Analytic Extension of Oppenheimer-Snyder to Nonuniform Dust

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Abstract

The crucial role of the Birkhoff theorem in Oppenheimer and Snyder's treatment of gravitationally contracting dust is psychologically difficult to reconcile with their exclusive utilization of position-independent dust energy density: on its face the latter excludes the existence any empty-space region whereas the Birkhoff theorem applies only to such regions. Their use of position-independent dust energy density was due to lack of known nonuniform-dust analytic solutions for spherically-symmetric "comoving coordinate systems". This article fills that gap for nonuniform dust which is initially momentarily stable, but in so doing reveals that its behavior in the "comoving coordinate system" always owes more to Newton than to Einstein: the metric periodically singularly contravenes the Principle of Equivalence due to its Newtonian-signature time cycloid component, and it manifests no trace of gravitational redshift. The now justifiable application of the Birkhoff theorem turns out to effect the physically correct inclusion of gravitational redshift; it singularly stretches the "comoving" dust system's time-cycloid first half-period to infinite time duration. The unfamiliar lesson here is that "comoving coordinate systems" are inherently bereft of crucial relativistic gravitational physics: this must be incorporated by a singular transformation to a physically more appropriate coordinate system.

Introduction

The manner in which Oppenheimer and Snyder approached the energy-conserving "gravitational collapse" of an initially stationary spherically-symmetric cloud of pressure-free "dust" [1, 2] raises some bewildering gravitational theoretical physics issues. Their overarching desire to achieve *an analytic solution* to this problem motivated Oppenheimer and Snyder to tackle it in a "comoving coordinate system" [3, 4], where *significant simplification* could be expected to result from the vanishing three-momenta of *all* of the dust particles *at all times* [5].

In *that* "coordinate system" the presumably "gravitationally contracting" dust cloud *cannot in fact experience any contraction whatsoever*, but instead *only experiences fluctuations in its local energy density* as this energy is dynamically swapped with the potential energy of the gravitational field which the dust cloud produces.

In *addition*, the metric tensor of such a "comoving coordinate system" very atypically has its g_{00} component *rigidly fixed to the value unity* [6] in spite of the fact that the local redshift factor which any static gravitational field produces is given by $(g_{00})^{-\frac{1}{2}}$ [7] and the fact that in the limit of such a static field's being weak the Newtonian gravitational potential ϕ which it produces is given by $\frac{1}{2}(g_{00} - 1)$ [8].

The foregoing considerations make it abundantly clear that "comoving coordinate systems" can *at best* be regarded as hopefully useful calculational contrivances *which are inherently at loggerheads in crucial ways with correct relativistic gravitational theoretical physics*. These "coordinate systems" apparently *systematically fail to account for gravitational redshift effects*, and consequently exhibit *physically disallowed metric manifestations which contravene the Principle of Equivalence* [9, 10, 11]. The *mapping* of a "comoving coordinate system" onto a more "standard" coordinate system, *namely one that doesn't fix the value of g_{00} to unity*, will be *singular* in certain space-time regions of that "comoving coordinate system" *which in fact are devoid of legitimate relativistic gravitational theoretical physics meaning*.

An energy-conserving "gravitationally collapsing" Oppenheimer-Snyder dust system whose initial condition has all the dust particles at relative rest is dynamically *in a bound state*: the behavior of such an energy-conserving bound state is typically *cyclic*, forever revisiting the parts of the phase space which it has previously touched. As a matter of fact, we shall see that in the "comoving coordinate system" both the local energy density of the dust and the local values of the metric components do *indeed* manifest cyclic behavior, which, however, is periodically *singular*; the function of time which *governs* this periodically singular cyclic behavior turns out to be *the very same* time cycloid which describes the periodically singular *Newtonian gravitational infall* from initial relative rest and nonzero relative separation *of a test particle toward a point mass*.

This astonishingly close relationship of dust "gravitational collapse" in the "comoving coordinate system" to the basic periodically singular *Newtonian* gravitational infall of a test particle *starkly underlines the relativistic gravitational physics deficiencies* of the "comoving coordinate system". Supplying the gravitational *redshift* which is simply *absent* from the "comoving coordinate system" stretches *the half-period*

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of the periodically *singular* time cycloid which governs its metric *into an infinite time interval*. Supplying the *nonzero dust particle three-momenta* that are simply *absent* from the "comoving coordinate system" increases the relativistic *effective gravitational attraction* between those dust particles, rendering the dust cloud's gravitational contraction *into a sphere smaller than a certain critical radius* incompatible with the system's *energy conservation*.

The overarching desire of Oppenheimer and Snyder to achieve *an analytic solution* to this energy-conserving sperically-symmetric dust "gravitational collapse" problem *not only* motivated them to choose to work in the "comoving coordinate system", it *also* caused them to make a *physically drastic* simplifying assumption, namely that the spherically-symmetric dust energy density $\rho(r,t)$ *is independent of position* r [1, 12]. Such dust unphysically uniformly permeates *all of space* and so has *infinite* total energy. Oppenheimer and Snyder's resulting "comoving coordinate system" metric not surprisingly turns out *even at the initial time* $t = t_0$ to exhibit at all sufficiently large values of r *eigenvalues that are incompatible with the Principle of Equivalence*. Oppenheimer and Snyder's subsequent elaborate application of the Birkhoff theorem [13] *is as well incompatible with strictly position-independent dust energy density* because the Birkhoff theorem *is valid only in empty space* [14], and the *existence* of any empty-space region is, of course, *precluded* by dust of *strictly position-independent energy density*.

In the following section we obviate any conceivable need for the physically problematic simplifying assumption of *strictly* position-independent dust energy density by *analytically solving* the spherically-symmetric "comoving coordinate system" Einstein equation for *arbitrary nonuniform initial dust energy density* $\rho(r,t_0)$, with *initial* "comoving coordinate system" metric conditions that are *identical* to the specific "gravitational collapse" ones of Oppenheimer and Snyder. Of course *arbitrary* spherically-symmetric nonuniform initial dust energy density *leaves plenty of scope for empty-space regions to exist*, which permits *unambiguously valid applications of the Birkhoff theorem*.

Solving the comoving-metric Einstein equation for nonuniform dust

The spherically-symmetric "comoving coordinate system" metric is given by the line element [15],

$$ds^2 = (cdt)^2 - U(r,t)dr^2 - V(r,t)((d\theta)^2 + (\sin\theta d\phi)^2), \tag{1}$$

where $U(r,t)$ is dimensionless, but $V(r,t)$ has the dimension of length squared.

The stress-energy tensor $T^{\mu\nu}$ of the dust in this "comoving coordinate system" *has only the single component* $T^{00} = \rho(r,t)$ [16], and of the four components of its covariant vector equation of continuity $(T^{\mu}{}_{\nu})_{;\mu} = 0$, *only* the time component *isn't* an identity; that component can be written in the form [17],

$$\partial(\rho V U^{\frac{1}{2}})/\partial t = 0, \tag{2a}$$

which implies that,

$$\rho(r,t) = \frac{\rho(r,t_0)V(r,t_0)(U(r,t_0))^{\frac{1}{2}}}{V(r,t)(U(r,t))^{\frac{1}{2}}}. \tag{2b}$$

Of the ten components of the covariant symmetric second-rank tensor Einstein equation for the metric of this dynamical dust system in the "comoving coordinate system", there are only *four* components which are neither identities nor redundant on their face, namely [18],

$$-\frac{1}{V} + \frac{1}{U}\left(\frac{V''}{2V} - \frac{U'V'}{4UV}\right) - \frac{1}{c^2}\left(\frac{\ddot{V}}{2V} + \frac{\dot{U}\dot{V}}{4UV}\right) = -\frac{4\pi G\rho}{c^4}, \tag{3a}$$

$$\frac{1}{U}\left(\frac{V''}{V} - \frac{U'V'}{2UV} - \frac{V'^2}{2V^2}\right) - \frac{1}{c^2}\left(\frac{\ddot{U}}{2U} - \frac{\dot{U}^2}{4U^2} + \frac{\dot{U}\dot{V}}{2UV}\right) = -\frac{4\pi G\rho}{c^4}, \tag{3b}$$

$$\frac{1}{c^2}\left(\frac{\ddot{U}}{2U} + \frac{\ddot{V}}{V} - \frac{\dot{U}^2}{4U^2} - \frac{\dot{V}^2}{2V^2}\right) = -\frac{4\pi G\rho}{c^4}, \tag{3c}$$

and,

$$\frac{\dot{V}'}{V} - \frac{V'\dot{V}}{2V^2} - \frac{\dot{U}V'}{2UV} = 0. \tag{3d}$$

We eliminate V'' by subtracting Eq. (3b) from twice Eq. (3a) to obtain,

$$-\frac{2}{V} + \frac{1}{U}\left(\frac{V'^2}{2V^2}\right) - \frac{1}{c^2}\left(\frac{\ddot{V}}{V} - \frac{\ddot{U}}{2U} + \frac{\dot{U}^2}{4U^2}\right) = -\frac{4\pi G\rho}{c^4}. \tag{4a}$$

We now eliminate both $\ddot{U}$ and $\dot{U}$ by subtracting Eq. (4a) from Eq. (3c) to obtain,

$$\tfrac{2}{V} - \tfrac{1}{U}\left(\tfrac{V'^2}{2V^2}\right) + \tfrac{1}{c^2}\left(\tfrac{2\ddot{V}}{V} - \tfrac{\dot{V}^2}{2V^2}\right) = 0. \tag{4b}$$

We can now solve Eq. (4b) for U in terms of a formal fraction which depends on V and its partial derivatives. Since U is dimensionless, we make *both* the numerator and denominator of this formal fraction dimensionless,

$$U = \tfrac{((V')^2/V)}{4+(1/c^2)[4\ddot{V}-((\dot{V})^2/V)]}. \tag{4c}$$

Turning our attention now to Eq. (3d), we note that after it is multiplied through by $-2(V/V')$ it can be written,

$$(\dot{U}/U) - 2(\dot{V}'/V') + (\dot{V}/V) = 0, \tag{4d}$$

which is straightforwardly seen to be equivalent to,

$$\partial((UV)/(V')^2)/\partial t = 0, \tag{4e}$$

and this in turn implies that,

$$U = C(r)((V')^2/V), \tag{4f}$$

where $C(r)$ *has no dependence on the time variable* t. We can now equate the right-hand side of Eq. (4f) to the right-hand side of Eq. (4c), and thus obtain the following second-order in time differential equation for V *alone*,

$$\ddot{V} - \tfrac{1}{4}((\dot{V})^2/V) = c^2((1/(4C(r))) - 1). \tag{4g}$$

We can as well insert Eq. (4f) into Eq. (2b), and thus eliminate $U(r,t)$ from $\rho(r,t)$ to obtain,

$$\rho(r,t) = \tfrac{\rho(r,t_0)(V(r,t_0))^{\frac{1}{2}}V'(r,t_0)}{(V(r,t))^{\frac{1}{2}}V'(r,t)}. \tag{5}$$

The dust energy density is required to initially *be at rest*, namely $\dot{\rho}(r,t_0) = 0$. From that and from Eq. (5) we see that one *of the initial conditions* which must apply to V with regard to its second-order in time differential equation that is given by Eq. (4g) is,

$$\dot{V}(r,t_0) = 0. \tag{6a}$$

We specify the *remaining* initial condition which is needed by Eq. (4g), the second-order in time differential equation for V, to be the *same* as that which was specified by Oppenheimer and Snyder [19], namely,

$$V(r,t_0) = r^2, \tag{6b}$$

which, inter alia, causes Eq. (5) to read,

$$\rho(r,t) = \rho(r,t_0)\left[\tfrac{2r^2}{(V(r,t))^{\frac{1}{2}}V'(r,t)}\right]. \tag{7}$$

It is convenient at this point to eliminate $C(r)$ from Eqs. (4f) and (4g) in favor of $(1-(1/(4C(r)))$, which we shall denote as $K(r)$. Therefore Eq. (4f) now reads,

$$U = (((V')^2/V)/(4(1-K(r)))), \tag{8a}$$

which together with the initial condition $V(r,t_0) = r^2$ of Eq. (6b) implies that,

$$U(r,t_0) = (1-K(r))^{-1}, \tag{8b}$$

while the second-order in time differential equation for V that is given by Eq. (4g) now reads,

$$\ddot{V} - \tfrac{1}{4}((\dot{V})^2/V) = -c^2K(r). \tag{8c}$$

In order to *evaluate* $K(r)$ we need *further information that is contained in* Eqs. (3a) through (3d) *which is not already implicit in* Eqs. (8c) and (8a). Because $K(r)$ *is independent of the time variable* t, the needed information ought to follow from enforcing *any one* of the *three* Eqs. (3a), (3b) or (3c) *at the initial time* t_0 (note that Eq. (3d) *doesn't qualify because it is already implicit in* Eq. (8a)). Of the left-hand sides of Eqs. (3a) through (3c), that of Eq. (3a) *is the easiest to evaluate at the initial time* t_0 because *no knowledge of* $\dot{U}(r, t_0)$ *or* $\ddot{U}(r, t_0)$ *is needed* (the single occurrence of $\dot{U}(r, t_0)$ on the left-hand side of Eq. (3a) at time t_0 is multiplied by the factor $\dot{V}(r, t_0)$, which is equal to zero in accord with Eq. (6a)).

The further information *that is actually needed* for the evaluation of the left-hand side of Eq. (3a) at time t_0 consists of $V(r, t_0) = r^2$, in accord with Eq. (6b), from which $V'(r, t_0) = 2r$ and $V''(r, t_0) = 2$ as well follow; $U(r, t_0) = (1/(1 - K(r)))$, in accord with Eq. (8b), from which $(U'(r, t_0)/U(r, t_0)) = (K'(r)/(1 - K(r)))$ as well follows; and $\ddot{V}(r, t_0) = -c^2 K(r)$, in accord with Eqs. (8c), (6a) and (6b). Therefore at time t_0 Eq. (3a) implies that,

$$-((rK'(r) + K(r))/(2r^2)) = -((4\pi G\rho(r, t_0))/c^4),$$

or,

$$(rK(r))' = ((8\pi G r^2 \rho(r, t_0))/c^4),$$

which determines $K(r)$ only up to an integration constant. But *the particular choice* of the integration constant that yields,

$$K(r) = ((8\pi G)/(c^4 r)) \int_0^r dr'(r')^2 \rho(r', t_0), \tag{8d}$$

is the one which *accords* with the Birkhoff theorem result that a spherically-symmetric metric inside an empty-space central region is that of flat space. That result follows from Eq. (8d) because if $\rho(r, t_0) = 0$ for $0 \le r \le r_0$, where $r_0 > 0$ then *it is as well true* that $K(r) = 0$ for $0 \le r \le r_0$, and *for those values of* r it is clear from Eq. (8c) and its initial conditions which are given by Eqs. (6b) and (6a) that we have the simple solution $V(r, t) = r^2$, which together with $K(r) = 0$ and Eq. (8a) *also* implies that $U(r, t) = 1$. These results for $V(r, t)$ and $U(r, t)$ show that the "comoving coordinate system" metric given by Eq.(1) is that of flat space for $0 \le r \le r_0$ when $\rho(r, t_0)$ is equal to zero for those values of r, which is in accord with the Birkhoff theorem.

A physically suggestive way to present the $K(r)$ of Eq. (8d) is,

$$K(r) = ((2GM(r))/(c^2 r)), \tag{8e}$$

where,

$$M(r) \stackrel{\text{def}}{=} (4\pi/c^2) \int_0^r dr'(r')^2 \rho(r', t_0), \tag{8f}$$

is *the cumulative effective mass from the origin to* r of the spherically-symmetric initial energy distribution $\rho(r, t_0)$.

Having obtained the definitive Eq. (8d) general result for $K(r)$, we now turn to the final issue of solving Eq. (8c) when its initial conditions for $V(r, t)$ are given by Eqs. (6b) and (6a). The nonlinear first-derivative term $-\frac{1}{4}((\dot{V})^2/V)$ of Eq. (8c) is readily transformed away by changing the dependent variable from V to $W = V^{\frac{3}{4}}$, i.e., $V = W^{\frac{4}{3}}$, with the result,

$$\ddot{W} = -\tfrac{3}{4} c^2 (K(r)/W^{\frac{1}{3}}),$$

whose initial conditions are given by $W(r, t_0) = r^{\frac{3}{2}}$ and $\dot{W}(r, t_0) = 0$. This second-order in time equation for W presents a distinctly Newtonian dynamical impression of purely radial motion in the presence of a (peculiar) central force. We therefore treat it accordingly, multiplying it through by the quintessential Newtonian dynamical integrating factor $2\dot{W}$ in order to carry out its dynamical first integration. Taking into account this equation's initial conditions, the result of its dynamical first integration is,

$$(\dot{W})^2 = -\left(\tfrac{3}{2}\right)^2 c^2 K(r)(W^{\frac{2}{3}} - r),$$

an equation form which virtually *begs* for its dependent variable to be changed to $\varrho = W^{\frac{2}{3}} = V^{\frac{1}{2}}$, i.e., $W = \varrho^{\frac{3}{2}}$. Since $\dot{W} = \frac{3}{2}\varrho^{\frac{1}{2}}\dot{\varrho}$, in terms of ϱ the above equation's form changes to,

$$(\dot{\varrho})^2 = -c^2 K(r)(1 - (r/\varrho)),$$

which is very close to the *time cycloid* equation form of Oppenheimer and Snyder [20]. It is, in fact, a minor matter to change the above equation to precisely that time cycloid form by the scaling change of the dependent variable to the dimensionless $R = (\varrho/r) = (V^{\frac{1}{2}}/r)$, i.e., $\varrho = rR$,

$$(\dot{R})^2 = (c^2/r^2)K(r)((1/R) - 1) = ((2GM(r))/r^3)((1/R) - 1), \tag{9a}$$

where the second equality follows from Eq. (8e). The initial condition,

$$R(r, t_0) = 1, \tag{9b}$$

is, of course, one of the initial conditions at $t = t_0$ of $((V(r,t))^{\frac{1}{2}}/r)$. The further "initial condition" $\dot{R}(r, t_0) = 0$ that corresponds to $\dot{V}(r, t_0) = 0$ follows from Eq. (9a) itself in conjunction with its Eq. (9b) initial condition.

A slightly simpler and more suggestive way to present Eq. (9a) is,

$$(\dot{R})^2 = (\omega(r))^2((1/R) - 1), \tag{9c}$$

where,

$$\omega(r) = (c/r)(K(r))^{\frac{1}{2}} = ((2GM(r))/r^3)^{\frac{1}{2}}. \tag{9d}$$

The time cycloid $R(r, t)$ of Eq. (9c) can't be expressed in terms of elementary functions, but *its inverse for its first half-period* (during which it falls from unity to zero strictly monotonically) *does* have an expression in terms of elementary functions,

$$\arccos((R(r,t))^{\frac{1}{2}}) + (R(r,t)(1 - R(r,t)))^{\frac{1}{2}} = \omega(r)(t - t_0) \text{ when } 0 \le (t - t_0) \le (\pi/(2\omega(r))). \tag{10a}$$

Differentiating both sides of the equality in Eq. (10a) with respect to time produces the the *specialized* time cycloid relation,

$$\dot{R}(r,t) = -\omega(r)((1/R(r,t)) - 1)^{\frac{1}{2}}, \tag{10b}$$

which is *consistent* with Eq. (9c) but *is itself only valid for the first half-period, namely*, $0 \le (t - t_0) < (\pi/(2\omega(r)))$. More generally, however, Eq. (9c) *itself* is obviously *consistent with*,

$$\dot{R}(r,t) = \pm[-\omega(r)((1/R(r,t)) - 1)^{\frac{1}{2}}], \tag{10c}$$

where *in fact* $\pm$ is the *plus* sign during the *odd* half-periods of the time cycloid $R(r, t)$ and is the *minus* sign during that time cycloid's *even* half-periods—*the time duration of a half-period of* $R(r,t)$ *is of course* $(\pi/(2\omega(r)))$. This *particular* choice of the $\pm$ sign ensures that $R(t, r)$ is not only *periodic in time* with time period $(\pi/\omega(r))$, but is also *continuous in time*, notwithstanding that its time derivative $\dot{R}(r, t)$ *diverges* at the *odd* half-period time points. Dividing both sides of Eq. (10c) by $\pm((1/R(r,t)) - 1)^{\frac{1}{2}}$ and then *integrating both sides of the result with respect to time* produces the *extension* of Eq. (10a) to,

$$\pm[\arccos((R(r,t))^{\frac{1}{2}}) + (R(r,t)(1 - R(r,t)))^{\frac{1}{2}}] = \omega(r)(t - t_0), \tag{10d}$$

where $\pm$ is the *plus* sign during the *odd* half-periods of the time cycloid $R(r,t)$ and is the *minus* sign during that time cycloid's *even* half-periods—the time duration of a half-period of $R(r,t)$ is $(\pi/(2\omega(r)))$.

Another useful relationship follows from differentiating both sides of Eq. (10d) with respect to r, namely,

$$rR'(r,t) = \pm[-r\omega'(r)(t - t_0)((1/R(r,t)) - 1)^{\frac{1}{2}}] = ((r\omega'(r))/\omega(r))(t - t_0)\dot{R}(r,t), \tag{10e}$$

where the second equality follows from Eq. (10c). The presence of the factor of $(t - t_0)$ on the right-hand side of these two equalities shows that $rR'(r, t)$ *departs* from the *periodicity in time* (with full-period time duration $(\pi/\omega(r))$) which is manifested by the time cycloid $R(r, t)$ and its time derivative $\dot{R}(r, t)$.

This fact *isn't* relevant to the *particular* "comoving coordinate system" metric function $V(r, t)$ because $R(r,t) = ((V(r,t))^{\frac{1}{2}}/r)$ implies that,

$$V(r,t) = r^2(R(r,t))^2, \tag{11a}$$

which is clearly periodic in time if $R(r, t)$ is periodic in time.

However, because Eq. (11a) implies that,

$$V'(r,t) = 2rR(r,t)(R(r,t) + rR'(r,t)),$$

the departure from periodicity in time of $rR'(r, t)$ *is* relevant to the "comoving coordinate system" metric function $U(r, t)$, since it involves $(V')^2$ according to Eq. (8a) above, namely,

$$U = (((V')^2/V)/(4(1 - K(r)))) = ((R + rR')^2/(1 - K(r))), \tag{11b}$$

which shows that the departure from periodicity in time of rR' does infect the particular "comoving coordinate system" metric function U.

The density $\rho(r,t)$ as well departs from periodicity in time since it involves V' according to Eq. (7) above, namely,

$$\rho(r,t) = ((2r^2\rho(r,t_0))/(V^{\frac{1}{2}}V')) = (\rho(r,t_0)/(R^2(R+rR'))), \tag{11c}$$

which shows that the departure from periodicity in time of rR' infects $\rho(r,t)$.

With the *exception* of these special metric and density departures from periodicity in time, which bring to mind the perihelion *precession* of basically *periodic* planetary orbits, the "comoving coordinate system" metric tends to strongly reflect *merely Newtonian* gravitational physics.

For example, if we go to the limit that the initial effective mass density $(\rho(r,t_0)/c^2)$ describes a point mass at $r = 0$, so that the cumulative mass $M(r)$ of Eq. (8f) is equal to some constant mass $M > 0$ for all $r > 0$, we see from Eq. (9a) that the basic "comoving coordinate system" metric time-cycloid equation becomes,

$$(\dot{R})^2 = ((2GM)/r^3)((1/R) - 1), \tag{12}$$

for all $r > 0$, with the initial condition $R(r,t_0) = 1$.

Now compare the behavior of the metric-related dimensionsless entity R of Eq. (12) with the behavior of a Newtonian test particle which falls *from initial rest* toward a point mass $M > 0$ located at the origin, from which that test particle *is initially separated by the radial distance* r_0. The Newtonian equation for that test particle's ensuing purely radial motion is,

$$\ddot{r} = -((GM)/r^2),$$

with the initial conditions $r(t_0) = r_0$ and $\dot{r}(t_0) = 0$. We multiply this equation of motion through by the usual Newtonian integrating factor $2\dot{r}$ to carry out its dynamical first integration, whose constant of integration of course accords with the two initial conditions given in the preceding sentence. The result is,

$$(\dot{r})^2 = (2GM)((1/r) - (1/r_0),$$

which when differentiated with respect to time produces the original Newtonian equation of motion $\ddot{r} = -((GM)/r^2)$ noted above, and which at time t_0 as well accords with the two initial conditions. If we now make the simple scaling change to the new dimensionless variable $R = (r/r_0)$ whose initial condition is $R(t_0) = 1$, so that $r = r_0 R$, our above dynamical first integration result becomes,

$$(\dot{R})^2 = ((2GM)/(r_0)^3)((1/R) - 1),$$

which is identical in form to Eq. (12), and whose initial condition for R at time t_0 *is the same as well*.

Thus the "comoving coordinate system" evidences *no hint at all* of departures from Newtonian gravitational physics *that are due to gravitational redshift*; it is *only* with respect to the existence of departures from *periodicity* (phenomena with a flavor reminiscent of planetary perihelion precession) that "comoving coordinate system" gravitational physics reveals clearly discernible *differences* from Newtonian gravitational physics. Therefore attribution of *relativistic gravitational physical legitimacy* to the "comoving coordinate system" *is a grave error*: specifically the underlying *radius-dependent Newtonian time cycloids* that are *intrinsic* to the "comoving coordinate system" metric *manifestly cause that metric to periodically violate the Metric Signature Theorem* [21] *which follows directly from the Principle of Equivalence*. This occurs at the *odd* cycloid half-periods $t = (2n+1)(\pi/(2\omega(r)))$, for $n = 0, 1, 2, \ldots$, at which times $R(r,t) = 0$ and therefore the diagonal-component metric function $V(r,t) = 0$ (see Eq. (11a)).

The occurrence of such singularity-inducing time-periodic metric anomalies that violate the Principle of Equivalence make it apparent *that only singular mappings* from the "comoving coordinate system" onto more "standard" coordinate systems (i.e., those *capable* of accommodating gravitational redshift because their g_{00} metric component *isn't* fixed to unity) *could possibly be compatible with the honoring of the Principle of Equivalence by those latter coordinate systems*. For example, *the gravitational redshift* which a more "standard" mapped-onto coordinate system manifests could be such that *all* times greater than or equal to the shortest of the time-cycloid half-periods of the "comoving" system *get singularly mapped to infinite time* in the more "standard" system: such a redshift singularity of the mapping *erases* all the time-periodic metric singularities of the "comoving" system which violate the Principle of Equivalence.

In the next section *we review in detail the development* of the singular Oppenheimer-Snyder mapping between the nonempty-space part of a particular "comoving coordinate system" metric and the "standard"-form metric—the intricate character of this mapping is beyond understanding in the absence of a comprehensive appreciation of the imperatives *that are connected with its development*. This Oppenheimer-Snyder mapping does indeed turn out *to singularly map to infinite time* all the "comoving coordinate system" times associated with its nonempty-space part that are greater than or equal to its time-cycloid half-period (the Oppenheimer-Snyder system has *only one* such time-cycloid half-period).

The technical way in which this occurs is that *the time part* of the Oppenheimer-Snyder mapping *is an integral* which *diverges* for all time values at and beyond the "comoving" system's time-cycloid half-period. In its region of *convergence* this integral presentation of the time part of the mapping can (with sufficient effort) be evaluated *analytically*. The resulting analytic expression, *albeit only applicable where the integral that produces it converges*, nevertheless *also* automatically provides irrelevant *purely mathematical analytic continuation into the region where its underlying integral diverges*. This *inapplicable* automatic analytic continuation of the integral into space-time regions *where it actually diverges* can bewilder an inattentive reader with nonexistent *apparent anomalies*. Thus it has been remarked that the time part of the Oppenheimer-Snyder mapping is physically inadmissible because one of its contributing terms can be a logarithm of negative argument [9, 11], when *that* is in fact *purely* a property of the time part's *inapplicable automatic analytic continuation to the region where the time part itself in fact diverges*. It is also asserted that physically admissible space-time mappings can't embrace divergences [9, 11], which is actually an essential corollary of the Principle of Equivalence, but one that *here* is completely *subverted* by the periodic violation of the Principle of Equivalence by the "comoving coordinate system's" metric-function time cycloids; in consequence *the only hope* for the Principle of Equivalence *not to be violated in the coordinate system that is mapped onto* is in fact for that mapping to be *singular*.

The singular Oppenheimer-Snyder mapping developed from scratch

Now that we have in hand from Eqs. (11a) and (11b) the full analytic details of the "comoving coordinate system" metric of Eq. (1) for spherically-symmetric radially-nonuniform dust that is initially started from rest, we need to map this metric onto a more "standard" one that *unlike* the "comoving coordinate system" metric *admits relativistic gravitational redshift by not fixing its g_{00} component to unity*. Because of their intention to take technical advantage of the Birkhoff theorem, Oppenheimer and Snyder chose to map the spherically-symmetric "comoving coordinates" (t, r, θ, ϕ), in terms of which the invariant line element ds^2 is given by Eq. (1), into spherically-symmetric "standard coordinates" [22] $(\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi})$, in terms of which the *same* invariant line element ds^2 is given by,

$$\begin{aligned} ds^2 &= B(\bar{r},\bar{t})(cd\bar{t})^2 - A(\bar{r},\bar{t})(d\bar{r})^2 - \bar{r}^2((d\bar{\theta})^2 + (\sin\bar{\theta}d\bar{\phi})^2) \\ &= (cdt)^2 - U(r,t)(dr)^2 - V(r,t)((d\theta)^2 + (\sin\theta d\phi)^2). \end{aligned} \tag{13a}$$

Inspection in Eq. (13a) *of the rightmost two terms* of the line element ds^2 in both its "standard" and its "comoving" form immediately reveals three very convenient mapping choices,

$$\bar{\theta} = \theta,\ \bar{\phi} = \phi \text{ and } \bar{r} = (V(r,t))^{\frac{1}{2}} = rR(r,t), \tag{13b}$$

where we have used the Eq. (11a) relation $V(r,t) = r^2(R(r,t))^2$. Next we obviously would like to obtain $\bar{t}$ as a function of r and t, just as has been done in Eq. (13b) for $\bar{r}$. Inspection of Eq. (13a), however, reveals that that task is completely entwined with the determination of B and A as functions of r and t; moreover $\bar{t}$ *itself doesn't occur in relations that can be extracted from* Eq. (13a), *only its partial derivatives* $(\partial\bar{t}/\partial t)$ *and* $(c(\partial\bar{t}/\partial r))$ *do*. We are thus faced with solving *both* simultaneous algebraic *and* first-order partial differential equations merely to obtain $\bar{t}(r,t)$! The path ahead thus appears long and tortuous, *enough so that the ensuing development is almost never actually presented in detail*. That is all the *more* reason to do so *here*, above all in view of the fact that the narrowly circumscribed and specialized nature *and* baffling intricacy of Oppenheimer and Snyder's singular result for $\bar{t}(r,t)$ is simply beyond comprehension if this journey *isn't* painstakingly pursued to its conclusion.

Presenting now in greater detail that part of Eq. (13a) *which isn't eliminated* by the three mapping choices of Eq. (13b),

$$B[(\partial\bar{t}/\partial t)(cdt) + c(\partial\bar{t}/\partial r)dr]^2 - A[(1/c)(\partial\bar{r}/\partial t)(cdt) + (\partial\bar{r}/\partial r)dr]^2 = (cdt)^2 - U(r,t)(dr)^2. \tag{13c}$$

Since the three bilinear differential forms $(cdt)^2$, $(2c\,dt\,dr)$ and $(dr)^2$ are linearly independent, Eq. (13c) produces *the three simultaneous equations*,

$$B(\partial\bar{t}/\partial t)^2 - A((1/c)(\partial\bar{r}/\partial t))^2 = 1, \tag{14a}$$

$$B(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)) - A((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r) = 0, \quad (14b)$$

$$B(c(\partial\bar{t}/\partial r))^2 - A(\partial\bar{r}/\partial r)^2 = -U. \quad (14c)$$

We now eliminate A and B from Eqs. (14) in order to obtain the partial differential equation for $\bar{t}$. Solving Eq. (14b) for A yields,

$$A = \frac{B(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r))}{((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r)}. \quad (15a)$$

We now insert this value of A into each one of Eqs. (14a) and (14c) and follow that by solving each one for $(1/B)$,

$$(1/B) = (\partial\bar{t}/\partial t)^2 - \left[\frac{((1/c)(\partial\bar{r}/\partial t))}{(\partial\bar{r}/\partial r)}\right](\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)), \quad (15b)$$

$$(1/B) = \left[\frac{(\partial\bar{r}/\partial r)}{U((1/c)(\partial\bar{r}/\partial t))}\right](\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)) - \left[\frac{1}{U}\right](c(\partial\bar{t}/\partial r))^2. \quad (15c)$$

Subtracting Eq. (15c) from Eq. (15b) reveals the vanishing of a homogeneous bilinear form in the two unknown partial derivatives $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$,

$$(\partial\bar{t}/\partial t)^2 - \left(\left[\frac{((1/c)(\partial\bar{r}/\partial t))}{(\partial\bar{r}/\partial r)}\right] + \left[\frac{(\partial\bar{r}/\partial r)}{U((1/c)(\partial\bar{r}/\partial t))}\right]\right)(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)) + \left[\frac{1}{U}\right](c(\partial\bar{t}/\partial r))^2 = 0. \quad (15d)$$

The homogeneous bilinear form in $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$ on the left-hand side of Eq. (15d) *factors* into the product of two homogeneous *linear* forms in $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$, namely,

$$\left((\partial\bar{t}/\partial t) - \left[\frac{((1/c)(\partial\bar{r}/\partial t))}{(\partial\bar{r}/\partial r)}\right](c(\partial\bar{t}/\partial r))\right)\left((\partial\bar{t}/\partial t) - \left[\frac{(\partial\bar{r}/\partial r)}{U((1/c)(\partial\bar{r}/\partial t))}\right](c(\partial\bar{t}/\partial r))\right) = 0. \quad (15e)$$

If the linear form in $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$ in the *first* factor on the left-hand side of Eq. (15e) vanished, then both Eq. (15b) and Eq. (15c) would yield that $(1/B)$ vanishes. Since we seek a non-pathological result for the unknown "standard" metric function B, we must assume that the *second* factor on the left-hand side of Eq. (15e) vanishes, which yields the following homogeneous linear first-order partial differential equation for the unknown time part $\bar{t}(r,t)$ of the mapping from spherically-symmetric "comoving" space-time coordinates to spherically-symmetric "standard" space-time coordinates,

$$((1/c)(\partial\bar{r}/\partial t))(\partial\bar{t}/\partial t) = (1/U)(\partial\bar{r}/\partial r)(c(\partial\bar{t}/\partial r)). \quad (15f)$$

Since $\bar{r}(r,t) = rR(r,t)$ from Eq. (13b) and $U(r,t) = (R(r,t) + rR'(r,t)))^2/(1 - K(r))$ from Eq. (11b), the Eq. (15f) partial differential equation for $\bar{t}(r,t)$ is in detail,

$$(r/c)\dot{R}(r,t)(\partial\bar{t}/\partial t) = \left[\frac{R(r,t)(1-K(r))}{(R(r,t)+rR'(r,t))^2}\right](c(\partial\bar{t}/\partial r)). \quad (15g)$$

The only well-known way to obtain an *analytic* solution to a homogeneous linear partial differential equation such as Eq. (15g) is by separation of variables. But insofar as R(r, t) depends on r as well as on t, it is apparent that Eq. (15g) doesn't lend itself to separation in the variables t and r. However, if the cumulative effective mass $M(r)$ defined by Eq. (8f) is proportional to r^3 for a range of values of r, then we see from Eqs. (9a) and (9b) that $R(r,t)$ will be independent of r over that range of values of r, and therefore Eq. (15g) *will* be analytically solvable by separation of variables for that range of values of r. That will be the case for a range of values of r of the form $0 < r \leq a$ whenever $\rho(r,t_0)$ *is constant as a function of* r *over that range of values of* r. If *in addition* $\rho(r,t_0)$ *vanishes altogether for* $r > a$, *the Birkhoff theorem will apply to that empty-space region*, which provides a constraint on the values of the unknown "standard" metric functions A and B of Eq. (13a) *at the boundary radius* $r = a$ *of that empty-space region*. Thus by going along with the Oppenheimer-Snyder *imperative* of finding an *analytic* solution, we find that we are channeled into treating exclusively the particular limited and narrowly circumscribed gravitational physics which they treated. It is convenient to characterize this physical configuration at the initial time t_0 by $M(a)$, which we denote as M. Then $\rho(r,t_0)$ has the value $((3Mc^2)/(4\pi a^3))$ for $0 \leq r \leq a$ and the value zero for $r > a$. From Eqs. (8f) and (9d) we further see that,

$$\omega = ((2GM)/a^3)^{\frac{1}{2}} \text{ for } 0 \leq r \leq a, \quad (16a)$$

and,

$$K(r) = (\omega/c)^2 r^2 \text{ for } 0 \leq r \leq a. \quad (16b)$$

Taking note of Eqs. (16a), (16b), (9a) and (9b) we also see that for $0 \le r \le a$, Eq. (15g) becomes,

$$(r/c)\dot{R}(t)(\partial\bar{t}/\partial t) = \left[\tfrac{(1-(\omega/c)^2r^2)}{R(t)}\right](c(\partial\bar{t}/\partial r)), \tag{16c}$$

where,

$$(\dot{R}(t))^2 = \omega^2((1/R(t)) - 1). \tag{16d}$$

Making the variable-separation Ansatz $\bar{t}(r,t) = (1/\omega)\alpha(r)\beta(t)$ in Eq. (16c) then yields,

$$R(t)\dot{R}(t)(d(ln(\beta(t)))/dt) = -p\omega^2 = (1-(\omega/c)^2r^2)(c^2/r)(d(ln(\alpha(r)))/dr), \tag{16e}$$

where p is an arbitrary dimensionless constant. The separated equation for $\alpha(r)$ is straightforward to solve, and yields,

$$\alpha(r) = C_1(1-(\omega/c)^2r^2)^{p/2}, \tag{16f}$$

where C_1 is an arbitrary dimensionless constant, but to solve the separated equation for $\beta(t)$ it must be borne in mind that from Eq. (16d), $\dot{R} = \mp\omega((1-R)/R)^{\frac{1}{2}}$ and likewise, $dt = dR/\dot{R} = \mp(1/\omega)(R/(1-R))^{\frac{1}{2}}dR$. Using these relations in Eq. (16e), one readily obtains,

$$\beta(t) = C_2(1-R(t))^p, \tag{16g}$$

where C_2 is an arbitrary dimensionless constant. From Eqs. (16f), (16g) and the variable-separation Ansatz above, we obtain,

$$\bar{t}(r,t) = (1/\omega)C[(1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-R(t))]^p, \tag{16h}$$

where C and p are arbitrary dimensionless constants. Since the Eq. (16c) partial differential equation is homogeneous and linear, *any linear combination of its solutions of the form of the solution given in* Eq. (16h) *is also a solution*. That fact leads us to expect that given *any arbitrary sufficiently smooth dimensionless function $\phi(u)$ of a single dimensionless variable u*, the form,

$$\bar{t}(r,t) = (1/\omega)\phi(u(r,t)),\ \text{where}\ u(r,t) \stackrel{\text{def}}{=} [(1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-R(t))], \tag{16i}$$

will be a solution of the Eq. (16c) partial differential equation. That this expectation is actually true is readily verified by substitution of Eq. (16i) into Eq. (16c).

In the region $0 \le r \le a$ we now have the Eq. (16i) general form of $\bar{t}(r,t)$ in addition to our previous knowledge that $\bar{r}(r,t) = rR(t)$, $U(r,t) = ((R(t))^2/(1-(\omega/c)^2r^2))$, and $\dot{R}(t) = \mp\omega((1-R(t))/R(t))^{\frac{1}{2}}$. This at long last permits us to use Eqs. (15b) and (15a) to obtain the unknown "standard"-form metric functions B and A. Requiring these "standard" B and A to adhere to the Birkhoff theorem at the empty-space boundary $r = a$ will then pin down $\bar{t}(r,t)$ *as a specific entity*, not merely the general form given by Eq. (16i). At that point *the singular nature of* $\bar{t}(r,t)$, the time part of the space-time mapping from *the relativistically unphysical* "comoving" coordinate system to the "standard" coordinate system, *will be manifest*.

First, however, we must have in hand the evaluated partial derivatives that are needed in Eqs. (15b) and (15a) to calculate the unknown "standard" metric functions B and A. By making use of Eq. (16i) for $\bar{t}(r,t)$ and the two relations $\bar{r}(r,t) = rR(t)$ and $\dot{R}(t) = \mp\omega((1-R(t))/R(t))^{\frac{1}{2}}$, we obtain the needed four partial derivatives,

$$(c(\partial\bar{t}/\partial r)) = -(\omega/c)r(1-(\omega/c)^2r^2)^{-\frac{1}{2}}(1-R(t))\phi'(u(r,t)), \tag{17a}$$

$$(\partial\bar{t}/\partial t) = \pm(1-(\omega/c)^2r^2)^{\frac{1}{2}}((1-R(t))/R(t))^{\frac{1}{2}}\phi'(u(r,t)), \tag{17b}$$

$$((1/c)(\partial\bar{r}/\partial t)) = \mp(\omega/c)r((1-R(t))/R(t))^{\frac{1}{2}}, \tag{17c}$$

$$(\partial\bar{r}/\partial r) = R, \tag{17d}$$

where $u(r,t) = ((1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-R(t)))$. Eqs. (17) together with Eq. (15b) for $(1/B)$ yield,

$$(1/B(r,t)) = ((1-R(t))/R(t)^2)(R(t)-(\omega/c)^2r^2)(\phi'(u(r,t)))^2,$$

or,

$$B(r,t) = [R(t)^2/((1-R(t))(R(t)-(\omega/c)^2r^2)(\phi'(u(r,t)))^2)]. \tag{18a}$$

Together with Eqs. (17) and (18a), Eq. (15a) for A then yields,

$$A(r,t) = (R(t)/(R(t)-(\omega/c)^2r^2)). \tag{18b}$$

Since Eq. (18b) for the particular "standard" metric function $A(r,t)$ *happens to have no dependence on the not yet determined dimensionless function* ϕ, we can straightaway *check* whether the behavior of $A(r,t)$ at the $r=a$ empty-space boundary accords with what would be expected from the Birkhoff theorem. First we absorb all dependence that $A(r,t)$ has on the "comoving" *time* t into the "standard" *spatial* radial coordinate $\bar{r}=rR(t)$ by everywhere *replacing* $R(t)$ by $(\bar{r}/r)$. As a result, $A(r,t)$ can be written,

$$A(r,\bar{r})=(1/(1-(((\omega/c)^2r^3)/\bar{r}))), \tag{19a}$$

and its value at the $r=a$ empty-space boundary is,

$$A(r=a,\bar{r})=(1/(1-(((\omega/c)^2a^3)/\bar{r})))=(1/(1-((2GM)/(c^2\bar{r})))), \tag{19b}$$

where in the second equality of Eq. (19b) we have used the fact pointed out in Eq. (16a) that $\omega=((2GM)/a^3)^{\frac{1}{2}}$ for $0\le r\le a$. This second equality shows that the metric function A at the $r=a$ empty-space boundary indeed adheres to the form that it is expected to from the Birkhoff theorem.

For the "standard" metric function $B(r,t)$ of Eq. (18a), Oppenheimer and Snyder worked out the the so far undetermined dimensionless $(\phi'(u(r,t)))^2$, where $u(r,t)=((1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-R(t)))$, by requiring that at the $r=a$ empty-space boundary $B(r=a,\bar{r})$ *also* adheres to the form that it is expected to from the Birkhoff theorem. With $\phi'(u(r,t))$ thus in hand, Oppenheimer and Snyder obtained $\bar{t}(r,t)=(1/\omega)\phi(u(r,t))$ via *integration* with respect to u of the $\phi'(u)$ which they worked out. This time part $\bar{t}(r,t)$ of the mapping from "comoving" to "standard" coordinates turns out *to diverge for* $0\le r\le a$ *at all sufficiently large "comoving" times* t; indeed its *divergence* occurs *even before* the "comoving" metric's time cycloid function $R(t)$ attains *the singular "comoving" metric value zero* at its *first* half-period time point $t=(t_0+(\pi/(2\omega)))$. The *divergent* singular character of the $\bar{t}(r,t)$ time part of the *mapping* from "comoving" to "standard" coordinates *thus eliminates all the time-periodic singlarities that are inherent to the quasi-Newtonian "comoving" metric by properly accounting for gravitational redshift, which the relativistically unphysical "comoving" metric inherently cannot do*. The divergent singular nature of $\bar{t}(r,t)$ is obviously *a physically essential antidote to the quasi-Newtonian periodic singularities of the relativistically unphysical "comoving" metric*.

To work out $(\phi'(u(r,t)))^2$, we first eliminate, in analogy with Eqs. (19) above, from the $B(r,t)$ given by Eq. (18a) its dependence on the "comoving" time t by replacing all occurrences of $R(t)$ by $\bar{r}/r$,

$$B(r,\bar{r})=[(\bar{r}/r)/((1-(\bar{r}/r))(1-(((\omega/c)^2r^2)/(\bar{r}/r)))(\phi'((1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-(\bar{r}/r))))^2)]. \tag{20a}$$

In order for $B(r,t)$ to adhere to the form at the empty-space boundary $r=a$ that it is expected to from the Birkhoff theorem, we must have that,

$$B(r=a,\bar{r})=(1-((2GM)/(c^2\bar{r})))=(1-(((\omega/c)^2a^2)/(\bar{r}/a))), \tag{20b}$$

where the second equality follows the Eq. (16a) result that $\omega=((2GM)/a^3)^{\frac{1}{2}}$ for $0\le r\le a$.

In order to make the algebra needed to achieve Eq. (20b) more transparent, we emulate Oppenheimer and Snyder's introduction of a *new* dimensionless function $S(r,\bar{r})$ which is linearly related to the *argument* $((1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-(\bar{r}/r)))$ of the dimensionless function ϕ' in Eq. (20a),

$$S(r,\bar{r})\overset{\text{def}}{=}1-(1-((\omega/c)^2a^2)^{-\frac{1}{2}}(1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-(\bar{r}/r))), \tag{20c}$$

but which, *unlike* that argument, *has an extremely convenient value at the empty-space boundary* $r=a$, namely,

$$S(r=a,\bar{r})=(\bar{r}/a). \tag{20d}$$

Using Eq. (20c) we can reexpress the *argument* $((1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-(\bar{r}/r)))$ of ϕ' in Eq. (20a) as $((1-(\omega/c)^2a^2)^{\frac{1}{2}}(1-S(r,\bar{r})))$, and therefore regard the function ϕ' of the argument which it has in Eq. (20a) to be a function ψ of the argument $S(r,\bar{r})$, namely,

$$\psi(S(r,\bar{r}))\overset{\text{def}}{=}\phi'((1-(\omega/c)^2a^2)^{\frac{1}{2}}(1-S(r,\bar{r})))=\phi'((1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-(\bar{r}/r))), \tag{20e}$$

where the last equality follows from Eq. (20c). Therefore in terms of $\psi(S(r,\bar{r}))$, Eq. (20a) can be rewritten as,

$$B(r,\bar{r})=[(\bar{r}/r)/((1-(\bar{r}/r))(1-(((\omega/c)^2r^2)/(\bar{r}/r)))(\psi(S(r,\bar{r})))^2)], \tag{20f}$$

which implies that,

$$B(r=a,\bar{r}) = [(\bar{r}/a)/((1-(\bar{r}/a))(1-(((\omega/c)^2a^2)/(\bar{r}/a)))(\psi(\bar{r}/a))^2)], \tag{20g}$$

where we have used the relation $S(r=a,\bar{r}) = (\bar{r}/a)$ given by Eq. (20d). Putting Eq. (20g) into the $r=a$ empty-space boundary Birkhoff theorem requirement of Eq. (20b) yields,

$$[(\bar{r}/a)/((1-(\bar{r}/a))(1-(((\omega/c)^2a^2)/(\bar{r}/a)))(\psi(\bar{r}/a))^2)] = (1-(((\omega/c)^2a^2)/(\bar{r}/a))), \tag{20h}$$

which determines that,

$$\psi(\bar{r}/a) = (1-(((\omega/c)^2a^2)/(\bar{r}/a)))^{-1}((\bar{r}/a)/(1-(\bar{r}/a)))^{\frac{1}{2}}. \tag{20i}$$

In light of Eq. (20d), Eq. (20i) will hold if,

$$\psi(S(r,\bar{r})) = (1-(((\omega/c)^2a^2)/S(r,\bar{r})))^{-1}(S(r,\bar{r})/(1-S(r,\bar{r})))^{\frac{1}{2}}, \tag{20j}$$

From the definition of $\psi(S(r,\bar{r}))$ in Eq. (20e), Eq. (20j) is equivalent to,

$$\phi'((1-(\omega/c)^2a^2)^{\frac{1}{2}}(1-S(r,\bar{r}))) = (1-(((\omega/c)^2a^2)/S(r,\bar{r})))^{-1}(S(r,\bar{r})/(1-S(r,\bar{r})))^{\frac{1}{2}}. \tag{20k}$$

Before we tackle the integration of Eq. (20k), it is convenient to replace $\bar{r}$ in $S(r,\bar{r})$, by $rR(t)$, which together with the Eq. (20c) definition of $S(r,\bar{r})$ defines $S(r,t)$, namely,

$$S(r,t) \stackrel{\text{def}}{=} S(r,\bar{r}=rR(t)) = 1-(1-(\omega/c)^2a^2)^{-\frac{1}{2}}(1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-R(t))). \tag{20l}$$

Therefore replacing $\bar{r}$ by $rR(t)$ in Eq. (20k) results in,

$$\phi'((1-(\omega/c)^2a^2)^{\frac{1}{2}}(1-S(r,t))) = (1-(((\omega/c)^2a^2)/S(r,t)))^{-1}(S(r,t)/(1-S(r,t)))^{\frac{1}{2}}. \tag{20m}$$

If we now define $u(r,t)$ as the *argument* of ϕ' in Eq. (20m), then,

$$u(r,t) \stackrel{\text{def}}{=} ((1-(\omega/c)^2a^2)^{\frac{1}{2}}(1-S(r,t))) = (1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-R(t)), \tag{20n}$$

where we used Eq. (20l) to obtain the second equality in Eq. (20n). Note that this second equality in Eq. (20n) *exactly corresponds* to the definition of $u(r,t)$ that was given in Eq. (16i). By inverting the definition of $u(r,t)$ as given in Eq. (20n) we as well obtain that,

$$S(r,t) = (1-((1-(\omega/c)^2a^2)^{-\frac{1}{2}}u(r,t))). \tag{20o}$$

By combining Eq. (20m) with Eqs. (20n) and (20o), we can write,

$$\phi'(u(r,t)) = (1-(((\omega/c)^2a^2)/s(u(r,t))))^{-1}(s(u(r,t))/(1-s(u(r,t))))^{\frac{1}{2}}, \tag{20p}$$

where,

$$s(u) \stackrel{\text{def}}{=} (1-((1-(\omega/c)^2a^2)^{-\frac{1}{2}}u)). \tag{20q}$$

The inverse of the linear function $s(u)$ of Eq. (20q) is,

$$u(s) = (1-(\omega/c)^2a^2)^{\frac{1}{2}}(1-s). \tag{20r}$$

Since $R(t_0) = 1$ we note from Eq. (20l) that $S(r,t_0) = 1$ and and from Eq. (20n) that $u(r,t_0) = 0$. Combining this with Eqs. (16i) and (20p) yields,

$$\bar{t}(r,t) = (1/\omega)\phi(u(r,t)) = (1/\omega)\int_0^{u(r,t)} du(s(u)/(s(u)-(\omega/c)^2a^2))(s(u)/(1-s(u)))^{\frac{1}{2}}, \tag{20s}$$

where $s(u)$ is given by Eq. (20q), and its inverse function $u(s)$ is given by Eq. (20r). We use the function $u(s)$ to change the integration variable in Eq. (20s) from u to s with the result,

$$\bar{t}(r,t) = (1/\omega)(1-(\omega/c)^2a^2)^{\frac{1}{2}}\int_{s(u(r,t))}^{1} ds(s/(s-(\omega/c)^2a^2))(s/(1-s))^{\frac{1}{2}}, \tag{20t}$$

where from Eqs. (20q), (20o), and (20l),

$$s(u(r,t)) = S(r,t) = 1 - (1 - (\omega/c)^2 a^2)^{-\frac{1}{2}}(1 - (\omega/c)^2 r^2)^{\frac{1}{2}}(1 - R(t))). \tag{20u}$$

The crucial property of the Eq. (20t) result for the time part $\bar{t}(r,t)$ of the mapping from the "comoving" coordinates to the "standard" coordinates *is that it diverges whenever*,

$$S(r,t) \le ((\omega/c)^2 a^2. \tag{20v}$$

Moreover, since from Eq. (20u),

$$R(t) - S(r,t) = [(1 - (\omega/c)^2 a^2)^{-\frac{1}{2}}(1 - (\omega/c)^2 r^2)^{\frac{1}{2}} - 1](1 - R(t)) \ge 0 \text{ for } r \le a,$$

it is the case that,

$$R(t) \ge S(r,t). \tag{20w}$$

Therefore the time part $\bar{t}(r,t)$ of the mapping from the "comoving" to the "standard" coordinates definitely diverges whenever,

$$R(t) \le ((\omega/c)^2 a^2), \tag{20x}$$

which indeed shows that *none* the unphysical quasi-Newtonian periodic time cycloid singularities of the "comoving" metric exist in the "standard" metric, where gravitational redshift is permitted to act instead of being artificially suppressed. Moreover, in terms of $\bar{r} = rR(t)$, the "standard" coordinate system radial coordinate, Eq. (20x) tells us that $\bar{t}(r,t)$ diverges whenever,

$$\bar{r} \le ((\omega/c)^2 a^2 r) \le ((\omega/c)^2 a^3) = ((2GM)/c^2), \tag{20y}$$

where we have again used the $\omega = ((2GM)/a^3)^{\frac{1}{2}}$ relation of Eq. (16a). Therefore in the "standard" coordinates, *the divergence of* $\bar{t}(r,t)$ *prevents the shrinking system from ever attaining its Schwarzschild radius* $((2GM)/c^2)$ *and forming an event horizon* [23]. In a nutshell, it is the *divergent* singular nature of the time part $\bar{t}(r,t)$ of the mapping from the "comoving" to the "standard" coordinates that *walls off* the *unphysical* features of the "comoving" metric, *preventing them from exerting any physically untoward effects whatsoever on the "standard" metric.*

The integral in the Eq. (20t) expression for $\bar{t}(r,t)$ can (with effort) be evaluated analytically *in the region where it doesn't diverge*, i.e., when,

$$S(r,t) > ((\omega/c)^2 a^2). \tag{21a}$$

The caveat that the analytic result for $\bar{t}(r,t)$ *applies only in the region described by* Eq. (21a) *must be kept strictly in mind* because the analytic result itself *automatically* provides *a completely inapplicable* and potentially extremely misleading *analytic continuation into the region where the underlying integral diverges*. That fact has indeed sometimes caused confusion in the past [9, 11].

We now sketch the main steps of the analytic evaluation of the $\bar{t}(r,t)$ *where it doesn't diverge*. To reduce expression bulk, we reexpress Eq. (20t) in a streamlined notation,

$$\omega\bar{t}_\alpha(S) = (1-\alpha)^{\frac{1}{2}} \int_S^1 ds(s/(s-\alpha)(s/(1-s))^{\frac{1}{2}}, \tag{21b}$$

where,

$$\alpha \stackrel{\text{def}}{=} ((\omega/c)^2 a^2), \tag{21c}$$

and,

$$S \stackrel{\text{def}}{=} S(r,t) = 1 - (1 - (\omega/c)^2 a^2)^{-\frac{1}{2}}(1 - (\omega/c)^2 r^2)^{\frac{1}{2}}(1 - R(t))). \tag{21d}$$

Note that the $\omega\bar{t}_\alpha(S)$ of Eq. (21b) *diverges whenever* $S \le \alpha$ *and is convergent only for* $S > \alpha$. The first step of its evaluation *in its region of convergence* is to change the integration variable to $v = ((1-s)/s)^{\frac{1}{2}}$, so that $s = (1/(1+v^2))$, $ds = -2dv(v/(1+v^2)^2)dv$ and $sds(s/(1-s))^{\frac{1}{2}} = -2dv(1/(1+v^2)^3)$. The upshot of this variable change is,

$$\omega\bar{t}_\alpha(S) = 2(1-\alpha)^{-\frac{1}{2}} \int_0^{((1-S)/S)^{\frac{1}{2}}} dv(1/(1+v^2)^2)(1/(1-(\alpha/(1-\alpha))v^2)). \tag{21e}$$

The next step is the three-term partial fraction expansion of the integrand,

$$\omega\bar{t}_\alpha(S) = 2(1-\alpha)^{\frac{1}{2}} \int_0^{((1-S)/S)^{\frac{1}{2}}} dv\{(1/(1+v^2)^2) + (\alpha/(1+v^2)) + ((\alpha^2/(1-\alpha))/(1-(\alpha/(1-\alpha))v^2))\}. \tag{21f}$$

In Eq. (21f) the third term *itself* requires a *further* elementary two-term partial fraction expansion,

$$\omega\bar{t}_\alpha(S) = 2(1-\alpha)^{\frac{1}{2}} \int_0^{((1-S)/S)^{\frac{1}{2}}} dv\{(1/(1+v^2)^2) + (\alpha/(1+v^2)) + [(\alpha^2/(2(1-\alpha)))[(1/(1+(\alpha/(1-\alpha))^{\frac{1}{2}}v)) + (1/(1-(\alpha/(1-\alpha))^{\frac{1}{2}}v))]]\}. \tag{21g}$$

The last three terms of the above integrand are elementary to integrate, moreover it is readily verified that the first term yields,

$$2(1-\alpha)^{\frac{1}{2}} \int_0^{((1-S)/S)^{\frac{1}{2}}} dv(1/(1+v^2)^2) = [(1-\alpha)^{\frac{1}{2}}[(S(1-S))^{\frac{1}{2}} + \arctan(((1-S)/S)^{\frac{1}{2}})]]. \tag{21h}$$

The entire result, *which, however, only applies when* $S > \alpha$, is therefore,

$$\omega\bar{t}_\alpha(S) = \{[(1-\alpha)^{\frac{1}{2}}[(S(1-S))^{\frac{1}{2}} + ((1+2\alpha)\arctan(((1-S)/S)^{\frac{1}{2}}))]] + [\alpha^{\frac{3}{2}}[\ln(1+(\alpha/(1-\alpha))^{\frac{1}{2}}((1-S)/S)^{\frac{1}{2}}) - \ln(1-(\alpha/(1-\alpha))^{\frac{1}{2}}((1-S)/S)^{\frac{1}{2}})]]\}. \tag{21i}$$

With regard to the range of applicability of Eq. (21i), we *reiterate that* $\omega\bar{t}_\alpha(S)$ *diverges for* $S \le \alpha$. The ostensible "issue" of logarithms of negative argument [9, 11] *is merely a distraction by the completely inapplicable* (but automatically mathematically feasible) *analytic continuation of the valid convergent result of* $\omega\bar{t}_\alpha(S)$ *for* $S > \alpha$ *into the region* $S < \alpha$ *where the actual integral expression of* Eq. (21b) *for* $\omega\bar{t}_\alpha(S)$ *clearly diverges*.

What is occurring here is "the vanquishing of singularity by singularity". "Comoving coordinate systems" have relativistically unphysical time-cycloid quasi-Newtonian metrics which exhibit no trace of gravitational redshift but manifest periodic singular violations of the Principle of Equivalence. They retain a degree of tortured mathematical/calculational relationship to valid relativistic gravitational physics, but this relationship *must necessarily be singular in character in order for it to be able to banish the relativistically unphysical time-cycloid metric singularities that are inherent to "comoving coordinate systems"*. That is the *reason* why the Eq. (21b) time part $\omega\bar{t}_\alpha(S)$ of the Oppenheimer-Snyder mapping *has divergence as its most prominent and physically relevant feature*.

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