Extreme Values of the Sequence of Independent and Identically Distributed random variables with Mixed Asymmetric Distributions*

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Abstract: In this paper, we derive the extreme value distributions of independent identically distributed random variables with mixed distributions of two and finite components, which include generalized logistic, asymmetric Laplace and asymmetric normal distributions.

Keywords: Asymmetric Laplace distribution; Asymmetric normal distribution; extreme value distribution; Generalized logistic distribution; mixed distribution.

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1 Introduction

Let $\{Y_n,\ n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with common distribution function $F(x)$, let $M_n = max\{Y_1, \cdots, Y_n\}$ denote the partial maximum. If there exist normalizing constants $a_n > 0,\ b_n \in R$ and non-degenerate distribution $G(x)$ such that

$$\lim_{n\to\infty} P(M_n \leq a_n x + b_n) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x) \tag{1.1}$$

for all continuity points of G, then G must belong to one of the following three classes:

$$H_{1,\ \alpha}(x) = \begin{cases} 0, & x < 0, \\ \exp\{-x^{-\alpha}\}, & x \geq 0, \end{cases}$$

$$H_{2,\ \alpha}(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & x < 0, \\ 1, & x \geq 0, \end{cases}$$

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for some $\alpha > 0$ and

$$H_{3,\ 0}(x) = \exp\{-e^{-x}\},\ x \in R.$$

If (1.1) holds, we say that F belongs to one of the max domain of attraction of G, denoted by $F \in D_{\max}(G)$. Criteria for $F \in D_{\max}(G)$ and the choice of normalizing constants, a_n and b_n, can be found in de Haan (1970), Galambos (1987), Leadbetter et al. (1983)and Resnick (1987).

Similarly, let $W_n = \min\{Y_1, \cdots, Y_n\}$ denote the partial minimum, there must exist normalizing constants $c_n > 0,\ d_n \in R$ and non-degenerate distribution $L(x)$ such that

$$\lim_{n\to\infty} P(W_n \le c_n x + d_n) = \lim_{n\to\infty}\{1 - [1 - F(c_n x + d_n)]^n\} = L(x) \tag{1.2}$$

for all continuity points of $L(x)$, then L must belong to one of the following three classes:

$$L_1(x) = \begin{cases} 1 - \exp\{-(-x)^{-\alpha}\}, & x < 0, \\ 1, & x \ge 0. \end{cases}$$

$$L_2(x) = \begin{cases} 0, & x < 0, \\ 1 - \exp\{-x^{\alpha}\}, & x \ge 0, \end{cases}$$

for some $\alpha > 0$ and

$$L_3(x) = 1 - \exp\{-e^{x}\},\ x \in R.$$

If (1.2) holds, we say that F belongs to one of the min domain of attraction of L, denoted by $F \in D_{\min}(L)$. Criteria for $F \in D_{\min}(L)$ and the choice of normalizing constants, c_n and d_n, can be found in Galambos (1987).

Finite mixtures of distributions have provided a mathematical-based approach to the statistical modeling of a wide variety of random phenomena (Yang and Ahujan (1998), Roederk (1994), and Lindsay (1995)). Meanwhile, some interesting problems, such as the choice of the distributions of mixed components and the number of components, estimation of the related parameters and hypotheses related to mixture distributions, are still incomplete (Figueiredo (2002), Nobile (1994), and Venturini et al. (2008)).

Finite mixture distribution is defined as follows. Let $X_1, X_2, ..., X_k$ be independent random variables and each with the distribution function $X_i \sim F_i(x),\ i = 1, 2, ..., k$. Define a new random variable Z by

$$Z = \begin{cases} X_1, & \text{with probability } p_1, \\ X_2, & \text{with probability } p_2, \\ ..., & ..., \\ X_k, & \text{with probability } p_k, \end{cases}$$

where $p_i \ge 0\ for\ 1 \le i \le k$ and $\sum_{i=1}^{k} p_i = 1$. It is easy to check that the distribution function of Z is given by

$$F(x) = p_1 F_1(x) + p_2 F_2(x) + \cdots + p_k F_k(x). \tag{1.3}$$

In particular, when $k = 2$,

$$F(x) = pF_1(x) + qF_2(x), \tag{1.4}$$

where $p + q = 1$.

It is interesting to consider the limiting distributions of maxima of i.i.d. random variables with common mixture distributions defined by (1.3) or (1.4). Mladenović gave the results for some mixed distributions of two components. The results of Mladenović (1999) show that the limiting distributions of maxima of i.i.d. random variables from finite mixture distributions may be one of the extreme value distributions. In this note, we study some asymmetric distributions to derive extreme value distributions of i.i.d. random variables with mixed distributions of two and finite components.

This paper is organized as follows: the definition of asymmetric distributions and some lemmas are given in Section 2. Main results are given in Section 3. Their proofs are deferred to Section 4.

2 Preliminaries

In order to derive the extreme value distribution of mixed asymmetric distributions, we firstly give the following definitions.

Definition 2.1. *Let X be a random variable having the generalized logistic distribution, written $X \sim F(x)$. The pdf and the cdf of $X \sim F(x)$ are given by*

$$f(t) = \frac{b\exp(-x/\sigma)}{\sigma\{1 + \exp(-x/\sigma)\}^{1+b}}$$

and

$$F(x) = \{1 + \exp(-x/\sigma)\}^{-b},$$

where $b/\sigma > 0$.

Definition 2.2. *Let X be a random variable having the asymmetric Laplace distribution, written $X \sim F(x)$. The characteristic function of $X \sim F(x)$ is given by*

$$\varphi(t) = \frac{1}{1 + \sigma^2 t^2 - i\mu t}, \ \sigma > 0, \ \ -\infty < \mu < \infty,$$

the characteristic function of difference of two independent exponential random variables, where $i = \sqrt{-1}$. So, the pdf and the cdf of $X \sim F(x)$ are:

$$f(x) = \frac{\kappa}{\sigma(1+\kappa^2)} \begin{cases} \exp(-\frac{\kappa x}{\sigma}), & x \geq 0, \\ \exp(\frac{x}{\sigma\kappa}), & x < 0. \end{cases}$$

and

$$F(x) = \begin{cases} 1 - \frac{1}{1+\kappa^2}\exp(-\frac{\kappa x}{\sigma}), & x \geq 0, \\ \frac{\kappa^2}{1+\kappa^2}\exp(\frac{x}{\sigma\kappa}), & x < 0, \end{cases}$$

where $\kappa = (2\sigma)/(\mu + (4\sigma^2 + \mu^2)^{1/2})$ and $\kappa/\sigma > 0$ (see Kozubowski and Podgorski (1999)).

Definition 2.3. *Let X be a random variable having the asymmetric normal distribution, written $X \sim F(x)$. The pdf of X is given by*

$$f(x) = \frac{2}{(2\pi)^{1/2}(\sigma_l + \sigma_r)} \begin{cases} \exp\{-\frac{(x-\theta)^2}{2\sigma_l^2}\}, & x \le \theta, \\ \exp\{-\frac{(x-\theta)^2}{2\sigma_r^2}\}, & x > \theta, \end{cases}$$

where $\sigma_l > 0, \sigma_r > 0$ and $\theta \in R$.

In order to prove our main results, we need the following lemmas which follow from Theorem 2.1.3 and Theorem 2.1.6 in Galambos (1987).

Lemma 2.1. *Let $\{X_n,\ n \ge 1\}$ be an i.i.d. random variable sequence with common distribution function $F(x)$, denote $x_0 = \sup\{y : F(y) < 1\}$ and $x_l = \inf\{y : F(y) > 0\}$ the right and left end point of $F(x)$. Assume that, for some finite a, $\int_a^{x_0}(1 - F(y))\,\mathrm{d}y < +\infty$. For $x_l < t < x_0$, define $g(t) = (1 - F(t))^{-1}\int_t^{x_0}(1 - F(y))\,\mathrm{d}y$. Assume that, for all real x,*

$$\lim_{t \uparrow x_0} \frac{1 - F(t + xg(t))}{1 - F(t)} = e^{-x}.$$

Then there exist $a_n > 0$ and b_n such that,

$$\lim_{n \to \infty} P(M_n \le a_n x + b_n) = \exp(-e^{-x}),$$

where the normalizing constants a_n and b_n can be chosen as $a_n = \inf\{x : 1 - F(x) \le n^{-1}\}$ and $b_n = g(a_n)$.

Lemma 2.2. *Assume that, for some finite a, $\int_{x_l}^{a} F(y)\,\mathrm{d}y < \infty$. For $t > x_l$, define $g(t) = F^{-1}(t)\int_{x_l}^{t} F(y)\,\mathrm{d}y$. Assume that, for all real x,*

$$\lim_{t \downarrow x_l} \frac{F(t + xg(t))}{F(t)} = e^{x}.$$

Then there are sequences $c_n > 0$ and d_n such that,

$$\lim_{n \to \infty} P(W_n \le c_n x + d_n) = 1 - \exp(-e^{x}),$$

where the normalizing constants c_n and d_n can be chosen as $c_n = \sup\{x : F(x) \le 1/n\}$ and $d_n = g(c_n)$.

The following lemma gives Mills-type ratios for the asymmetric normal distribution.

Lemma 2.3. *Let X be a continuous random variable with the asymmetric normal distribution whose culumative distribution function is $F(x)$ and probability density function is $f(x)$ (see Definition 2.3.). For any $\sigma_l > 0,\ \sigma_r > 0$ and $\theta \in R$, we have:*

$$\frac{F(x)}{f(x)} \sim -\frac{\sigma_l^2}{x}$$

as $x \to -\infty$ and

$$\frac{1-F(x)}{f(x)} \sim \frac{\sigma_r^2}{x}$$

as $x \to +\infty$.

Proof. The proofs of the two statements are similar. We only prove the first. For any $\sigma_l > 0,\ \sigma_r > 0,\ x \leq \theta$, the cdf of asymmetric normal distribution is

$$F(x) = \frac{2\sigma_l}{(2\pi)^{1/2}(\sigma_l+\sigma_r)} \int_{(\theta-x)/\sigma_l}^{+\infty} \exp(-\frac{t^2}{2})\,\mathrm{d}t$$

Since

$$x^{-1}(1+x^{-2})^{-1}(2\pi)^{-1/2}e^{-x^2/2} < 1 - \int_{-\infty}^{x} (2\pi)^{-1/2}\exp(-\frac{t^2}{2})\,\mathrm{d}t < x^{-1}(2\pi)^{-1/2}e^{-x^2/2}$$

for all $x > 0$, we have

$$\frac{\sigma_l}{\theta-x}\{1+(\frac{\sigma_l}{\theta-x})^2\}^{-1}\exp\{-\frac{1}{2}(\frac{\theta-x}{\sigma_l})^2\} < \int_{(\theta-x)/\sigma_l}^{\infty} \exp(-\frac{t^2}{2})\,\mathrm{d}t < \frac{\sigma_l}{\theta-x}\exp\{-\frac{1}{2}(\frac{\theta-x}{\sigma_l})^2\}.$$

Therefore

$$\frac{\sigma_l^2}{\theta-x}\{1+(\frac{\sigma_l}{\theta-x})^2\}^{-1} < \frac{F(x)}{f(x)} < \frac{\sigma_l^2}{\theta-x}.$$

We obtain $F(x)/f(x) \sim -\sigma_l^2/x$ as $x \to -\infty$. □

3 Main results

In this section, we will obtain our main results. First of all, we give the results for some mixed distributions of two components, then extend the results to the case of general finite mixed distributions.

Theorem 3.1. *Let $\{Z_n,\ n \geq 1\}$ be a sequence of i.i.d. random variables with common mixed generalized logistic distribution $F(x)$ determined by (1.4). Let $M_n = \max\{Z_1, \cdots, Z_n\}$ denote the partial maximum. If $\sigma_1 \geq \sigma_2,\ b_1, b_2 \in R$, then for every real number x the equality*

$$\lim_{n\to\infty} P(M_n \leq a_n x + b_n) = \exp\{-\exp(-x)\} \tag{3.1}$$

holds with

$$a_n = \sigma_1, \quad b_n = \sigma_1(\log n + \log(pb_1)).$$

Remark 3.1. *As* $k = 1, \sigma = 1$, Z_i *follows* $F(x) = \{1 + \exp(-x)\}^{-b}$, *and* $p = 1$, *(3.1) holds with* $a_n = 1$ *and* $b_n = \log n + \log b$

Remark 3.2. *As* $k = 1$, Z_i *follows* $F(x) = \{1+\exp(-x/\sigma)\}^{-b}$, *and* $p = 1$, *(3.1) holds with* $a_n = \sigma$ *and* $b_n = \sigma(\log n + \log b)$.

Theorem 3.2. *Let* $\{Z_n,\ n \geq 1\}$ *be a sequence of i.i.d. random variables with common mixed asymmetric Laplace distribution* $F(x)$ *determined by (1.4). Let* $M_n = \max\{Z_1, \cdots, Z_n\}$ *denote the partial maximum. If* $\kappa_1/\sigma_1 < \kappa_2/\sigma_2$, *then for* $x \geq 0$, *the equality*

$$\lim_{n\to\infty} P(M_n \leq a_n x + b_n) = \exp\{-\exp(-x)\} \tag{3.2}$$

holds with

$$a_n = \frac{\sigma_1}{\kappa_1}, \qquad b_n = \frac{\sigma_1}{\kappa_1}\log n + \frac{\sigma_1}{\kappa_1}\log\frac{p}{1+\kappa_1^2}.$$

Let $W_n = \min\{Z_1, \cdots, Z_n\}$ *denote the partial minimum. If* $\kappa_1\sigma_1 > \kappa_2\sigma_2$, *then for* $x < 0$, *the equality*

$$\lim_{n\to\infty} P(W_n \leq c_n x + d_n) = 1 - \exp\{-e^x\} \tag{3.3}$$

holds with

$$c_n = \kappa_1\sigma_1, \qquad d_n = -\kappa_1\sigma_1\log n - \kappa_1\sigma_1\log\frac{p\kappa_1^2}{1+\kappa_1^2}.$$

Theorem 3.3. *Let* $\{Z_n,\ n \geq 1\}$ *be a sequence of i.i.d. random variables with common mixed asymmetric normal distribution* $F(x)$ *determined by (1.4). Let* $M_n = \max\{Z_1, \cdots, Z_n\}$ *denote the partial maximum. If* $\theta_1 > \theta_2$, $\sigma_{r1} \geq \sigma_{r2}$, *then for* $x > \theta_1$, *the equality (3.2) holds with*

$$a_n = \frac{\sigma_{r1}}{(2\log n)^{1/2}},\ b_n = \theta_1 + \sigma_{r1}(2\log n)^{1/2} - \frac{\sigma_{r1}}{(2\log n)^{1/2}}\left(\frac{1}{2}\log(2\log n) - \log\frac{2p\sigma_{r1}}{(2\pi)^{1/2}(\sigma_{r1}+\sigma_{l1})}\right).$$

Let $W_n = \min\{Z_1, \cdots, Z_n\}$ *denote the partial minimum. If* $\theta_1 < \theta_2$, $\sigma_{l1} \geq \sigma_{l2}$, *then for* $x < \theta_1$, *the equality (3.3) holds with*

$$c_n = \frac{\sigma_{l1}}{(2\log n)^{1/2}},\ d_n = \theta_1 - \sigma_{l1}(2\log n)^{1/2} + \frac{\sigma_{l1}}{(2\log n)^{1/2}}\left(\frac{1}{2}\log(2\log n) - \log\frac{2p\sigma_{l1}}{(2\pi)^{1/2}(\sigma_{r1}+\sigma_{l1})}\right).$$

Next, we will give the results for some mixed distributions of finite components.

Corollary 3.1. *Let* $\{Y_n,\ n \geq 1\}$ *be a sequence of i.i.d. random variables with common mixed generalized logistic distribution* $F(x)$ *determined by (1.3). Let* $M_n = \max\{Y_1, \cdots, Y_n\}$ *denote the partial maximum. Then for all* $x \in R$, *the equality (3.2) holds, the normalized constants* a_n *and* b_n *are given by*

$$a_n = \sigma_c, \quad b_n = \sigma_c\log n + \sigma_c\log(p_c b_c),$$

where σ_c *denotes the maximum* $\{\sigma_1, \cdots, \sigma_k\}$, b_c *is a corresponding constant such that* $\sigma_c = \max\{\sigma_1, \cdots, \sigma_k\}$, *and* p_c *is the corresponding weight.*

Corollary 3.2. *Let $\{Y_n,\ n \geq 1\}$ be a sequence of i.i.d. random variables with common mixed asymmetric Laplace distribution $F(x)$ determined by (1.3). Let $M_n = \max\{Y_1, \cdots, Y_n\}$ denote the partial maximum. Then for $x \geq 0$, the equality (3.2) holds with*

$$a_n = \frac{\sigma_c}{\kappa_c}, \quad b_n = \frac{\sigma_c}{\kappa_c} \log n + \frac{\sigma_c}{\kappa_c} \log \frac{p_c}{1+\kappa_c^2},$$

where $\sigma_i/\kappa_i \neq \sigma_j/\kappa_j$, for $i \neq j$, σ_c/κ_c denotes the minimum of $\{\sigma_1/\kappa_1, \cdots, \sigma_k/\kappa_k\}$, and p_c is the corresponding weight.

Let $W_n = \min\{Y_1, \cdots, Y_n\}$ denote the partial minimum. For $x < 0$, the equality (3.3) holds with

$$c_n = \sigma_0\kappa_0, \quad d_n = -(\sigma_0\kappa_0)\log n - (\sigma_0\kappa_0)\log \frac{p_0\kappa_0^2}{1+\kappa_0^2},$$

where $\sigma_i\kappa_i \neq \sigma_j\kappa_j$, for $i \neq j$, $\sigma_0\kappa_0$ denotes the maximum of $\{\sigma_1\kappa_1, \cdots, \sigma_k\kappa_k\}$, and p_0 is the corresponding weight.

Corollary 3.3. *Let $\{Y_n,\ n \geq 1\}$ be a sequence of i.i.d. random variables with common mixed asymmetric normal distribution $F(x)$ determined by (1.3). Let $M_n = \max\{Y_1, \cdots, Y_n\}$ denote the partial maximum. If $x > \theta$, then the equality (3.2) holds with*

$$a_n = \frac{\sigma_r}{(2\log n)^{1/2}}, \quad b_n = \theta + \sigma_r(2\log n)^{1/2} - \frac{\sigma_r}{(2\log n)^{1/2}}\left(\frac{1}{2}\log(2\log n) - \log\frac{2p_c\sigma_r\delta}{(2\pi)^{1/2}(\sigma_r+\sigma_l)}\right).$$

Where $\sigma_r = \max\{\sigma_{r1}, \cdots, \sigma_{rk}\}$, and $\theta = \max\{\theta_n : \sigma_{rn} = \sigma_r,\ 1 \leq n \leq m\}$, m is an integer such that $\sigma_{rn} = \sigma_r,\ 1 \leq n \leq m$, $\sigma_l = \max\{\sigma_{ln} : \sigma_{rn} = \sigma_r,\ 1 \leq n \leq m\}$, and p_c *is the corresponding weight,* $\delta = \sum_{i_j} p_{i_j}/p_c$ and $i_j \in \{n : \sigma_{rn} = \sigma_r$ *and* $\theta_n = \theta\}$.

Let $W_n = \min\{Y_1, \cdots, Y_n\}$ denote the partial minimum. If $x < \theta_0$, then the equality (3.3) holds with

$$c_n = \frac{\sigma_{l0}}{(2\log n)^{1/2}}, \quad d_n = \theta_0 - \sigma_{l0}(2\log n)^{1/2} + \frac{\sigma_{l0}}{(2\log n)^{1/2}}\left(\frac{1}{2}\log(2\log n) - \log\frac{2p_0\sigma_{l0}\delta_0}{(2\pi)^{1/2}(\sigma_{l0}+\sigma_{r0})}\right).$$

Where $\sigma_{l0} = \max\{\sigma_{l1}, \cdots, \sigma_{lk}\}$, and $\theta_0 = \max\{\theta_n : \sigma_{ln} = \sigma_{l0},\ 1 \leq n \leq m\}$, m is an integer such that $\sigma_{ln} = \sigma_{l0},\ 1 \leq n \leq m$, $\sigma_{r0} = \max\{\sigma_{rn} : \sigma_{ln} = \sigma_{l0},\ 1 \leq n \leq m\}$, and p_0 is the corresponding weight, and $\delta_0 = \sum_{i_j} p_{i_j}/p_0$ and $i_j \in \{n : \sigma_{ln} = \sigma_{l0}$ *and* $\theta_n = \theta_0\}$.

4 Proof

Proof of Theorem 3.1. By (1.4), we have

$$\begin{aligned} 1 - F(t) &= p\{1 - \{1 + \exp(-\frac{t}{\sigma_1})\}^{-b_1}\} + q\{1 - \{1 + \exp(-\frac{t}{\sigma_2})\}^{-b_2}\} \\ &= p\{b_1 \exp(-\frac{t}{\sigma_1}) - \frac{(b_1(b_1+1))}{2}\exp(-\frac{2t}{\sigma_1}) + o(\exp(-\frac{2t}{\sigma_1}))\} \\ &+ q\{b_2 \exp(-\frac{t}{\sigma_2}) - \frac{(b_2(b_2+1))}{2}\exp(-\frac{2t}{\sigma_2}) + o(\exp(-\frac{2t}{\sigma_2}))\} \end{aligned}$$

$$
\begin{aligned}
&= pb_1 \exp(-\frac{t}{\sigma_1})\{1 - \frac{(b_1+1)}{2}\exp(-\frac{t}{\sigma_1}) + \frac{qb_2}{pb_1}\exp((\frac{1}{\sigma_1} - \frac{1}{\sigma_2})t) \\
&- \frac{qb_2(b_2+1)}{2pb_1}\exp((\frac{1}{\sigma_1} - \frac{2}{\sigma_2})t) + o(1)\} \\
&= pb_1 \exp(-\frac{t}{\sigma_1})(1 + o(1)),\ t \to \infty, \qquad (4.1)
\end{aligned}
$$

where $\sigma_1 \geq \sigma_2,\ b_1, b_2 \in R$.

By (4.1), we have

$$
1 - F(t) = c(t)\exp\{-\int_0^t (1/f(u))\,\mathrm{d}u\}
$$

for $t \in (0, x_0)$ and

$$
\lim_{t \to x_0} c(t) = pb_1 > 0
$$

where $f(u) = \sigma_1 > 0,\ 0 < u < x_0$, and f is absolutely continuous on $(0,\ x_0)$ with density $f'(u)$ and $\lim_{u \uparrow x_0} f'(u) = 0$.

Hence by Proposition 1.4 in Resnick (1987), we conclude that $F(x) \in D(\Lambda)$.

The constants a_n and b_n can be determined as follows: let us first determine the constant u_n, such that $1 - F(u_n) \sim n^{-1}e^{-x}$, as $n \to \infty$ i.e.

$$
1 - pF_1(u_n) - qF_2(u_n) \sim \frac{1}{n}e^{-x}, as \quad n \to \infty,
$$

by (4.1), we obtain

$$
pb_1 \exp(-\frac{u_n}{\sigma_1}) \sim \frac{1}{n}e^{-x}, as \quad n \to \infty.
$$

Therefore

$$
npb_1 \exp(-\frac{u_n}{\sigma_1})e^x \to 1, as \quad n \to \infty.
$$

$$
\log n + \log(pb_1) - \frac{1}{\sigma_1}u_n + x \to 0, as \quad n \to \infty.
$$

Thus,

$$
u_n = \sigma_1 x + \sigma_1(\log n + \log(pb_1)) + o(1).
$$

Now letting $a_n = \sigma_1,\ and\ \ b_n = \sigma_1(\log n + \log(pb_1))$, we obtain the desired result. □

Proof of Theorem 3.2. For $t \geq 0$, by (1.4), we obtain

$$
\begin{aligned}
1 - F(t) &= 1 - p\{1 - \frac{1}{1+\kappa_1^2}\exp(-\frac{\kappa_1 t}{\sigma_1})\} - q\{1 - \frac{1}{1+\kappa_2^2}\exp(-\frac{\kappa_2 t}{\sigma_2})\} \\
&= \frac{p}{1+\kappa_1^2}\exp(-\frac{\kappa_1 t}{\sigma_1}) + \frac{q}{1+\kappa_2^2}\exp(-\frac{\kappa_2 t}{\sigma_2}) \\
&= \frac{p}{1+\kappa_1^2}\exp(-\frac{\kappa_1 t}{\sigma_1})\{1 + \frac{q(1+\kappa_1^2)}{p(1+\kappa_2^2)}\exp((\frac{\kappa_1}{\sigma_1} - \frac{\kappa_2}{\sigma_2})t)\}
\end{aligned}
$$

$$= \frac{p}{1+\kappa_1^2} \exp(-\frac{\kappa_1 t}{\sigma_1})(1+o(1)), \quad as\ t \to \infty, \tag{4.2}$$

where $\kappa_1/\sigma_1 < \kappa_2/\sigma_2$.

By (4.2), we have

$$1 - F(t) = c(t) \exp\{-\int_0^t (1/f(u))\, \mathrm{d}u\}$$

for $t \in (0, x_0)$ and

$$\lim_{t \to x_0} c(t) = \frac{p}{1+\kappa_1^2} > 0$$

where $f(u) = \sigma_1/\kappa_1 > 0,\ 0 < u < x_0$, and f is absolutely continuous on $(0,\ x_0)$ with density $f'(u)$ and $\lim_{u \uparrow x_0} f'(u) = 0$.

Hence by Proposition 1.4 in Resnick (1987), we conclude that $F(x) \in D(\Lambda)$.

The constants a_n and b_n can be determined as follows: let us first determine the constant u_n, such that $1 - F(u_n) \sim n^{-1}e^{-x}$, as $n \to \infty$ i.e.

$$1 - pF_1(u_n) - qF_2(u_n) \sim \frac{1}{n} e^{-x}, \quad n \to \infty,$$

by (4.2), we obtain

$$\frac{p}{1+\kappa_1^2} \exp(-\frac{\kappa_1 u_n}{\sigma_1}) \sim \frac{1}{n} e^{-x}, as \quad n \to \infty.$$

Therefore

$$n \frac{p}{1+\kappa_1^2} \exp(-\frac{\kappa_1 u_n}{\sigma_1}) e^x \to 1, as \quad n \to \infty,$$

$$\log n + \log \frac{p}{1+\kappa_1^2} - \frac{\kappa_1}{\sigma_1} u_n + x \to 0, as \quad n \to \infty.$$

Hence

$$u_n = \frac{\sigma_1}{\kappa_1} x + \frac{\sigma_1}{\kappa_1} \log n + \frac{\sigma_1}{\kappa_1} \log \frac{p}{1+\kappa_1^2}.$$

Now letting $a_n = \sigma_1/\kappa_1,\ and\ b_n = (\sigma_1/\kappa_1)\log n + (\sigma_1/\kappa_1)\log(p/(1+\kappa_1^2))$. We obtain the result.

For $t < 0$, by (1.4), we obtain

$$\begin{aligned} F(x) &= \frac{p\kappa_1^2}{1+\kappa_1^2} \exp(\frac{t}{\sigma_1\kappa_1}) + \frac{q\kappa_2^2}{1+\kappa_2^2} \exp(\frac{t}{\sigma_2\kappa_2}) \\ &= \frac{p\kappa_1^2}{1+\kappa_1^2} \exp(\frac{t}{\sigma_1\kappa_1}) \{1 + \frac{q\kappa_2^2(1+\kappa_1^2)}{p\kappa_1^2(1+\kappa_2^2)} \exp((\frac{1}{\sigma_2\kappa_2} - \frac{1}{\sigma_1\kappa_1})t)\} \\ &= \frac{p\kappa_1^2}{1+\kappa_1^2} \exp(\frac{t}{\sigma_1\kappa_1})(1+o(1)), \quad as\ t \to -\infty, \end{aligned} \tag{4.3}$$

where $\kappa_1\sigma_1 > \kappa_2\sigma_2$.

For $g(t) = \sigma_1\kappa_1$, we get

$$\begin{aligned}\frac{F(t + xg(t))}{F(t)} &= \frac{\frac{p\kappa_1^2}{1+\kappa_1^2}\exp(\frac{1}{\sigma_1\kappa_1}(t + xg(t)))(1 + o(1))}{\frac{p\kappa_1^2}{1+\kappa_1^2}\exp(\frac{1}{\sigma_1\kappa_1}t)(1 + o(1))} \\ &\to \exp(x),\ as\ t \to -\infty.\end{aligned}$$

Hence by Lemma 2.2, we conclude that $F(x) \in D(L_3)$.

The constants c_n and d_n can be determined as follows: let us first determine the constant v_n, such that $F(v_n) \sim n^{-1}e^x$, as $n \to \infty$ i.e.

$$pF_1(v_n) + qF_2(v_n) \sim \frac{1}{n}e^x, as \quad n \to \infty,$$

by (4.3), we obtain

$$\frac{p\kappa_1^2}{1+\kappa_1^2}\exp(\frac{v_n}{\sigma_1\kappa_1}) \sim \frac{1}{n}e^x, as \quad n \to \infty.$$

Therefore

$$n\frac{p\kappa_1^2}{1+\kappa_1^2}\exp(\frac{v_n}{\sigma_1\kappa_1})e^{-x} \to 1, as \quad n \to \infty,$$

$$\log n + \log\frac{p\kappa_1^2}{1+\kappa_1^2} + \frac{1}{\sigma_1\kappa_1}v_n - x \to 0, as \quad n \to \infty.$$

Hence

$$v_n = \sigma_1\kappa_1 x - \sigma_1\kappa_1\log n - \sigma_1\kappa_1\log\frac{p\kappa_1^2}{1+\kappa_1^2} + o(1).$$

Now letting $c_n = \sigma_1\kappa_1,\ and\ d_n = -\sigma_1\kappa_1\log n - \sigma_1\kappa_1\log(p\kappa_1^2/(1+\kappa_1^2))$. We obtain the result.

This completes the proof. □

***Proof of Theorem 3.3*.** For $t > \theta_1$, by (1.4), we obtain $1 - F(t) = p(1 - F_1(t)) + q(1 - F_2(t))$.

By Lemma 2.3, we have

$$\frac{1 - F_i(t)}{f_i(t)} \sim \frac{\sigma_{ri}^2}{t},\ i = 1,\ 2,\ as\ t \to \infty. \tag{4.4}$$

So,

$$\begin{aligned}1 - F(t) &\sim \frac{p\sigma_{r1}^2}{t}f_1(t) + \frac{q\sigma_{r2}^2}{t}f_2(t) \\ &= \frac{2p\sigma_{r1}^2}{(2\pi)^{1/2}(\sigma_{l1} + \sigma_{r1})}\frac{1}{t}\exp\{-\frac{(t-\theta_1)^2}{2\sigma_{r1}^2}\} + \frac{2q\sigma_{r2}^2}{(2\pi)^{1/2}(\sigma_{l2} + \sigma_{r2})}\frac{1}{t}\exp\{-\frac{(t-\theta_2)^2}{2\sigma_{r2}^2}\} \\ &= \frac{2p\sigma_{r1}^2}{(2\pi)^{1/2}(\sigma_{l1} + \sigma_{r1})}\frac{1}{t}\exp\{-\frac{(t-\theta_1)^2}{2\sigma_{r1}^2}\}(1 + o(1)),\end{aligned} \tag{4.5}$$

as $t \to \infty$, where $\theta_1 > \theta_2, \quad \sigma_{r1} \geq \sigma_{r2}$.

By (4.5), we obtain

$$\frac{1-F(t+xg(t))}{1-F(t)} = \exp\{-\frac{(t-\theta_1+xg(t))^2-(t-\theta_1)^2}{2\sigma_{r1}^2}\}\frac{t}{t+xg(t)}(1+o(1))$$
$$= \exp\{-\frac{xg(t)(t-\theta_1)}{\sigma_{r1}^2}\}\exp\{-\frac{x^2g^2(t)}{2\sigma_{r1}^2}\}\frac{t}{t+xg(t)}(1+o(1)).$$

For $g(t) = \sigma_{r1}^2/(t-\theta_1)$, we get

$$\frac{1-F(t+xg(t))}{1-F(t)} = e^{-x}\exp\{-\frac{x^2\sigma_{r1}^2}{2(t-\theta_1)^2}\}\frac{t}{t+(x\sigma_{r1}^2)/(t-\theta_1)}(1+o(1))$$
$$\to e^{-x}, \ as\ t \to \infty.$$

Hence by Lemma 2.1, we conclude that $F(x) \in D(\Lambda)$.

The constants a_n and b_n can be determined as follows: let us first determine the constant u_n, such that $1-F(u_n) \sim n^{-1}e^{-x}$, as $n \to \infty$ i.e.

$$1-pF_1(u_n)-qF_2(u_n) \sim \frac{1}{n}e^{-x}, as \quad n \to \infty,$$

By (4.4), we obtain

$$1-pF_1(u_n)-qF_2(u_n) \sim \frac{p\sigma_{r1}^2}{u_n}f_1(u_n)+\frac{q\sigma_{r2}^2}{u_n}f_2(u_n) \sim \frac{1}{n}e^{-x}, as \quad n \to \infty.$$

Let us denote: $v_n = (u_n-\theta_1)/\sigma_{r1},\ and\ w_n = (u_n-\theta_2)/\sigma_{r2}$. For large values of n the equality $v_n < w_n$ holds true, and

$$\frac{p\sigma_{r1}^2}{\sigma_{r1}v_n+\theta_1}f_1(u_n)+\frac{q\sigma_{r2}^2}{\sigma_{r2}w_n+\theta_2}f_2(u_n)$$
$$= \frac{2p\sigma_{r1}^2}{(2\pi)^{1/2}(\sigma_{l1}+\sigma_{r1})}\frac{1}{\sigma_{r1}v_n+\theta_1}e^{-v_n^2/2}+\frac{2q\sigma_{r2}^2}{(2\pi)^{1/2}(\sigma_{l2}+\sigma_{r2})}\frac{1}{\sigma_{r2}w_n+\theta_2}e^{-w_n^2/2}$$
$$= \frac{2p\sigma_{r1}^2}{(2\pi)^{1/2}(\sigma_{l1}+\sigma_{r1})}\frac{1}{\sigma_{r1}v_n+\theta_1}e^{-v_n^2/2}(1+\frac{q\sigma_{r2}^2(\sigma_{l1}+\sigma_{r1})}{p\sigma_{r1}^2(\sigma_{l2}+\sigma_{r2})}e^{-(w_n^2-v_n^2)/2}). \tag{4.6}$$

Let $s_n = w_n^2 - v_n^2 = (u_n-\theta_2)^2/\sigma_{r2}^2-(u_n-\theta_1)^2/\sigma_{r1}^2$. If $\sigma_{r1} > \sigma_{r2} > 0$, then $s_n = (1/\sigma_{r2}^2 - 1/\sigma_{r1}^2)u_n^2 + Au_n + B \to \infty,\ as\ n \to \infty$, where $A = 2\theta_1/\sigma_{r1}^2 - 2\theta_2/\sigma_{r2}^2,\ B = \theta_2^2/\sigma_{r2}^2 - \theta_1^2/\sigma_{r1}^2$.

If $\sigma_{r1} = \sigma_{r2} = \sigma\ and\ \theta_1 > \theta_2$, then $s_n = [2u_n(\theta_1-\theta_2)+\theta_2^2-\theta_1^2]/\sigma^2 \to \infty,\ as\ n \to \infty$. In both cases we have the following asymptotic equality

$$\frac{p\sigma_{r1}^2}{\sigma_{r1}v_n+\theta_1}f_1(u_n)+\frac{q\sigma_{r2}^2}{\sigma_{r2}w_n+\theta_2}f_2(u_n) = \frac{2p\sigma_{r1}^2}{(2\pi)^{1/2}(\sigma_{l1}+\sigma_{r1})}\frac{1}{\sigma_{r1}v_n+\theta_1}e^{-v_n^2/2}(1+o(1)),$$

as $n \to \infty$.

Hence, the constant u_n should be determined from the $u_n = \sigma_{r1}v_n + \theta_1$ and

$$\frac{p\sigma_{r1}^2}{\sigma_{r1}v_n + \theta_1} f_1(u_n) \sim \frac{1}{n}e^{-x}, \ as\ n \to \infty.$$

$$np\sigma_{r1}^2 e^x \frac{f_1(u_n)}{\sigma_{r1}v_n + \theta_1} \to 1, \ as\ n \to \infty$$

$$\log n + \log(p\sigma_{r1}^2) + x + \log \frac{2}{(2\pi)^{1/2}(\sigma_{l1} + \sigma_{r1})} - \frac{v_n^2}{2} - \log(\sigma_{r1}v_n + \theta_1) \to 0.\ as\ n \to \infty \quad (4.7)$$

By (4.7), we have $v_n^2/(2\log n) \to 1, \ as\ n \to \infty$. Hence

$$\log v_n = \frac{1}{2}\log(2\log n) + o(1). \quad (4.8)$$

The relation (4.7) can also be rewritten in the form

$$\begin{aligned} \frac{v_n^2}{2} &= x + \log n + \log \frac{2p\sigma_{r1}^2}{(2\pi)^{1/2}(\sigma_{l1} + \sigma_{r1})} - \log(\sigma_{r1}v_n + \theta_1) + o(1) \\ &= x + \log n + \log \frac{2p\sigma_{r1}^2}{(2\pi)^{1/2}(\sigma_{l1} + \sigma_{r1})} - \log v_n - \log \sigma_{r1} + o(1) \end{aligned} \quad (4.9)$$

Now we substitute the value of $\log v_n$ from (4.8) into (4.9). We obtain

$$\frac{v_n^2}{2} = x + \log n - \frac{1}{2}\log(2\log n) + \log \frac{2p\sigma_{r1}}{(2\pi)^{1/2}(\sigma_{l1} + \sigma_{r1})} + o(1)$$

$$v_n^2 = 2\log n\{1 + \frac{x - \frac{1}{2}\log(2\log n) + \log \frac{2p\sigma_{r1}}{(2\pi)^{1/2}(\sigma_{l1}+\sigma_{r1})}}{\log n} + o(\frac{1}{\log n})\}$$

Since $(1+x)^{1/2} = 1 + (1/2)x + o(x), \ as\ x \to 0$, we get

$$\begin{aligned} v_n &= (2\log n)^{1/2}\{1 + \frac{x - \frac{1}{2}\log(2\log n) + \log \frac{2p\sigma_{r1}}{(2\pi)^{1/2}(\sigma_{l1}+\sigma_{r1})}}{\log n} + o(\frac{1}{\log n})\}^{1/2} \\ &= (2\log n)^{1/2}\{1 + \frac{x - \frac{1}{2}\log(2\log n) + \log \frac{2p\sigma_{r1}}{(2\pi)^{1/2}(\sigma_{l1}+\sigma_{r1})}}{2\log n} + o(\frac{1}{\log n})\} \\ &= (2\log n)^{1/2} + \frac{x}{(2\log n)^{1/2}} - \frac{1}{(2\log n)^{1/2}}(\frac{1}{2}\log(2\log n) - \log \frac{2p\sigma_{r1}}{(2\pi)^{1/2}(\sigma_{l1} + \sigma_{r1})}) + o(\frac{1}{(\log n)^{1/2}}). \end{aligned}$$

Since $u_n = \theta_1 + \sigma_{r1}v_n$, we have

$u_n = \frac{\sigma_{r1}}{(2\log n)^{1/2}}x + \theta_1 + \sigma_{r1}(2\log n)^{1/2} - \frac{\sigma_{r1}}{(2\log n)^{1/2}}(\frac{1}{2}\log(2\log n) - \log \frac{2p\sigma_{r1}}{(2\pi)^{1/2}(\sigma_{l1}+\sigma_{r1})}) + o((\log n)^{-1/2})$.

Now letting $a_n = \sigma_{r1}/(2\log n)^{1/2},\ and\ b_n = \theta_1 + \sigma_{r1}(2\log n)^{1/2} - (\sigma_{r1}/(2\log n)^{1/2})(1/2\log(2\log n) - \log(2p\sigma_{r1}/((2\pi)^{1/2}(\sigma_{r1} + \sigma_{l1}))))$, we obtain the desired results.

The proof of $t < \theta_1$ is similar to the proof of $t > \theta_1$ by Lemma 2.2 and Lemma 2.3. The proof is completed. □

Proof of Corollary 3.1. By (1.3), we have

$$\begin{aligned}
1-F(t) &= 1-p_1F_1(t)-\cdots-p_{c-1}F_{c-1}(t)-p_cF_c(t)-p_{c+1}F_{c+1}(t)-\cdots-p_kF_k(t)\\
&= p_1\{1-\{1+\exp(-\frac{t}{\sigma_1})\}^{-b_1}\}+\cdots+p_{c-1}\{1-\{1+\exp(-\frac{t}{\sigma_{c-1}})\}^{-b_{c-1}}\}\\
&+p_c\{1-\{1+\exp(-\frac{t}{\sigma_c})\}^{-b_c}\}+p_{c+1}\{1-\{1+\exp(-\frac{t}{\sigma_{c+1}})\}^{-b_{c+1}}\}+\cdots\\
&+p_k\{1-\{1+\exp(-\frac{t}{\sigma_k})\}^{-b_k}\}\\
&= p_1\{b_1\exp(-\frac{t}{\sigma_1})-\frac{(b_1(b_1+1))}{2}\exp(-\frac{2t}{\sigma_1})+o(\exp(-\frac{2t}{\sigma_1}))\}+\cdots\\
&+p_{c-1}\{b_{c-1}\exp(-\frac{t}{\sigma_{c-1}})-\frac{(b_{c-1}(b_{c-1}+1))}{2}\exp(-\frac{2t}{\sigma_{c-1}})+o(\exp(-\frac{2t}{\sigma_{c-1}}))\}\\
&+p_c\{b_c\exp(-\frac{t}{\sigma_c})-\frac{(b_c(b_c+1))}{2}\exp(-\frac{2t}{\sigma_c})+o(\exp(-\frac{2t}{\sigma_c}))\}\\
&+p_{c+1}\{b_{c+1}\exp(-\frac{t}{\sigma_{c+1}})-\frac{(b_{c+1}(b_{c+1}+1))}{2}\exp(-\frac{2t}{\sigma_{c+1}})+o(\exp(-\frac{2t}{\sigma_{c+1}}))\}+\cdots\\
&+p_k\{b_k\exp(-\frac{t}{\sigma_k})-\frac{(b_k(b_k+1))}{2}\exp(-\frac{2t}{\sigma_k})+o(\exp(-\frac{2t}{\sigma_k}))\}\\
&= p_cb_c\exp(-\frac{t}{\sigma_c})\{\frac{p_1b_1}{p_cb_c}\exp((\frac{1}{\sigma_c}-\frac{1}{\sigma_1})t)-\frac{p_1b_1(b_1+1)}{2p_cb_c}\exp((\frac{1}{\sigma_c}-\frac{2}{\sigma_1})t)\\
&+o(\exp((\frac{1}{\sigma_c}-\frac{2}{\sigma_1})t))+\cdots+\frac{p_{c-1}b_{c-1}}{p_cb_c}\exp((\frac{1}{\sigma_c}-\frac{1}{\sigma_{c-1}})t)\\
&-\frac{p_{c-1}b_{c-1}(b_{c-1}+1)}{2p_cb_c}\exp((\frac{1}{\sigma_c}-\frac{2}{\sigma_{c-1}})t)+o(\exp((\frac{1}{\sigma_c}-\frac{2}{\sigma_{c-1}})t))+\cdots\\
&+1-\frac{b_c+1}{2}\exp(-\frac{t}{\sigma_c})+o(\exp(-\frac{t}{\sigma_c}))+\cdots+\frac{p_{c+1}b_{c+1}}{p_cb_c}\exp((\frac{1}{\sigma_c}-\frac{1}{\sigma_{c+1}})t)\\
&-\frac{p_{c+1}b_{c+1}(b_{c+1}+1)}{2p_cb_c}\exp((\frac{1}{\sigma_c}-\frac{2}{\sigma_{c+1}})t)+o(\exp((\frac{1}{\sigma_c}-\frac{2}{\sigma_{c+1}})t))+\cdots\\
&+\frac{p_kb_k}{p_cb_c}\exp((\frac{1}{\sigma_c}-\frac{1}{\sigma_k})t)-\frac{p_kb_k(b_k+1)}{2p_cb_c}\exp((\frac{1}{\sigma_c}-\frac{2}{\sigma_k})t)+o(\exp((\frac{1}{\sigma_c}-\frac{2}{\sigma_k})t))\}\\
&= p_cb_c\exp(-\frac{t}{\sigma_c})(1+o(1)),\ \ as\ t\to\infty, \qquad (4.10)
\end{aligned}$$

where $\sigma_c=\max\{\sigma_1,\cdots,\sigma_k\}$, b_c is a corresponding constant such that $\sigma_c=\max\{\sigma_1,\cdots,\sigma_k\}$, and p_c is the corresponding weight.

By (4.10), we have

$$1-F(t)=\tilde{c}(t)\exp\{-\int_0^t(1/f(u))\,\mathrm{d}u\}$$

for $t\in(0,x_0)$ and

$$\lim_{t\to x_0}\tilde{c}(t)=p_cb_c>0$$

where $f(u) = \sigma_c > 0,\ 0 < u < x_0$, and f is absolutely continuous on $(0,\ x_0)$ with density $f'(u)$ and $\lim_{u\uparrow x_0} f'(u) = 0$.

Hence by Proposition 1.4 in Resnick (1987), we conclude that $F(x) \in D(\Lambda)$.

The rest of proof is similar to the proof of Theorem 3.1, we can get

$$u_n = \sigma_c x + \sigma_c \log n + \sigma_c \log(p_c b_c) + o(1).$$

The result follows. We complete the proof. □

Proof of Corollary 3.2. For $t \geq 0$, by (1.3), we have

$$\begin{aligned}
1 - F(t) &= 1 - p_1 F_1(t) - \cdots - p_{c-1} F_{c-1}(t) - p_c F_c(t) - p_{c+1} F_{c+1}(t) - \cdots - p_k F_k(t) \\
&= 1 - p_1\{1 - \frac{1}{1+\kappa_1^2} \exp(-\frac{\kappa_1 t}{\sigma_1})\} - \cdots - p_{c-1}\{1 - \frac{1}{1+\kappa_{c-1}^2} \exp(-\frac{\kappa_{c-1} t}{\sigma_{c-1}})\} \\
&- p_c\{1 - \frac{1}{1+\kappa_c^2} \exp(-\frac{\kappa_c t}{\sigma_c})\} - p_{c+1}\{1 - \frac{1}{1+\kappa_{c+1}^2} \exp(-\frac{\kappa_{c+1} t}{\sigma_{c+1}})\} - \cdots \\
&- p_k\{1 - \frac{1}{1+\kappa_k^2} \exp(-\frac{\kappa_k t}{\sigma_k})\} \\
&= \frac{p_1}{1+\kappa_1^2} \exp(-\frac{\kappa_1 t}{\sigma_1}) + \cdots + \frac{p_{c-1}}{1+\kappa_{c-1}^2} \exp(-\frac{\kappa_{c-1} t}{\sigma_{c-1}}) + \frac{p_c}{1+\kappa_c^2} \exp(-\frac{\kappa_c t}{\sigma_c}) \\
&+ \frac{p_{c+1}}{1+\kappa_{c+1}^2} \exp(-\frac{\kappa_{c+1} t}{\sigma_{c+1}}) + \cdots + \frac{p_k}{1+\kappa_k^2} \exp(-\frac{\kappa_k t}{\sigma_k}) \\
&= \frac{p_c}{1+\kappa_c^2} \exp(-\frac{\kappa_c t}{\sigma_c})\{\frac{p_1(1+\kappa_c^2)}{p_c(1+\kappa_1^2)} \exp((\frac{\kappa_c}{\sigma_c} - \frac{\kappa_1}{\sigma_1})t) + \cdots + \frac{p_{c-1}(1+\kappa_c^2)}{p_c(1+\kappa_{c-1}^2)} \exp((\frac{\kappa_c}{\sigma_c} - \frac{\kappa_{c-1}}{\sigma_{c-1}})t) \\
&+ 1 + \frac{p_{c+1}(1+\kappa_c^2)}{p_c(1+\kappa_{c+1}^2)} \exp((\frac{\kappa_c}{\sigma_c} - \frac{\kappa_{c+1}}{\sigma_{c+1}})t) + \cdots + \frac{p_k(1+\kappa_c^2)}{p_c(1+\kappa_k^2)} \exp((\frac{\kappa_c}{\sigma_c} - \frac{\kappa_k}{\sigma_k})t)\} \\
&= \frac{p_c}{1+\kappa_c^2} \exp(-\frac{\kappa_c t}{\sigma_c})(1 + o(1)),\ as\ t \to \infty. \qquad (4.11)
\end{aligned}$$

where $\sigma_i/\kappa_i \neq \sigma_j/\kappa_j$, for $i \neq j$, σ_c/κ_c denotes the minimum of $\{\sigma_1/\kappa_1, \cdots, \sigma_k/\kappa_k\}$, and p_c is the corresponding weight.

By (4.11), we have

$$1 - F(t) = \tilde{c}(t) \exp\{-\int_0^t (1/f(u))\, \mathrm{d}u\}$$

for $t \in (0, x_0)$ and

$$\lim_{t \to x_0} \tilde{c}(t) = \frac{p_c}{1+\kappa_c^2} > 0$$

where $f(u) = \sigma_c/\kappa_c > 0,\ 0 < u < x_0$, and f is absolutely continuous on $(0,\ x_0)$ with density $f'(u)$ and $\lim_{u\uparrow x_0} f'(u) = 0$.

Hence by Proposition 1.4 in Resnick (1987), we conclude that $F(x) \in D(\Lambda)$.

The proof of $t < 0$ is similar to the proof of $t \geq 0$ by Lemma 2.2. The proof is completed. □

Proof of Corollary 3.3**.** For $t > \theta$, by (1.3), we have

$$\begin{aligned}
1 - F(t) &= 1 - p_1F_1(t) - p_2F_2(t) - \cdots - p_kF_k(t) \\
&= p_1(1 - F_1(t)) + p_2(1 - F_2(t)) \cdots + p_k(1 - F_k(t)) \\
&\sim \frac{p_1\sigma_{r1}^2}{t} f_1(t) + \frac{p_2\sigma_{r2}^2}{t} f_2(t) + \cdots + \frac{p_k\sigma_{rk}^2}{t} f_k(t) \\
&= \sum\nolimits_{i=1}^{k} \frac{2}{(2\pi)^{1/2}} \frac{p_i\sigma_{ri}^2}{(\sigma_{li} + \sigma_{ri})t} \exp\{-\frac{(t-\theta_i)^2}{2\sigma_{ri}^2}\} \\
&= \frac{2}{(2\pi)^{1/2}} \frac{p_c\sigma_r^2}{(\sigma_l + \sigma_r)t} \exp\{-\frac{(t-\theta)^2}{2\sigma_r^2}\} \sum\nolimits_{i=1}^{k} \frac{p_i\sigma_{ri}^2(\sigma_l + \sigma_r)}{p_c\sigma_r^2(\sigma_{li} + \sigma_{ri})} \exp\{-\frac{(t-\theta_i)^2}{2\sigma_{ri}^2} + \frac{(t-\theta)^2}{2\sigma_r^2}\} \\
&= \frac{2}{(2\pi)^{1/2}} \frac{p_c\sigma_r^2}{(\sigma_l + \sigma_r)t} \exp\{-\frac{(t-\theta)^2}{2\sigma_r^2}\}\{ \sum_{i:\sigma_{ri}=\sigma_r, \theta_i=\theta} \frac{p_i\sigma_{ri}^2(\sigma_l + \sigma_r)}{p_c\sigma_r^2(\sigma_{li} + \sigma_{ri})} \exp\{-\frac{(t-\theta_i)^2}{2\sigma_{ri}^2} \\
&+ \frac{(t-\theta)^2}{2\sigma_r^2}\} + \sum_{i:\sigma_{ri}\neq\sigma_r \ or \ \sigma_{ri}=\sigma_r, \ \theta_i\neq\theta} \frac{p_i\sigma_{ri}^2(\sigma_l + \sigma_r)}{p_c\sigma_r^2(\sigma_{li} + \sigma_{ri})} \exp\{-\frac{(t-\theta_i)^2}{2\sigma_{ri}^2} + \frac{(t-\theta)^2}{2\sigma_r^2}\}\},
\end{aligned} \tag{4.12}$$

where $\sigma_r = \max\{\sigma_{r1}, \cdots, \sigma_{rk}\}$, and $\theta = \max\{\theta_n : \sigma_{rn} = \sigma_r, \ 1 \le n \le m\}$, m is an integer such that $\sigma_{rn} = \sigma_r, \ 1 \le n \le m$, $\sigma_l = \max\{\sigma_{ln} : \sigma_{rn} = \sigma_r, \ 1 \le n \le m\}$, and p_c is the corresponding weight.

Notice for $\sigma_{ri} \neq \sigma_r \ or \ \sigma_{ri} = \sigma_r \ and \ \ \theta_i \neq \theta$, one can check

$$-\frac{(t-\theta_i)^2}{2\sigma_{ri}^2} + \frac{(t-\theta)^2}{2\sigma_r^2} = \frac{1}{\sigma_r^2}(\theta_i - \theta)t + \frac{1}{2}(\frac{\theta^2}{\sigma_r^2} - \frac{\theta_i^2}{\sigma_{ri}^2}) \to -\infty, \ as \ t \to \infty.$$

Hence, by (4.12), we have

$$1 - F(t) \sim \frac{2p_c\sigma_r^2\delta}{(2\pi)^{1/2}(\sigma_l + \sigma_r)} \frac{1}{t} \exp\{-\frac{(t-\theta)^2}{2\sigma_r^2}\}, \ as \ t \to \infty,$$

where $\delta = \sum_{i_j} p_{i_j}/p_c$ and $i_j \in \{n : \sigma_{rn} = \sigma_r \ and \ \theta_n = \theta\}$.

For $g(t) = \sigma_r^2/(t - \theta)$, we have

$$\begin{aligned}
\frac{1 - F(t + xg(t))}{1 - F(t)} &= \exp\{-\frac{(t - \theta + xg(t))^2 - (t-\theta)^2}{2\sigma_r^2}\} \frac{t}{t + xg(t)}(1 + o(1)) \\
&= \exp\{-\frac{xg(t)(t-\theta)}{\sigma_r^2}\} \exp\{-\frac{x^2g^2(t)}{2\sigma_r^2}\} \frac{t}{t + xg(t)}(1 + o(1)) \\
&= e^{-x} \exp\{-\frac{x^2\sigma_r^2}{2(t-\theta)^2}\} \frac{t}{t + (x\sigma_r^2)/(t-\theta)}(1 + o(1)) \\
&\to e^{-x}, \ as \ t \to \infty.
\end{aligned}$$

Hence by Lemma 2.1 we conclude that $F(x) \in D(\Lambda)$. The proof of the rest is similar to the proof of Theorem 3.3, we have

$$u_n = \frac{\sigma_r}{(2\log n)^{1/2}} x + \theta + \sigma_r(2\log n)^{1/2} - \frac{\sigma_r}{(2\log n)^{1/2}} (\frac{1}{2}\log(2\log n) - \log\frac{2p_c\sigma_r\delta}{(2\pi)^{1/2}(\sigma_l + \sigma_r)}) + o(\frac{1}{(\log n)^{1/2}}).$$

The result follows.

The proof of $t < \theta_0$ is similar to the proof of $t > \theta$ by Lemma 2.2 and Lemma 2.3. The proof is completed. $\square$

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