Extremely efficient acceptance-rejection method for simulating uncorrelated Nakagami fading channels

Luca Martino[1], David Luengo[2],

[1] Universidad Carlos III de Madrid, Madrid, Spain

[2] Universidad Politécnica de Madrid, Madrid, Spain

Abstract

Multipath fading is one of the most common distortions in wireless communications. The simulation of a fading channel typically requires drawing samples from a Rayleigh, Rice or Nakagami distribution. The Nakagami-m distribution is particularly important due to its good agreement with empirical channel measurements, as well as its ability to generalize the well-known Rayleigh and Rice distributions. In this paper, a simple and extremely efficient rejection sampling (RS) algorithm for generating independent samples from a Nakagami-m distribution is proposed. This RS approach is based on a novel proposal density composed of three pieces of well-known densities from which samples can be drawn easily and efficiently. The proposed method is valid for any combination of parameters of the Nakagami distribution, without any restriction in the domain and without requiring any adjustment from the final user. Simulations for several parameter combinations show that the proposed approach attains acceptance rates above 90% in all cases, outperforming all the RS techniques currently available in the literature.

Index Terms

Multipath fading, Nakagami random variables, rejection sampling.

I. INTRODUCTION

The Nakagami-m probability density function (PDF) was proposed in 1960 by Nakagami as an empirical model for the amplitude of the received samples in wireless radio communications subject to multipath fading [1]. This PDF is characterized by two parameters: the fading or shape parameter, m, which indicates the fading depth, and the average received power, Ω. Nakagami's

fading model has been widely used to describe the wireless fading channel due to its good agreement with empirical channel measurements for some urban multipath environments [2], [3]. Moreover, the Nakagami PDF can be used to generalize or approximate several situations and densities common in wireless communications: worse-than-Rayleigh fading for $0.5 \leq m < 1$, Rayleigh fading for $m = 1$, and less severe fading (e.g., Rice fading) for $m > 1$.

On the one hand, several schemes for simulating the correlated Nakagami fading channel have been proposed [4]–[8], but all of them present limitations that may restrict their use in some practical situations. On the other hand, the generation of independent Nakagami RVs is also frequently required, e.g., to simulate the performance of channel estimators, systems operating under slow fading conditions or independent fading branches in diversity systems [9]. Moreover, several approaches for the generation of bivariate or multivariate Nakagami random variables (RVs) are based on drawing a sequence of independent samples first and then performing some transformation [8], [10]. Hence, several simple acceptance-rejection methods using different proposal densities with increasing accuracy have been recently proposed [9], [11], [12]. Currently, the best results are provided by [12] using a Gaussian PDF as the proposal density. This approach achieves acceptance rates close to one for small values of m, that fall down to 70% for $m \geq 4$ without truncation and 80% truncating the proposal in the range $[0, 4\Omega]$. Unfortunately, this truncation prevents their approach from drawing samples from the tail of the Nakagami PDF, which can be important for some applications, such as co-channel interference problems [6].

In this paper we build on this work, designing an extremely efficient acceptance-rejection method for Nakagami RVs using a piecewise monotonic proposal density composed of three different pieces: two truncated Gaussian PDFs and a decaying exponential PDF. Our proposal density is based on well-known PDFs, from which samples can be easily and efficiently drawn [13], is valid for arbitrary average power and fading factors, and provides a very good fit of the target PDF, obtaining acceptance rates above 90% in all cases, which are the best ones ever reported in the literature. Furthermore, unlike previous approaches (e.g., [9], [12]), we attain this high acceptance rate without any truncation (i.e., our proposal provides truly Nakagami RVs) and without requiring any adjustment from the final user (i.e., the algorithm proceeds automatically once the parameters of the Nakagami distribution are provided).

The rest of the paper is organized as follows. First of all, Section II briefly reviews the standard RS algorithm and the Nakagami PDF. Then, Section III details the construction of the novel proposal density specifically designed for the Nakagami PDF. In Section IV, the practical implementation of the novel RS technique is introduced. In Section V, we derive an expression for the theoretical acceptance rate of the proposed RS method and provide some considerations about the optimal construction of the proposal density. Section VI shows numerical results for several parameters of the Nakagami target PDF. Finally, Section VII provides the conclusions and future lines.

II. REJECTION SAMPLING AND BACKGROUND

Rejection sampling (RS) is a classical technique for generating samples from an arbitrary target PDF, $p_o(x) = C_p p(x)$, known up to a proportionality constant C_p, using an alternative simpler proposal PDF, $\pi_o(x)$, such that

$$K\pi_o(x) \geq p(x), \quad \text{for some} \quad K > 0. \tag{1}$$

Note that the application of RS requires the knowledge of a suitable value K in order to satisfy the inequality above. Indeed, RS works by drawing samples from the proposal, $x' \sim \pi_o(x)$, and accepting or rejecting them on the basis of the ratio $p(x')/[K\pi_o(x')]$. The *standard RS algorithm*, which allows us to draw samples exactly from the target PDF, is the following:

1) Draw $x' \sim \pi_o(x)$ and $w' \sim \mathcal{U}([0,1])$.

2) If $w' \leq \frac{p(x')}{K\pi_o(x')}$, then x' is accepted. Otherwise, x' is discarded.

3) Repeat steps 1–2 until the desired number of samples has been obtained from the target.

The key performance measure for RS is the *acceptance rate* (i.e., the average number of candidate samples accepted out of the total number of samples generated),

$$\eta_a = \int_{-\infty}^{+\infty} p_a(x)\pi_o(x)dx = \int_{-\infty}^{+\infty} \frac{p(x)}{K\pi_o(x)}\pi_o(x)dx = \frac{1}{C_p K},$$

where $p_a(x) = \frac{p(x)}{K\pi_o(x)}$. The acceptance rate depends on how close the proposal is to the target PDF: in the ideal case that $\pi_o(x) = p_o(x)$, then we can set $K = 1/C_p$ and $\eta_a = 1$ (note that $0 \leq \eta_a \leq 1$). The acceptance rate determines the efficiency of an RS algorithm, since the

number of samples required on average to obtain N valid samples is $N_a = N/\eta_a$. Hence, low values of η_a lead to a large number of samples being required on average, thus wasting time and computational resources.

A. Nakagami Target PDF

In this work we concentrate on developing an extremely efficient RS algorithm for generating independent unidimensional Nakagami-m random variables. The Nakagami target PDF is $p_o(x) = C_p \, p(x)$, with normalizing constant

$$C_p = \frac{2m^m}{\Omega^m \Gamma(m)}, \tag{2}$$

and unnormalized target function

$$p(x) = x^{2m-1} \exp\left(-\frac{mx^2}{\Omega}\right) \qquad \text{for} \quad x \geq 0, \quad m \geq \frac{1}{2}, \quad \Omega > 0, \tag{3}$$

where $\Gamma(x)$ indicates the gamma function. Given a random variable $X \sim p_o(x)$, then $\Omega = \mathrm{E}\{X^2\}$ (with $\mathrm{E}\{\cdot\}$ denoting the mathematical expectation) represents the average received power, and $m = \frac{(E[X^2])^2}{\mathrm{Var}\{X^2\}} = \frac{\Omega^2}{\mathrm{Var}\{X^2\}}$, (with $\mathrm{Var}\{\cdot\}$ denoting the variance), is a fading parameter that characterizes the fading depth of the channel: the smaller the value of m the higher the fading depth. Note that m is a real number, greater than $\frac{1}{2}$. In the limit case of $m = \frac{1}{2}$, then $p_o(x)$ coincides with an half-Gaussian pdf.

III. PROPOSAL DENSITY FOR EFFICIENT REJECTION SAMPLING OF NAKAGAMI PDFS

As the proposal density, in this work, we consider a simple piecewise monotonic approximation composed of three PDFs from which samples can be easily drawn. The motivation behind this proposal comes from the shape of the Nakagami PDF, which can be seen in Figure 1. On the one hand, the Nakagami target PDF is unimodal, but asymmetric w.r.t. the mode. Thus, we use two different pieces for the proposal PDF on the left and right hand side of its mode in order to better accommodate the different decay rates of the target on both sides. On the other hand, providing a good proposal for the tail is critical in order to obtain a good acceptance probability [13]. Hence, we introduce a third piece of the proposal PDF that provides a good fit of the target as $x \to \infty$. From Figure 1, it can be seen that the number of intervals (three) is not arbitrarily

chosen, but corresponds to the natural choice given the shape of the target, thus providing the optimal trade-off between performance and computational cost. The full proposal PDF is then given by $\pi_o(x) = C_\pi \pi(x)$, i.e., $\pi(x) \propto \pi_o(x)$, with

$$\pi(x) = \pi_1(x)\mathbb{I}_1(x) + \pi_2(x)\mathbb{I}_2(x) + \pi_3(x)\mathbb{I}_3(x), \tag{4}$$

where $\pi_i(x)$ ($1 \leq i \leq 3$) is an exponential type function,

$$\pi_i(x) = \beta_i \exp\left(-\alpha_i(x - \mu_i)^{\gamma_i}\right), \tag{5}$$

and $\mathbb{I}_i(x)$ ($1 \leq i \leq 3$) is an indicator function, that determines whether x belongs to an interval $E_i = [e_{i-1}, e_i)$ or not, i.e.,

$$\mathbb{I}_i(x) = \begin{cases} 1, & x \in E_i, \\ 0, & x \notin E_i. \end{cases} \tag{6}$$

The intervals used for the target PDF are the left hand side of the mode, $E_1 = [0, e_1)$, its right hand side, $E_2 = [e_1, e_2)$, and the tail, $E_3 = [e_2, \infty)$. In the first two intervals, we use a truncated Gaussian density (i.e., $\gamma_1 = \gamma_2 = 2$), whereas in the third interval we use a decaying exponential (i.e., $\gamma_3 = 1$). The determination of the interval limits (e_1 and e_2), the remaining parameters (α_i, β_i and μ_i for $1 \leq i \leq 3$), and the normalizing constant, C_π, in such way that we obtain a proposal function suitable for applying the RS method, i.e., $K = 1/C_\pi$, so that $K\pi_o(x) = \pi(x)$ and

$$\pi(x) \geq p(x), \tag{7}$$

is detailed in the following section. We build an hat function $\pi(x)$ for the target $p(x)$.

A. Proposal around the mode

First of all, note that, since the first two pieces of the proposal PDF are two truncated Gaussian densities defined in $E_1 = [0, e_1)$ and $E_2 = [e_1, e_2)$, the optimum choice for e_1 is clearly the mode of the Nakagami target PDF. Differentiating $p(x)$ and equating it to zero, it is straightforward to show that this mode is given by

$$x_{\max} = \sqrt{\frac{\Omega(2m - 1)}{2m}}, \tag{8}$$

and we set $e_1 = x_{\max}$. Then, in order to have $\pi(x)$ as close as possible to $p(x)$ within E_1 and E_2, while ensuring that $\pi(x) \geq p(x)$, we must have $\pi_1(x_{\max}) = \pi_2(x_{\max}) = p(x_{\max})$. This implies setting $\mu_1 = \mu_2 = x_{\max}$ and

$$\beta_1 = \beta_2 = p(x_{\max}) = \exp\left(-\frac{2m-1}{2}\right)\left(\frac{\Omega(2m-1)}{2m}\right)^{\frac{2m-1}{2}}. \tag{9}$$

Finally, in order to specify completely the proposal within the first two intervals we need to determine α_1 and α_2. This can be easily done by noting that, $\pi(x) \geq p(x)$ within E_1 and E_2 implies that

$$\ln \pi_i(x) = \ln p(x_{\max}) - \alpha_i(x - x_{\max})^2 \geq \ln p(x), \tag{10}$$

for any $x \in E_i$ with $i \in \{1, 2\}$. Hence, in order to obtain a valid proposal we must choose

$$\alpha_i \leq L(x), \tag{11}$$

for any $x \in E_i$ with $i \in \{1, 2\}$, and

$$L(x) = \frac{1}{(x - x_{\max})^2} \ln \frac{p(x_{\max})}{p(x)} = \frac{(2m-1)\ln(x_{\max}/x)}{(x - x_{\max})^2} + \frac{m}{\Omega}\frac{x + x_{\max}}{x - x_{\max}}. \tag{12}$$

It is easy to check that $L(x)$ is a strictly decreasing function, with $L(0) \to \infty$ and $L_\infty = \lim_{x \to \infty} L(x) = m/\Omega$. Therefore, in order to obtain the best possible fit between the proposal and the target (while ensuring that $\pi(x) \geq p(x)$) within the range covered by each piece of the proposal, we must set $\alpha_1 = L(e_1)$ and $\alpha_2 = L(e_2)$. Regarding α_1, recalling that $e_1 = x_{\max}$, with $x_{\max}$ given by (8), it can be obtained explicitly as

$$\alpha_1 = \lim_{x \to x_{\max}} L(x) = \lim_{x \to x_{\max}} \frac{(2m-1)[\ln x_{\max} - \ln x] + \frac{m}{\Omega}(x^2 - x_{\max}^2)}{(x - x_{\max})^2}. \tag{13}$$

Applying L'Hôpital's rule twice to remove the indeterminacies in the limit, (13) becomes

$$\alpha_1 = \lim_{x \to x_{\max}} \frac{-\frac{2m-1}{x} + \frac{2mx}{\Omega}}{2(x - x_{\max})} = \lim_{x \to x_{\max}} \frac{2m-1}{2x^2} + \frac{m}{\Omega} = \frac{2m}{\Omega}.$$

With respect to α_2, it is obtained similarly, evaluating $L(x)$, as given by (12), at $x = e_2$:

$$\alpha_2 = L(e_2) = \frac{1}{(e_2 - x_{\max})^2} \ln \frac{p(x_{\max})}{p(e_2)}, \tag{14}$$

with e_2 given by (20), as discussed in the sequel. Note that the conditions for α_i, given by (11) and (12), were derived in [12]. However, in [12] a single Gaussian is used for the proposal,

$\pi(x) = p(x_{\max}) \exp(-\alpha(x - x_{\max})^2)$. Hence, in order to obtain a valid proposal they need to set $\alpha = L_\infty = m/\Omega$, thus achieving a looser fit of the target and lower acceptance rates. In order to improve the efficiency of their approach, they propose to truncate the target PDF within the range $[0, 4\Omega]$. This allows them to use $\alpha = L(4\Omega)$, thus improving the efficiency of the RS approach at the expense of generating truncated Nakagami random variables, which may produce misleading results in the simulation of wireless communication channels, especially for large values of m and small values of Ω. As an alternative, here we introduce a third piece in the proposal that allows us to attain higher acceptance rates without resorting to truncation.

B. Proposal for the tail

The last piece, the truncated exponential PDF, is used to obtain a good approximation of the tail of the Nakagami target PDF. The proposal considered is based on the fact that the function

$$V(x) = \ln p(x) = (2m - 1)\ln x - \frac{m}{\Omega}x^2, \tag{15}$$

is convex for any $m \geq 0.5$ and $\Omega > 0$, since $\ddot{V}(x) = -\frac{2m-1}{x^2} - \frac{2m}{\Omega} < 0$ for $x \geq 0$, with $\ddot{V}(x)$ denoting the second derivative of $V(x)$. Furthermore, setting $\beta_3 = 1$ we have $\ln \pi_3(x) = -\alpha_3(x - \mu_3)$ for $x \geq e_2$, and we can guarantee that $\pi_3(x) \geq p(x)$ for $x \geq e_2$ simply by adjusting $\ln \pi_3(x)$ to become the tangent line to $V(x)$ at $x = e_2$. It is straightforward to show that this tangent line is obtained setting

$$\alpha_3 = \dot{V}(e_2) = \frac{2m - 1}{e_2} - \frac{2m}{\Omega}e_2, \tag{16}$$

with $\dot{V}(e_2)$ indicating the first derivative of $V(x)$ evaluated at $x = e_2$, and

$$\alpha_3\mu_3 = V(e_2) - \dot{V}(e_2)e_2 = (2m - 1)(\ln e_2 - 1) + \frac{m}{\Omega}e_2^2, \tag{17}$$

so that finally

$$\mu_3 = \frac{(2m - 1)(\ln e_2 - 1) + \frac{m}{\Omega}e_2^2}{\frac{2m-1}{e_2} - \frac{2m}{\Omega}e_2}. \tag{18}$$

C. Suggested choice of e_2

The last element required for the complete definition of the proposal is the limit between the second and third intervals, e_2. A discussion about the optimal choice e_2^* is given in Section V and Appendix A. Unfortunately, a closed-form expression for e_2^* cannot be found. However, an efficient sub-optimal approximation can be easily obtained by noting that:

1) The point e_2 should correspond to the beginning of the tail of the Nakagami PDF.
2) The right hand side tail of a density is necessarily convex by definition.

Therefore, we argue that e_2^* must satisfy the inequality $e_2^* \geq x_{in}$, with x_{in} denoting the largest inflection point in the Nakagami PDF, which guarantees that $p(x)$ is convex for $x > x_{in}$. It can be easily found as the largest solution of $\frac{d^2 p(x)}{dx^2} = 0$, i.e.,

$$x_{in} = \frac{1}{2}\sqrt{\frac{\Omega(4m + \sqrt{16m - 7} - 1)}{m}}. \tag{19}$$

Moreover, although setting $e_2 = x_{in}$ provides satisfactory results, we have found out empirically (by an exhaustive regression study) that an approximation of the optimal value e_2^* is given by

$$e_2 = \hat{e}_2^* = x_{in} + \frac{\Omega}{4m} + am^b + c, \tag{20}$$

with $a = -0.8$, $b = 0.1$ and $c = 1.2$. This approximation is simple enough to be used in practice, and provides an improvement in acceptance probability of $4 - 9$ % w.r.t. using $e_2 = x_{in}$. The loss w.r.t. the approximately optimal solution derived numerically is usually less than 2%. See Section V and Appendix A for a comparison with the optimal value e_2^* and the alternative choice $e_2 = x_{in}$ (see Figures 2 and 9).[1]

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[1] Indeed, we have only noticed differences above 2% for very large values of Ω and small values of m, e.g., $\Omega = 100$ and $m \leq 2$, where the difference in acceptance probability can rise up to 4 % (see Figure 9).

IV. IMPLEMENTATION OF THE PROPOSED RS SCHEME

A. Drawing samples from the proposal PDF

First of all, we recall that the unnormalized proposal function is given by

$$\pi(x) = \begin{cases} \pi_1(x) = \beta_1 \exp(-\alpha_1(x - \mu_1)^{\gamma_1}), & 0 \le x < e_1; \\ \pi_2(x) = \beta_2 \exp(-\alpha_2(x - \mu_2)^{\gamma_2}), & e_1 \le x < e_2; \\ \pi_2(x) = \beta_3 \exp(-\alpha_3(x - \mu_3)^{\gamma_3}), & x \ge e_2. \end{cases} \tag{21}$$

The parameters required to construct this proposal function, which have been derived in the previous sections in order to be an hat function w.r.t. $p(x)$, i.e., $\pi(x) \ge p(x)$, are summarized in Table I. Note that all the parameters can be easily calculated and stored, the whole process is automatic (i.e., it can be performed automatically given the values of m and Ω, without requiring the user to adjust manually any parameter) and has to be performed only once before drawing all the samples required. As an example, Figure 1 shows the target, $p(x)$, our proposal, $\pi(x)$, and the proposal used in [12] for an unbounded domain, which fits the target PDF in a much looser way, thus leading to worse acceptance rates.

i	β_i	α_i	μ_i	γ_i	e_i
1	$x_{\max}^{2m-1} \exp\left(-\frac{m x_{\max}^2}{\Omega}\right)$	$\frac{2m}{\Omega}$	$x_{\max}$	2	$x_{\max}$
2	$x_{\max}^{2m-1} \exp\left(-\frac{m x_{\max}^2}{\Omega}\right)$	$\frac{1}{(e_2 - x_{\max})^2} \ln \frac{p(x_{\max})}{p(e_2)}$	$x_{\max}$	2	$x_{in} + \frac{\Omega}{4m} - 0.8m^{0.1} + 1.2$
3	1	$\frac{2m-1}{e_2} - \frac{2m}{\Omega} e_2$	$\frac{(2m-1)(\ln e_2 - 1) + \frac{m}{\Omega} e_2^2}{\frac{2m-1}{e_2} - \frac{2m}{\Omega} e_2}$	1	—

TABLE I

PARAMETERS REQUIRED TO CONSTRUCT THE PROPOSAL PDF FOR THE REJECTION SAMPLING ALGORITHM.

Once the proposal PDF, $\pi_o(x) \propto \pi(x)$, has been defined, one of the three truncated functions with non-overlapping supports must be selected with probabilities proportional to their areas, which can be easily obtained in a closed form. For the first region we have

$$A_1 = \frac{p(x_{\max})}{2} \sqrt{\frac{\pi}{\alpha_1}} \operatorname{erf}\left(\sqrt{\alpha_1} x_{\max}\right), \tag{22}$$
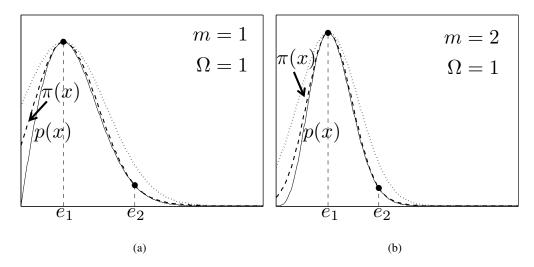
Fig. 1. Target function $p(x)$ (continuous line), our proposal function $\pi(x)$ (dashed line) and the proposal used in [12] for the unbounded domain (dotted line). **(a)** Construction for $m = 1$ and $\Omega = 1$. **(b)** Construction for $m = 2$ and $\Omega = 1$.

where erf (x) denotes the error function.[2] Similarly, for the second region we have

$$A_2 = \frac{p(x_{\max})}{2}\sqrt{\frac{\pi}{\alpha_2}}\mathrm{erf}\left(\sqrt{\alpha_2}(e_2 - x_{\max})\right),\tag{23}$$

and for the last region

$$A_3 = \frac{1}{\alpha_3}\exp(-\alpha_3(e_2 - \mu_3)).\tag{24}$$

Finally, samples must be drawn from the selected piece of the proposal PDF. For the truncated Gaussians there are many techniques available in the literature (see e.g. [14], [15]) that allow us to draw samples efficiently, whereas samples can also be obtained easily from the truncated exponential using the inversion method [13].

B. Proposed RS scheme

In this section, we outline in details the novel RS algorithm. Let us define the normalized version of each piece, $\bar{\pi}_i(x) = \frac{1}{A_i}\pi_i(x)$. Hence, the complete proposed RS approach is the following:

1) Given m and Ω construct the proposal PDF, given by (21), using the parameters in Table I.

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[2]It is important to remark that the error function can be always evaluated with the accuracy required.

2) Compute the selection probability for each piece of the proposal, $P_i = A_i/(A_1 + A_2 + A_3)$ for $1 \leq i \leq 3$.

3) Select the i-th interval of the proposal ($1 \leq i \leq 3$) with probability P_i.

4) Draw $x' \sim \bar{\pi}_i(x)$ and $w' \sim \mathcal{U}([0, 1])$.

5) If $w' \leq \frac{p(x')}{\pi_i(x')}$, then x' is accepted. Otherwise, x' is discarded.

6) Repeat steps 3–5 until the desired number of samples has been obtained from the target.

V. ACCEPTANCE RATE OF THE PROPOSED SCHEME AND OPTIMAL CHOICE e_2^*

In this section, we show the expressions for the acceptance rate of the proposed algorithm. First of all, note that the acceptance rate can be expressed as

$$\eta_a(e_2) = \int_0^\infty p_a(x)\pi_o(x) \, dx,$$

$$= \int_0^\infty \frac{p(x)}{\pi(x)}\pi_o(x) \, dx,$$

$$= C_\pi(e_2) \int_0^\infty p(x) \, dx = \frac{C_\pi(e_2)}{C_p}, \tag{25}$$

where $p_a(x) = \frac{p(x)}{\pi(x)}$ denotes the probability of accepting a sample x drawn from $\pi_o(x)$,[3] since we have built $\pi(x)$ such as $\pi(x) \geq p(x)$ for any value of x, C_p is given by Eq. (2), and

$$\frac{1}{C_\pi(e_2)} = \int_0^\infty \pi(x) \, dx,$$

$$= \int_0^{x_{\max}} \pi_1(x) \, dx + \int_{x_{\max}}^{e_2} \pi_2(x) \, dx + \int_{e_2}^\infty \pi_3(x) \, dx,$$

$$= A_1 + A_2(e_2) + A_3(e_2). \tag{26}$$

Hence, the acceptance rate finally becomes

$$\eta_a(e_2) = \frac{C_p^{-1}}{A_1 + A_2(e_2) + A_3(e_2)}. \tag{27}$$

Then the optimum value, e_2^*, can be obtained by minimizing the discrepancy between the target and the proposal within E_2 and E_3, i.e., maximizing the acceptance rate η_a in these two intervals.

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[3]Recall also that $K = \frac{1}{C_\pi(e_2)}$ by design.

Since $\pi(x) \geq p(x) \ \forall x > 0$ by construction, and $p(x)$ is fixed, e_2^* is given by

$$
\begin{aligned}
e_2^* &= \arg\max_{e_2} \ \{\eta_a(e_2)\}, \\
&= \arg\min_{e_2} \ \{A_2(e_2) + A_3(e_2)\}, \\
&= \arg\min_{e_2} \left\{ \int_{x_{\max}}^{e_2} p(x_{\max}) \exp(-\alpha_2(x - x_{\max})^2)dx + \int_{e_2}^{\infty} \exp(-\alpha_3(x - \mu_3))dx \right\}, \\
&= \arg\min_{e_2} \left\{ \frac{p(x_{\max})}{2} \sqrt{\frac{\pi}{\alpha_2}} \mathrm{erf}\left(\sqrt{\alpha_2}(e_2 - x_{\max})\right) + \frac{1}{\alpha_3} \exp(-\alpha_3(x - \mu_3)) \right\}, \quad (28)
\end{aligned}
$$

where $\eta_a(e_2)$ denotes the acceptance rate considering E_2 and E_3 expressed as a function of e_2, $A_2(e_2)$ and $A_3(e_2)$ are the areas of the second and third pieces of the hat function given by (22) and (23) respectively, and $\mathrm{erf}(x)$ denotes the well-known error function [16], [17]:

$$
\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) \ dt. \quad (29)
$$

Unfortunately, a closed-form expression for e_2^* cannot be found, since e_2 appears in the limits of the integrals and several parameters (α_2, α_3 and μ_3) also depend on it. An approximately optimal solution can be found through grid search and/or numerical integration (see also Appendix A).

By setting $e_2 = e_2^*$ the optimum value of the acceptance rate, $\eta_a(e_2^*)$, would be obtained. Unfortunately, we cannot obtain an analytical expression for e_2^*, but we can easily provide a lower bound by using $e_2 = x_{in}$, with x_{in} given by (19), or $e_2 = \hat{e}_2^*$ with $\hat{e}_2^*$ given by (20). Fig. 2 displays the optimum acceptance rate and these two lower bounds as a function of m for different values of Ω, showing that $\eta_a(\hat{e}_2^*)$ is quite close to $\eta_a(e_2^*)$, especially for the frequently used normalized value $\Omega = 1$.

VI. Simulation Results

A. Histograms of the generated samples

First of all, in order to analyze the performance of the proposed algorithm we have generated $N = 5 \cdot 10^5$ samples using the hat function given by (21). Fig. 3 depicts two examples of the Nakagami PDF, $p_o(x) \propto p(x)$: for $m = 0.6$ and $\Omega = 1$ in Fig. 3(a), and for $m = 2$ and $\Omega = 1$ in Fig. 3(b). Fig. 3 also displays the normalized histogram obtained using the samples generated by the RS algorithm. In both cases the histogram closely resembles the target density, both around
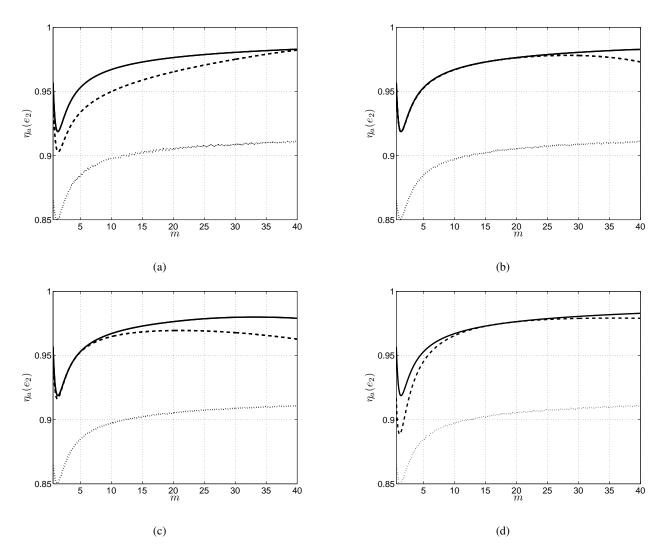
Fig. 2. Optimum acceptance rate $\eta_a(e_2^*)$ [continuous line] and two lower bounds, $\eta_a(x_{in})$ obtained using $e_2 = x_{in}$ [dashed line] and $\eta_a(\hat{e}_2^*)$ obtained using $e_2 = \hat{e}_2^*$ [dotted line], as a function of m for different values of Ω. **(a)** $\Omega = 0.1$. **(b)** $\Omega = 1$. **(c)** $\Omega = 10$. **(d)** $\Omega = 100$.

the mode and the tails, showing that our approach is able to produce samples from the true PDF without resorting to a truncated approximation, as required by certain methods to improve the efficiency (e.g., the ones proposed in [9], [12]).

B. Approximation of the cumulative distribution

In order to confirm the good performance of our approach for the tails, Fig. 4 displays the complementary cumulative distribution function (CCDF), $\bar{F}(x) = 1 - F(x)$, for the same two

examples in logarithmic scale. Fig. 4 shows the good match between the generated samples and the Nakagami CCDF, with the discrepancies for $\bar{F}(x) < 10^{-4}$ due to the limited number of samples available. Indeed, the RS algorithm guarantees that samples are drawn from the target PDF as long as $\pi(x) \geq p(x)$ for any value of x [13], [18]. Thus, in order to obtain a better approximation of the tails, all what is required is a larger number of samples.
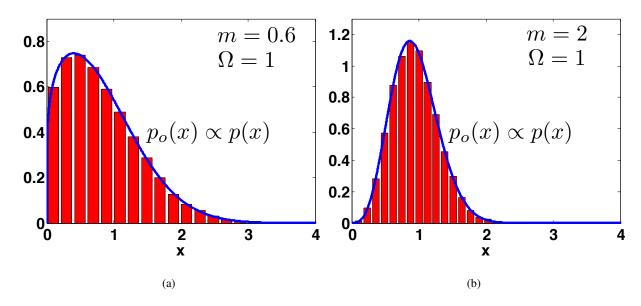

Fig. 3. Normalized histogram of the generated samples ($N = 5 \cdot 10^5$) and the normalized Nagakami PDF, $p_o(x) \propto p(x)$. **(a)** $m = 0.6$, $\Omega = 1$. **(b)** $m = 2$, $\Omega = 1$.

C. Acceptance Rate: comparison with [12]

We compute on the acceptance rate η_a (i.e., the number of samples accepted out of the total number of samples generated), which is the efficiency measure commonly used to characterize RS algorithms. Figure 5 shows the estimated acceptance rate $\widehat{\eta}_a$ (obtained drawing $N = 10^6$ samples from the Nakagami PDF; clearly, the number N_p of samples from π_o is a random variable, and varies at each run)[4] for several values of the fading parameters m and Ω. We compare with the approach described in [12], which is the most efficient one currently available.

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[4] In order to generate N samples from the target via the RS technique, we need to draw $N_p \geq N$ from the proposal PDF $\pi_o(x)$. Hence, the simplest approximation of acceptance rate can be computed as $\widehat{\eta}_a = \frac{N}{N_p}$. More refined approximations are discussed in Appendix B. For instance, we consider $\widehat{\eta}_a = \frac{1}{N_p} \sum_{i=1}^{N_p} \frac{p(x^{(i)})}{\pi(x^{(i)})}$, where $x^{(i)} \sim \pi_o(x)$.
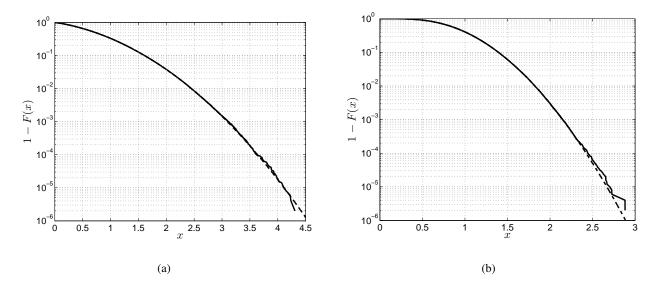
Fig. 4. Empirical complementary cumulative distribution function (CCDF) obtained using $N = 5 \cdot 10^5$ (continuous line) and theoretical Nagakami CCDF (dashed line). **(a)** $m = 0.6$, $\Omega = 1$. **(b)** $m = 2$, $\Omega = 1$.

It can be seen that our technique is extremely efficient, providing the best results ever reported in the literature, with acceptance rates above 90% in all cases and up to 97% in some cases, whereas the efficiency of [12] falls down to 80% for $m \geq 4$ in the truncated case (not shown), and to 70% without truncation. This means that in order to generate N samples our approach will never need more than $1.11 \times N$ iterations on average for any combination of the fading parameters, whereas the approach in [12] will need $1.25 \times N$ iterations on average for $m \geq 4$ in the truncated case, and $1.43 \times N$ without truncation.

D. Acceptance Rate: comparison with random samples generators from a Gamma PDF

Let us consider the Gamma density, $g_o(x) = C_1 g(x)$, with $G_1 = \frac{\beta^\alpha}{\Gamma(\alpha)}$,

$$g(x) = x^{\alpha-1} \exp\left(-\beta x\right), \qquad x \geq 0, \tag{30}$$

where $\alpha > 0$ is the shape parameter, $\beta > 0$ is the rate parameter and $\Gamma(\alpha)$ is the Gamma function. Note that, if we are able to draw samples from $g_o(x)$, then we can generate samples from the Nakagami PDF $p_o(x)$, indeed, if $X \sim g_o(x)$ we have that $Z = \sqrt{X} \sim p_o(x)$ with $m = \alpha$ and $\Omega = \frac{\alpha}{\beta}$. Thus, the problem is converted into finding an efficient Gamma random number generator.
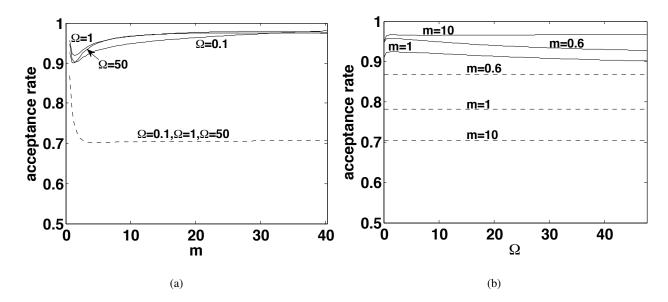
Fig. 5. Acceptance rates $\widehat{\eta}_a$ obtained using the proposed proposal (continuous line) and the one from [12] for an unbounded domain (dashed line). **(a)** Fixing Ω ($\Omega \in \{0.1, 1, 50\}$) and varying m. **(b)** Fixing m ($m \in \{0.6, 1, 10\}$) and varying Ω.

When α is an integer, an exact direct sampler is available (using α uniform random variables and a logarithmic transformation) [13], [19]. However, for the general case when α is not an integer, the problem of generating a Gamma random variable X is usually solved using also a RS scheme. Several RS methods have been introduced [13], [19]. We compare the proposed technique with the more efficient RS algorithms in literature for drawing from a Gamma PDF [20], [21], [22]. Namely, we use the RS techniques [20], [21], [22] for drawing N samples $x^{(i)}$ from the Gamma PDF and then obtain $z^{(i)} = \sqrt{x^{(i)}}$. Thus, we also apply our technique for generating directly $z^{(i)}$ for the corresponding Nakagami density. For the sake of simplicity we consider $\beta = 1$, since, given $\widetilde{X} \sim G(\alpha, 1)$, it can be easily shown that $X = \frac{1}{\beta}\widetilde{X} \sim G(\alpha, \beta)$. Therefore, after the square root transformation, we will have samples distributed according a Nakanmi PDF with parameters $m = \alpha$ and $\Omega = \alpha = m$. The results are provided in Figure 6. The proposed RS technique (shown with solid line) again obtains the greatest acceptance rate.

E. Variance of the acceptance rate estimator $\widehat{\eta}_a$

Let us define the random variable $Z = \frac{p(X)}{\pi(X)}$ with $X \sim \pi_o(x)$ (see also Appendix B), and the estimator of the acceptance rate $\widehat{\eta}_a = \frac{1}{N_p} \sum_{i=1}^{N_p} \frac{p(x^{(i)})}{\pi(x^{(i)})}$ with $x^{(i)} \sim \pi_o(x)$. Then, we have
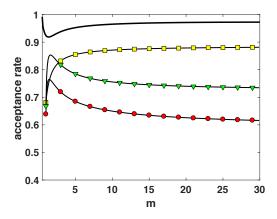
Fig. 6. Acceptance rates $\widehat{\eta}_a$ as function of m (recall that $\alpha = m$, $\beta = 1$ so that $\Omega = m$) obtained using the method introduced in this work (continuous line) and three alternative RS samplers based on a Gamma random variable generation, [22] (squares), [21] (triangles) and [20] (circles).

$E[Z] = \eta_a$ and

$$\text{var}[\widehat{\eta}_a] = \frac{1}{N_p^2} \sum_{i=1}^{N_p} \text{var}\left[\frac{p(x^{(i)})}{\pi(x^{(i)})}\right] = \frac{1}{N_p}\text{var}[Z]. \tag{31}$$

Observe that Z is an estimator of η_a using only sample. Furthermore, note that $\text{var}[Z]$ is a measure of the discrepancy between π and $p(x)$ (a smaller variance corresponds to a smaller discrepancy). Hence, the value of $\text{var}[Z]$ can be considered another index of performance. We have computed approximately $\text{var}[Z]$ (with $N = 10^6$) with the proposed method and the technique in [12], for different values of m and Ω of the Nakagami density.[5] The results are shown in Figure 7. The proposed algorithm always provides a smaller value of $\text{var}[Z]$.

VII. CONCLUSIONS AND FUTURE LINES

In this paper we have proposed an automatic rejection sampling (RS) algorithm to generate independent samples from Nakagami random variables, which are required for simulating the Nakagami-m fading channel, with arbitrary fading parameters. Our approach is based on a novel proposal density composed of three pieces: two truncated Gaussians around the mode and

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[5] Given $\text{var}[Z]$, we can obtain $\text{var}[\widehat{\eta}_a]$ as $\frac{1}{N_p}\text{var}[Z]$. Since $N_p \geq N = 10^6$, then we have directly an upper bound, $\text{var}[\widehat{\eta}_a] \leq \frac{1}{N}\text{var}[Z]$.
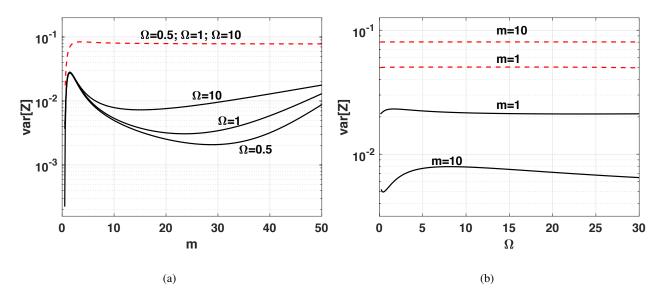
Fig. 7. The curve of the var$[Z]$ in log-scale, where $Z = \frac{p(X)}{\pi(X)}$ and $X \sim \pi_o(x)$ (see Eq. (31)) obtained with the proposed method (continuous line) and with the technique in [12] for an unbounded domain (dashed line). **(a)** Fixing $\Omega \in \{0.5, 1, 10\}$ and varying m. **(b)** Fixing $m \in \{1, 10\}$ and varying Ω.

an exponential for the tail. The resulting algorithm is simple and extremely efficient, providing acceptance rates above 90% for any value of the fading parameters (m and Ω), which are the best ones ever reported in the literature. Furthermore, for certain values of m and Ω the proposed RS scheme attains acceptance rates up to 97%, thus providing virtually exact sampling (i.e., sampling without any rejection). Moreover, unlike some previous approaches, these high acceptance rates are obtained without any truncation of the domain (i.e., we always provide samples from the true target density) and without requiring any adjustment from the final user. Future research lines include extending the method to the generation of multiple correlated Nakagami RVs. Another possible extension is the design of an adaptive scheme for adjusting online the choice of e_2, in the same fashion of the techniques introduced in [23], [24].

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APPENDIX

A. Gradient descent approach for obtaining the optimal e_2^*

Since the cost function to be minimized, $J(e_2) = A_2(e_2) + A_3(e_2)$, is convex, the optimal acceptance rate can always be attained through a gradient descent minimization algorithm [25], [26]. Mathematically, we can find an extremely close approximation to e_2^* iteratively as

$$\hat{e}_2[n+1] = \hat{e}_2[n] - \rho_n \nabla J(\hat{e}_2[n]), \tag{32}$$

where ρ_n is the step size parameter and $\nabla J(\hat{e}_2[n])$ denotes the gradient of $J(e_2)$ evaluated at $e_2 = \hat{e}_2[n]$, which is provided in Appendix A1. Note that, although the expressions for this gradient are quite involved, this process has to be performed only once, during the initialization stage of the algorithm. Therefore, the increase in acceptance rate obtained may be worth the effort when a large number of samples have to be drawn, as it often happens in the evaluation of wireless communication systems under fading conditions.

Fig. 8 shows two examples of the performance of the gradient descent algorithm using $\hat{e}_2[0] = x_{in}$ and $\rho_n = 0.2 \times 0.999^n$. In both cases the optimum value of e_2 (obtained numerically through a grid search) is attained after a moderate number of iterations (around 500 and 1600 respectively for a stopping condition $|\hat{e}_2[n+1] - \hat{e}_2[n]| < 10^{-5}$). On the one hand, in the first case the acceptance rate is only improved marginally w.r.t. using the value of e_2 given by (20) (0.02 % improvement), as shown in Fig. 9(a), so the optimization is not worth the effort. On the other hand, in the second case the acceptance rate is improved substantially w.r.t. using $e_2 = \hat{e}_2^*$ (3.85 % improvement), as shown in Fig. 9(b), so the optimization is clearly worth the effort when the number of samples to be drawn is large enough. Note that in both cases the improvement in acceptance rate of using either $e_2 = \hat{e}_2^*$ or the gradient descent w.r.t. setting $e_2 = x_{in}$ as given by (19) is remarkable: around 6.71 % in the first case and up to 7.73 % in the second one.

1) Gradient of the cost function: The cost function that has to be minimized in order to obtain the optimum value e_2^*, is

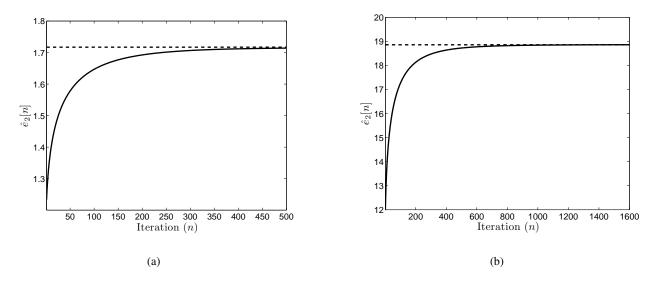$$J(e_2) = A_2(e_2) + A_3(e_2), \tag{33}$$

(a)

(b)

Fig. 8. Convergence of the gradient descent algorithm using $\hat{e}_2[0] = x_{in}$ and $\rho_n = 0.2 \times 0.999^n$ (continuous line) to the optimum value e_2^* obtained numerically through a grid search (dashed line). **(a)** $m = 1.5$ and $\Omega = 1$. **(b)** $m = 0.8$ and $\Omega = 100$.
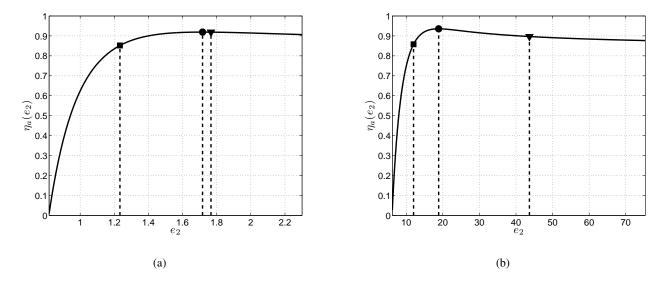

(a)

(b)

Fig. 9. Average acceptance rate as a function of e_2 and acceptance rates obtained using $e_2 = x_{in}$ as given by (19) (square), $e_2 = \hat{e}_2^*$ as given by (20) [triangle] and the e_2 obtained iteratively using the gradient descent algorithm shown in (32) (circle). **(a)** $m = 1.5$ and $\Omega = 1$. **(b)** $m = 0.8$ and $\Omega = 100$.

where $A_2(e_2)$ and $A_3(e_2)$ are given by (23) and (24) respectively. Hence, the gradient w.r.t. e_2 is simply

$$\nabla J(e_2) = \nabla A_2(e_2) + \nabla A_3(e_2). \tag{34}$$

The first gradient in (34) is given by

$$\nabla A_2(e_2) = \frac{p(x_{\max})}{2}\sqrt{\frac{\pi}{\alpha_2}}\left[-\mathrm{erf}\left(\sqrt{\alpha_2}(e_2 - x_{\max})\right)\frac{\nabla\alpha_2(e_2)}{2\alpha_2} + \nabla\mathrm{erf}\left(\sqrt{\alpha_2}(e_2 - x_{\max})\right)\right], \quad (35)$$

where

$$\nabla\alpha_2(e_2) = \frac{\alpha_3(e_2) - 2\alpha_2(e_2 - x_{\max})}{(e_2 - x_{\max})^2} \quad (36)$$

and

$$\nabla\mathrm{erf}\left(\sqrt{\alpha_2}(e_2 - x_{\max})\right) = \frac{2}{\sqrt{\pi}}\left(\frac{(e_2 - x_{\max})\nabla\alpha_2(e_2)}{2\sqrt{\alpha_2}} + \sqrt{\alpha_2}\right)\exp(-\alpha_2(e_2 - x_{\max})^2). \quad (37)$$

Regarding the second gradient, it is given by

$$\nabla A_3(e_2) = -\left(1 + \frac{\nabla\alpha_3(e_2)}{\alpha_3^2}\right)\exp(-\alpha_3(e_2 - \mu_3)), \quad (38)$$

where

$$\nabla\alpha_3(e_2) = \frac{2m - 1}{e_2^2} + \frac{2m}{\Omega}. \quad (39)$$

B. Computation and variance of the approximated acceptance rate $\widehat{\eta}_a$

Let us recall that the acceptance rate is defined as

$$\eta_a = \int_0^\infty p_a(x)\pi_o(x) \, dx,$$
$$= \int_0^\infty \frac{p(x)}{\pi(x)}\pi_o(x) \, dx. \quad (40)$$

where $\pi_o(x)$ is the normalized proposal PDF and $\pi(x) \propto \pi_o(x)$. Since, in order to generate N samples from the target via the RS technique, we need to draw $N_p \geq N$ from the proposal PDF, a first estimator of the acceptance rate is $\widehat{\eta}_a = \frac{N}{N_p}$. Observe that the Monte Carlo approximation of the integral defining η_a is

$$\widehat{\eta}_a = \frac{1}{N_p}\sum_{i=1}^{N_p}\frac{p(x^{(i)})}{\pi(x^{(i)})}, \quad \text{with} \quad x^{(i)} \sim \pi_o(x). \quad (41)$$

This is a Rao-Blackwellised version of the previous estimator $\widehat{\eta}_a = \frac{N}{N_p}$. Given the expression in Eq. (41) and since $x^{(i)}$ are i.i.d. (realizations of a random variable $X \sim \pi_o(x)$), note that $E[\widehat{\eta}_a] = \eta_a$ and

$$\text{var}[\widehat{\eta}_a] = \frac{1}{N_p^2} \sum_{i=1}^{N_p} \text{var}\left[\frac{p(x^{(i)})}{\pi(x^{(i)})}\right] = \frac{1}{N_p} \text{var}\left[\frac{p(X)}{\pi(X)}\right], \quad X \sim \pi_o(x). \tag{42}$$

Thus, if we define a random variable $Z = \frac{p(X)}{\pi(X)}$, then we have $E[\widehat{\eta}_a] = E[Z] = \eta_a$ and $\text{var}[\widehat{\eta}_a] = \frac{1}{N_p}\text{var}[Z]$. Note that $\text{var}[Z]$ is itself a measure of the performance of the RS sampler since smaller value of $\text{var}[Z]$ means that the hat function $\pi(x)$ is closer to $p(x)$ (hence the density of Z accumulates probability mass close to 1). Finally, note that in the proposed RS method, we can easily compute the theoretical value of the acceptance rate η_a, using Eq. (25) i.e., $\eta_a = \frac{C_\pi}{C_p}$, where $\frac{1}{C_\pi} = \int_0^\infty \pi(x) \, dx = A_1 + A_2 + A_3$ and $C_p = \frac{2m^m}{\Omega^m \Gamma(m)}$.