A Proof of the Collatz Conjecture

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Abstract

If every positive integer is able to be operated into 1 by set operational rule of the Collatz conjecture, then begin with 1, we can get all positive integers after pass infinite many operations by another operational rule on the contrary to the set operational rule. Thereby, substitute some unknown proof of the former proposition by a poof of the latter proposition, in addition apply mathematical induction, then this is exactly an effective way for proving Collatz conjecture, thus we use it in this article justly.

Keywords

Mathematical induction, Classify integers, A bunch of integers' chains, Rightward operational rule, Leftward operational rule, Operational routes.

Basic Concepts

The Collatz conjecture is also known variously as 3n+1 conjecture, the Ulam conjecture, Kakutani problem, the Thwaites conjecture, Hasse's algorithm or the Syracuse problem, etc.

The Collatz conjecture asserts that take any positive integer n, if n is an even number, then divide n by 2 to obtain an integer n/2; if n is an odd number, then multiply n by 3 and add 1 to obtain an even number 3n+1.

Repeat the above process indefinitely, then no matter which positive integer you start with, you will always eventually reach a result of 1.

We consider the way of doing for aforesaid two steps as the leftward operational rule for any positive integer given. Also consider another operational rule on the contrary to the leftward operational rule as the rightward operational rule for any positive integer got. We consider two such operational rules as set each other's contrary operational rules.

The rightward operational rule states that for any positive integer n, if n is an odd number, then multiply n by 2 to obtain an even number 2n. If n is an even number, then, on the one hand, multiply n by 2 to obtain an even number 2n. On the other hand, if the difference of n minus 1 is able to be divided by 3 and obtain an odd number, then must operate the step as such, and proceed from here to operate; if it is not so, do not the step.

We regard a segment within an operational course by aforementioned either operational rule as an operational route. If either end of the segment is positive integer P, then we term further it P-operational route.

Begin with 1, proceed to operate every positive integer successively got by the rightward operational rule, then it automatically made a bunch of operational routes which consist of positive integers plus arrows. We reckon the bunch of operational routes as a bunch of integers' chains.

Since an origin of each integer at the bunch of integers' chains is unique, thus, each integer therein likewise is unique. Therefore, any segment of the bunch of integers' chains is none of the repeat.

Comparatively speaking, in greater limits, integers on the left of the bunch of integers' chains are smaller integers as compared with the right. Overall, odd numbers and even numbers from left to right at the bunch of integers' chains are getting greater and greater, up to infinity. Please, see a beginning of the bunch of integers' chains as first illustration.

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                                …              …     452→…
                              168↑          680↑   226↑→75→…
                             84↑          340↑→113↑       227→…
                           42↑          170↑           682↑→…
       1                  21↑          85↑           341↑
1→2→4↑→8→16↓→32→64↑→128→256↑→512→1024↑→2048→…
              5→10↓→20→40↓→80→160↓→320→…
                   3↓       13↓       53→106↓→212→…      …
                    6↓       26↓           35→70↓→……   30↑
                     12↓       52↓→17↓          23→46↑→15↑
                      24↓       104↓  34↓→11→22↓→44→…
                        …          …   68↓        7↓
                                        136↓→…   14↓
                                          45↓       28↓→…
                                           …          9→18→…
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First Illustration

Annotation: signs ↓ and ↑ must rightwards tilt, but since every page is narrow.

No matter which positive integer, so long as it is able to be operated into 1 by the leftward operational rule, then it is surely at the bunch of integers' chains; conversely, it is so too, .

That is to say, positive integers got at the bunch of integers' chains by rightward operational rule and positive integers which can operate into 1 by the leftward operational rule are one-to-one correspondence.

Therefore, whether a positive integer suits the conjecture? It needs merely to be ascertained that whether it is at the bunch of integers' chains.

If every positive integer is able to be operated into 1 by the leftward operational rule, then, there are all positive integers at the bunch of integers' chains after pass infinite many operations by the rightward operational rule. Conversely, if there are all positive integers at the bunch of integers' chains after pass infinite many operations by the rightward operational rule, then every positive integer is able to be operated into 1 by the leftward operational rule.

Consequently, we have intention to prove that the bunch of integers' chains contains all positive integers by mathematical induction.

Nevertheless, we need also to determine first an axiom when use the mathematical induction to prove the aforementioned proposition.

Axiom. For any positive integer P, if there is a positive integer $L < P$ at a P-operational route or at another operational route which intersects with the P-operational route, and known L suits the conjecture, then P suits the conjecture too, of course P is surely at the bunch of integers' chains.

For example, when $P= 31+3^2\eta$, and $\eta \geq 0$, a P-operational route is $27+2^3\eta \to 82+3*2^3\eta \to 41+3*2^2\eta \to 124+3^2*2^2\eta \to 62+3^2*2\eta \to 31+3^2\eta > 27+2^3\eta$, so $31+3^2\eta$ suit the conjecture.

For example, when $P=5+12\mu$, and $\mu \geq 0$, a P-operational route is $5+12\mu \to 16+36\mu \to 8+18\mu \to 4+9\mu < 5+12\mu$, so $5+12\mu$ suit the conjecture.

For example, when $P=63+3*2^8\varphi$, and $\varphi \geq 0$, a P-operational route intersects with another are $63+3*2^8\varphi \to 190+3^2*2^8\varphi \to 95+3^2*2^7\varphi \to$

$286+3^3*2^7\varphi\rightarrow143+3^3*2^6\varphi\rightarrow430+3^4*2^6\varphi\rightarrow215+3^4*2^5\varphi\rightarrow646+3^5*2^5\varphi\rightarrow$ $323+3^5*2^4\varphi\rightarrow970+3^6*2^4\varphi\rightarrow485+3^6*2^3\varphi\rightarrow1456+3^7*2^3\varphi\rightarrow728+3^7*2^2\varphi$ $\rightarrow364+3^7*2\varphi\rightarrow182+3^7\varphi\rightarrow\ldots$

$\uparrow121+3^6*2\varphi\leftarrow242+3^6*2^2\varphi\leftarrow484+3^6*2^3\varphi\leftarrow161+3^5*2^3\varphi\leftarrow322+3^5*2^4\varphi$ $\leftarrow107+3^4*2^4\varphi\leftarrow214+3^4*2^5\varphi\leftarrow71+3^3*2^5\varphi\leftarrow142+3^3*2^6\varphi\leftarrow47+3^2*2^6\varphi<$ $63+3*2^8\varphi$, so $63+3*2^8\varphi$ suit the conjecture.

The Proof

Let us set about the proof that the bunch of integers' chains contains all positive integers by mathematical induction, thereinafter.

1. From the above-listed first illustration, you can directly see and count that begin with 1, operate positive integers got successively by the rightward operational rule, we got a part of positive integers including consecutive positive integers from 1 to 18, and they formed a beginning of the bunch of integers' chains.

2. Suppose that after further operate positive integers got successively by the rightward operational rule, there are consecutive positive integers $\leq n$ within positive integers got successively at the bunch of integers' chains, where n is a natural number ≥18.

3. Prove that after continue to operate positive integers successively got by the rightward operational rule, we can get consecutive positive integers $\leq 2n$ within positive integers successively got at the bunch of integers' chains.

Divide positive integers from left to right at the number axis into segments according to their number $2^X n$ per segment, so as to accord with the proof, where X≥0 from small to large. Please, see second illustration.

1———n———2n——————4n—————————-8n-→

Second Illustration

Proof. Since there are consecutive positive integers ≤ n within positive integers got at the bunch of integers' chains, then multiply every positive integer ≤ n by 2 by the rightward operational rule, we obtain all even numbers between n and 2n+1, irrespective of repeated even numbers ≤ n.

Since every positive integer ≤ n is at the bunch of integers' chains, consequently all even numbers between n and 2n+1 are at the bunch of integers' chains according to the preceding axiom.

Next we must seek an origin of each kind of odd numbers between n and 2n+1, whether each such origin is able to be traced to be smaller than the kind of odd numbers.

First, let us divide consecutive odd numbers between n and 2n+1 into two kinds, i.e. 5+4k and 7+4k, where k is a natural number, then any odd number between n and 2n+1 must belong in one of two such kinds. Now we list two such kinds of odd numbers in correspondence with k, below.

k: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 …

5+4k: 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57, 61, 65, 69 …

7+4k: 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55, 59, 63, 67, 71 …

From 5+4k→16+12k→8+6k→4+3k<5+4k, 5+4k suit the conjecture, and 5+4k are at the bunch of integers' chains according to the preceding axiom.

For 7+4k, we again divide them into three kinds, i.e. 11+12c, 15+12c and 19+12c, where c is a natural number or 0. From 7+8c→22+24c→11+12c > 7+8c, 11+12c suit the conjecture, and 11+12c are at the bunch of integers' chains according to the preceding axiom.

By now we list remainder two kinds of odd numbers in correspondence with c, next operate each kind by the leftward operational rule orderly, and reach conclusions at certain operational branches.

c: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 …

15+12c: 15, 27, 39, 51, 63, 75, 87, 99, 111, 123, 135, 147, 159 …

19+12c: 19, 31, 43, 55, 67, 79, 91, 103, 115, 127, 139, 151, 163 …

15+12c→46+36c→23+18c→70+54c→35+27c ♣

d=2e+1: 29+27e (**1**) e=2f: 142+486f→71+243f ♥
♣35+27c↓→c=2d+1: 31+27d↑→d=2e: 94+162e→47+81e↑→e=2f+1:64+81f (**2**)
c=2d: 106+162d→53+81d↓→d=2e+1:67+81e↓→e=2f+1:74+81f (**3**)
d=2e:160+486e♦ e=2f: 202+486f→101+243f ♠

g=2h+1: 200+243h (**4**) **…**
♥ 71+243f↓→f=2g+1:157+243g↑→g=2h: 472+1458h→236+729h↑→ **…**
f=2g: 214+1458g→107+729g↓→g=2h+1: 418+729h↓→**…**
g=2h: 322+4374h→**…** **…**

g=2h: 86+243h (**5**)
♠101+243f↓→f=2g+1:172+243g↑→g=2h+1:1246+1458h→**…**
F=2g: 304+1458g→152+729g ↓→**…**
…

…
♦160+486e→80+243e↓→e=2f+1: 970+1458f→485+729f↑→ … …
e=2f:40+243f↓→f=2g+1:850+1458g→ 425+729g↑→ …
f=2g: 20+243g↓→g=2h:10+243h (**6**) …
g=2h+1:880+1458h→440+729h↑→ …

Alphabet c, d, e, f, g … in the above-listed operational routes expresses respectively each of natural numbers or 0, similarly hereinafter. Also there are ♣↔♣, ♥↔♥, ♠↔♠, and ♦↔♦.

We conclude several partial conclusions from above-listed a bunch of operational routes for 15+12c, as the follows.

From c =2d+1 and d =2e+1, get c=2d+1=2(2e+1)+1=4e+3, and 15+12c = 15+12(4e+3)=51+48e >29+27e where mark (1), so 15+12c suit the conjecture where c=4e+3 according to the preceding axiom.

From c=2d+1, d=2e, and e=2f+1, get c =2d+1=4e+1=4(2f+1)+1=8f+5, and 15+12c=15+12(8f+5)=75+96f > 64+81f where mark (2), so 15+12c suit the conjecture where c=8f+5 according to the preceding axiom.

From c=2d, d=2e+1 and e=2f+1, get c=2d= 4e+2= 4(2f+1)+2=8f+6, and 15+12c=15+12(8f+6)=87+96f >74+81f where mark (3), so 15+12c suit the conjecture where c=8f+6 according to the preceding axiom.

From c=2d+1, d=2e, e=2f, f=2g+1 and g=2h+1, get c=2d+1=4e+1= 8f+1=8(2g+1)+1=16g+9=16(2h+1)+9=32h+25, and 15+12c=15+12(32h+ 25)=315+384h >200+243h where mark (4), so 15+12c suit the conjecture where c=32h+25 according to the preceding axiom.

From c=2d, d=2e+1, e=2f, f=2g+1 and g=2h, get c=2d=2(2e+1)= 4e+2=8f+2=8(2g+1)+2=16g+10=32h+10, and 15+12c=15+12(32h+10) =

135+384h > 86+243h where mark (5), so 15+12c suit the conjecture where c=32h+10 according to the preceding axiom.

From c=2d, d=2e, e=2f, f=2g and g=2h, get c=2d =32h, and 15+12c = 15+12(32h)=15+384h > 10+243h where mark (6), so 15+12c suit the conjecture where c=32h according to the preceding axiom.

Secondly we operate 19+12c by the leftward operational rule, below.

19+12c→58+36c→ 29+18c→ 88+54c→ 44+27c ♣

d=2e: 11+27e (**α**) e=2f:37+81f (**β**)
♣ 44+27c↓→c=2d: 22+27d↑→d=2e+1:148+162e→74+81e↑→e=2f+1:466+486f ♥
c=2d+1: 214+162d→107+81d↓→d=2e:322+486e ♠
d=2e+1:94+81e↓→e=2f:47+81f (γ)
e=2f+1:516+486f ♦

g=2h: 129+243h (**δ**) **…**
f=2g+1:258+243g↑→g=2h+1:1504+1458h→752+729h↑→**…**
♥466+486f→233+243f↑→f=2g:700+1458g→350+729g↓→g=2h+1:3238+4374h↓
g=2h: 175+729h↓→**…** **…**
…

g=2h+1:172+243h (**ε**)
f=2g: 101+243g↑→g=2h: 304+1458h→**…**
e=2f+1:202+243f↑→f=2g+1:1336+1458g→**…**
♠322+486e→161+243e↑→e=2f:484+1458f→**…**

♦516+486f→258+243f↓→f=2g+1:1504+1458g→**…**
f=2g: 129+243g↓→g=2h:388+1458h→**…**
g=2h+1:186+243h (ζ)

Alphabet c, d, e, f, g, h **…** in the above-listed operational routes expresses respectively each of natural numbers or 0, similarly hereinafter. Also there are ♣↔♣, ♥↔♥, ♠↔♠, ♦↔♦.

We also conclude several partial conclusions from above-listed a bunch of operational routes for 19+12c, as the follows.

From c=2d, d=2e, get c=2d=4e, and 19+12c= 19+12(4e)= 19+48e > 11+27e where mark (α), so 19+12c suit the conjecture where c=4e according to the preceding axiom.

From $c=2d$, $d=2e+1$ and $e=2f$, get $c=2d=2(2e+1)=4e+2=8f+2$, and $19+12c=19+12(8f+2)=43+96f > 37+81f$ where mark (β), so $19+12c$ suit the conjecture where $c=8f+2$ according to the preceding axiom.

From $c=2d+1$, $d=2e$, and $e=2f$, get $c=2d+1= 4e+1= 8f+1$, and $19+12c= 19+12(8f+1)= 31+96f > 47+81f$ where mark (γ), so $19+12c$ suit the conjecture where $c=8f+1$ according to the preceding axiom.

From $c=2d$, $d=2e+1$, $e=2f+1$, $f=2g+1$ and $g=2h$, get $c=2d=2(2e+1)= 4e+2= 4(2f+1)+2 = 8f+6=8(2g+1)+6 = 16g+14 = 32h+14$, and $19+12c = 19+ 12(32h+14)=187+384h > 129+243h$ where mark (δ), so $19+12c$ suit the conjecture where $c=32h+14$ according to the preceding axiom.

From $c=2d+1$, $d=2e$, $e=2f+1$, $f=2g$ and $g=2h+1$, get $c=2d+1=4e+1= 4(2f+1)+1 = 8f+5 = 16g+5 = 16(2h+1)+5 = 32h+21$, and $19+12c = 19+ 12(32h+21) = 271+384h > 172+243h$ where mark (ε), so $19+12c$ suit the conjecture where $c=32h+21$ according to the preceding axiom.

From $c=2d+1$, $d=2e+1$, $e=2f+1$, $f=2g$ and $g=2h+1$, get $c=2d+1= 2(2e+1) +1=4e+3=4(2f+1)+3=8f+7=16g+7=16(2h+1)+7=32h+23$, and $19+12c= 19+12(32h+23)=295+384h>186+243h$ where mark (ζ), so $19+12c$ suit the conjecture where $c=32h+23$ according to the preceding axiom.

Let χ=d, e, f, g, h … etc, at two bunches of operational routes for $15+12c$ and $19+12c$, the operation where an unknown number$=2\chi$ and the operation where the unknown number $=2\chi+1$ synchronize at every step.

And begin with a greater result at every step, it will continue to operate. If another smaller result at any step suits not the conjecture, then it must too continue to operate. If the smaller result is smaller than its beginning 15+12c or 19+12c, then the operational route must stop at here.

Or rather, on the one hand, two kinds of operations for an unknown number $=2\chi+1$ plus the unknown number $=2\chi$ are always continuously progress and branch up to infinitely progress and branch, and the number of expressions which continuously operate are getting more and more, and their values are getting greater and greater, up to infinite many infinity. On the other hand, they uninterruptedly stop operations of part branches in the course of infinite operations because operational results at those steps arrived to suit the conjecture, and expressions which stop operate are such kinds, namely their constant terms and coefficients of χ are always smaller positive integers as compared with expressions whose unknown number's sequence are not in front of now χ.

Since for an expression which denotes positive integers, both operate it as an even number into a half of value of the expression, and operate it as an odd number into three times of value of the expression, then add 1. Thus for an incremental result and a reductive result at every step, there is only at most one reductive result to suit the conjecture.

Consequently operating 15+12c and operating 19+12c will proceed infinitely. In other words, 15+12c and 19+12c must be divided respectively into infinite many kinds, just enable every kind to suit the conjecture via operations at infinite many time by the leftward operational rule, yet a stop of operations of each kind of 15+12c plus 19+12c relies only on a kind of c to decide.

This notwithstanding, since early arisen kinds which suit the conjecture are all such expressions whose constant terms and coefficients of χ are smaller positive integers, thus when χ is orderly equal to 0, 1, 2, …, such expressions are exactly smaller positive odd numbers which belong in 15+12c and in 19+12c respectively, for example, 51+48e, 75+96f, 87+96f, 315+384h, 135+384h, 15+384h, 19+48e, 43+96f, 31+96f, 187+384h, 271+384h and 295+384h at the above-listed two bunches of operational routes, etc. It is obvious that these smaller positive odd numbers contain smaller consecutive odd numbers of 15+12c plus 19+12c inevitably.

What we need are merely odd numbers of 15+12c plus 19+12c between n and 2n+1, yet it is not all of 15+12c plus 19+12c. Evidently odd numbers of 15+12c plus 19+12c between n and 2n+1 are smaller odd numbers within kindred unproved odd numbers. Thus we can find each of them according to a certain value of c after operating 15+12c and operating 19+12c at finite times to suit the conjecture. Of course, when find such 15+12c and 19+12c, they are just proven to suit the conjecture.

Sum up the proof above-mentioned, first we have proven all even numbers between n and 2n+1 at the bunch of integers' chains by the rightward operational rule. After that, divide consecutive odd numbers between n and 2n+1 into two kinds, i.e. 5+4k and 7+4k, and have proven 5+4k to suit the conjecture by the leftward operational rule. Next again divide 7+4k into three kinds, i.e. 11+12c, 15+12c and 19+12c, and have proven 11+12c to suit the conjecture by the leftward operational rule. For final remainder two kinds of 15+12c plus 19+12c between n and 2n+1, we have proven too them to suit the conjecture via finite operations by the leftward operational rule. In a word, we have proven every odd number between n and 2n+1 to suit the conjecture by the leftward operational rule, so all odd number between n and 2n+1 are at the bunch of integers' chains according to the preceding axiom.

Altogether, we have proven that all consecutive integers from 1 to 2n within integers got successively are at the bunch of integers' chains via finite operations by set each other's contrary operational rules.

Since we can get all consecutive integers between n and 2n+1 from consecutive integers got from 1 to n via finite operations by set each other's contrary operational rules, likewise we can get all consecutive integers between 2n and 4n+1 from consecutive integers got from 1 to 2n via finite operations by set each other's contrary operational rules, according to the foregoing way of doing. Thus we can get consecutive integers from 1 to 4n, and they all suit the conjecture, then they all are at the bunch of integers' chains.

If we divide consecutive positive integers from small to great into segments according to integers $2^X n$ per segment, where $X \geq 0$ from small to large, and $n \geq 18$, then after get consecutive integers from 1 to $2^X n$, can further get consecutive integers from 1 to 2^{X+1}according to the foregoing way of making the proof.

X begins with 0, next it is orderly endowed with 1, 2, 3,… In pace with which values of X are getting greater and greater, consecutive integers from 1 to $2^X n$ are getting more and more, and their values are getting greater and greater too. When X reaches infinity, we get all positive integers, and all positive integers are at the bunch of integers' chains.

That is to say, the bunch of integers' chains contains all positive integers, provided operate positive integers got progressively at infinite many times by set each other's contrary operational rules.

Consequently begin with any positive integer to operate by the leftward operational rule, eventually reach a result, and the result is always 1, according to one-to-one correspondence between positive integers at the bunch of integers' chains and positive integers which can operate into 1 by the leftward operational rule.

The proof was thus brought to a close. As a consequence, the Collatz conjecture is tenable.