Extension of Maxwell's Equations for Charge Creation and Annihilation

Hideki Mutoh*

Link Research Corporation

Odawara, Kanagawa 250-0055, Japan

(Dated: July 7, 2014)

Abstract

Extension of Maxwell's equations is proposed to realize charge creation and annihilation. The proposed equations includes Nakanishi-Lautrup (NL) field, which was introduced to construct Lorentz covariant electromagnetic field model for quantum electrodynamics (QED). The necessity of the extension of Maxwell's equations is shown by the comparison of current values given by Maxwell's and the proposed equations in the simple structure consisting of a silicon sphere surrounded by SiO_2. Maxwell's equations give unreasonable currents in SiO_2, although the proposed equations give reasonable result. The electromagnetic field energy density is increased by existence of NL field.

KEYWORDS: Maxwell's equations, charge conservation, Nakanishi-Lautrup field, charge creation

* hideki.mutoh@nifty.com

I. INTRODUCTION

Maxwell's equations have been believed as the fundamental equations to describe electromagnetic field since J. C. Maxwell found the equations in 1865[1]. In early 1930s, E. Fermi proposed modified electromagnetic field model for quantum electrodynamics (QED) [2–4], where he assumed that 4-D vector potential satisfy d' Alembert equation even in the case except Lorenz gauge condition. Gupta and Bleuler gave subsidiary conditions to Fermi's model in 1950[5, 6]. In 1960s, Nakanishi and Lautrup proposed the auxiliary field called Nakanishi-Lautrup (NL) field[7–10] to construct Lorentz covariant electromagnetic field model for QED. It is now included in the model of QED and Yang-Mills theory[11–14]. However, these models have not been reflected to classical electromagnetism. Recently, we found that the electromagnetic field model including a Lorentz scalar field, which is equivalent to NL field with Feynman gauge, can easily treat creation and annihilation of positive and negative charge pairs, although it is difficult for Maxwell's equations to treat them[15–17]. In this paper, the necessity of the extension of Maxwell's equations is shown by the comparison of current values given by Maxwell's and the proposed equations with a simple structure.

Maxwell's equations are given by

$$\mathbf{J} = \nabla \times \mathbf{H} - \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \tag{1}$$

$$\rho = \varepsilon \nabla \mathbf{E}, \tag{2}$$

$$\nabla \times \mathbf{E} + \mu \frac{\partial \mathbf{H}}{\partial t} = 0, \tag{3}$$

$$\nabla \mathbf{H} = 0, \tag{4}$$

where $\mathbf{J}$ and ρ are current and charge density, ε and μ are permittivity and permeability, $\mathbf{E}$ and $\mathbf{H}$ are electric and magnetic field, respectively. Eqs. (1) and (2) directly give the following equation of the charge conservation,

$$\nabla \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \tag{5}$$

The creation and annihilation of positive and negative charge pairs are ordinarily described by the following equation, which is given by semiconductor physics[18–20],

$$\nabla \mathbf{J}_p + \frac{\partial \rho_p}{\partial t} = -\nabla \mathbf{J}_n - \frac{\partial \rho_n}{\partial t} = G, \tag{6}$$

where ρ_p and ρ_n are positive and negative charge concentration, $\mathbf{J}_p$ and $\mathbf{J}_n$ are positive and negative charge current density, and G is charge creation-annihilation rate. Since Maxwell's equations satisfy the principle of superposition[21], positive and negative charges must individually satisfy Eqs. (1) and (2). Therefore, positive charges satisfy

$$\mathbf{J}_p = \nabla \times \mathbf{H}_p - \varepsilon \frac{\partial \mathbf{E}_p}{\partial t}, \tag{7}$$

and

$$\rho_p = \varepsilon \nabla \mathbf{E}_p, \tag{8}$$

where $\mathbf{E}_p$ and $\mathbf{H}_p$ denote electric and magnetic field induced by positive charges, respectively. Eqs. (7) and (8) directly give

$$\nabla \mathbf{J}_p + \frac{\partial \rho_p}{\partial t} = 0, \tag{9}$$

which contradicts (6) in the case of $G \neq 0$. Since this situation is same for negative charges, it is difficult for Maxwell's equations to treat creation and annihilation of charge pairs.

II. MODIFICATION OF MAXWELL'S EQUATIONS

In order to solve the above problem, we introduce Nakanishi-Lautrup field B and a gauge parameter α. The Lagrangian density of the electromagnetic field $\mathcal{L}_{EM}$ is given by[10]

$$\mathcal{L}_{EM} = -\frac{1}{4} F^{\nu\lambda} F_{\nu\lambda} + B \partial^\nu A_\nu + \frac{1}{2} \alpha B^2 - \mu J^\nu A_\nu, \tag{10}$$

where J^ν and A^ν denote 4-D current $(c\rho, \mathbf{J})$ and 4-D vector potential $(\psi/c, \mathbf{A})$, respectively, and $F^{\nu\lambda}$ is given by

$$F^{\nu\lambda} = \partial^\nu A^\lambda - \partial^\lambda A^\nu. \tag{11}$$

The above Lagrangian density gives the following equations.

$$\mu J_\nu = \Box A_\nu - \partial_\nu \partial^\lambda A_\lambda - \partial_\nu B, \tag{12}$$

$$\partial^\nu A_\nu + \alpha B = 0, \tag{13}$$

$$\pi^\nu = \frac{\partial \mathcal{L}_{EM}}{\partial(\partial_0 A_\nu)} = (B, -\mathbf{E}/c), \tag{14}$$

where π^ν denotes 4-D canonical momentum density and $\Box$ is d'Alembertian defined by $\Box \equiv \partial_0^2 - \nabla^2$.

Since $\mathbf{E}$ and $\mathbf{H}$ are written by

$$\mathbf{E} = -\nabla\psi - \frac{\partial \mathbf{A}}{\partial t}, \tag{15}$$

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A}, \tag{16}$$

Eqs. (1) and (2) are rewritten by Eqs. (12), (15) and (16) as

$$\mathbf{J} = \nabla \times \mathbf{H} - \varepsilon\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu}\nabla B, \tag{17}$$

$$\rho = \varepsilon\nabla\mathbf{E} - \varepsilon\frac{\partial B}{\partial t}. \tag{18}$$

Then, the charge creation-annihilation rate is given by

$$G = \nabla\mathbf{J} + \frac{\partial \rho}{\partial t} = -\frac{1}{\mu}\Box B. \tag{19}$$

The above relation enable us to treat creation and annihilation of positive and negative charge pairs. It should be noticed that $G = 0$ needs not $B = 0$ but $\Box B = 0$. Although $\Box B = 0$ is assumed in QED[12, 13], we assume $\Box B \neq 0$ in the region of $G \neq 0$. The above model is a natural extension from 3-D to 4-D field for the complex electromagnetic field $\mu\mathbf{H} + i\mathbf{E}/c$. Maxwell's equations, given by Eqs. (1), (2), (3), (4), (15), and (16), can be written by using 3-D complex field as

$$\begin{pmatrix} \mu H_x + \frac{i}{c}E_x \\ \mu H_y + \frac{i}{c}E_y \\ \mu H_z + \frac{i}{c}E_z \end{pmatrix} = \begin{pmatrix} -i\partial_0 & -\partial_z & \partial_y & -\partial_x \\ \partial_z & -i\partial_0 & -\partial_x & -\partial_y \\ -\partial_y & \partial_x & -i\partial_0 & -\partial_z \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \\ \frac{i}{c}\psi \end{pmatrix}, \tag{20}$$

$$\mu \begin{pmatrix} J_x \\ J_y \\ J_z \\ ic\rho \end{pmatrix} = \begin{pmatrix} i\partial_0 & -\partial_z & \partial_y \\ \partial_z & i\partial_0 & -\partial_x \\ -\partial_y & \partial_x & i\partial_0 \\ \partial_x & \partial_y & \partial_z \end{pmatrix} \begin{pmatrix} \mu H_x + \frac{i}{c}E_x \\ \mu H_y + \frac{i}{c}E_y \\ \mu H_z + \frac{i}{c}E_z \end{pmatrix}. \tag{21}$$

The model including NL field, given by Eqs. (3), (4), (13), (15), (16), (17), and (18), can be written by using 4-D complex field as

$$\begin{pmatrix} \mu H_x + \frac{i}{c}E_x \\ \mu H_y + \frac{i}{c}E_y \\ \mu H_z + \frac{i}{c}E_z \\ -\alpha B \end{pmatrix} = \begin{pmatrix} -i\partial_0 & -\partial_z & \partial_y & -\partial_x \\ \partial_z & -i\partial_0 & -\partial_x & -\partial_y \\ -\partial_y & \partial_x & -i\partial_0 & -\partial_z \\ \partial_x & \partial_y & \partial_z & -i\partial_0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \\ \frac{i}{c}\psi \end{pmatrix}, \tag{22}$$

$$\mu \begin{pmatrix} J_x \\ J_y \\ J_z \\ ic\rho \end{pmatrix} = \begin{pmatrix} i\partial_0 & -\partial_z & \partial_y & -\partial_x \\ \partial_z & i\partial_0 & -\partial_x & -\partial_y \\ -\partial_y & \partial_x & i\partial_0 & -\partial_z \\ \partial_x & \partial_y & \partial_z & i\partial_0 \end{pmatrix} \begin{pmatrix} \mu H_x + \frac{i}{c}E_x \\ \mu H_y + \frac{i}{c}E_y \\ \mu H_z + \frac{i}{c}E_z \\ -B \end{pmatrix}. \tag{23}$$

When the coordinate system has velocity v along x-axis, the Lorentz transformation of $\mathbf{A}, \psi, \mathbf{H}, \mathbf{E}, B, \mathbf{J}$, and ρ are given by

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \\ \frac{i}{c}\psi' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{i\beta}{\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-i\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \\ \frac{i}{c}\psi \end{pmatrix}, \tag{24}$$

$$\begin{pmatrix} \mu H'_x + \frac{i}{c}E'_x \\ \mu H'_y + \frac{i}{c}E'_y \\ \mu H'_z + \frac{i}{c}E'_z \\ -B' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-\beta^2}} & \frac{-i\beta}{\sqrt{1-\beta^2}} & 0 \\ 0 & \frac{i\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu H_x + \frac{i}{c}E_x \\ \mu H_y + \frac{i}{c}E_y \\ \mu H_z + \frac{i}{c}E_z \\ -B \end{pmatrix}, \tag{25}$$

$$\begin{pmatrix} J'_x \\ J'_y \\ J'_z \\ ic\rho' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{i\beta}{\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-i\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix} \begin{pmatrix} J_x \\ J_y \\ J_z \\ ic\rho \end{pmatrix}, \tag{26}$$

where β denotes v/c. Therefore, $\mathbf{A}, \psi, \mathbf{H}, \mathbf{E}, J$, and ρ have same transformation as original Maxwell's equations, and B is not changed by Lorentz transformation. Although Eq. (13) does not satisfy the gauge invariance, if a scalar function Λ satisfies $\Box\Lambda = 0$, $\mathbf{E}$, $\mathbf{H}$, and B are not changed by the transformation of

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda, \tag{27}$$

$$\psi' = \psi - \frac{\partial\Lambda}{\partial t}. \tag{28}$$

Next we consider about the electromagnetic field energy including NL field. By using Eqs. (3), (4), (14), (17) and (18), $cJ^\nu\pi_\nu$ is written by

$$\begin{aligned} cJ^\nu\pi_\nu = \mathbf{J}\mathbf{E} + c^2\rho B = -\nabla\left(\mathbf{E}\times\mathbf{H} - \frac{1}{\mu}B\mathbf{E}\right) \\ -\frac{\partial}{\partial t}\left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2} + \frac{B^2}{2\mu}\right). \end{aligned} \tag{29}$$

Since the above equation is regarded as the continuity equation for energy density, $\mathbf{J}\mathbf{E}+c^2\rho B$ is energy annihilation rate, $\mathbf{E}\times\mathbf{H}-B\mathbf{E}/\mu$ is the energy flow vector, and $(\varepsilon E^2+\mu H^2+B^2/\mu)/2$ is the energy density. The NL field induces the additional energy density of $B^2/2\mu$. Recently, we found that NL field causes confinement of charge creation and annihilation centers, which means charge conservation for this model[17].

III. COMPARISON OF CURRENT VALUES BY MAXWELL'S AND MODIFIED EQUATIONS

Now we compare the calculation results given by Maxwell's and the proposed equations including NL field, using a simple structure. Fig. 1 shows an example structure consisting of a silicon sphere with radius R surrounded by SiO_2 under illumination or in a heating chamber, where

$$\mathbf{J} = \mathbf{J}_p + \mathbf{J}_n = 0, \tag{30}$$

$$\rho = \rho_p + \rho_n = 0, \tag{31}$$

$$\mathbf{E} = \mathbf{E}_p + \mathbf{E}_n = 0, \tag{32}$$

$$\mathbf{H} = \mathbf{H}_p + \mathbf{H}_n = 0. \tag{33}$$

$\mathbf{E}_n$ and $\mathbf{H}_n$ denote electric and magnetic field induced by negative charges, respectively. Since this structure has spherical symmetry, the magnetic field does not exist[21],

$$\mathbf{H}_p = \mathbf{H}_n = 0. \tag{34}$$

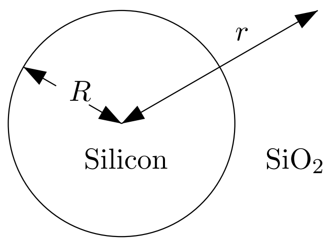

FIG. 1. A silicon sphere with radius R surrounded by SiO_2 under illumination or in a heating chamber.

Then, the NL fields B and scalar potential ψ also satisfy

$$B = B_p + B_n = 0, \tag{35}$$

$$\psi = \psi_p + \psi_n = 0. \tag{36}$$

where B_p and ψ_p are induced by holes and B_n and ψ_p are induced by electrons. It is assumed that the hole and electron charge density ρ_p and ρ_n in the silicon generated by light or thermal energy increase linearly with time as

$$\rho_p = -\rho_n = \begin{cases} \rho_0(1+\frac{t}{\tau}) & (r \leq R) \\ 0 & (r > R), \end{cases} \tag{37}$$

where the light or the heater is switched on at $t = 0$, ρ_0 is the charge density at $t = 0$, and the charge density increases with the charge creation rate of ρ_0/τ. Using spherical coordinate system and Gauss's law, the electric field has only radial component as

$$E_p = -E_n = \begin{cases} \frac{\rho_0 r}{3\varepsilon_{Si}}(1+\frac{t}{\tau}) & (r \leq R) \\ \frac{\rho_0 R^3}{3\varepsilon_{ox} r^2}(1+\frac{t}{\tau}) & (r > R), \end{cases} \tag{38}$$

where ε_{si} and ε_{ox} are permittivity of silicon and SiO_2, respectively. Then ψ_p and ψ_n are given by

$$\psi_p = -\psi_n = \begin{cases} \frac{\rho_0\{(2\varepsilon_{Si}+\varepsilon_{ox})R^2-\varepsilon_{ox}r^2\}}{6\varepsilon_{Si}\varepsilon_{ox}}(1+\frac{t}{\tau}) & (r \leq R) \\ \frac{\rho_0 R^3}{3\varepsilon_{ox} r}(1+\frac{t}{\tau}) & (r > R), \end{cases} \tag{39}$$

In the case of original Maxwell's equations, the radial component of the current J_p and J_n out of the sphere are needed by Eqs. (1), (34), and (38) as

$$J_p = -J_n = \begin{cases} -\frac{\rho_0 r}{3\tau} & (r \leq R) \\ -\frac{\rho_0 R^3}{3\tau r^2} & (r > R). \end{cases} \tag{40}$$

The above result does not describe the real condition, because the hole and electron currents cannot exist in SiO_2. Maxwell's equations cannot increase charge concentration without current because of the charge conservation of Eq. (5). If we consider the NL field B_p and B_n for the charge pairs creation with assuming $\alpha = 0$ and $\nabla \mathbf{A} = 0$, they are given by

$$B_p = -B_n = \begin{cases} -\frac{\mu\rho_0\{(2\varepsilon_{Si}+\varepsilon_{ox})R^2-\varepsilon_{ox}r^2\}}{6\varepsilon_{ox}\tau} & (r \leq R) \\ -\frac{\mu\rho_0 R^3}{3\tau r} & (r > R). \end{cases} \tag{41}$$

Since the radial component of the gradient of B_p and B_n are given by

$$(\nabla B_p)_r = -(\nabla B_n)_r = \begin{cases} \frac{\mu\rho_0 r}{3\tau} & (r \leq R) \\ \frac{\mu\rho_0 R^3}{3\tau r^2} & (r > R), \end{cases} \tag{42}$$

the positive and negative charge current density J_p and J_n in and out of the sphere are given by Eqs. (17), (34), (38), and (41) as

$$J_p = -J_n = -\varepsilon\frac{\partial E_p}{\partial t} + \frac{1}{\mu}(\nabla B_p)_r = \begin{cases} 0 & (r \leq R) \\ 0 & (r > R). \end{cases} \tag{43}$$

There is no current in and out of the sphere. Then the charge creation-annihiltion rate G is given by

$$G = -\frac{1}{\mu}\Box B_p = \begin{cases} \frac{\rho_0}{\tau} & (r \leq R) \\ 0 & (r > R). \end{cases} \tag{44}$$

The electromagnetic field model including NL field gives the reasonable result.

IV. CONCLUSION

In conclusion, the proposed electromagnetic field equations including NL field realizes charge creation and annihilation. They can be described by quite simple formula using 4D vectors and differential operator matrices and satisfy Lorentz covariance. The necessity of the extension of Maxwell's equations was shown by the comparison of current values given by Maxwell's and the proposed equations in a simple structure. The electromagnetic field energy density is increased by existence of NL field.

---

[1] J. C. Maxwell, Phil. Trans. R. Soc. Lond. **155**, 459 (1865).

[2] E. Fermi., Rev. Mod. Phys **4**, 87 (1932).

[3] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, 1962).

[4] F. Mandl and G. Shaw, *Quantum Field Theory 2nd ed.* (Wiley, 2010).

[5] S. N. Gupta, Proc. Phys. Soc. **63**, 681 (1950).

[6] K. Bleuler, Helv. Phys. Acta **23**, 567 (1950).

[7] N. Nakanishi, Progr. Theor. Phys. **35**, 1111 (1966).

[8] N. Nakanishi, Progr. Theor. Phys. Suppl. **51**, 1 (1972).

[9] B. Lautrup, K. Danske Vidensk, Selk. Mat.-fis. Medd. **35**, no. 11 (1967).

[10] N. Nakanishi, *Ba-No-Ryoshiron (Quantum Field Theory)* (Baifukan, in Japanese, 1975).

[11] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

[12] I. J. R. Aitchison, *An Informal Introduction to Gauge Field Theories* (Cambridge University Press, 1982).

[13] T. Kugo, *Gauge-Ba-No-Ryoshiron (Quantum Theory of Gauge field)* (Baifukan, in Japanese, 1989).

[14] K. Kondo, *Gauge-Ba-No-Ryosiron-Nyumon (Introduction to Quantum Theory of Gauge field)* (Science-Sha, in Japanese, 2013).

[15] H. Mutoh, J. Institute of Image Information and Television Engineers **67**, J89 (2013), https://www.jstage.jst.go.jp/article/itej/67/3/67_J89/_pdf.

[16] H. Mutoh, "Proceedings of the 2013 international image sensor workshop," (2013), https://www.imagesensors.org/Past%20Workshops/2013%20 Workshop/2013%20Papers/07-01_052-Mutoh_paper.pdf.

[17] H. Mutoh, viXra (hep) , 1403.0118 (2014), http://vixra.org/pdf/1403.0118v1.pdf.

[18] S. M. Sze, *Physics of Semiconductor Devices* (Wiley, New York, 1981).

[19] A. S. Grove, *Physics and Technology of Semiconductor Devices* (Wiley, New York, 1967).

[20] S. Selberherr, *Analysis and Simulation of Semiconductor Devices* (Springer, Wien, 1984).

[21] R. P. Feynman, R. B. Leighton, and M. L. Sands, *Feynman Lectures on Physics, 2* (Addison-Wesley, 1965).