are not filters and ideals?

82. Let S = {RG be the special fuzzy set semigroup semiring where G = $\{Z_{30}, \times\}$ be the semigroup and R = {[0, 1), max, min} be the semiring.

Study questions (i) to (viii) of problem 81 for this S.

83. If G is replaced by $\{Z_{19}, \times\}$.

Study questions (i) to (viii) of problem 81 for this S.

84. Let S = {RG / R = {[0, 1), min, max} and G = $\{Z_{15}, \times\} \times \{C(Z_6), \times\} \times \{\langle Z_{12} \cup I\rangle\}$} be the semiring.

Study questions (i) to (viii) of problem 81 for this S.

85. Let S = {RG / R = {[0, 1), min, max} and G = $S_3 \times D_{2,7} \times (Z_{18}, \times)$ be the semigroup} be the special fuzzy set semigroup semiring of the group G over the ring R.

Study questions (i) to (viii) of problem 81 for this S.

FURTHER READING

1. Birkhoff. G, *Lattice theory*, 2nd Edition, Amer-Math Soc. Providence RI 1948.
2. Castillo J., *The Smarandache semigroup*, International Conference on Combinatorial Methods in Mathematics, II meeting of the project 'Algebra, Geometria e Combinatoria', Faculdade de Ciencias da Universidade do Porto, Portugal, 9-11 July 1998.
3. Padilla, Raul, *Smarandache algebraic structures*, Bull. Pure Appl. Sci., **17E**, 199-121 (1998).
4. Vasantha Kandasamy, W. B., *A note on the modular group ring of a finite p-group*, Kyungpook Math. J., **26**, 163-166 (1986).
5. Vasantha Kandasamy, W.B., *Smarandache Rings*, American Research Press, Rehoboth, 2002.
6. Vasantha Kandasamy, W.B., *Smarandache Semigroups*, American Research Press, Rehoboth, 2002.
7. Vasantha Kandasamy, W.B., *Smarandache Semirings, Semifields and Semivector spaces*, American Research Press, Rehoboth, 2002.
8. Vasantha Kandasamy, W.B. and Florentin Smarandache, *Algebraic Structures using [0, n)*, Educational Publisher Inc, Ohio, 2013.

INDEX

ABOUT THE AUTHORS

Dr.W.B.Vasantha Kandasamy is a Professor in the Department of Mathematics, Indian Institute of Technology Madras, Chennai. In the past decade she has guided 13 Ph.D. scholars in the different fields of non-associative algebras, algebraic coding theory, transportation theory, fuzzy groups, and applications of fuzzy theory of the problems faced in chemical industries and cement industries. She has to her credit 646 research papers. She has guided over 100 M.Sc. and M.Tech. projects. She has worked in collaboration projects with the Indian Space Research Organization and with the Tamil Nadu State AIDS Control Society. She is presently working on a research project funded by the Board of Research in Nuclear Sciences, Government of India. This is her 92nd book.

On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

She can be contacted at vasanthakandasamy@gmail.com
Web Site: http://mat.iitm.ac.in/home/wbv/public_html/
or http://www.vasantha.in

Dr. Florentin Smarandache is a Professor of Mathematics at the University of New Mexico in USA. He published over 75 books and 200 articles and notes in mathematics, physics, philosophy, psychology, rebus, literature. In mathematics his research is in number theory, non-Euclidean geometry, synthetic geometry, algebraic structures, statistics, neutrosophic logic and set (generalizations of fuzzy logic and set respectively), neutrosophic probability (generalization of classical and imprecise probability). Also, small contributions to nuclear and particle physics, information fusion, neutrosophy (a generalization of dialectics), law of sensations and stimuli, etc. He got the 2010 Telesio-Galilei Academy of Science Gold Medal, Adjunct Professor (equivalent to Doctor Honoris Causa) of Beijing Jiaotong University in 2011, and 2011 Romanian Academy Award for Technical Science (the highest in the country). Dr. W. B. Vasantha Kandasamy and Dr. Florentin Smarandache got the 2012 New Mexico-Arizona and 2011 New Mexico Book Award for Algebraic Structures. He can be contacted at smarand@unm.edu

Authors in this book have introduced the new algebraic structures using the semi-open fuzzy interval [0, 1). Such study is innovative and interesting. Several open conjectures are given. Fuzzy semirings using min-max operations of infinite order are introduced and studied.

US $35.00
Educational Publisher Inc.

ISBN 9781599732596
90000 >
9 781599 732596