SOME STATIONARY SEQUENCES

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§1. Define a sequence a_n by $a_1 = a$ and $a_{n+1} = P(a_n)$, where P is a polynomial with real coefficients. For which a values, and for which P polynomials will this sequence be constant after a certain rank?

In this note, the author answers this question using as reference F. Lazebnik & Y. Pilipenko's E 3036 problem from A. M. M., Vol. 91, No. 2/1984, p. 140.

An interesting property of functions admitting fixed points is obtained.

§2. Because a_n is constant after a certain rank, it results that a_n converges. Hence, $(\exists) e \in \mathbb{R} : e = P(e)$, that is the equation $P(x) - x = 0$ admits real solutions. Or P admits fixed points $((\exists) x \in \mathbb{R} : P(x) = x)$.

Let $e_1, ..., e_m$ be all real solutions of this equation. It constructs the recurrent set E as follows:

1) $e_1, ..., e_m \in E$;

2) if $b \in E$ then all real solutions of the equation $P(x) = b$ belong to E;

3) no other element belongs to E, then the obtained elements from the rule 1) or 2), applying for a finite number of times these rules.

We prove that this set E, and the set A of the "a" values for which a_n becomes constant after a certain rank are indistinct, "$E \subseteq A$".

1) If $a = e_i,\ 1 \le i \le m$, then $(\forall) n \in \mathrm{N}^*$ $a_n = e_i = \text{constant}$.

2) If for $a = b$ the sequence $a_1 = b,\ a_2 = P(b)$ becomes constant after a certain rank, let x_0 be a real solution of the equation $P(x) - b = 0$, the new formed sequence: $a_1^{'} = x_0,\ a_2^{'} = P(x_0) = b,\ a_3^{'} = P(b)...$ is indistinct after a certain rank with the first one, hence it becomes constant too, having the same limit.

3) Beginning from a certain rank, all these sequences converge towards the same limit e (that is: they have the same e value from a certain rank) are indistinct, equal to e.

"$A \le E$"

Let "a" be a value such that: a_n becomes constant (after a certain rank) equal to e. Of course $e \in e_1, ..., e_m$ because $e_1, ..., e_m$ are the single values towards these sequences can tend.

If $a \in \; e_1, ..., e_m$, then $a \in E$.

Let $a \notin \; e_1, ..., e_m$, then $(\exists) n_0 \in \mathrm{N} : a_{n_0+1} = P(a_{n_0}) = e$, hence we obtain a applying the rules 1) or 2) a finite number of times. Therefore, because $e \in \; e_1, ..., e_m$ and the equation $P(x) = e$ admits real solutions we find a_{n_0} among the real solutions of this equation: knowing a_{n_0} we find a_{n_0-1} because the equation $P(a_{n_0-1}) = a_{n_0}$ admits real solutions (because $a_{n_0} \in E$ and our method goes on until we find $a_1 = a$ hence $a \in E$.

Remark. For $P(x) = x^2 - 2$ we obtain the E 3036 Problem (A. M. M.).

Here, the set E becomes equal to

$$\pm 1, 0, \pm 2 \;\bigcup \left\{ \underbrace{\pm\sqrt{2 \pm \sqrt{2 \pm \sqrt{... \pm 2}}}}_{n_0 \ times}, \;\; n \in \mathrm{N}^* \right\} \bigcup \left\{ \underbrace{\pm\sqrt{2 \pm \sqrt{... \sqrt{2 \pm \sqrt{3}}}}}_{n_0 \ times}, \;\; n \in \mathrm{N} \right\}.$$

Hence, for all $a \in E$ the sequence $a_1 = a, \; a_{n+1} = a_n^2 - 2$ becomes constant after a certain rank, and it converges (of course) towards –1 or 2:

$$(\exists) n_0 \in \mathrm{N}^* : \; (\forall) n \geq n_0 \qquad a_n = -1$$

or

$$(\exists) n_0 \in \mathrm{N}^* : \; (\forall) n \geq n_0 \qquad a_n = 2 .$$

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