Cyclic Nature of Energy-Conserving “Gravitational Collapse”

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Abstract

The “collapse” of a solely gravitationally-interacting, energy-conserving dynamical system necessarily involves the time evolution of a bound state of that system. An archetypal feature of energy-conserving bound state time evolution is its cyclicity, its predilection to forever revisit the parts of phase space it has previously touched. Thus it isn’t surprising that the energy-conserving position-independent dust density gravitational model of Oppenheimer and Snyder produces a Robertson-Walker metric that is time-periodic, specifically time-cycloidal. In fact a mere pair of Newtonian point masses, starting from relative rest at nonzero separation, also execute a specifically time-cycloidal linear gravitational trajectory. Relativistic upgrade of that model causes the two particles to respect a minimum mutual separation and thus a speed limit of $0.866c$, subtly changing shape details of the basic Newtonian cycloid in time. But no credible evidence is found that energy-conserving “gravitational collapse” can be other than cyclic in character: Oppenheimer and Snyder erroneously scuppered their time-cycloidal Robertson-Walker metric by forgetting that dust of position-independent density is necessarily present in all of space, which leaves no physical scope for their “application” of the Birkhoff theorem.

Introduction

It has for many years been asserted that nonreversing gravitational contraction of a system cannot occur unless that system sheds part of its energy, e.g., by electromagnetic or other radiation [1, 2]. This assertion implicitly rejects the correctness of the Oppenheimer-Snyder picture of the *permanent* gravitational collapse of an energy-conserving spherically-symmetric cloud of dust particles that only interact gravitationally [3, 4]. The *associated formation* in the Oppenheimer-Snyder picture *of a gravitational event horizon* has as well been criticized on the basis that the Principle of Equivalence requires that any such dust particle’s geodesic trajectory always remains timelike, whereas the infinite redshift at a gravitational event horizon would *change* the nature there of a dust particle’s geodesic trajectory from timelike to lightlike [1, 5].

The *mathematically simplest part* of the Oppenheimer-Snyder calculation occurs in the comoving coordinates of the spherically-symmetric dust cloud *after* it has been assumed that the density of the dust *is completely independent of position* and that the two unknown functions of the spherically-symmetric comoving metric tensor each factor into a function of the radial coordinate r times a function of the time t [6]. The upshot of these assumptions and the Einstein equation is that the comoving metric tensor takes on the Robertson-Walker form [7],

$$ds^2 = (cdt)^2 - (R(t))^2\left[(1-((\omega r)/c)^2)^{-1}dr^2 + r^2((d\theta)^2 + (\sin\theta d\phi)^2)\right], \tag{1a}$$

and the completely position-independent, time-dependent energy density of the dust is given by,

$$\rho(t) = \rho(t=0)/(R(t))^3, \tag{1b}$$

where the dimensionless Robertson-Walker metric function $R(t)$ satisfies $R(t=0)=1$. Furthermore, the Einstein equation together with the requirement that,

$$\dot{\rho}(t=0) = 0, \tag{1c}$$

which, of course, implies that,

$$\dot{R}(t=0) = 0, \tag{1d}$$

produces the particular Friedmann equation variant [8],

$$(\dot{R}(t))^2 = \omega^2[(1/R(t)) - 1], \tag{1e}$$

where the Robertson-Walker metric constant ω^2 is given by,

$$\omega^2 = ((8\pi G\rho(t=0))/(3c^2)). \tag{1f}$$

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The solution of the Eq. (1e) Friedmann equation variant with the initial condition $R(t = 0) = 1$ is *implicitly* given by,

$$t = (2\omega)^{-1}[\arccos(2R(t) - 1) + 2(R(t)(1 - R(t)))^{\frac{1}{2}}], \tag{1g}$$

as can be verified by noting that Eq. (1g) implies both that $t(R = 1) = 0$ and,

$$dt/dR = -(\omega)^{-1}(R/(1 - R))^{\frac{1}{2}} \;\Rightarrow\; \dot{R}(t) = -\omega[(1/R(t)) - 1]^{\frac{1}{2}}, \tag{1h}$$

from which Eq. (1e) immediately follows. Differentiation of Eq. (1e) additionally yields,

$$\ddot{R}(t) = -\tfrac{1}{2}(\omega/R(t))^2. \tag{1i}$$

From Eqs. (1h), (1i) and (1g) it is readily seen that $R(t)$ decreases monotonically and increasingly rapidly from its initial value of unity at $t = 0$ to zero at $t = (\pi/(2\omega))$; in fact Eq. (1g) *is the inverse of the monotonic half-period of a cycloid* [9]. The *remaining* half-period of $R(t)$ follows from noting that $R(-t) = R(t)$ is consistent with Eq. (1e) and with continuity at $t = 0$.

As a matter of mathematics, there is no question that the Eq. (1e) Friedmann equation variant and its initial condition $R(t = 0) = 1$ define *a unique continuous cycloid* $R(t)$ of *period* $T = (\pi/\omega)$, although $R(t)$ *isn't differentiable at the discrete points* $t = \pm(n + \frac{1}{2})T,\ n = 0, 1, 2, \ldots,$ *where it is equal to zero*.

Oppenheimer and Snyder *strangely neglected to ponder the possibility* that this unique *continuous periodic solution* of Eq. (1e) *might portray cyclic dynamical behavior which has kinship to familiar orbital gravitational physics*.

They then compounded this error *of insufficient reflection on the nature of their result* by inattention to *the consequences* of their own drastic assumption that the energy density of the dust is completely independent of position. Oblivious to having made that assumption, *which precludes the existence of any region where space is empty*, they proceeded to "apply" the Birkhoff theorem to regions outside of essentially *arbitrary* radius values, notwithstanding that the Birkhoff theorem *only* applies to spherically symmetric gravitational fields *in regions where space is empty* [10]. Therefore the part of the Oppenheimer-Snyder calculation which goes *beyond* the solution of Eq. (1e) must be eschewed.

Giving that misadventure its due berth, we instead point out the astonishing fact that an elementary "gravitationally collapsing" two-body Newtonian system turns out to be described by *precisely the same* periodic cycloidal Eq. (1e). We then upgrade this elementary Newtonian two-body "gravitationally collapsing" system by introducing basic relativistic modifications, including that gravity interacts with a particle's *effective* mass, which *includes* its kinetic energy, rather than with *only* its rest mass. The interesting result is that *not only*, as in the Newtonian case, are all separations between the two bodies that are *larger* than the initial "at relative rest" separation energetically disallowed, *in addition* a range of sufficiently *small separations* between the two bodies *are as well energetically disallowed*. That precludes manifestation of the Newtonian case's periodically *infinite* kinetic energy (offset by simultaneously *negative infinite* potential energy) from occurring in the relativistic case, which eliminates infinite slopes from the periodic cusps of the relativistic solution (infinite slopes are a *fixture* of the periodic cusps of cycloidal functions, such as the Newtonian "collapse" cycloid). There is as well a reduction in the relativistic solution's *period* in comparison with that of the cycloidal Newtonian solution. But the relativistic modifications produce *not a trace* of putative Oppenheimer-Snyder *permanent* gravitational collapse and gravitational event horizon formation, which of course seem incompatible with *the apparently inherent cyclic nature of the bound states* of solely-gravitationally-interacting, energy-conserving dynamical systems.

"Gravitationally collapsing" two-identical-particle systems

It is obviously much easier to deal with the mutual gravitational infall of just two identical particles than with the gravitational infall of an entire dust-cloud fluid. A crucial psychological dividend of such relative simplicity is the automatic absence *of any calculation-related temptation* to make physically problematic *further* "simplifying assumptions", e.g., the complete independence of position of the dust-cloud energy density, which was inconsistently embraced by Oppenheimer and Snyder (two particles amount to almost *the antithesis* of that uniform-density-everywhere assumption).

For their "gravitational collapse" infall, we specifically *start* the two identical particles *at relative rest* at time $t = 0$, separated by the distance $2a$ along the line joining them, which yields for this two-particle system's initial energy E,

$$E = 2mc^2 - ((Gm^2)/(2a)), \tag{2a}$$

which we of course *take to be conserved*. For the case where this system is treated nonrelativistically, it is most convenient to carry out the calculations in terms of the particle mass m, but when the system is treated relativistically, using the conserved initial energy E is more convenient. In anticipation of the relativistic treatment of this two-particle system, we solve Eq. (2a) for mc^2 in terms of E,

$$mc^2 = E\left(1 + \sqrt{1 - ((GE)/(2ac^4))}\right)^{-1}. \tag{2b}$$

In the center of momentum system of these two identical particles, at a subsequent time $t > 0$ they will have equal and opposite momenta and velocities, directed along the line joining them, experience equal and opposite gravitational force directed along that line, and have equal kinetic energies and speeds. Taking the midpoint of the line joining the two particles as the origin of the relevant one-dimensional coordinate system for treating their mutual gravitational infall, at time $t > 0$ one particle will have the coordinate $x(t)$ while the other particle has the coordinate $-x(t)$, so the nonrelativistic gravitational potential energy of this two-particle system will be $-((Gm^2)/(2|x(t)|))$. The nonrelativistic kinetic energy of this two-particle system at time $t > 0$ is $2[\frac{1}{2}m(\dot{x}(t))^2] = m(\dot{x}(t))^2$. Thus for the purpose of treating this gravitationally infalling two-particle system nonrelativistically, conservation of its energy implies that,

$$E = 2mc^2 - ((Gm^2)/(2a)) = 2mc^2 + m(\dot{x}(t))^2 - ((Gm^2)/(2|x(t)|)). \tag{2c}$$

The second equality of Eq. (2c) yields the first-order differential equation,

$$(\dot{x}(t))^2 = ((Gm)/2)[(1/|x(t)|) - (1/a)], \tag{2d}$$

with the initial condition $|x(t = 0)| = a$ in accord with the discussion above Eq. (2a). This initial condition together with Eq. (2d) implies that $\dot{x}(t = 0) = 0$, fulfilling the initial "two identical particles at relative rest" requirement that is also mentioned in the discussion above Eq. (2a).

If we now define the dimensionless variable $R(t) = (|x(t)|/a)$, so that $|x(t)| = aR(t)$, Eq. (2d) becomes,

$$(\dot{R}(t))^2 = \omega^2[(1/R(t)) - 1], \tag{2e}$$

with the initial condition $R(t = 0) = 1$ and $\omega^2 = ((Gm)/(2a^3))$. Eq. (2e) and its initial condition *formally completely correspond* to the particular Friedmann equation variant of Eq. (1e) *that features so very prominently in the Oppenheimer-Snyder calculation*. But since Eqs. (2d) and (2e) manifestly pertain here *to the degenerate one-dimensional Newtonian orbit* of the mutual gravitational infall of two identical particles that start from relative rest, *there can here be no doubt that the physically correct solution for $R(t)$ is the unique continuous periodic cycloid in time t that is defined by* Eq. (2e) (i.e., Eq. (1e))—see the discussion in the *first* paragraph which *begins after* Eq. (1i).

The astonishingly parallel degenerate-orbit Eq. (2e) to Oppenheimer and Snyder's Eq. (1e) Friedmann equation variant *underlines just how profoundly regrettable it was* that Oppenheimer and Snyder *flatly failed* to ponder the possibility that the unique continuous periodic cycloidal solution of their Eq. (1e) *might portray cyclic dynamical behavior which has kinship to familiar orbital gravitational physics*.

We next turn our attention to a *relativistic* treatment of the gravitational infall from relative rest of the two identical particles. Instead of illogically attempting in a gravitational context to deal *ad hoc* with relativistic-particle *kinetic energy*, we adopt the more holistic viewpoint that a relativistic particle in motion *has an effective mass value which systematically replaces its rest mass value* whose use would be physically appropriate in the limit that it had vanishing speed. If at time $t > 0$ our two gravitationally interacting particles both had vanishing speed in their center of momentum reference frame, the resulting energy in that reference frame of that two-particle system would be,

$$2mc^2 - ((Gm^2)/(2|x(t)|)),$$

in which expression we now *systematically replace* each occurrence of the particle rest mass m by the particle *effective mass* $(m\gamma)$, where, of course,

$$\gamma = (1 - (\dot{x}(t)/c)^2)^{-\frac{1}{2}}. \tag{3a}$$

Carrying out this systematic replacement of m by $(m\gamma)$ in our above expression for the energy of a system of two zero-speed particles, followed by requiring energy conservation, yields the two equations,

$$2mc^2 - ((Gm^2)/(2a)) = E = 2(mc^2\gamma) - ((G(mc^2\gamma)^2)/(2|x(t)|c^4)). \tag{3b}$$

We have already solved the first of these two equations for mc^2 in terms of the conserved energy E and a (where $a = |x(t = 0)|$); the result is given by Eq. (2b). We next solve the second of these two equations for $(mc^2\gamma)$ in terms of E and $|x(t)|$. Combining those two results so as to eliminate the particle rest energy mc^2 yields γ in terms of E, $|x(t)|$ and a,

$$\gamma = \left(1 + \sqrt{1 - ((GE)/(2ac^4))}\right)\left(1 + \sqrt{1 - ((GE)/(2|x(t)|c^4))}\right)^{-1}. \tag{3c}$$

The initial condition $|x(t = 0)| = a$ together with Eq. (3c) implies that $\gamma(t = 0) = 1$, which fulfills the initial "two identical particles at relative rest" requirement. From Eq. (3a) we note that $(\dot{x}(t))^2 = c^2(1 - (1/\gamma)^2)$, using which we calculate $(\dot{x}(t))^2$ from Eq. (3c),

$$(\dot{x}(t))^2 = \frac{((GE)/(4c^2))[(1/|x(t)|) - (1/a)]}{\left(1+\sqrt{1-((GE)/(2ac^4))}\right)^2}\left[2 + \left(\frac{4}{\sqrt{1-((GE)/(2ac^4))}+\sqrt{1-((GE)/(2|x(t)|c^4))}}\right)\right]. \tag{3d}$$

Since it would usually be expected that a, the initial $t = 0$ value of $|x(t)|$, satisfies $a \gg ((GE)/c^4)$, an adequate approximation to Eq. (3d) is normally given by,

$$(\dot{x}(t))^2 \approx ((GE)/(4c^2))[(1/|x(t)|) - (1/a)]\left[\tfrac{1}{2} + \frac{1}{1+\sqrt{1-((GE)/(2|x(t)|c^4))}}\right]. \tag{3e}$$

We note from Eqs. (3d) and (3e) that *not only* is it energetically disallowed for $|x(t)|$ to be *greater* than a, as was true in the nonrelativistic Newtonian case (see Eq. (2d) above); it is *as well* energetically disallowed for $|x(t)|$ to be *less* than $((GE)/(2c^4))$. That clearly *prevents* the occurrence in this relativistic case of the nonrelativistic Newtonian cycloidal-trajectory manifestation of periodically *infinite* kinetic energy (offset by simultaneously *negative infinite* potential energy): from Eq. (3e) it is seen that the maximum possible value of $(\dot{x}(t))^2$ is approximately $\frac{3}{4}c^2$, which corresponds to a maximum possible value of γ of approximately 2 (as can also be noted directly from Eq. (3c)). The relativistic truncation from a to $[a - ((GE)/(2c^4))]$ of the *length* of the periodic trajectory cycle which the relativistic trajectory function $|x(t)|$ traces out thus renders finite *the maximum slopes* of the periodic *cusps* which occur in this relativistic trajectory function $|x(t)|$; the periodic *strictly cycloidal cusps* which occur *in the nonrelativistic Newtonian cycloidal trajectory function* $|x(t)|$ described by the Eq. (2d) differential equation in contrast of course *have unbounded slopes*.

There is as well a reduction in the relativistic trajectory function's *period* in comparison with the period of the cycloidal Newtonian trajectory function, due *both* to the relativistic truncation of the *length* of the periodic trajectory cycle which the relativistic trajectory function $|x(t)|$ traces out *and* to a slightly increased *speed* of the remaining part of that truncated periodic trajectory cycle: the dimensionless rightmost factor in square brackets in Eq. (3e) has no nonrelativistic Newtonian cycloidal counterpart in Eq. (2d), and its value is marginally greater than unity.

But the relativistic modifications which have been made *produce no trace whatsoever* of putative Oppenheimer-Snyder *permanent* gravitational collapse and gravitational event horizon formation. Such "phenomena" seem incompatible with *the inherently cyclic nature of the bound states* of solely-gravitationally-interacting, energy-conserving dynamical systems. As much could readily have dawned on Oppenheimer and Snyder *themselves* the moment they set their eyes on the particular *cycloidal* Friedmann equation variant given by Eq. (1e), if only they had possessed *some background familiarity* with nonrelativistic Newtonian "gravitational collapse" theory, as encapsulated by *the likewise cycloidal* Eq. (2d).

References

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