A Clifford-Gravity Based Cosmology, Dark Matter and Dark Energy

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Abstract

A Clifford-Gravity based model is exploited to build a generalized action (beyond the current ones used in the literature) and arrive at relevant numerical results which are consistent with the presently-observed de Sitter accelerating expansion of the universe driven by a very small vacuum energy density $\rho_{obs} \sim 10^{-120}(M_P)^4$ (M_P is the Planck mass) and provide promising dark energy/matter candidates in terms of the 16 scalars corresponding to the degrees of freedom associated with a $Cl(3,1)$-algebra valued scalar field $\mathbf{\Phi}$ in four dimensions.

Keywords: Clifford Algebras, Gravity, Yang-Mills, Grand Unification, Inflation, Cosmology, Strings.

1 Introduction

Clifford, Division, Exceptional and Jordan algebras are deeply related and essential tools in many aspects in Physics [1], [2], [3] . A Clifford $Cl(5,C)$ Unified Gauge Field Theory formulation of Conformal Gravity and $U(4) \times U(4) \times U(4)$ Yang-Mills in $4D$ was recently reviewed in [4] along with its implications for the Pati-Salam group $SU(4) \times SU(2)_L \times SU(2)_R$, and $Trinification$ GUT models of 3 fermion generations based on the group $SU(3)_C \times SU(3)_L \times SU(3)_R$.

In the past years, the Extended Relativity Theory in C-spaces (Clifford spaces) and Clifford-Phase spaces were developed [5], [6]. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory whose generalized coordinates are Clifford polyvector-valued quantities which incorporate the lines, areas, volumes, and hyper-volumes degrees of freedom associated with the collective dynamics of particles, strings, membranes, p-branes (closed p-branes) moving in a D-dimensional target spacetime background.

We learnt from Special Relativity that the concept of simultaneity is also relative. By the same token, we have shown in [12] that the concept of spacetime locality is *relative* due to the *mixing* of area-bivector coordinates with spacetime vector coordinates under generalized Lorentz transformations in C-space. In the most general case, there will be mixing of all polyvector valued coordinates. This was the motivation to build a unified theory of all extended objects, p-branes, for all values of p subject to the condition $p + 1 = D$.

In [8] we explored the many novel physical consequences of Born's Reciprocal Relativity theory [9], [10], [11] in flat phase-space and generalized the theory to the curved phase-space scenario. We provided six specific novel physical results resulting from Born's Reciprocal Relativity and which are *not* present in Special Relativity. These were : momentum-dependent time delay in the emission and detection of photons; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations. We finalized by constructing a Born reciprocal general relativity theory in curved phase-spaces which required the introduction of a complex Hermitian metric, torsion and nonmetricity.

Recently, novel physical consequences of the Extended Relativity Theory in C-spaces (Clifford spaces) were explored in [12]. The latter theory provides a very different physical explanation of the phenomenon of "relativity of locality" than the one described by the Doubly Special Relativity (DSR) framework. Furthermore, an elegant *nonlinear* momentum-addition law was derived in order to tackle the "soccer-ball" problem in DSR. Neither derivation in C-spaces requires a *curved* momentum space nor a deformation of the Lorentz algebra. While the constant (energy-independent) speed of photon propagation is always compatible with the generalized photon dispersion relations in C-spaces, another important consequence was that the generalized C-space photon dispersion relations allowed also for energy-dependent speeds of propagation while still *retaining* the Lorentz symmetry in ordinary spacetimes, while breaking the *extended* Lorentz symmetry in C-spaces. This does *not* occur in DSR nor in other approaches, like the presence of quantum spacetime foam.

The aim of this work is to exploit the Clifford symmetry to build a generalized action beyond the current ones used in the literature and arrive at relevant numerical results which are consistent with the presently-observed de Sitter accelerating expansion of the universe driven by a very small vacuum energy density $\rho_{obs} \sim 10^{-120}(M_P)^4$ (M_P is the Planck mass) and provide promising dark energy/matter candidates in terms of the 16 scalars corresponding to the degrees of freedom associated with a $Cl(3,1)$-algebra valued scalar field $\mathbf{\Phi}$ in four dimensions.

2 Clifford Gravity Cosmology and Dark Energy

We begin by explaining the relationship between Clifford-algebra-valued Gauge Field Theories and Conformal Gravity. By fixing some of the gauge symmetries and imposing some constraints one recovers ordinary gravity. Let us show how the conformal algebra in four dimensions admits a Clifford algebra realization; i.e. the generators of the conformal algebra can be expressed in terms of the Clifford algebra basis generators. The conformal algebra in four dimensions $so(4,2)$ is isomorphic to $su(2,2)$.

Let $\eta_{ab} = (-,+,+,+)$ be the Minkowski spacetime (flat) metric in $D = 3+1$-dimenisons. The epsilon tensors are defined as $\epsilon_{0123} = -\epsilon^{0123} = 1$, The real Clifford $Cl(3,1,R)$ algebra associated with the tangent space of a $4D$ spacetime $\mathcal{M}$ is defined by the anticommutators

$$\{ \Gamma_a, \Gamma_b \} \equiv \Gamma_a \Gamma_b + \Gamma_b \Gamma_a = \ = 2 \eta_{ab} \tag{2.1a}$$

such that

$$[\Gamma_a, \Gamma_b] = 2\Gamma_{ab}, \quad \Gamma_5 = -i \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3, \quad (\Gamma_5)^2 = 1; \quad \{\Gamma_5, \Gamma_a\} = 0; \tag{2.1b}$$

$$\Gamma_{abcd} = \epsilon_{abcd} \Gamma_5; \quad \Gamma_{ab} = \frac{1}{2} (\Gamma_a \Gamma_b - \Gamma_b \Gamma_a). \tag{2.2a}$$

$$\Gamma_{abc} = \epsilon_{abcd} \Gamma_5 \Gamma^d; \quad \Gamma_{abcd} = \epsilon_{abcd} \Gamma_5. \tag{2.2b}$$

$$\Gamma_a \Gamma_b = \Gamma_{ab} + \eta_{ab}, \quad \Gamma_{ab} \Gamma_5 = \frac{1}{2} \epsilon_{abcd} \Gamma^{cd}, \tag{2.2c}$$

$$\Gamma_{ab} \Gamma_c = \eta_{bc} \Gamma_a - \eta_{ac} \Gamma_b + \epsilon_{abcd} \Gamma_5 \Gamma^d \tag{2.2d}$$

$$\Gamma_c \Gamma_{ab} = \eta_{ac} \Gamma_b - \eta_{bc} \Gamma_a + \epsilon_{abcd} \Gamma_5 \Gamma^d \tag{2.2e}$$

$$\Gamma_a \Gamma_b \Gamma_c = \eta_{ab} \Gamma_c + \eta_{bc} \Gamma_a - \eta_{ac} \Gamma_b + \epsilon_{abcd} \Gamma_5 \Gamma^d \tag{2.2f}$$

$$\Gamma^{ab} \Gamma_{cd} = \epsilon^{ab}{}_{cd} \Gamma_5 - 4\delta^{[a}_{[c} \Gamma^{b]}{}_{d]} - 2\delta^{ab}_{cd}. \tag{2.2g}$$

$$\delta^{ab}_{cd} = \frac{1}{2} (\delta^a_c \delta^b_d - \delta^a_d \delta^b_c). \tag{2.2.h}$$

the generators $\Gamma_{ab}, \Gamma_{abc}, \Gamma_{abcd}$ are defined as usual by a signed-permutation sum of the anti-symmetrizated products of the gammas. A representation of the $Cl(3,1)$ algebra exists where the generators

$$\mathbf{1}; \ \Gamma_1, \ \Gamma_2, \ \Gamma_3, \ \Gamma_4 = -i\Gamma_0; \ and \ \Gamma_5 \tag{2.3}$$

are Hermitian; while the generators $\Gamma_a \Gamma_5$ and Γ_{ab} for $a, b = 1, 2, 3, 4$ are anti-Hermitian. Using eqs-(2.1-2.3) allows to write the $Cl(3,1)$ algebra-valued one-form as

$$\mathbf{A} = \left(a_\mu \mathbf{1} + b_\mu \Gamma_5 + e^a_\mu \Gamma_a + f^a_\mu \Gamma_a \Gamma_5 + \frac{1}{4} \omega^{ab}_\mu \Gamma_{ab} \right) dx^\mu. \tag{2.4}$$

The physical significance of the field components $a_\mu, b_\mu, e^a_\mu, f^a_\mu, \omega^{ab}_\mu$ in eq-(2.4) will be explained below.

The Clifford-valued gauge field A_μ transforms according to $A'_\mu = U^{-1} A_\mu U + U^{-1}\partial_\mu U$ under Clifford-valued gauge transformations. The Clifford-valued field strength is $F = dA + [A, A]$ so that F transforms covariantly $F' = U^{-1} F U$. Decomposing the field strength in terms of the Clifford algebra generators gives

$$F_{\mu\nu} = F^1_{\mu\nu} \mathbf{1} + F^5_{\mu\nu} \Gamma_5 + F^a_{\mu\nu} \Gamma_a + F^{a5}_{\mu\nu} \Gamma_a \Gamma_5 + \frac{1}{4} F^{ab}_{\mu\nu} \Gamma_{ab}. \tag{2.5}$$

the Clifford-algebra-valued 2-form field strength is $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ where $\partial_\mu A_\nu = \frac{\partial A_\nu}{\partial x^\mu}$. The field-strength components are given by

$$F^1_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \tag{2.6a}$$

$$F^5_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu + 2 e^a_\mu f_{\nu a} - 2 e^a_\nu f_{\mu a} \tag{2.6b}$$

$$F^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^{ab}_\mu e_{\nu b} - \omega^{ab}_\nu e_{\mu b} + 2 f^a_\mu b_\nu - 2 f^a_\nu b_\mu \tag{2.6c}$$

$$F^{a5}_{\mu\nu} = \partial_\mu f^a_\nu - \partial_\nu f^a_\mu + \omega^{ab}_\mu f_{\nu b} - \omega^{ab}_\nu f_{\mu b} + 2 e^a_\mu b_\nu - 2 e^a_\nu b_\mu \tag{2.6d}$$

$$F^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu + \omega^{ac}_\mu \omega_{\nu c}{}^{b} + 4\left(e^a_\mu e^b_\nu - f^a_\mu f^b_\nu\right) - \mu \longleftrightarrow \nu. \tag{2.6e}$$

At this stage we may provide the relation among the $Cl(3,1)$ algebra generators and the the conformal algebra $so(4,2) \sim su(2,2)$ in $4D$. It is well known to the experts that the operators of the Conformal algebra can be written in terms of the Clifford algebra generators as

$$P_a = \frac{1}{2}\Gamma_a (1 - \Gamma_5); \quad K_a = \frac{1}{2}\Gamma_a (1 + \Gamma_5); \quad D = -\frac{1}{2}\Gamma_5, \quad L_{ab} = \frac{1}{2}\Gamma_{ab}. \tag{2.7}$$

P_a ($a = 1,2,3,4$) are the translation generators; K_a are the conformal boosts; D is the dilation generator and L_{ab} are the Lorentz generators. The total number of generators is respectively $4+4+1+6 = 15$. From the above realization of the conformal algebra generators (2.7), the explicit evaluation of the commutators yields

$$[P_a, D] = P_a; \quad [K_a, D] = -K_a; \quad [P_a, K_b] = -2 g_{ab} D + 2 L_{ab}$$

$$[P_a, P_b] = 0; \quad [K_a, K_b] = 0; .... \tag{2.8}$$

which is consistent with the $su(2,2) \sim so(4,2)$ commutation relations. We should notice that the K_a, P_a generators in (2.7) are both comprised of Hermitian Γ_a *and* anti-Hermitian $\pm\Gamma_a\Gamma_5$ generators, respectively. The dilation D operator is Hermitian, while the Lorentz generator L_{ab} is anti-Hermitian. The fact that Hermitian and anti-Hermitian generators are required is consistent with the fact that $U(2,2)$ is a pseudo-unitary group as we shall see bellow.

Having established this one can infer that the real-valued tetrad V^a_μ field (associated with translations) and its real-valued partner $\tilde{V}^a_\mu$ (associated with conformal boosts) can be defined in terms of the real-valued gauge fields e^a_μ, f^a_μ as follows

$$e^a_\mu \ \Gamma_a \ + \ f^a_\mu \ \Gamma_a \Gamma_5 \ = \ V^a_\mu \ P_a \ + \ \tilde{V}^a_\mu \ K_a \tag{2.9}$$

From eq-(2.7) one learns that eq-(2.9) leads to

$$e^a_\mu \ - \ f^a_\mu \ = \ V^a_\mu; \quad e^a_\mu \ + \ f^a_\mu \ = \ \tilde{V}^a_\mu \ \Rightarrow$$

$$e^a_\mu \ = \ \frac{1}{2} \ (V^a_\mu + \tilde{V}^a_\mu), \quad f^a_\mu \ = \ \frac{1}{2} (\tilde{V}^a_\mu - V^a_\mu). \tag{2.10}$$

The components of the torsion and conformal-boost curvature of conformal gravity are given respectively by the linear combinations of eqs-(2.6c, 2.6d)

$$F^a_{\mu\nu} \ - \ F^{a5}_{\mu\nu} \ = \ \widetilde{F}^a_{\mu\nu}[P]; \quad F^a_{\mu\nu} \ + \ F^{a5}_{\mu\nu} \ = \ \widetilde{F}^a_{\mu\nu}[K] \ \Rightarrow$$

$$F^a_{\mu\nu} \ \Gamma_a \ + \ F^{a5}_{\mu\nu} \ \Gamma_a \ \Gamma_5 \ = \ \widetilde{F}^a_{\mu\nu}[P] \ P_a \ + \ \widetilde{F}^a_{\mu\nu}[K] \ K_a. \tag{2.11a}$$

Inserting the expressions for e^a_μ, f^a_μ in terms of the vielbein V^a_μ and $\tilde{V}^a_\mu$ given by (2.10), yields the standard expressions for the Torsion and conformal-boost curvature, respectively

$$\widetilde{F}^a_{\mu\nu}[P] \ = \ \partial_{[\mu} \ V^a_{\nu]} \ + \ \omega^{ab}_{[\mu} \ V_{\nu]b} \ - \ V^a_{[\mu} \ b_{\nu]}, \tag{2.11b}$$

$$\widetilde{F}^a_{\mu\nu}[K] \ = \ \partial_{[\mu} \ \tilde{V}^a_{\nu]} \ + \ \omega^{ab}_{[\mu} \ \tilde{V}_{\nu]b} \ + 2 \ \tilde{V}^a_{[\mu} \ b_{\nu]}, \tag{2.11b}$$

The Lorentz curvature in eq-(2.6e) can be recast in the standard form as

$$F^{ab}_{\mu\nu} = \mathcal{R}^{ab}_{\mu\nu} \ = \ \partial_{[\mu} \ \omega^{ab}_{\nu]} \ + \ \omega^{ac}_{[\mu} \ \omega^{b}_{\nu]c} \ + \ 2(\ V^a_{[\mu} \ \tilde{V}^b_{\nu]} \ + \ \tilde{V}^a_{[\mu} \ V^b_{\nu]} \ ). \tag{2.11c}$$

The components of the curvature corresponding to the Weyl dilation generator given by $F^5_{\mu\nu}$ in eq-(2.6b) can be rewritten as

$$F^5_{\mu\nu} \ = \ \partial_{[\mu} \ b_{\nu]} \ + \ \frac{1}{2} \ (\ V^a_{[\mu} \ \tilde{V}_{\nu]a} \ - \ \tilde{V}^a_{[\mu} \ V_{\nu]a} \ ). \tag{2.11d}$$

and the Maxwell curvature is given by $F^1_{\mu\nu}$ in eq-(2.6a). A re-scaling of the vielbein V^a_μ/l and $\tilde{V}^a_\mu/l$ by a length scale parameter l is necessary in order to endow the curvatures and torsion in eqs-(2.11) with the proper dimensions of $length^{-2}, length^{-1}$, respectively.

To sum up, the real-valued tetrad gauge field V^a_μ (that gauges the translations P_a) and the real-valued conformal boosts gauge field $\widetilde{V}^a_\mu$ (that gauges the conformal boosts K_a) of conformal gravity are given, respectively, by the linear combination of the gauge fields $e^a_\mu \ \mp \ f^a_\mu$ associated with the $\Gamma_a, \ \Gamma_a \ \Gamma_5$ generators of the Clifford algebra $Cl(3,1)$ of the tangent space of spacetime $\mathcal{M}^4$ after performing a Wick rotation $-i \ \Gamma_0 = \Gamma_4$.

Gauge invariant actions involving Yang-Mills terms of the form $\int Tr(F \wedge^* F)$ and theta terms of the form $\int Tr(F \wedge F)$ are straightforwardly constructed. For example, a $SO(4,2)$ gauge-invariant action for conformal gravity is [15]

$$S \ = \ \int \ d^4x \ \epsilon_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ \mathcal{R}^{ab}_{\mu\nu} \ \mathcal{R}^{cd}_{\rho\sigma} \qquad (2.12)$$

where the components of the Lorentz curvature 2-form $\mathcal{R}^{ab}_{\mu\nu} dx^\mu \wedge dx^\nu$ are given by eq-(2.11c) after re-scaling the vielbein V^a_μ / l and $\tilde{V}^a_\mu / l$ by a length scale parameter l in order to endow the curvature with the proper dimensions of $length^{-2}$.

The conformal boost symmetry can be fixed by choosing the gauge $b_\mu = 0$ because under infinitesimal conformal boosts transformations the field b_μ transforms as $\delta b_\mu = -2 \ \xi^a \ e_{a\mu} = -2 \ \xi_\mu$; i.e the parameter ξ_μ has the same number of degrees of feedom as b_μ. After further fixing the dilational gauge symmetry, setting the torsion to zero (which constrains the spin connection $\omega^{ab}_\mu (V^a_\mu)$ to be of the Levi-Civita form given by a function of the vielbein V^a_μ), and eliminating the $\tilde{V}^a_\mu$ field algebraically via its (non-propagating) equations of motion, the expression in eq-(2.12) leads to the de Sitter group $SO(4,1)$ invariant Macdowell-Mansouri-Chamseddine-West action (MMCW) [14] (suppressing spacetime indices for convenience)

$$S \ = \ \int \ (\ R^{ab}(\omega) \ - \frac{1}{l^2} \ V^a \wedge V^b \ ) \ \wedge \ (\ R^{cd}(\omega) \ - \frac{1}{l^2} \ V^c \wedge V^d \ ) \ \epsilon_{abcd}. \quad (2.13)$$

The action (2.13) is comprised of 3 terms. One term is the topological invariant Gauss-Bonnet term $\int R^{ab}(\omega) \wedge R^{cd}(\omega) \epsilon_{abcd}$. The standard Einstein-Hilbert gravitational action term is given by $-\frac{1}{l^2} \int R^{ab}(\omega) \wedge V^c \wedge V^d \epsilon_{abcd}$, and the cosmological constant term $\frac{1}{l^4} \int V^a \wedge V^b \wedge V^c \wedge V^d \epsilon_{abcd}$. l is the de Sitter space's throat size; i.e. l^2 is proportional to the square of the Planck scale (the Newtonian coupling constant).

The familiar Einstein-Hilbert gravitational action can also be obtained from a coupling of gravity to a scalar field like it occurs in a Brans-Dicke-Jordan theory of gravity

$$S \ = \ \frac{1}{2} \int \ d^4x \ \sqrt{g} \ \phi \ D^c_\mu \ D^\mu_c \ \phi \ =$$

$$\frac{1}{2} \int \ d^4x \ \sqrt{g} \ \phi \ \left(\ \frac{1}{\sqrt{g}} \partial_\nu (\sqrt{g} \ g^{\mu\nu} \ D^c_\mu \phi) \ + \ b^\mu \ (D^c_\mu \phi) \ + \ \frac{1}{6} \ R \ \phi \ \right). \quad (2.14a)$$

where the conformally covariant derivative acting on a scalar field ϕ of Weyl weight one is

$$D^c_\mu \phi \ = \ \partial_\mu \ - \ b_\mu \ \phi \qquad (2.14b)$$

Fixing the conformal boosts symmetry by setting $b_\mu = 0$ and the dilational symmetry by setting $\phi = constant$ leads to the Einstein-Hilbert action for ordinary gravity.

We proceed next with the cosmological applications by introducing the Clifford-valued scalar field (a hyper-complex valued scalar) defined as

$$\mathbf{\Phi} \ = \ \Phi^A \ \Gamma_A \ = \ \phi \ \mathbf{1} + \phi^a \ \gamma_a + \frac{1}{2!} \ \phi^{ab} \ \gamma_{ab} + \frac{1}{3!} \ \phi^{abc} \ \gamma_{abc} + \frac{1}{4!} \ \phi^{abcd} \ \gamma_{abcd} \quad (2.15)$$

Now we can propose the most general action as an *extension* of the MMCW action displayed in eq-(2.13) and given by

$$\mathbf{S} \;=\; \int d^4x\, \epsilon^{\mu\nu\rho\sigma} \;<\; \mathbf{F}_{\mu\nu}\, \mathbf{F}_{\rho\sigma}\, \mathbf{\Phi} \;>\;\; = \;\; \int d^4x\, \epsilon^{\mu\nu\rho\sigma} \;<\; F^A_{\mu\nu}\, F^B_{\rho\sigma}\, \Phi^C\, \Gamma_A\, \Gamma_B\, \Gamma_C > \tag{2.16}$$

The bracket operation $< ..... >$ denotes extracting the Clifford scalar part of the geometric product of Clifford-valued quantities. It is the analog of taking the trace of a matrix product. The most general action can be decomposed into several pieces $\mathbf{S} = S_1 + S_2 + S_3 + S_4 + S_5$. Defining $\phi^{abcd} \;=\; \epsilon^{abcd}\, \phi^5 = \epsilon^{abcd}\, \varphi$ we have

$$S_5 \;=\; \int d^4x\, \epsilon^{\mu\nu\rho\sigma} \;<\; F^A_{\mu\nu}\, F^B_{\rho\sigma}\, \phi^{abcd}\, \Gamma_A\, \Gamma_B\, \gamma_{abcd} > \;\; =$$
$$\int d^4x\, \epsilon_{abcd}\, \epsilon^{\mu\nu\rho\sigma}\, \varphi\, \left(a_{51} F^{ab}_{\mu\nu}\, F^{cd}_{\rho\sigma} \; + a_{52}\, F^a_{\mu\nu}\, F^{bcd}_{\rho\sigma} \; + a_{53}\, F_{\mu\nu}\, F^{abcd}_{\rho\sigma} \right) \; +$$
$$\int d^4x\, \epsilon_{abcd}\, \epsilon^{\mu\nu\rho\sigma}\, \varphi\, \left(a_{54} F^{ab}_{\mu\nu e}\, F^{ecd}_{\rho\sigma} \; + a_{55} F^a_{\mu\nu e}\, F^{ebcd}_{\rho\sigma} \; + a_{56} F^{ab}_{\mu\nu e f}\, F^{efcd}_{\rho\sigma} \right) \tag{2.17}$$

One can rewrite (2.17) in differential form notation as

$$S_5 \;=\; \int \epsilon_{abcd}\, \varphi\, \left(a_{51}\, F^{ab} \wedge F^{cd} \; + a_{52}\, F^a \wedge F^{bcd} \; + a_{53}\, F \wedge F^{abcd} \right) +$$
$$\int \epsilon_{abcd}\, \varphi\, \left(a_{54}\, F^{ab}{}_{e} \wedge F^{ecd} \; + a_{55}\, F^a{}_{e} \wedge F^{ebcd} \; + a_{56}\, F^{ab}{}_{ef} \wedge F^{efcd} \right) \tag{2.18}$$

One can recognize that the MMCW action (2.13) is contained in one *piece* of S_5 and given by

$$S_{MMCW} \;\subset\; \int d^4x\, \epsilon_{abcd}\, \epsilon^{\mu\nu\rho\sigma}\, \varphi\, \left(F^{ab}_{\mu\nu}\, F^{cd}_{\rho\sigma} \right) \tag{2.19}$$

when $\varphi = 1$ as described by eqs-(2.6e, 2.11). One should notice that when the scalar field φ is *not* constant the expression

$$\int d^4x\, \sqrt{g}\, \varphi\, (\, R_{\mu\nu\rho\sigma}\, R_{\mu\nu\rho\sigma} \; - 4\, R_{\mu\nu}\, R^{\mu\nu} \; + \; R^2 \,) \tag{2.20}$$

is no longer equal to the Gauss-Bonnet topological invariant due to the key $\varphi(x)$ factor and such terms will now contribute to the equations of motion.

The term $\epsilon_{abcd} F^a \wedge F^{bcd}$ in (2.18) can be rewritten as $F^a \wedge \tilde{F}_a$, while the term $\epsilon_{abcd} F \wedge F^{abcd} = F \wedge \tilde{F}$, etc.... The components $F^{bcd} = F^{bcd}_{\mu\nu} dx^\mu \wedge dx^\nu$, $F^{abcd} = F^{abcd}_{\mu\nu} dx^\mu \wedge dx^\nu$, etc. ... are all given by eqs-(2.4,2.5,2.6) after taking into account the relations among the Clifford algebra generators (gamma matrices) in eqs-(2.1, 2.2). The other terms in the action are

$$S_1 \; = \; \int d^4x \; \epsilon^{\mu\nu\rho\sigma} \; < \; F^A_{\mu\nu} \; F^B_{\rho\sigma} \; \phi \; \Gamma_A \; \Gamma_B \; \mathbf{1} \; > \; =$$

$$\int d^4x \; \epsilon^{\mu\nu\rho\sigma} \; \phi \; \left(\; a_{11} \; F_{\mu\nu} \; F_{\rho\sigma} \; + a_{12} \; F^a_{\mu\nu} \; F_{a \; \rho\sigma} \; + a_{13} \; F^{ab}_{\mu\nu} \; F_{ab \; \rho\sigma} \; \right) \; +$$

$$\int d^4x \; \epsilon^{\mu\nu\rho\sigma} \; \phi \; \left(\; a_{14} \; F^{abc}_{\mu\nu} \; F_{abc \; \rho\sigma} \; + a_{15} \; F^{abcd}_{\mu\nu} \; F_{abcd \; \rho\sigma} \; \right) \tag{2.21}$$

One can rewrite (2.21) in differential form notation as

$$S_1 \; = \; \int \phi \; \left(\; a_{11} \; F \; \wedge \; F \; + a_{12} \; F^a \; \wedge \; F_a \; + a_{13} \; F^{ab} \; \wedge \; F_{ab} \right) \; +$$

$$\int \phi \; \left(\; a_{14} \; F^{abc} \; \wedge F_{abc} \; + a_{15} \; F^{abcd} \; \wedge \; F_{abcd} \right) \tag{2.22}$$

$$S_3 = \int d^4x \; \epsilon^{\mu\nu\rho\sigma} \; < \; F^A_{\mu\nu} \; F^B_{\rho\sigma} \; \phi^{ab} \; \Gamma_A \; \Gamma_B \; \gamma_{ab} \; > \; =$$

$$\int \phi_{ab} \; \left(\; a_{31} \; F^a \wedge F^b \; + a_{32} \; F^{ab} \wedge F \; + a_{33} \; F^a_c \wedge F^{cb} \; \right) \; +$$

$$\int \phi_{ab} \; \left(\; a_{34} \; F^a_{cd} \wedge F^{cdb} \; + a_{35} \; F^a_{cde} \wedge F^{cdeb} \; \right) \tag{2.23}$$

$$S_2 = \int d^4x \; \epsilon^{\mu\nu\rho\sigma} \; < \; F^A_{\mu\nu} \; F^B_{\rho\sigma} \; \phi^a \; \Gamma_A \; \Gamma_B \; \gamma_a \; > \; =$$

$$\int \phi_a \; \left(\; a_{21} \; F^a \wedge F \; + a_{22} \; F^a_{\;\;b} \; \wedge \; F^b \; + a_{23} \; F^a_{bc} \wedge F^{bc} \; + a_{24} \; F^a_{bcd} \wedge F^{bcd} \; \right) \tag{2.24}$$

$$S_4 = \int d^4x \; \epsilon^{\mu\nu\rho\sigma} \; < \; F^A_{\mu\nu} \; F^B_{\rho\sigma} \; \phi^{abc} \; \Gamma_A \; \Gamma_B \; \gamma_{abc} \; > \; =$$

$$\int \phi_{abc} \; \left(\; a_{41} \; F^{abc} \wedge F \; + a_{42} \; F^{ab} \wedge F^c \; + a_{43} \; F^{abc}{}_d \wedge F^d \; \right) \; +$$

$$\int \phi_{abc} \; \left(\; a_{44} \; F^{ab}{}_d \wedge F^{dc} \; + a_{45} \; F^{ab}{}_{de} \wedge F^{dec} \; \right) \tag{2.25}$$

the way to obtain the numerical coefficients a_{ij} is explained in the Appendix.

It is essential to introduce dynamics for the *dimensionless* Clifford-valued scalar field $\mathbf{\Phi}$ otherwise a variation of the action (2.16) with respect to the $\mathbf{\Phi}$ field will trivially constraint the action to zero since in this case $\mathbf{\Phi}$ will act as a Lagrange multiplier. The scalar field contribution to the action for the signature $(-,+,+,+)$ is

$$S[\mathbf{\Phi}] \; = \; \int d^4x \; \sqrt{g} \; < \; - \; \frac{1}{2l^2} \; (D_\mu \; \mathbf{\Phi}^\dagger) \; (D^\mu \; \mathbf{\Phi}) \; - \frac{1}{l^4} \; V(\mathbf{\Phi}) \; > \tag{2.26a}$$

The dagger operation $\mathbf{\Phi}^\dagger$ denotes the reversal operation and is obtained by reversing the order of the Clifford generators. For example, $(\gamma_a \wedge \gamma_b)^\dagger = \gamma_b \wedge \gamma_a$, $(\gamma_a \wedge \gamma_b \wedge \gamma_c)^\dagger = \gamma_c \wedge \gamma_b \wedge \gamma_a$, etc ..... so that

$$< (D_\mu \mathbf{\Phi}^\dagger) (D^\mu \mathbf{\Phi}) > = (D_\mu \phi) (D^\mu \phi) + (D_\mu \phi_a) (D^\mu \phi^a) + (D_\mu \phi_{ab}) (D^\mu \phi^{ab}) +$$

$$(D_\mu \phi_{abc}) (D^\mu \phi^{abc}) + (D_\mu \phi_{abcd}) (D^\mu \phi^{abcd}) \quad (2.26b)$$

where we have omitted combinatorial numerical factors for convenience.

The potential, for example, may be given by a polynomial $V(\mathbf{\Phi}) = \sum_{n=0} a_n \mathbf{\Phi}^n$ or a more complicated function. Upon taking the Clifford scalar part of the potential one has $< V(\mathbf{\Phi}) >= \mathcal{V}(\phi, \phi^a, \phi^{ab}, \phi^{abc}, \phi^{abcd})$ which is a complicated (polynomial, for example) expression given in terms of the 16 scalars. For simplicity we shall choose the analog of a quartic Higgs-like potential given by

$$\mathcal{V} = \frac{1}{l^4} \lambda (|\Phi^A \Phi_A| - \mathbf{v}^2)^2$$

$$\Phi^A \Phi_A = \phi^2 + \phi^a \phi_a + \frac{1}{2!} \phi^{ab} \phi_{ab} + \frac{1}{3!} \phi^{abc} \phi_{abc} + \frac{1}{4!} \phi^{abcd} \phi_{abcd} \quad (2.27)$$

the reason one must take the absolute value in $|\Phi^A \Phi_A|$ is because the Clifford scalar norm $\Phi^A \Phi_A$ is *not* positive definite since the 16-dimensional quadratic form has a split $(8, 8)$ signature [7] when the tangent space metric η_{ab} is Minkowskian $diag(-1, +1, +1, +1)$.

The gauge covariant derivative acting on the Clifford-valued scalar $\mathbf{\Phi}$ is defined as

$$(D_\mu \Phi^A) \Gamma_A = (\partial_\mu \Phi^A) \Gamma_A + [A_\mu^B \Gamma_B, \Phi^C \Gamma_C] \Rightarrow$$

$$D_\mu \Phi^A = (\partial_\mu \Phi^A) + A_\mu^B \Phi^C < [\Gamma_B, \Gamma_C] \Gamma^A > = (\partial_\mu \Phi^A) + A_\mu^B \Phi^C f_{BC}^A \quad (2.28a)$$

where we have written the commutator Clifford algebra as $[\Gamma_B, \Gamma_C] = f_{BC}^A \Gamma_A$ and whose structure constants are displayed in the Appendix. Under infinitesimal $Cl(3, 1)$ gauge transformations the Clifford-valued scalar $\mathbf{\Phi}$ field transforms as

$$\delta \Phi^C = f_{AB}^C \xi^A \Phi^B, \quad \xi = \xi^A \Gamma_A = \tilde{\xi} \mathbf{1} + \xi^a \gamma_a + \frac{1}{2} \xi^{ab} \gamma_{ab} +$$

$$\frac{1}{3!} \xi^{abc} \gamma_{abc} + \frac{1}{4!} \xi^{abcd} \gamma_{abcd} \quad (2.28b)$$

and the gauge covariant derivative transforms as well $\delta(D_\mu \Phi^C) = f_{AB}^C \xi^A D_\mu \Phi^B$.

To sum up, the action $\mathbf{S} + S[\mathbf{\Phi}]$ given by eqs-(2.16-2.26) is comprised of (i) φ times the MMCW Lagrangian (2.13) that contains the Einstein-Hilbert and cosmological constant terms. (ii) Extra terms quadratic in the curvature and torsion. (iii) A coupling of curvature and torsion terms. (iv) kinetic and potential terms for a multiplet of 16 spacetime scalar fields $\phi, \phi^a, \phi^{ab}, \phi^{abc}, \phi^{abcd}$ that from the tangent space point of view behave as a scalar, vector, antisymmetric tensors of rank two and three and a pseudo-scalar field, respectively. (v) Non-minimal couplings of the scalars and curvature and torsion terms. (vi) terms involving the field strengths associated with conformal boosts, a dilational (Weyl gauge field) and a $U(1)$ Maxwell-like generator as displayed by eqs-(2.6, 2.11). A review of conformal (super) gravity can be found in [15].

Our action displayed by eqs-(2.16-2.26) is a more complex generalization of the $f(R,T)$ modified gravity models involving powers of curvature and torsion [22]. It is also a more general extension of the cosmological models based on Brans-Dicke-Jordan gravity [21] and non-minimally coupled Einstein-Electroweak theory [19]. It contains many more terms than a $U(2,2) = SU(2,2)\times U(1)$ gauge theory (conformal gravity and Maxwell theory) combined with the kinetic and potential terms of a multiplet of 16 scalar fields (corresponding to a 4×4 matrix-valued scalar in the 16-dimensional adjoint representation of $U(2,2)$).

Solving the equations of motion of the action $\mathbf{S} + S[\mathbf{\Phi}]$ after performing a variation with respect to all the fields is a very cumbersome project that requires a Clifford computer algebra package and which is beyond the scope of this work. Fixing and/or breaking some of the gauge symmetries will simplify things. Let us *truncate* the action given in eqs-(2.16,2.26) by *freezing* all the components of $\mathbf{\Phi}$ to zero except φ so that the following Higgs-like potential $\mathcal{V}$

$$\mathcal{V} \ = \ \frac{1}{l^4}\ \lambda\ (\varphi^2 - \mathbf{v}^2)^2, \quad \lambda > 0 \tag{2.29}$$

is minimized to zero when $\varphi_o = \mathbf{v}$. Focusing solely on the terms in eq-(2.19) and the Higgs potential in eq-(2.26a), we have (i) φ times the { Gauss-Bonnet terms, the Einstein-Hilbert action, and the cosmological constant }; and (ii) the *effective* potential energy density given by the scalar potential minus the running cosmological "constant" term

$$U_{eff} \ = \ \frac{1}{l^4}\ \lambda\ (\ \varphi^2 \ - \ \mathbf{v}^2\ )^2 - \ \frac{\varphi}{l^4} \tag{2.30}$$

Let us define the reduced Planck mass by $M_P^2 = (1/8\pi L_P^2)$ and equate the Planck energy density $\frac{1}{4}M_P^4$ to the value of U_{eff} when $\varphi = 0$ in eq-(2.30)

$$U_{eff}(\varphi = 0) \ = \ \frac{1}{l^4}\ (\ \lambda\ \mathbf{v}^4) \ = \ \frac{1}{4}\ M_P^4 \ = \ \frac{1}{(16\pi)^2 L_P^4} \tag{2.31}$$

By equating the value of the effective potential energy density at $\varphi = \varphi_*$ to the present-day observed vacuum energy density one has

$$U_{eff}(\varphi_*) \ = \ \frac{1}{l^4}\ \lambda\ (\ \varphi_*^2 \ - \ \mathbf{v}^2\ )^2 - \ \frac{\varphi_*}{l^4} \ = \ \rho_{obs} \ \sim \ \frac{1}{L_P^2\ R_H^2} \ =$$

$$(\frac{L_P}{R_H})^2\ \frac{1}{L_P^4} \ \sim \ 10^{-120}\ M_P^4 \tag{2.32}$$

where L_P and R_H are the Planck and Hubble scale, respectively. The ratio $(\frac{L_P}{R_H})^2$ is chosen to be of the order of 10^{-120}. Matching the present-day value of the Newtonian coupling constant with the running coupling appearing in the Einstein-Hilbert term in eq-(2.19), when $\varphi = \varphi_*$, gives

$$\frac{\varphi_*}{l^2} \;=\; \frac{1}{16\pi G} \;=\; \frac{1}{2}\;\frac{1}{8\pi L_P^2} \;=\; \frac{1}{2}\;M_P^2 \tag{2.33}$$

It is interesting to note that negative values of φ furnish a negative coupling G that would correspond to a repulsive gravitational regime. For the time being we shall focus in the case where $\varphi \geq 0$.

Finally, from eqs-(2.30, 2.31, 2.33) one arrives at the following numerical results for the $l, \mathbf{v}, \lambda$ parameters of the Higgs-like potential (2.29)

$$l \;\simeq\; R_H, \quad \mathbf{v} \;\simeq\; \frac{1}{16\pi}\;(\frac{R_H}{L_P})^2, \quad \lambda \;\simeq\; (16\pi)^2\;(\frac{L_P}{R_H})^4 \tag{2.34}$$

and $\varphi_* \simeq \mathbf{v}$.

From the plot of the graph U_{eff}/ρ_{obs} versus φ one learns that $\varphi_* < \varphi_o = \mathbf{v}$ but its value is *very* close to $\mathbf{v}$. Since the throat size of the present de Sitter accelerating universe $l = R_H$ agrees with the value for l obtained in eq-(2.34) this is sign of consistency. The value of $\varphi_* + \epsilon$ is the *crossover* point when the effective potential energy density (2.30) switches from *positive* to *negative* values as φ increases (assuming it increases with the flow of time). Anti de Sitter spacetime has a constant negative energy density and positive pressure (attractive force) ; whereas de Sitter spacetime has a constant positive energy density and negative pressure (repulsive force). In our most simplified scenario, the universe has not entered yet the phase of negative energy density where its expansion may halt, and begin to contract until the point φ_{**} , when it will crossover again into a positive energy density epoch of perpetual expansion.

Our results obtained above are compatible with a very rapid de Sitter inflationary phase in the very early universe because of the very large initial value of the (positive) energy density. An extensive and recent review (with a vast number of references) about cosmological inflation and its realization in quantum field theory and in string theory can be found [17]. Furthermore, our results are also consistent with the present-day de Sitter accelerating universe with a very small value of the vacuum energy density (2.32) due to the very large value of the Hubble scale. More recently, the authors [20] have argued that the so-called cosmological constant fine-tuning problem (why the cosmological constant observed today is so much smaller than the Planck scale or why the universe is accelerating at present) can be solved with the help of Higgs inflation by simply assuming a variable cosmological "constant" during the inflation epoch. This is compatible with our findings.

To sum up, in our simplified scenario all the parameters $l, \mathbf{v}, \lambda$ of the Higgs-like scalar potential (2.29) are given in terms of the *two* fundamental scales, L_P, R_H (a lower and upper scale) by eq-(2.34) which allows us to reproduce the extremely small observed vacuum energy density (2.32) and the current value of the Newtonian gravitational coupling (2.33).

The fact that a running Newtonian coupling in eq-(2.33) leads to $G = \frac{l^2}{16\pi\varphi} \to \infty$ when $\varphi \to 0$, at the Big Bang singularity for example, does not mean that the Einstein-Hilbert action necessarily collapses to zero, because one may have $R = \infty$ at the singularity such that the ratio $R/16\pi G$ might still be well defined. In order to study the behavior of the scalar φ as a function of x^μ, one has to determine the spacetime dynamics of $\varphi(x^\mu)$ which is obtained by performing a variation of the truncated action with respect to φ, and yielding a very complex equation of the form

$$\frac{1}{l^2}\ D_\mu D^\mu \varphi\ -\frac{1}{l^4}\ \frac{\partial V(\varphi)}{\partial\varphi}\ +\ \epsilon_{abcd}\ \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}\ (a_{51}\ F^{ab}_{\mu\nu}\ F^{cd}_{\nu\sigma}\ +\ ......)\ +$$

$$\frac{1}{l^2}\ (A^a_\mu A^\mu_a\ +\ A^{abc}_\mu A^\mu_{abc})\ \varphi\ =\ 0 \tag{2.35}$$

the last terms in (2.35) stem from the contribution $[\mathbf{A}_\mu, \mathbf{\Phi}]^2$ to the $(D_\mu\Phi^A)(D^\mu\Phi_A)$ terms in the truncated action.

One cannot solve eq-(2.35) without performing a variation of the action with respect to the remaining gauge fields. In the most general case, one has to study the full spacetime dynamics of *all* the gauge fields involved in the non-truncated action, with the key contribution of the kinetic and potential terms $(D_\mu\Phi_A)(D^\mu\Phi^A)$, $\mathcal{V}(\Phi^A)$ for *all* the scalars, to see whether or not there is a dynamical evolution of the 16 scalar fields that is consistent with the extremely small value of the vacuum energy density observed today, and associated with a de Sitter accelerated phase of expansion. The throat size of the de Sitter solution is $l = R_H$.

Fermionic matter terms and gauge fields of the Standard and GUT Models should be taken into account in the most general theory. A de Sitter, Anti de Sitter and Minkowski vacuum spacetime solution is also consistent with a breaking of the $SU(2,2) \sim SO(4,2)$ conformal symmetry down to the de Sitter $SO(4,1)$, Anti de Sitter $SO(3,2)$ and Minkowski $SO(4)$ one. Recently, the authors [18] studied the problem of obtaining de Sitter and inflationary vacua from dimensional reduction of double field theory (DFT) on non-geometric string backgrounds. They also considered a new class of effective potentials that admit Minkowski and de Sitter minima.

Before embarking into the study of the full action comprised of eqs-(2.16-2.26), one can start instead with the simpler Clifford-gravity inspired action

$$S\ =\ \int d^4x\ \sqrt{g}\ \Big(\ \varphi\ [\ R_{\mu\nu\rho\sigma}\ R_{\mu\nu\rho\sigma}\ -4\ R_{\mu\nu}\ R^{\mu\nu}\ +\ R^2\ ]\ -\frac{\varphi}{l^2}\ R\ +\frac{\varphi}{l^4}\ \Big)\ -$$

$$\int d^4x\ \sqrt{g}\ (\ \frac{1}{2l^2}\ (\partial_\mu\varphi)\ (\partial^\mu\varphi)\ +\ \frac{1}{l^4}\ V(\varphi)\ ) \tag{2.36}$$

as a testing ground for cosmological scenarios. An even simpler action was the Weyl invariant action investigated in [23] where the source of dark energy was identified with a dilaton-like scalar field θ of dimensions $length^{-1}$ that is required to implement Weyl (scale) invariance of the action

$$S \;=\; \frac{1}{16\pi} \int d^4x \;\sqrt{g}\; \left(\;-\;\theta^2\; \mathcal{R}_{Weyl} \;-\; \frac{1}{2}\; g^{\mu\nu}\; (D_\mu \theta)\; (D_\nu \theta) \;-\; V(\theta) \right) \tag{2.37a}$$

under the Weyl scalings

$$\theta' = e^{-\Omega}\;\theta;\;\;\; g'_{\mu\nu} = e^{2\Omega} g_{\mu\nu},\;\;\; \mathcal{R}'_{Weyl} = e^{-2\Omega}\mathcal{R}_{Weyl},\;\;\; V(\theta') = e^{-4\Omega}V(\theta)$$

$$D_\mu \theta = \partial_\mu \theta - A_\mu \theta \;\rightarrow\; (D_\mu\theta)' = e^{-\Omega} D_\mu \theta,\;\;\; A'_\mu = A_\mu - \partial_\mu \Omega, ... \tag{2.37b}$$

the Weyl symmetry naturally selects a quartic potential $V \sim \theta^4$. It was shown in [23] how the action was related to a Brans-Dicke-Jordan model whose ω parameter had its critical value $\omega = -3/2$ and leading to the observed constant vacuum energy density when the scalar field θ was scaled to a constant such that $(\theta_o)^2 = 1/G$. To conclude, we believe that Clifford-gravity-based cosmology is a promising avenue to understand the origins of the very small presently observed value of the vacuum energy density, and the 16 scalar fields corresponding to the Clifford-valued scalar $\mathbf{\Phi}$ in four-dimensions could be plausible dark energy/matter candidates.

APPENDIX

In this Appendix we shall write the (anti) commutator relations for the Clifford algebra generators and explain how to obtain the numerical coefficients a_{ij} in eqs-(2.16-2.25).

$$\frac{1}{2}\;\{\;\gamma_a,\;\gamma_b\;\} \;=\; g_{ab}\;\mathbf{1};\;\; \frac{1}{2}\;[\;\gamma_a,\;\gamma_b\;] \;=\; \gamma_{ab} \;=\; -\;\gamma_{ba},\;\; a,b = 1,2,3,\cdots,m \tag{A.1}$$

$$[\;\gamma_a,\;\gamma_{bc}\;] \;=\; 2\;g_{ab}\;\gamma_c \;-2\;g_{ac}\;\gamma_b,\;\;\; \{\;\gamma_a,\;\gamma_{bc}\;\} \;=\; 2\;\gamma_{abc} \tag{A.2}$$

$$[\;\gamma_{ab},\;\gamma_{cd}\;] \;=\; -\;2\;g_{ac}\;\gamma_{bd} \;+2\;g_{ad}\;\gamma_{bc} \;-2\;g_{bd}\;\gamma_{ac} \;+2\;g_{bc}\;\gamma_{ad} \tag{A.3}$$

In general one has [16]

$$pq = \mathbf{odd},\;\; [\gamma_{m_1 m_2 .... m_p},\;\gamma^{n_1 n_2 .... n_q}] \;=\; 2\;\gamma^{n_1 n_2 .... n_q}_{m_1 m_2 .... m_p} - \frac{2p!q!}{2!(p-2)!(q-2)!}\;\delta^{[n_1 n_2}_{[m_1 m_2}\;\gamma^{n_3 .... n_q]}_{m_3 ...... m_p]} +$$

$$\frac{2p!q!}{4!(p-4)!(q-4)!}\;\delta^{[n_1 .... n_4}_{[m_1 .... m_4}\;\gamma^{n_5 .... n_q]}_{m_5 ...... m_p]} \;-\; .... \tag{A.4}$$

$$pq = \mathbf{even}, \ \{ \gamma_{m_1 m_2 .... m_p}, \ \gamma^{n_1 n_2 .... n_q} \} \ = \ 2 \gamma^{n_1 n_2 .... n_q}_{m_1 m_2 .... m_p} - \frac{2p!q!}{2!(p-2)!(q-2)!} \delta^{[n_1 n_2}_{[m_1 m_2} \gamma^{n_3 .... n_q]}_{m_3 ...... m_p]} +$$

$$\frac{2p!q!}{4!(p-4)!(q-4)!} \delta^{[n_1 .... n_4}_{[m_1 .... m_4} \gamma^{n_5 .... n_q]}_{m_5 ...... m_p]} - \ .... \qquad (A.5)$$

$$pq = \mathbf{even}, \ [\gamma_{m_1 m_2 .... m_p}, \ \gamma^{n_1 n_2 .... n_q}] \ = \ \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \delta^{[n_1}_{[m_1} \gamma^{n_2 .... n_q]}_{m_2 .... m_p]} -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \delta^{[n_1 n_2 n_3}_{[m_1 m_2 m_3} \gamma^{n_4 .... n_q]}_{m_4 ...... m_p]} + \ .... \qquad (A.6)$$

$$pq = \mathbf{odd}, \ \{ \gamma_{m_1 m_2 .... m_p}, \ \gamma^{n_1 n_2 .... n_q} \} \ = \ \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \delta^{[n_1}_{[m_1} \gamma^{n_2 .... n_q]}_{m_2 .... m_p]} -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \delta^{[n_1 n_2 n_3}_{[m_1 m_2 m_3} \gamma^{n_4 .... n_q]}_{m_4 ...... m_p]} + \ .... \qquad (A.7)$$

The generalized Kronecker delta is defined as the determinant

$$\delta^{a_1 a_2 ..... a_k}_{b_1 b_2 ..... b_k} \equiv det \begin{pmatrix} \delta^{a_1}_{b_1} & \cdots & \cdots & \delta^{a_1}_{b_k} \\ \delta^{a_2}_{b_1} & \cdots & \cdots & \delta^{a_2}_{b_k} \\ ------------ & & & -------------- \\ \delta^{a_k}_{b_1} & \cdots & \cdots & \delta^{a_k}_{b_k} \end{pmatrix} \qquad (A.8)$$

These equations are all that is need to evaluate the numerical coefficients of the action provided by eqs-(2.16-2.26). For instance if one wishes to extract the scalar part of the Clifford geometric product of $< \gamma_{mnp} \gamma_{rst} \gamma_{uv} >$, all one needs is to extract the *bivector* part of the product

$$\gamma_{mnp} \ \gamma^{rst} \ = \ \frac{1}{2} \ [\gamma_{mnp}, \ \gamma^{rst}] \ + \frac{1}{2} \ \{\gamma_{mnp}, \gamma^{rst}\} \ =$$

$$\frac{1}{2} \ (\ 2 \ \gamma_{mnp}^{\ \ \ \ rst} \ - 36 \ \delta_{[mn}^{\ \ [rs} \ \gamma_{p}^{\ t]} \ ) \ + \frac{1}{2} \ (\ 18 \ \delta^{[r}_{[m} \ \gamma_{np]}^{\ \ st]} \ - 12 \ \delta^{\ rst}_{[mnp]} \ ) \qquad (A.9)$$

From eq-(A.9) one learns that its *bivector* piece is

$$- \ \frac{1}{2} \ 36 \ \delta_{[mn}^{\ \ [rs} \ \gamma_{p}^{\ t]} \qquad (A.10)$$

and whose contraction with γ_{uv} will bring up the scalar part as follows

$$< \ \gamma^{pt} \ \gamma_{uv} > \ = \ - 4 \ \delta^{\ pt}_{[uv]} \qquad (A.11)$$

In this fashion one extracts the scalar part of the Clifford triple geometric product of generators and obtains the numerical coefficients a_{ij} in the action displayed by eqs-(2.16-2.26).

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