

## Exploring Prime Numbers and Modular Functions III:

## On the Exponential of Prime Number via Dedekind Eta Function

Edigles Guedes

edigles.guedes@gmail.com

December 7, 2013

*Every way of a man seems right to himself, but the Lord is the tester of hearts.**Proverbs 21:2*

ABSTRACT. The main objective this paper is to develop asymptotic formulas for the exponential of prime number, using Dedekind eta function.

## 1. INTRODUCTION

As consequence of the prime number theorem, I get the asymptotic formula for the  $n$ th prime number, denoted by  $p_n$ :

$$(1) \quad p_n \sim n \ln n.$$

M. Pervouchine, in *Mémoires de la Société physico-mathématique de Kasan*, [1, page 848] deduced that

$$(2) \quad \frac{p_n}{n} \approx \ln n + \ln \ln n - 1 + \frac{5}{12 \ln n} + \frac{1}{24 \ln \ln n}.$$

On the other hand, Ernest Cesáro, in *Sur une formule empirique de M. Pervouchine* [1, page 849], I encounter the formula

$$(3) \quad \frac{p_n}{n} \approx \ln n + \ln \ln n - 1 + \frac{\ln \ln n - 2}{\ln n} - \frac{(\ln \ln n)^2 - 6 \ln \ln n + 11}{2(\ln n)^2},$$

in modern notation,

$$(4) \quad \frac{p_n}{n} = \ln n + \ln \ln n - 1 + \frac{\ln \ln n - 2}{\ln n} - \frac{(\ln \ln n)^2 - 6 \ln \ln n + 11}{2(\ln n)^2} + o\left(\frac{1}{(\ln n)^2}\right).$$

On the other hand, the Dedekind eta function was introduced by Richard Dedekind, in 1877, and is defined in the half-plane  $\mathbb{H} = \{\tau: \Im(\tau) > 0\}$  by the equation [2, page 47]

$$(5) \quad \eta(\tau) := e^{\pi i \tau / 12} \prod_{k=1}^{\infty} (1 - e^{2\pi i k \tau}).$$

The infinite product has the form  $\prod(1 - x^n)$ , where  $x = e^{2\pi i \tau}$ . If  $\tau \in \mathbb{H}$ , then  $|x| < 1$ , so the product converges absolutely and is nonzero.

In [2, page 48], I have

$$(6) \quad -\frac{1}{2} \ln y = \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k y}} - \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k / y}} - \frac{\pi}{12} \left(y - \frac{1}{y}\right).$$

Hence,

$$(7) \quad \ln y = 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/y}} - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k y}} + \frac{\pi}{6} \left( y - \frac{1}{y} \right)$$

and

$$(8) \quad y = \frac{6 \ln y}{\pi} - \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/y}} + \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k y}} + \frac{1}{y},$$

for  $y > 0$ .

In this paper, I prove, among other things, that

$$p_n \approx n \left( 1 - \frac{1}{\ln n} - \frac{2}{(\ln n)^2} \right) \left( \frac{6 \ln \ln n}{\pi} + \frac{1}{\ln n} + \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k/\ln n} - 1} \right) \\ + n \left( 1 + \frac{1}{\ln n} + \frac{3}{(\ln n)^2} - \frac{\ln \ln n}{2(\ln n)^2} \right) \left[ \frac{\pi}{6} \left( \ln n - \frac{1}{\ln n} \right) - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k/\ln n} - 1} \right] - \frac{11n}{2(\ln n)^2}.$$

## 2. THEOREM

LEMMA 1. *I have*

$$\frac{2n^2 p_{e^n}}{e^n} \approx 2n(n^2 - n - 2) + (2n^2 + 2n + 6 - \ln n) \ln n - 11,$$

where  $p_n$  denotes the  $n$ th prime number defined by  $p_n: \mathbb{N} \rightarrow \mathbb{R}^+$ .

*Proof.* Multiplying (3) by  $2(\ln n)^2$ , I obtain

$$\frac{2p_n(\ln n)^2}{n} \approx 2(\ln n)^3 + 2(\ln n)^2 \ln \ln n - 2(\ln n)^2 + 2 \ln n (\ln \ln n - 2) - (\ln \ln n)^2 + 6 \ln \ln n - 11,$$

thereupon, letting  $n \rightarrow e^n$ , I find

$$\frac{2n^2 p_{e^n}}{e^n} \approx 2n^3 + 2n^2 \ln n - 2n^2 + 2n(\ln n - 2) - (\ln n)^2 + 6 \ln n - 11,$$

$$\frac{2n^2 p_{e^n}}{e^n} \approx 2n(n^2 - n - 2) + (2n^2 + 2n + 6 - \ln n) \ln n - 11. \square$$

COROLLARY 1. *I have*

$$\frac{2n^2 p_{\lfloor e^n \rfloor}}{\lfloor e^n \rfloor} \approx 2n(n^2 - n - 2) + (2n^2 + 2n + 6 - \ln n) \ln n - 11,$$

where  $p_n$  denotes the  $n$ th prime number defined by  $p_n: \mathbb{N} \rightarrow \mathbb{R}^+$  and  $\lfloor n \rfloor$  denotes the floor function.

*Proof.* I note that  $\lfloor e^n \rfloor \approx e^n$ , arising from the definition of the floor function.  $\square$

THEOREM 1. *For  $n$  sufficiently large, then*

$$p_n \approx \left[ \frac{12 \ln \ln n}{\pi} + \frac{2}{\ln n} - \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{n^{2\pi k} - 1} + \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k/\ln n} - 1} \right] \left[ n \left( \frac{1}{2} - \frac{1}{2 \ln n} - \frac{1}{(\ln n)^2} \right) \right]$$

$$+ \left[ \frac{\pi}{6} \left( \ln n - \frac{1}{\ln n} \right) + 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{n^{2\pi k} - 1} - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k / \ln n} - 1} \right] \left\{ n \left[ 1 + \frac{1}{\ln n} + \frac{3}{(\ln n)^2} - \frac{\ln \ln n}{2(\ln n)^2} \right] \right\} \\ - \frac{11n}{2(\ln n)^2},$$

where  $p_n$  denotes the  $n$ th prime number.

*Proof.* I put (7) and (8) into Lemma 1, as following

$$\frac{2n^2 p_e^n}{e^n} \approx \left[ \frac{12 \ln n}{\pi} - \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/n}} + \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k n}} + \frac{2}{n} \right] (n^2 - n - 2) \\ + (2n^2 + 2n + 6 - \ln n) \left[ 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/n}} - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k n}} + \frac{\pi}{6} \left( n - \frac{1}{n} \right) \right] - 11.$$

Dividing both members of above equation by  $2n^2 + 2n + 6 - \ln n$

$$\frac{2n^2 p_e^n}{e^n (2n^2 + 2n + 6 - \ln n)} \\ \approx \left[ \frac{12 \ln n}{\pi} - \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/n}} + \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k n}} + \frac{2}{n} \right] \left( \frac{n^2 - n - 2}{2n^2 + 2n + 6 - \ln n} \right) \\ + \left[ 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k/n}} - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k n}} + \frac{\pi}{6} \left( n - \frac{1}{n} \right) \right] - \frac{11}{2n^2 + 2n + 6 - \ln n}.$$

I return to replace  $n$  by  $\ln n$  in the previous equation

$$(9) \frac{2(\ln n)^2 p_n}{n[2(\ln n)^2 + 2 \ln n + 6 - \ln \ln n]} \\ \approx \left[ \frac{12 \ln \ln n}{\pi} - \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k / \ln n}} + \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - n^{2\pi k}} + \frac{2}{\ln n} \right] \left[ \frac{(\ln n)^2 - \ln n - 2}{2(\ln n)^2 + 2 \ln n + 6 - \ln \ln n} \right] \\ + \left[ 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k / \ln n}} - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - n^{2\pi k}} + \frac{\pi}{6} \left( \ln n - \frac{1}{\ln n} \right) \right] \\ - \frac{11}{2(\ln n)^2 + 2 \ln n + 6 - \ln \ln n}.$$

Multiplying both members of (9) by  $\frac{n[2(\ln n)^2 + 2 \ln n + 6 - \ln \ln n]}{2(\ln n)^2}$ , I encounter

$$p_n \approx \left[ \frac{12 \ln \ln n}{\pi} - \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k / \ln n}} + \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - n^{2\pi k}} + \frac{2}{\ln n} \right] \left[ n \left( \frac{1}{2} - \frac{1}{2 \ln n} - \frac{1}{(\ln n)^2} \right) \right] \\ + \left[ 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - e^{2\pi k / \ln n}} - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1 - n^{2\pi k}} + \frac{\pi}{6} \left( \ln n - \frac{1}{\ln n} \right) \right] \left\{ n \left[ 1 + \frac{1}{\ln n} + \frac{3}{(\ln n)^2} - \frac{\ln \ln n}{2(\ln n)^2} \right] \right\} \\ - \frac{11n}{2(\ln n)^2},$$

consequently,

$$\begin{aligned}
p_n \approx & \left[ \frac{12 \ln \ln n}{\pi} + \frac{2}{\ln n} - \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{n^{2\pi k} - 1} + \frac{24}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k / \ln n} - 1} \right] \left[ n \left( \frac{1}{2} - \frac{1}{2 \ln n} - \frac{1}{(\ln n)^2} \right) \right] \\
& + \left[ \frac{\pi}{6} \left( \ln n - \frac{1}{\ln n} \right) + 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{n^{2\pi k} - 1} - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k / \ln n} - 1} \right] \left\{ n \left[ 1 + \frac{1}{\ln n} + \frac{3}{(\ln n)^2} - \frac{\ln \ln n}{2(\ln n)^2} \right] \right\} \\
& - \frac{11n}{2(\ln n)^2}.
\end{aligned}$$

This completes the Theorem.  $\square$

COROLLARY 2. For  $n$  sufficiently large, then

$$\begin{aligned}
p_n \approx & n \left( 1 - \frac{1}{\ln n} - \frac{2}{(\ln n)^2} \right) \left( \frac{6 \ln \ln n}{\pi} + \frac{1}{\ln n} + \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k / \ln n} - 1} \right) \\
& + n \left( 1 + \frac{1}{\ln n} + \frac{3}{(\ln n)^2} - \frac{\ln \ln n}{2(\ln n)^2} \right) \left[ \frac{\pi}{6} \left( \ln n - \frac{1}{\ln n} \right) - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k / \ln n} - 1} \right] - \frac{11n}{2(\ln n)^2}
\end{aligned}$$

where  $p_n$  denotes the  $n$ th prime number.

*Proof.* I note that, for  $n \in \mathbb{N}_{>1}$ ,

$$(11) \quad \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{n^{2\pi k} - 1} \downarrow 0.$$

From Theorem 1 and (11), I obtain

$$\begin{aligned}
p_n \approx & n \left( 1 - \frac{1}{\ln n} - \frac{2}{(\ln n)^2} \right) \left( \frac{6 \ln \ln n}{\pi} + \frac{1}{\ln n} + \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k / \ln n} - 1} \right) \\
& + n \left( 1 + \frac{1}{\ln n} + \frac{3}{(\ln n)^2} - \frac{\ln \ln n}{2(\ln n)^2} \right) \left[ \frac{\pi}{6} \left( \ln n - \frac{1}{\ln n} \right) - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k / \ln n} - 1} \right] - \frac{11n}{2(\ln n)^2}
\end{aligned}$$

This completes the proof.  $\square$

CONJECTURE: I observe that, numerically,  $p_n$  is better approximate if I add the term  $\frac{5n}{12 \ln n} + \frac{n}{24 \ln \ln n}$ , so

$$\begin{aligned}
p_n \approx & n \left( 1 - \frac{1}{\ln n} - \frac{2}{(\ln n)^2} \right) \left( \frac{6 \ln \ln n}{\pi} + \frac{1}{\ln n} + \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k / \ln n} - 1} \right) \\
& + n \left( 1 + \frac{1}{\ln n} + \frac{3}{(\ln n)^2} - \frac{\ln \ln n}{2(\ln n)^2} \right) \left[ \frac{\pi}{6} \left( \ln n - \frac{1}{\ln n} \right) - 2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{e^{2\pi k / \ln n} - 1} \right] - \frac{11n}{2(\ln n)^2} \\
& + \frac{5n}{12 \ln n} + \frac{n}{24 \ln \ln n}.
\end{aligned}$$

REFERENCES

[1] Cesàro, Ernest, *Sur une formule empirique de M. Pervouchine*, *Comptes rendus hebdomadaires des séances de l'Académie des sciences* **119**: 848-849, (1894). (French)

[2] Apostol, Tom M., *Modular functions and Dirichlet series in number theory*, Springer Verlag, 2000.

καιδικαίς