Novel Physical Consequences of the Extended Relativity in Clifford Spaces

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Abstract

Novel physical consequences of the Extended Relativity Theory in C-spaces (Clifford spaces) are explored. The latter theory provides a very different physical explanation of the phenomenon of "relativity of locality" than the one described by the Doubly Special Relativity (DSR) framework. Furthermore, an elegant *nonlinear* momentum-addition law is derived in order to tackle the "soccer-ball" problem in DSR. Neither derivation in C-spaces requires a *curved* momentum space nor a deformation of the Lorentz algebra. While the constant (energy-independent) speed of photon propagation is always compatible with the generalized photon dispersion relations in C-spaces, another important consequence is that these generalized photon dispersion relations allow also for energy-dependent speeds of propagation while still *retaining* the Lorentz symmetry in ordinary spacetimes, while breaking the *extended* Lorentz symmetry in C-spaces. This does *not* occur in DSR nor in other approaches, like the presence of quantum spacetime foam. We conclude with some comments on the quantization program and the key role that quantum Clifford-Hopf algebras might have in the future developments since the latter q-Clifford algebras naturally contain the κ-deformed Poincare algebras which are essential ingredients in the formulation of DSR.

Keywords : Clifford algebras; Extended Relativity in Clifford Spaces; Doubly Special Relativity; Quantum Clifford-Hopf algebras.

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*Dedicated to the memory of Carlos Sanchez-Robles, gifted musician

1 Clifford Algebras

In the past years, the Extended Relativity Theory in C-spaces (Clifford spaces) and Clifford-Phase spaces were developed [1], [2]. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory whose generalized coordinates are Clifford polyvector-valued quantities which incorporate the lines, areas, volumes, and hyper-volumes degrees of freedom associated with the collective dynamics of particles, strings, membranes, p-branes (closed p-branes) moving in a D-dimensional target spacetime background. C-space Relativity permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing. Our theory has 2 fundamental parameters : the speed of a light c and a length scale which can be set equal to the Planck length. The role of "photons" in C-space is played by *tensionless* branes. An extensive review of the Extended Relativity Theory in Clifford spaces can be found in [1]. The polyvector valued coordinates $x^\mu, x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, ...$ are now linked to the basis vectors generators γ^μ, bi-vectors generators $\gamma_\mu \wedge \gamma_\nu$, tri-vectors generators

$\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$, ... of the Clifford algebra, including the Clifford algebra unit element (associated to a scalar coordinate). These polyvector valued coordinates can be interpreted as the quenched-degrees of freedom of an ensemble of p-loops associated with the dynamics of closed p-branes, for $p = 0, 1, 2, ..., D-1$, embedded in a target D-dimensional spacetime background.

The C-space polyvector-valued momentum is defined as $\mathbf{P} = d\mathbf{X}/d\Sigma$ where $\mathbf{X}$ is the Clifford-valued coordinate corresponding to the $Cl(1,3)$ algebra in four-dimensions, for example,

$$\mathbf{X} \;=\; s\,\mathbf{1} \;+\; x^\mu\,\gamma_\mu \;+\; x^{\mu\nu}\,\gamma_\mu \wedge \gamma_\nu \;+\; x^{\mu\nu\rho}\,\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \;+\; x^{\mu\nu\rho\tau}\,\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\tau \quad (1)$$

where we have omitted combinatorial numerical factors for convenience in the expansion (1). It can be generalized to any dimensions, including $D = 0$. The component s is the Clifford scalar component of the polyvector-valued coordinate and $d\Sigma$ is the infinitesimal C-space proper "time" interval which is *invariant* under $Cl(1,3)$ transformations which are the Clifford-algebra extensions of the $SO(1,3)$ Lorentz transformations [1]. One should emphasize that $d\Sigma$, which is given by the square root of the quadratic interval in C-space

$$(d\Sigma)^2 \;=\; (ds)^2 \;+\; dx_\mu\,dx^\mu \;+\; dx_{\mu\nu}\,dx^{\mu\nu} \;+\; \ldots \quad (2)$$

is *not* equal to the proper time Lorentz-invariant interval $d\tau$ in ordinary spacetime $(d\tau)^2 = g_{\mu\nu}dx^\mu dx^\nu = dx_\mu dx^\mu$. In order to match units in all terms of eqs-(1,2) suitable powers of a length scale (say Planck scale) must be introduced. For convenience purposes it is can be set to unity. For extensive details of the generalized Lorentz transformations (poly-rotations) in flat C-spaces and references we refer to [1].

Let us now consider a basis in C-space given by

$$E_A = \gamma,\ \gamma_\mu,\ \gamma_\mu \wedge \gamma_\nu,\ \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho,\ ... \quad (3)$$

where γ is the unit element of the Clifford algebra that we label as $\mathbf{1}$ from now on. In (3) when one writes an r-vector basis $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge ... \wedge \gamma_{\mu_r}$ we take the indices in "lexicographical" order so that $\mu_1 < \mu_2 < .... < \mu_r$. An element of C-space is a Clifford number, called also *Polyvector* or *Clifford aggregate* which we now write in the form

$$X \ = \ X^A E_A \ = \ s\,\mathbf{1} \ + \ x^\mu \gamma_\mu \ + \ x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + ... \tag{4}$$

A C-space is parametrized not only by 1-vector coordinates x^μ but also by the 2-vector coordinates $x^{\mu\nu}$, 3-vector coordinates $x^{\mu\nu\alpha}$, ..., called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops,..., onto the coordinate planes . By p-loop we mean a closed p-brane; in particular, a 1-loop is closed string. In order to avoid using the powers of the Planck scale length parameter L_p in the expansion of the polyvector X (in order to match units) we can set it to unity to simplify matters. In a *flat* C-space the basis vectors E^A, E_A are *constants*. In a *curved* C-space this is no longer true. Each E^A, E_A is a function of the C-space coordinates

$$X^A \ = \ \{ \ s, \ x^\mu, \ x^{\mu_1\mu_2}, \ ....., \ x^{\mu_1\mu_2....\mu_D} \ \} \tag{5}$$

which include scalar, vector, bivector,..., p-vector,... coordinates in the underlying D-dim base spacetime and whose corresponding C-space is 2^D-dimensional since the Clifford algebra in D-dim is 2^D-dimensional.

Defining

$$E^A \equiv \gamma^A, \ \ \mathcal{J}^{AB} \equiv \ \frac{1}{2} \, (\gamma^A \otimes \gamma^B - \gamma^B \otimes \gamma^A), \ \ \mathcal{J}^A \ \equiv \ \frac{1}{2} \, (\gamma^A \otimes \mathbf{1} - \mathbf{1} \otimes \gamma^A) \ \neq \ 0 \tag{6}$$

for arbitrary polyvector valued indices $A, B, ...$ and after using the relations

$$[\ \gamma^A \otimes \gamma^B, \ \gamma^C \otimes \gamma^D \ ] \ = \ \frac{1}{2} \, [\ \gamma^A, \ \gamma^C \ ] \otimes \{ \gamma^B, \ \gamma^D \ \} \ + \ \frac{1}{2} \, \{ \ \gamma^A, \ \gamma^C \ \} \otimes [\ \gamma^B, \ \gamma^D \ ] \tag{7}$$

$$\{ \ \gamma^A \otimes \gamma^B, \ \gamma^C \otimes \gamma^D \ \} \ = \ \frac{1}{2} \, [\ \gamma^A, \ \gamma^C \ ] \otimes [\ \gamma^B, \ \gamma^D \ ] \ + \ \frac{1}{2} \, \{ \ \gamma^A, \ \gamma^C \ \} \otimes \{ \ \gamma^B, \ \gamma^D \ \} \tag{8}$$

yields, for example, the commutator relation involving the boost generator $\mathcal{J}^{01}$ (along the X_1 direction) and the area-boost generator $\mathcal{J}^{0\ 12}$ (along the bivector X_{12} direction) in C-space

$$[\ \mathcal{J}^{0\ 12}, \ \mathcal{J}^{01} \ ] \ = \ \frac{1}{4} \, [\ \gamma^0 \otimes \gamma^{12} - \gamma^{12} \otimes \gamma^0, \ \gamma^{01} \otimes \mathbf{1} - \mathbf{1} \otimes \gamma^{01} \ ] \ =$$

$$- \ \frac{1}{8} \, g^{11} \, (\gamma^{20} \otimes \gamma^0 - \gamma^0 \otimes \gamma^{20}) \ - \ \frac{1}{8} \, g^{00} (\gamma^1 \otimes \gamma^{12} - \gamma^{12} \otimes \gamma^1) \tag{9}$$

The (anti) commutators of all the gamma generators are explicitly given in the Appendix. One requires to use the expressions in the Appendix in order to arrive at the last terms of eq-(9). Hence, from the definitions in eqs-(6) one learns that

$$[\ \mathcal{J}^{0\ 12}, \ \mathcal{J}^{01} \ ] \ = \ \frac{1}{4} \, g^{00} \ \mathcal{J}^{12\ 1} \ + \frac{1}{4} \, g^{11} \ \mathcal{J}^{02\ 0} \tag{10}$$

A careful study reveals that the commutators obtained in eq-(10) (after using the expressions in eqs-(7,8) and in those in the Appendix) *do not obey* the relations

$$[\ \mathcal{J}^{AB},\ \mathcal{J}^{C}\ ]\ \sim\ -\ G^{AC}\ \mathcal{J}^{B}\ +\ G^{BC}\ \mathcal{J}^{A} \tag{11}$$

$$[\ \mathcal{J}^{AB},\ \mathcal{J}^{CD}\ ]\ \sim\ -\ G^{AC}\ \mathcal{J}^{BD}\ +\ G^{AD}\ \mathcal{J}^{BC}\ -\ G^{BD}\ \mathcal{J}^{AC}\ +\ G^{BC}\ \mathcal{J}^{AD} \tag{12}$$

where the C-space metric is chosen to be $G^{AB} = 0$ when the *grade* $A \neq$ *grade* B. And for the same-grade metric components $g^{[a_1a_2...a_k]\ [b_1b_2...b_k]}$ of G^{AB}, the metric can decomposed into its irreducible factors as antisymmetrized sums of products of η^{ab} given by the following *determinant* [16]

$$G^{AB}\ \equiv\ det\ \begin{pmatrix} \eta^{a_1b_1} & \ldots & \ldots & \eta^{a_1b_k} \\ \eta^{a_2b_1} & \ldots & \ldots & \eta^{a_2b_k} \\ ------ & ------ & ------ & ------ \\ \eta^{a_kb_1} & \ldots & \ldots & \eta^{a_kb_k} \end{pmatrix} \tag{13}$$

The spacetime signature is chosen to be $(-,+,+,....,+)$.

It would be tempting to suggest that the C-space generalization of the Poincare algebra could be given by the commutators in eq-(12) and

$$[\ \mathcal{J}^{AB},\ P^{C}\ ]\ \sim\ -\ G^{AC}\ P^{B}\ +\ G^{BC}\ P^{A},\ \ [P^A,\ P^B]\ =\ 0 \tag{14}$$

where P^A is the poly-momentum and $\mathcal{J}^{AB}$ are the generalized Lorentz generators. A more careful inspection suggests that this *is not* the case. The actual commutators are more complicated as displayed by eq-(10). One always must use the relations in eqs-(7,8) and in the Appendix in order to determine the $[\mathcal{J}^{AB}, \mathcal{J}^{CD}], [\mathcal{J}^{AB}, \mathcal{J}^{C}]$ commutators. In this way one will ensure that the Jacobi identities are satisfied.

Let us provide some examples of the generalized Lorentz relativistic transformations in C-space. Performing an area-boost transformation along the bivector X_{12} direction and followed by a boost transformation along the X_1 direction one arrives after some laborious but straightforward algebra at

$$X_0''\ =\ (\ X_0\ cosh\beta\ + L^{-1}\ X_{12}\ sinh\beta\ )\ cosh\alpha\ +\ X_1\ sinh\alpha \tag{15a}$$

$$X_1''\ =\ (\ X_0\ cosh\beta\ + L^{-1}\ X_{12}\ sinh\beta\ )\ sinh\alpha\ +\ X_1\ cosh\alpha \tag{15b}$$

$$X_{12}''\ =\ L\ X_0\ sinh\beta\ +\ X_{12}\ cosh\beta \tag{15c}$$

where α is the standard Lorentz boost parameter and β is the area-boosts one. Due to the identities $cosh^2\alpha - sinh^2\alpha = 1$ and $cosh^2\beta - sinh^2\beta = 1$, a straightforward algebra leads to

$$-\ (X_0'')^2\ +\ (X_1'')^2\ + L^{-2}\ (X_{12}'')^2\ =\ -\ (X_0)^2\ +\ (X_1)^2\ + L^{-2}\ (X_{12})^2 \tag{16}$$

which is a consequence of the invariance of the norm in C-space [1]

$$< \mathbf{X}^\dagger \, \mathbf{X} > \; = \; X_A \, X^A \; = \; s^2 + X_\mu \, X^\mu + X_{\mu_1\mu_2} \, X^{\mu_1\mu_2} + ... \, X_{\mu_1\mu_2.....\mu_D} \, X^{\mu_1\mu_2....\mu_D} \quad (17)$$

where $\mathbf{X}^\dagger$ denotes the reversal operation obtained by reversing the order of the gamma generators in the wedge products. The symbol $< .... >$ denotes taking the scalar part in the Clifford geometric product.

In the particular case when the spacetime dimension is chosen to be $D = 3$ for simplicity, one has in addition to the transformations provided by eqs-(15) that the other remaining coordinates remain invariant under boosts along the X_1 direction and area-boosts along X_{12},

$$X''_2 \; = \; X_2, \;\; X''_{01} \; = \; X_{01}, \;\; X''_{02} \; = \; X_{02}, \;\; X''_{012} \; = \; X_{012} \quad (18)$$

the Clifford scalar parts of the polyvectors are trivially invariant $s'' = s$ as they should.

Performing, instead, a boost transformation along the X_1 direction and then followed by an area-boost transformation along the bivector X_{12} direction one arrives at

$$X''_0 \; = \; (\; X_0 \; cosh\alpha \; + \; X_1 \; sinh\alpha \;) \; cosh\beta \; + L^{-1} \; X_{12} \; sinh\beta \quad (18a)$$

$$X''_1 \; = \; X_0 \; sinh\alpha \; + \; X_1 \; cosh\alpha \quad (18b)$$

$$X''_{12} \; = \; X_{12} \; cosh\beta \; + L \; (\; X_0 \; cosh\alpha \; + \; X_1 \; sinh\alpha \;) \; sinh\beta \quad (18c)$$

straightforward algebra leads again to

$$- \; (X''_0)^2 \; + \; (X''_1)^2 \; + L^{-2} \; (X''_{12})^2 \; = \; - \; (X_0)^2 \; + \; (X_1)^2 \; + L^{-2} \; (X_{12})^2 \quad (19)$$

We may notice the *mixing* of polyvector valued coordinates under generalized Lorentz transformations in C-space. Stringy (area coordinates) $X^{\mu\nu}$ and point particle coordinates X^μ in eqs-(15,18) appear mixed with each other under the C-space transformations.

Because $[\; \mathcal{J}^{0 \; 12}, \;\; J^{01} \;] \;\; \neq \; 0$ the *order* in which one performs the generalized boosts transformations matters. In ordinary Relativity the commutator of two boosts $[M^{0i}, M^{0j}] \sim \eta^{00} \; M^{ij}$ gives a rotation. This is the reasoning behind the Thomas precession. In C-space, one will arrive at different results if one first performs an area-boost followed by an ordinary boost, compared if we perform an ordinary boost followed by an area boost. This is the reason why eqs-(15) $differ$ from eqs-(18) although both of them lead to the same invariance property of the C-space interval (17) .

The C-space *rotations* mixing the area-bivector X^{12} with the X^1 vector component are of the form

$$X'_1 \; = \; X_1 \; cos\theta \; - L^{-1} \; X_{12} \; sin\theta; \;\; X'_{12} \; = \; L \; X_1 \; sin\theta \; + \; X_{12} \; cos\theta \quad (20)$$

such that

$$L^{-2} \; (X'_{12})^2 \; + \; (X'_1)^2 \; = \; L^{-2} \; (X_{12})^2 \; + \; (X_1)^2 \quad (21)$$

Due to the fact that $g^{11} = g^{22} = 1$ this explains why $(X_{12})^2$ appears with a plus sign in all the above equations. The spacetime signature is chosen to be $(-, +, +, ..., +)$.

We shall provide next a very different physical explanation of the phenomenon of Relativity of Locality than the one described in [12] and which does *not* rely on the nature of *curved* momentum space. Let us look at the transformations in eqs-(18) and compare the values of the coordinates of two events $\mathbf{1}, \mathbf{2}$ in C-space in the primed and double-primed framed of references, respectively. Denoting the coordinate intervals between the two events in different frames of reference by $\Delta X_0 = X_0(\mathbf{2}) - X_0(\mathbf{1})$, $\Delta X'_0 = X'_0(\mathbf{2}) - X'_0(\mathbf{1})$, $\Delta X''_0 = X''_0(\mathbf{2}) - X''_0(\mathbf{1})$, etc.... one gets from eqs-(18)

$$\Delta X''_0 \ = \ (\Delta X_0 \ cosh\alpha + \Delta X_1 \ sinh\alpha) \ cosh\beta \ + \ \frac{\Delta X_{12}}{L} \ sinh\beta \qquad (22a)$$

$$\Delta X''_1 \ = \ \Delta X_0 \ sinh\alpha + \Delta X_1 \ cosh\alpha \qquad (22b)$$

$$\Delta X''_{12} \ = \ L \ (\Delta X_0 \ cosh\alpha + \Delta X_1 \ sinh\alpha) \ sinh\beta \ + \ \Delta X_{12} \ cosh\beta \qquad (22c)$$

An immediate consequence of the above relations is that *locality* in ordinary spacetime is not an invariant concept, it becomes *relative*. In particular, from eqs-(22) one learns that if one has two ordinary spacetime events obeying the locality condition in the unprimed reference frame $\Delta X_0 = \Delta X_1 = 0$, in the double primed frame we have that

$$\Delta X''_1 \ = \ 0, \ \ \mathbf{but} \ \ \Delta X''_0 \ = \ \frac{\Delta X_{12}}{L} \ sinh\beta \ = \ \frac{\Delta X''_{12}}{L \ cosh\beta} \ sinh\beta \ = \ \frac{\Delta X''_{12}}{L} \ tanh\beta \ \neq \ 0 \qquad (23)$$

If we repeat the same arguments for the transformations in eqs-(15), instead of eqs-(18), one arrives when $\Delta X_0 = \Delta X_1 = 0$ at

$$\Delta X''_0 \ = \ \frac{\Delta X_{12}}{L} \ sinh\beta \ cosh\alpha \ \neq \ 0$$

$$\Delta X''_1 \ = \ \frac{\Delta X_{12}}{L} \ sinh\beta \ sinh\alpha \ \neq \ 0 \qquad (24)$$

Therefore from eqs-(23, 24) we can conclude that the invariant notion of spacetime *locality* is now lost due to the nonvanishing contribution of the area-coordinates interval $\Delta X_{12} \neq 0$ in C-space. Therefore, we have seen how the concept of spacetime locality is *relative* and does *not* rely on the nature of *curved* momentum space like it does in [12]. If one imposes full locality in the full C-space this would require to set $\Delta X_{12} = 0$ in eqs-(22) leading in this *restrictive* case to a locality in spacetime.

We learnt from Special Relativity that the concept of simultaneity is also relative. This can also be seen simply by setting $\beta = 0$ and $\Delta X_0 = 0$ in eqs-(22) leading to $\Delta X''_0 = \Delta X_1 \ sinh\alpha \neq 0$ when $\Delta X_1 \neq 0$. For example, if two doors of a train (separated by a distance ΔX_1) open/close simultaneously in one frame of reference, they will *not* open/close simultaneously in another frame of reference. Hence, the concept of simultaneity is *relative* in Special Relativity due to the *mixing* of spatial and temporal coordinates under Lorentz transformations. By the same token, we have shown in the above examples that the concept of spacetime locality is *relative* due to the *mixing* of area-bivector coordinates with spacetime vector coordinates under generalized Lorentz transformations in C-space. In the most general case, there will be mixing of all polyvector valued coordinates. This was the motivation to build a unified theory of all extended objects, p-branes, for all values of p subject to the condition $p + 1 = D$.

In [4] we explored the many novel physical consequences of Born's Reciprocal Relativity theory [6], [8], [9] in flat phase-space and generalized the theory to the curved phase-space scenario. We provided six specific novel physical results resulting from Born's Reciprocal Relativity and which are *not* present in Special Relativity. These were : momentum-dependent time delay in the emission and detection of photons; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations. We finalized by constructing a Born reciprocal general relativity theory in curved phase-spaces which required the introduction of a complex Hermitian metric, torsion and nonmetricity.

In particular, if two photons of different momentum P_1, P_2 are emitted simultaneously in a given reference frame, $\Delta T = 0$, there is a time delay in the emission times as measured with respect to an accelerated frame of reference. The time delay experienced in the accelerated frame of reference, corresponding to pure acceleration boosts represented by the ξ parameter, was shown to be given by [4]

$$\Delta T' \ = \ \frac{P_2 - P_1}{\mathbf{b}} \ sinh\xi \tag{24}$$

where $\mathbf{b}$ is the maximal proper force sustained by a fundamental particle and was given by $(m_P c^2/L_P)$. m_P is the Planck mass. L_P is the Planck scale. A momentum dependent delay in the emission times of photons will also cause a time delay in their detection as measured with respect to an accelerated frame of reference. Coincidentally, one could also write $\mathbf{b}$ in terms of an upper mass M_U (Universe mass), and the Hubble scale R_H as $\mathbf{b} = (M_U c^2/R_H)$ reflecting some sort of large/small scale duality and compatibility with Mach's principle [4].

We should emphasize that *no* spacetime foam was introduced, nor Lorentz invariance was broken, in order to explain the time delay in the photon emission/arrival in eq-(24). In the conventional approaches of DSR (Double Special Relativity) where there is a Lorentz invariance breakdown [12], a longer wavelength photon (lower energy) experiences a smoother spacetime than a shorter wavelength photon (higher energy) because the higher energy photon experiences more of the graininess/foamy structure of spacetime at shorter scales. Consequently, the less energetic photons will move faster (less impeded) than the higher energetic ones and will arrive at earlier times.

However, in our case above [4] the time delay is entirely due to the very nature of Born's Reciprocal Relativity when one looks at pure acceleration (force) boosts transformations of the phase space coordinates in $flat$ phase-space. No *curved* momentum space is required as it happens in [12]. The condition $\Delta T' > 0$ in eq-(24) implies also that higher momentum (higher energy) photons will take longer to arrive than the lower momentum (lower energy) ones.

Another novel consequence of C-space Relativity deals with the concept of mass. The true quadratic Casimir invariant is now given by

$$\mathbf{P}^2 \;=\; P^2 \;+\; P_\mu \; P^\mu \;+\; P_{\mu_1\mu_2} \; P^{\mu_1\mu_2} \;+\; \ldots \; P_{\mu_1\mu_2....\mu_D} \; P^{\mu_1\mu_2.....\mu_D} \;=\; \mathbf{M}^2 \qquad (25)$$

where as usual one must introduce a dimensionful parameter of length or mass in the expansion in (25) in order to match physical units. P is the Clifford scalar component of the poly-momentum $\mathbf{P}$. The usual quantity $P_\mu P^\mu = m^2$ identified with the ordinary mass-squared is *no* longer an *invariant* quantity. It is now a variable. $\mathbf{M}^2$ is now the proper C-space invariant *mass* quantity. This has important consequences for the so-called "soccer-ball" problem [12], [13].

Let us focus in the $D = 2$ case to simplify matters. Eq-(25) gives in this case after introducing a length scale parameter L to match units

$$(P)^2 \;-\; (P_0)^2 \;+\; (P_1)^2 \;-\; (LP_{01})^2 \;=\; \mathcal{M}^2 \qquad (26)$$

Choosing $P^2 = \mathcal{M}^2$ and after dividing by $(LP_{01})^2$ leads to

$$(\frac{P_1}{LP_{01}})^2 \;-\; (\frac{P_0}{LP_{01}})^2 \;=\; 1 \qquad (27)$$

Multiplying by m^2 (which is no longer an invariant quantity) gives

$$(\frac{mP_1}{LP_{01}})^2 \;-\; (\frac{mP_0}{LP_{01}})^2 \;=\; m^2 \;\Rightarrow\; (\pi_1)^2 \;-\; (\pi_2)^2 \;=\; m^2 \qquad (28)$$

where the nonlinearly-defined momenta π_0, π_1 obeying the ordinary dispersion relation in eq-(28) are defined as

$$\pi_0 \;\equiv\; \frac{mP_0}{LP_{01}}, \quad \pi_1 \;\equiv\; \frac{mP_1}{LP_{01}} \qquad (29)$$

Due to the definition (29) one can infer the *nonlinear* addition law for π_0, π_1 derived from a *linear* addition law in C-space. Given two poly-momentum variables in C space :

$$\mathbf{P} \;=\; P \; \mathbf{1} \;+\; P_0 \; \gamma^0 \;+ P_1 \; \gamma^1 \;+ L \; P_{01} \; \gamma^0 \wedge \gamma^1 \qquad (30a)$$

$$\mathbf{Q} \;=\; Q \; \mathbf{1} \;+\; Q_0 \; \gamma^0 \;+ Q_1 \; \gamma^1 \;+ L \; Q_{01} \; \gamma^0 \wedge \gamma^1 \qquad (30b)$$

The *linear* addition law in C-space is

$$\mathbf{P} + \mathbf{Q} \;=\; (P + Q) \; \mathbf{1} \;+\; (P_0 + Q_0) \; \gamma^0 \;+ (P_1 + Q_1) \; \gamma^1 \;+ L \; (P_{01} + Q_{01}) \; \gamma^0 \wedge \gamma^1 \qquad (31)$$

From which we can derive the *nonlinear* addition law from the definitions in eq-(29) associated with the *linear* addition law in C-space provided by eq-(31)

$$(\pi \oplus \xi)_0 \;=\; \pi_0 \oplus \xi_0 \;=\; \frac{m(P_0 + Q_0)}{L(P_{01} + Q_{01})} \;\neq\; \frac{mP_0}{LP_{01}} \;+\; \frac{mQ_0}{LQ_{01}} \;=\; \pi_0 \;+\; \xi_0 \qquad (32a)$$

$$(\pi \oplus \xi)_1 \;=\; \pi_1 \oplus \xi_1 \;=\; \frac{m(P_1+Q_1)}{L(P_{01}+Q_{01})} \;\neq\; \frac{mP_1}{LP_{01}} \;+\; \frac{mQ_1}{LQ_{01}} \;=\; \pi_1 \;+\; \xi_1 \qquad (32a)$$

From these nonlinear addition laws we can deduce that the nonlinear addition law for N identical particles is

$$\pi_0 \oplus \pi_0 \oplus \;\ldots\ldots\; \oplus \pi_0 \;=\; \frac{m(P_0+P_0+\;\ldots\ldots\;+P_0)}{L(P_{01}+P_{01}+\;\ldots..+P_{01})} \;=\; \frac{N\; m\; P_0}{N\; L\; P_{01}} \;=\; \frac{mP_0}{LP_{01}} \qquad (33a)$$

$$\pi_1 \oplus \pi_1 \oplus \;\ldots\ldots\; \oplus \pi_1 \;=\; \frac{m(P_1+P_1+\;\ldots\ldots\;+P_1)}{L(P_{01}+P_{01}+\;\ldots..+P_{01})} \;=\; \frac{N\; m\; P_1}{N\; L\; P_{01}} \;=\; \frac{mP_1}{LP_{01}} \qquad (33a)$$

The immediate physical consequence of eqs-(33a, 33b) is that the nonlinear addition law of N identical particles will *not* exceed a given momentum bound. Namely, if the π_0, π_1 momenta are bound by the Planck momentum, the nonlinear addition of N identical particles will also be bound by the Planck momentum values. To sum up, C-space provides a $different$ physical realization of the nonlinear addition law in order to tackle the "soccer-ball" problem in DSR [13], [12] and which does not involve a *curved* momentum space.

Superluminal particles were studied within the framework of the Extended Relativity theory in Clifford spaces (C-spaces) in [5]. In the simplest scenario, it was found that it is the contribution of the Clifford scalar component P of the poly-vector-valued momentum $\mathbf{P}$ which is responsible for the superluminal behavior in ordinary spacetime due to the fact that the effective mass $\sqrt{\mathcal{M}^2 - P^2}$ can be imaginary (tachyonic). However from the point of view of C-space there is no superluminal behaviour (tachyonic) because the true physical mass still obeys $\mathcal{M}^2 > 0$. As discussed in detailed by [1], [3] one can have tachyonic (superluminal) behavior in ordinary spacetime while having non-tachyonic behavior in C-space. Hence from the C-space point of view there is no violation of causality nor the Clifford-extended Lorentz symmetry. The analog of "photons" in C-space are *tensionless* strings and branes [1].

Long ago [17] we showed how the quadratic Casimir invariant in C-space given by eq-(25) leads to modified wave equations, dispersion laws and to the generalizations of the stringy-uncertainty principle relations. The on-shell mass condition for a massless polyparticle in the 2^4-dimensional C-space corresponding to a Clifford algebra in $D = 4$, can be rewritten in terms of the polyvector valued components of a wave polyvector $\mathbf{K}$, after setting $L = 1, \hbar = c = 1$ for simplicity, as

$$k^2 \;+\; K_\mu K^\mu \;+\; K_{\mu_1\mu_2} K^{\mu_1\mu_2} \;+\; \ldots\ldots \;+\; K_{\mu_1\mu_2\ldots.\mu_4} K^{\mu_1\mu_2\ldots\mu_4} \;=\; \mathcal{M}^2 \;=\; 0 \qquad (34)$$

A particular *slice* through the 2^4-dimensional C-space can be taken by imposing the set of algebraic conditions

$$k^2 \;=\; 0, \quad K_{\mu_1\mu_2} K^{\mu_1\mu_2} \;=\; \lambda_1\; (K_\mu K^\mu)^2 \;=\; \lambda_1\; K^4 \qquad (35a)$$

$$K_{\mu_1\mu_2\mu_3}\, K^{\mu_1\mu_2\mu_3} \;=\; \lambda_2\, (K_\mu K^\mu)^3 \;=\; \lambda_2\, K^6,\;\; K_{\mu_1\mu_2\mu_3\mu_4}\, K^{\mu_1\mu_2\mu_3\mu_4} \;=\; \lambda_3\, (K_\mu K^\mu)^4 \;=\; \lambda_3\, K^8 \tag{35b}$$

where the λ's are numerical parameters. Since k is the Clifford scalar part of the wave polyvector it is invariant under C-space transformations. Hence the condition $k^2 = 0$ will not break the C-space symmetry. However the other slice conditions in eqs-(35a, 35b) will *break* the generalized (extended) Lorentz symmetry in C-space because these conditions are *not* preserved under the most general C-space transformations as described earlier. There will be only the residual standard Lorentz symmetry (in ordinary spacetime) remaining which preserves these conditions/constraints in eqs-(35a, 35b).

Inserting the conditions of eqs-(35) into eq-(34), after setting $k^2 = 0$, yields the modified dispersion law

$$K^2\; (\; 1\; + \lambda_1\; K^2\; + \lambda_2\; K^4\; + \lambda_3\; K^6\;)\; =\; \mathcal{M}^2 - k^2\; =\; 0 \tag{36}$$

Upon writing explicitly

$$K^2\; =\; K_\mu\; K^\mu\; =\; |\vec{K}|^2\; -\; (K_0)^2\; =\; |\vec{K}|^2\; -\; (\omega)^2 \tag{37}$$

and solving the algebraic equation for ω in terms of $|\vec{K}|$ obtained from eq-(36) leads to $\omega = \omega(|\vec{K}|)$. Finally, the group velocity (after reinstating c) is given by

$$c(|\vec{K}|)\; =\; \frac{\partial\omega(|\vec{K}|)}{\partial|\vec{K}|}\; =\; c\; +\; ... \tag{38}$$

The group velocity might be greater, smaller or equal to c. From eq-(36) one can deduce immediately that one solution is $K^2 = |\vec{K}|^2 \; - \; (\omega)^2 = 0 \Rightarrow \omega = |\vec{K}| \Rightarrow \frac{\partial\omega(|\vec{K}|)}{\partial|\vec{K}|} = 1$ (in $c = 1$ units) and as expected massless particles move at the speed of light. However, there are *other* solutions to eq-(36) besides the trivial one leading to *energy* dependent speed of propagation. Setting $K^2 = Z$ leads to a cubic equation inside the parenthesis of eq-(36)

$$1\; + \lambda_1\; Z\; + \lambda_2\; Z^2\; + \lambda_3\; Z^3\; =\; 0 \tag{39}$$

that can be solved exactly in terms of the λ's parameters giving 3 roots $z_i(\lambda_1, \lambda_2, \lambda_3),\; i = 1, 2, 3$. The roots can be all real, or one real and a pair of complex conjugate roots. In the former case we have (after reinstating c and adjusting the proper units for z_i) the particular solutions are

$$K^2\; =\; c^2\; |\vec{K}|^2\; -\; (\omega)^2\; =\; z_i(\lambda_1, \lambda_2, \lambda_3), \quad \Rightarrow\; \omega\; =\; \sqrt{c^2|\vec{K}|^2\; - z_i}\; \Rightarrow$$

$$c(|\vec{K}|) = \frac{\partial\omega(|\vec{K}|)}{\partial|\vec{K}|}\; =\; c\; \frac{c\; |\vec{K}|}{\sqrt{c^2\; |\vec{K}|^2\; - z_i}}\; =\; c\; \frac{\sqrt{(\omega)^2 + z_i}}{\omega}\quad i = 1, 2, 3 \tag{40}$$

Therefore, from eq-(40) one has an *energy* dependent speed of propagation that can be superluminal if $z_i > 0$, or subluminal if $z_i < 0$, in the case one has 3 real roots to the cubic equation (39). In order to make contact with experiment, gamma ray bursts for example, the simplest scenario is obtained after inserting the proper units in eq-(36). This requires to replace

$$\lambda_1 \rightarrow \lambda_1 \ L^2, \quad \lambda_2 \rightarrow \lambda_2 \ L^4, \quad \lambda_3 \rightarrow \lambda_3 \ L^6 \tag{41}$$

For very small scales L of the order of the Planck length L_P the leading order terms in eq-(36) becomes $K^2 \ (\ 1 \ + \lambda_1 \ L^2 \ K^2) \simeq 0$ and the solution to the latter equation besides the trivial one $K^2 = 0$ is

$$K^2 \ = \ c^2 \ |\vec{K}|^2 \ - \ (\omega)^2 \ = \ z_1 \ = \ - \ \frac{1}{\lambda_1 L^2} \ \Rightarrow$$

$$c(|\vec{K}|) \ = \ c \ \frac{c \ |\vec{K}|}{\sqrt{c^2 \ |\vec{K}|^2 \ + \ \frac{1}{\lambda_1 L^2}}} \ = \ c \ \frac{\sqrt{(\omega)^2 \ - \ \frac{1}{\lambda_1 L^2}}}{\omega} \ < \ c \tag{42a}$$

Upon reabsorbing the parameter λ_1 into a rescaled L as $\lambda_1 L^2 = \tilde{L}^2$, one finally arrives at the energy-dependent speed (in units of $\hbar = 1$) given by

$$c \ \frac{\sqrt{(\omega)^2 \ - \ \tilde{L}^{-2}}}{\omega} \ < \ c \tag{42b}$$

after assuming that L is a very small scale of the order of the Planck length. This does *not* necessarily mean that the rescaled length $\tilde{L}$ has to be that small. For instance in Pavlopoulos's modified dispersion relations [14] the length scale was estimated to be of the order of $10^{-13} cm$.

To ensure that the terms inside the square root are positive (no imaginary solutions in eqs-(42)) one must have bounds on the ω-domains to ensure that $(\omega)^2 \ - \ \tilde{L}^{-2} \ \geq 0$. For very small lengths $\tilde{L}$ (very low values of λ_1) compared to the Planck length one will have very high-energy domains for ω compared to the Planck energy when these effects (energy-dependent photon speed) are seen. And vice versa, for very large lengths $\tilde{L}$ (very high values of λ_1) compared to the Planck length one will have very low- energy domains ω compared to the Planck energy one. These would be the experimental signals which could in principle help us find the values of $\tilde{L}$, and in turn, determine the parameter λ_1 when L is identified with the Planck length.

One should add that after differentiating $c^2 \ |\vec{K}|^2 \ - \ (\omega)^2 \ = \ z_i$ in eq-(40) gives

$$2 \ c^2 \ |\vec{K}| \ d|\vec{K}| \ = \ 2 \ \omega \ d\omega \ \Rightarrow \ c^2 \ = \ \frac{\omega}{|\vec{K}|} \ \frac{d\omega}{d|\vec{K}|} \tag{43}$$

leading always to the standard relation $v_{group} \ v_{phase} \ = c^2$ between group and phase velocities for all the possible solutions. The above results were all obtained by setting the Clifford scalar part k of the wave polyvector to zero. The calculations in the simplest $D = 2$ case when $k^2 \neq 0$ can be found in [5] leading also to the possibility of superluminal propagation.

Thus the key *novel* results one obtains from our analysis of wave propagation in C-space when $k^2 = 0$ are :

1. Irrespective of the solutions found in eqs-(40,42) the standard dispersion relation $K^2 = c^2|\vec{K}|^2 - (\omega)^2 = 0$ is *always* a solution to eq-(36) giving a constant speed of photon propagation. This is a valid solution to choose whether or not an energy-dependent photon speed is found.

2 . Because the *modified* dispersion relation in eq-(36) *is Lorentz invariant* since the proper norm $K^2 = c^2|\vec{K}|^2 - (\omega)^2$ is Lorentz invariant, one is able to arrive at the energy-dependent speed of propagation $c(|\vec{K}|)$ in eqs-(40,42) while still *retaining* the Lorentz symmetry. This does *not* occur in DSR nor in other approaches.

To our knowledge, Pavlopoulos [14] was the first to propose *modifications* to the wave equation involving an invariant length scale L_0 and leading to modified dispersion relations which are very different than the ones discussed in this work based on the Extended Relativity Theory in Clifford Spaces. Other modified dispersion relations were proposed by Fujiwara [15]. For a relative recent analysis of the observations of gamma-ray bursts (GRBs) based on the dispersion laws proposed by Pavlopoulos [14]. It appears that spectral time lags exist between higher-energy gamma rays photons and lower-energy photons which vary with the energy difference and time (distance) traveled.

Einstein's gravity in Riemannian spacetimes was extended to *curved* Clifford spaces in [16] and relations to Lovelock-Lanczos higher curvature gravity were found. Born's Reciprocal Relativity principle [6] in phase spaces, based on a maximal velocity (speed of light) and maximal proper force (that can be postulated to be $m_P c^2/L_P$), was extended to Clifford Phase Spaces in [2]. It required the introduction of a maximal (Hubble) and minimal (Planck) scales. The idea of maximal acceleration was proposed by [7] later on.

Quantization might be studied from the perspective of Noncommutative Geometry by introducing noncommutative spacetime coordinates [21]. As emphasized by the authors [18], conformal symmetry represents the fundamental spacetime symmetry, and it contains the Poincare and de Sitter geometries as particular cases, besides describing massless particles and field symmetries. In order to investigate modifications of the relativistic kinematics at sufficiently high energy, quantum deformations of the Poincare algebra were introduced by [19], [20] and followed by the doubly special relativity (DSR) [12] , which contains two observer-independent parameters, the light velocity and the Planck length.

The DSR framework coincides with the algebraic structure of the *kappa*-deformation of the Poincare algebra , where the deformation parameter κ is related to the Planck mass. The conformal transformations, κ-deformed Poincare algebras, and a quantum κ-deformed Poincare symmetry (with algebraic and co-algebraic structures) were formulated together with the respective Hopf algebra relations, in the context of quantum Clifford-Hopf algebras, by [18]. The subject of quantum Clifford-Hopf algebras is vast [22]. Since they contain κ-deformed Poincare algebras as the authors [18] have shown, it is warranted to *extend* the formalism of quantum Clifford-Hopf algebras to Noncommutative Clifford spaces. Noncommutative coordinates in Clifford spaces were studied in [23].

To finalize we should also mention that the theory of Scale Relativity proposed by Nottale [11] based on a minimal observational length-scale, the Planck scale, as there is in Special Relativity a maximum speed, the speed of light, deserves to be looked within the Clifford algebraic perspective. Nottale aims to unify Quantum Physics and Relativity Theory by introducing explicitly the scales of *observation* in physical equations as characterizing the "state of scale" of the coordinate system. This is made possible by the fundamental relative nature of scales: only scale *ratios* have a physical meaning, never an absolute scale, in the same way as there exists no absolute velocity, but only velocity differences. The immediate application to this work lies in the key fact that in eqs-(41,42) the parameter λ_1 was defined as the ratio of two scales $\tilde{L}/L$ when L is the Planck scale. Also, in Scale Relativity, the composition of two scale changes is inferior to the product of these two scales. Similarly, in special relativity, the composition of two speeds is inferior to the sum of those two speeds. This is very reminiscent of the nonlinear addition laws for the momenta found in eqs-(32,33).

APPENDIX

In this Appendix we shall write the (anti) commutator relations for the Clifford algebra generators.

$$\frac{1}{2}\{\gamma_a,\ \gamma_b\} = g_{ab}\,\mathbf{1};\quad \frac{1}{2}[\gamma_a,\ \gamma_b] = \gamma_{ab} = -\gamma_{ba},\quad a,b = 1,2,3,\cdots,m \qquad (A.1)$$

$$[\gamma_a,\ \gamma_{bc}] = 2\,g_{ab}\,\gamma_c - 2\,g_{ac}\,\gamma_b,\quad \{\gamma_a,\ \gamma_{bc}\} = 2\,\gamma_{abc} \qquad (A.2)$$

$$[\gamma_{ab},\ \gamma_{cd}] = -2\,g_{ac}\,\gamma_{bd} + 2\,g_{ad}\,\gamma_{bc} - 2\,g_{bd}\,\gamma_{ac} + 2\,g_{bc}\,\gamma_{ad} \qquad (A.3)$$

In general one has [10]

$$pq = \mathbf{odd},\ [\gamma_{m_1m_2....m_p},\ \gamma^{n_1n_2....n_q}] = 2\,\gamma^{n_1n_2....n_q}_{m_1m_2....m_p} - \frac{2p!q!}{2!(p-2)!(q-2)!}\,\delta^{[n_1n_2}_{[m_1m_2}\,\gamma^{n_3....n_q]}_{m_3......m_p]} +$$

$$\frac{2p!q!}{4!(p-4)!(q-4)!}\,\delta^{[n_1....n_4}_{[m_1....m_4}\,\gamma^{n_5....n_q]}_{m_5......m_p]} - \ldots\ldots\ldots\ldots \qquad ()$$

$$pq = \mathbf{even},\ \{\gamma_{m_1m_2....m_p},\ \gamma^{n_1n_2....n_q}\} = 2\,\gamma^{n_1n_2....n_q}_{m_1m_2....m_p} - \frac{2p!q!}{2!(p-2)!(q-2)!}\,\delta^{[n_1n_2}_{[m_1m_2}\,\gamma^{n_3....n_q]}_{m_3......m_p]} +$$

$$\frac{2p!q!}{4!(p-4)!(q-4)!}\,\delta^{[n_1....n_4}_{[m_1....m_4}\,\gamma^{n_5....n_q]}_{m_5......m_p]} - \ldots\ldots\ldots\ldots \qquad ()$$

$$pq = \mathbf{even},\ [\gamma_{m_1m_2....m_p},\ \gamma^{n_1n_2....n_q}] = \frac{(-1)^{p-1}2p!q!}{1!(p-1)!(q-1)!}\delta^{[n_1}_{[m_1}\,\gamma^{n_2....n_q]}_{m_2....m_p]} -$$

$$\frac{(-1)^{p-1}2p!q!}{3!(p-3)!(q-3)!}\,\delta^{[n_1n_2n_3}_{[m_1m_2m_3}\,\gamma^{n_4....n_q]}_{m_4......m_p]} + \ldots\ldots. \qquad ()$$

$$pq = \mathbf{odd}, \quad \{ \gamma_{m_1 m_2 .... m_p}, \ \gamma^{n_1 n_2 .... n_q} \} \ = \ \frac{(-1)^{p-1} 2 p! q!}{1!(p-1)!(q-1)!} \delta^{[n_1}_{[m_1} \ \gamma^{n_2 .... n_q]}_{m_2 .... m_p]} \ -$$

$$\frac{(-1)^{p-1} 2 p! q!}{3!(p-3)!(q-3)!} \ \delta^{[n_1 n_2 n_3}_{[m_1 m_2 m_3} \ \gamma^{n_4 .... n_q]}_{m_4 ...... m_p]} \ + \ .... \qquad ()$$

The generalized Kronecker delta is defined as the determinant

$$\delta^{a_1 a_2 ..... a_k}_{b_1 b_2 ..... b_k} \ \equiv \ det \begin{pmatrix} \delta^{a_1}_{b_1} & \ldots & \ldots & \delta^{a_1}_{b_k} \\ \delta^{a_2}_{b_1} & \ldots & \ldots & \delta^{a_2}_{b_k} \\ ---------- & & ------------ & \\ \delta^{a_k}_{b_1} & \ldots & \ldots & \delta^{a_k}_{b_k} \end{pmatrix} \qquad ()$$

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