Examples of Products of Distributions

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1 Introduction

This paper is an extract of the paper in ref. [1]. For further informations, a more formal approach to the topic and clarifications on the notation refer to the above mentioned article.

Products of distributions are usually handled by means of Colombeau algebras (see [2] and [3]). The method I propose in the [1] is much more elementary. However, I am not a professional mathematician and therefore the correctness of my method should be evaluated by an expert of the subject.

2 Initial discussion

Aim of this paragraph is to provide the reader with an elementary introduction to the product of distributions developed in [1]. To keep the discussion simple, we start from a specific example which is also the obvious starting point for defining products of distributions, namely $\delta^2(x)$.

Given any function $f \in C^0$, a possible way to define the Dirac delta function is by means of the limit of a sequence of functions as follows:

$$\lim_{n\to\infty} nf(nx) = A\delta(x) \tag{1}$$

where $A = \int_{-\infty}^{+\infty} f(x)dx$ is the amplitude of the delta. Now, we suggest that the most straightforward way to define the $\delta^2(x)$ is also by means of the limit of a sequence of functions which elements are precisely the square of the elements of the sequence defined above:

$$\lim_{n\to\infty} n^2 f^2(nx) = B\delta^2(x) \tag{2}$$

Unfortunately we do not know how to evaluate B which is the amplitude of the δ^2. To be constituent with the (1), we may think that where $B = \int_{-\infty}^{+\infty} f^2(x)dx$. However, given any f, to have a consistent definition, we should have $B = A^2$ which is not always the case. Even worst, given A, if we pick a function f such that $\int_{-\infty}^{+\infty} f(x)dx = A$, B depends from the choice of f.

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Let us see how to overcome the above issues. Given any function $g \in C^0$, we will call the limit of the following sequence:

$$\lim_{n\to\infty} n^2 g(nx) \tag{3}$$

a δ^2-like generalised functions.

We notice a very interesting property of the δ^2-like generalised functions defined above. Given any $g_1 \in C^0$ and the relevant h_1 generalised function defined as:

$$h_1 = \lim_{n\to\infty} n^2 g_1(nx) \tag{4}$$

if we choose a second function $g_2 = \alpha^2 g_1(\alpha x)$, with $\alpha > 0$, and we consider the relevant generalised function h_2 defined as:

$$h_2 = \lim_{n\to\infty} n^2 g_2(nx) = \lim_{n\to\infty} n^2 \alpha^2 g_1(n\alpha x) \tag{5}$$

then, using the notation $A(g) = \int_{-\infty}^{+\infty} g(x)dx$, we see that by a changing of the scaling of g_1 by a factor of α, we increase the amplitude of g_1 by α^2 and we shrink its shape by α so the net effect is to change the integral by a factor of α and therefore we have:

$$A(g_2) = \alpha A(g_1) \tag{6}$$

at the same time we have also:

$$\begin{aligned} h_1 = \lim_{n\to\infty} n^2 g_1(nx) &= \lim_{n\to\infty} (n\alpha)^2 g_1(n\alpha x) \\ &= \lim_{n\to\infty} n^2 g_2(nx) = h_2 \end{aligned} \tag{7}$$

which shows clearly that h_1 and h_2 are the same generalised function because, in the (x, y) plane and for n that goes to infinity, the two sequences of functions shrink (along x) and grow (along y) in the same way. For example, if α is an integer, the sequence for h_2 is a sub-sequence of the one for h_1.

The key point here is that the integral of the functions g_i is not a good criteria for determining the amplitude of s δ^2-like generalised functions. We may say that there is a degree of freedom in defining the same δ^2 (i.e. the scaling factor of f^2) that has an impact on $A(f^2)$. We propose that, to determine the amplitude of a δ^2-like generalised function, we may compare it with a separate reference δ^2-like generalised function in order to remove the dependency from the scaling factor. For example we may use, as a reference function, the (2) itself. Let us see how to do that.

Suppose we want to evaluate the product $u(x)\delta'(x)$, with $u(x)$ the Heaviside function, which is known in the literature to be a δ^2-like function having amplitude $-\delta^2(x)$ (compare with [4]). To be consistent with the (2), we have in this case:

$$u(x)\delta'(x) = \lim_{n\to\infty} n^2 (f(nx))^{(-1)} f'(nx) = B\delta^2(x) \tag{8}$$

where $A(f) = 1$ (i.e. δ function of amplitude 1) and:

$$f(x)^{(-1)} = \int_{-\infty}^{x} f(\tau)d\tau \tag{9}$$

we mesure the amplitude of the (8) with respect to the δ^2 given by the (2) meaning that we set $B = A((f(x))^{(-1)}f'(x))/A(f^2(x))$. We have:

$$u(x)\delta'(x) = \frac{\overbrace{\int (f(x))^{(-1)}f'(x)dx}^{\text{the product } u\delta'}}{\underbrace{\int f^2(x)dx}_{\text{ref. func. } \delta^2}}\delta^2(x) = -\delta^2(x) \tag{10}$$

Where the above result is independent from f because for any possible f we choose, integrating by parts, we have:

$$\int_{-\infty}^{+\infty} (f(x))^{(-1)}f'(x)dx = \overbrace{\Big[(f(x))^{(-1)}f(x)\Big]_{-\infty}^{+\infty}}^{\text{equal to zero}} - \int_{-\infty}^{+\infty} (f(x))^2 dx \tag{11}$$

Of corse, with the above definition of product, if we want to evaluate δ^2 itself, we have $B = A(f^2)/A(f^2) = 1$ which is consistent.

So, to sum up, we define the product of the δ with itself to be δ^2, which is a mathematical object with its own right to exist outside D', and, by the above method, we evaluate all the δ^2-like product of distributions with respect of the reference function given by it.

In order to evaluate all possible products of distributions, we define a whole set of reference generalised functions as follows:

Definition. *Let $f(x) \in C^p$ be any function such that $\int_{-\infty}^{+\infty} f(x)dx = 1$. We define the generalised functions $\eta^{p,q}$, with $q > p$ to be the following limit:*

$$\eta^{p,q}(x) = \lim_{n\to\infty} n^q \frac{d^p}{dx^p}(f(nx))^{q-p} \quad \textit{with}\, p, q \in \mathbb{Z} \tag{12}$$

What kind of generalised function are the $\eta^{p,q}$? If the sequence of distributions $f_n = n^q f^{(p)}(nx)$, in the (12), converges to $\eta^{p,q}$, then $\frac{f_n}{n^{q-p-1}}$ converges to $\delta^{(p)}$. So, with an abuse of notation, we may say that:

$$\eta^{p,q} = A\frac{\delta^{(p)}}{n^{p-q+1}} \quad \textit{with}\, A\, \textit{depending on}\, f \tag{13}$$

The $\eta^{p,q}$ are therefore the limit of sequences of functions that are shaped like $\delta^{(p)}$ and that, when we take the limit, grow at a lower or faster rate with respect to it (according to the sign of p-q+1). Moreover, we will call p the order and q the growing index of the generalised function.

The (12) tells us what is the real nature of the $\eta^{p,q}$ and that we may rename them as for the following table:

$\eta^{p,q}$	p=-1	p=0	p=1	p=2	p=3
q=5		$\cdots$	$\cdots$	$\cdots$	$\cdots$
q=4		$\cdots$	$\frac{d}{dx}(\delta^3(x))$	$\frac{d^2}{dx^2}(\delta^2(x))$	$\cdots$
q=3		$\delta^3(x)$	$\frac{d}{dx}(\delta^2(x))$	$\delta''(x)$	
q=2	$\cdots$	$\delta^2(x)$	$\delta'(x)$		
q=1	$(\delta^2(x))^{(-1)}$	$\delta(x)$			
q=0	$u(x)$				

Figure 1 : η functions

Finally, we say that a function $f \in C^0$ is a function of order p if it is possible to find a function g such that $0 < |A(g)| < \infty$ and $g^{(p)} = f$.

The following proposition applies:

Proposition. *Given any function $f \in C^m$ with $m \in \mathbb{N}$, $\int_{-\infty}^{+\infty} f(x)dx = 1$ and $f(x) \geq 0$ for each $x \in \mathbb{R}$, the product of k generalised functions, having generating function $f_i = \frac{d^{p_i}}{dx^{p_i}}(f(x))^{q_i-p_i}$ with orders $p_i < m$ and growing indexes $q_i \in \mathbb{Z}$:*

$$h = \eta^{p_1,q_1}\eta^{p_2,q_2}\cdots\eta^{p_k,q_k} \tag{14}$$

is a representatives of the following generalised function:

$$h \sim \frac{a_p\left(f_*\right)}{a_p\left(\frac{d^p}{dx^p}f^{q-p}\right)}\eta^{p,q} = \frac{\int_{-\infty}^{+\infty} x^p f_* dx}{\int_{-\infty}^{+\infty} x^p \frac{d^p}{dx^p} f^{q-p} dx}\eta^{p,q} \tag{15}$$

where $f_* = f_1 f_2 \cdots f_k$, $p < m$ is the order of the function f_* and $q = q_1 q_2 \cdots q_k$, provided that the condition $q > p$ is verified.
Moreover, the amplitude evaluated above is independent from f.
In particular, if $q = p + 1$, the above product h is an element of D' and it is equal to:

$$h = \frac{\int_{-\infty}^{+\infty} x^p f_* dx}{\int_{-\infty}^{+\infty} x^p \frac{d^p}{dx^p} f dx}\delta^{(p)} \tag{16}$$

In the next paragraph we give some examples of product of distributions evaluated using the method described above. For the definition of the a_p and the b_p coefficients, the structure of a generalised function and the notation $R(\eta^{p,q})$, present in the next paragraph, refer to [1].

3 Equalities and examples of products in D'

By using the above defined product, we can prove interesting equalities involving products among elements of D'. We will see some examples in this paragraph.

Example 6.1: Evaluate the following product:

$$u(x)\delta'(x) \tag{17}$$

We use the proposition above. Before we start we need to choose the function f. In this example we need C^1 class functions, we choose the most simple one which is a triangular window centred in the origin with base 2 and hight 1:

$$f(x) = (x+1)u(x+1) - 2xu(x) + (x-1)u(x-1) \tag{18}$$

we have $q = q_1 + q_2 = 2$ and $f_*(x) = f^{(-1)}(x)f^{(1)}(x)$ and therefore:

$$u(x)\delta'(x) = \lim_{n\to\infty} n^2 \, f^{(-1)}(nx)f^{(1)}(nx) \tag{19}$$

We can now evaluate all the coefficients of the structure of our generalised function:

$$\begin{aligned} b_0 &= \frac{\int_{-\infty}^{+\infty} f_*(x)dx}{\int_{-\infty}^{+\infty} f^2(x)dx} = \frac{-\frac{2}{3}}{\frac{2}{3}} = -1 \quad \text{coeff. of } \eta^{0,2} = \delta^2 \\ b_1 &= a_1 = \int_{-\infty}^{+\infty} xf_*(x)dx = \tfrac{1}{2} \quad \text{coeff. of } \eta^{1,2} = \delta' \end{aligned} \tag{20}$$

where $b_1 = a_1$ because for $p = 1$, $p + 1 = q$ and therefore, the coefficient a_1 is independent from f. We have:

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) + R\left(\eta^{2,2}\right) \tag{21}$$

We may also express $u(x)\delta'(x)$ as an equality among products of elements of D' (compare with [4]), by ignoring the higher order terms:

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) \tag{22}$$

There is a second way to get to the same result. By using the proposition above we evaluate the the product of $u(x)\delta(x)$. We have:

$$u(x)\delta(x) \to n\; f^{(-1)}(nx)f(nx) \to q = 1 \tag{23}$$

From which we have:

$$u(x)\delta(x) = \frac{1}{2}\delta(x) + R\left(\eta^{1,1}\right) \tag{24}$$

We use the Leibniz rule, which we know to work with our definition of product. By taking the derivatives of both sides of the above equality we have:

$$\delta^2(x) + u(x)\delta'(x) = \frac{1}{2}\delta'(x) + R\left(\eta^{2,2}\right) \tag{25}$$

as expected.
Example 6.2: Evaluate the following product:

$$u(x)\delta''(x) \tag{26}$$

We use the proposition above. Before we start we need to choose the function f. In this example we need C^1 class functions, we choose again the (18) of the previous example.
We have $q = q_1 + q_2 = 3$ and $f_*(x) = f^{(-1)}(x) f^{(2)}(x)$. and therefore:

$$u(x)\delta^{''}(x) = \lim_{n\to\infty} n^3 \ f^{(-1)}(nx) f^{(2)}(nx) \tag{27}$$

We can now evaluate all the coefficients of the structure of our generalised function:

$$\begin{aligned} a_0 &= \textstyle\int_{-\infty}^{+\infty} f_*(x)dx = 0 & \text{coeff. of } \eta^{0,3} = \delta^3 \\ b_1 &= \frac{\int_{-\infty}^{+\infty} x f_*(x)dx}{\int_{-\infty}^{+\infty} x \frac{d}{dx} f^2(x)dx} = -\frac{3}{2} & \text{coeff. of } \eta^{1,3} = (\delta^2)' \\ b_2 = a_2 &= \textstyle\int_{-\infty}^{+\infty} f_*(x)x^2 dx = \frac{1}{2} & \text{coeff. of } \eta^{2,3} = \delta^{''} \end{aligned} \tag{28}$$

where $b_2 = a_2$ because for $p = 2$, $p + 1 = q$ and therefore, the coefficient a_2 is independent from f. We have:

$$u(x)\delta^{''}(x) = -\frac{3}{2}\eta^{1,3} + \frac{1}{2}\delta^{''} + R\left(\eta^{3,3}\right) \tag{29}$$

We see that $u(x)\delta^{''}(x) \notin D'$ since its component $\delta^{''}$ is negligible with respect of $\eta^{1,3}$ and therefore $u(x)\delta^{''}(x) \sim -\frac{3}{2}\eta^{1,3}$.

Example 6.3: Evaluate the following product:

$$\delta(x)\delta'(x) \tag{30}$$

We use the proposition above. Before we start we need to choose the function f. In this example we need C^1 class functions, we choose once again the (18) of the previous example.
We have $q = q_1 + q_2 = 3$ and $f_*(x) = f(x) f^{(1)}(x)$. and therefore:

$$\delta(x)\delta'(x) = \lim_{n\to\infty} n^3 \ f(nx) f^{(1)}(nx) \tag{31}$$

We can now evaluate all the coefficients of the structure of our generalised function:

$$\begin{aligned} a_0 &= \textstyle\int_{-\infty}^{+\infty} f_*(x)dx = 0 & \text{coeff. of } \eta^{0,3} = \delta^3 \\ b_1 &= \frac{\int_{-\infty}^{+\infty} f_*(x)x dx}{\int_{-\infty}^{+\infty} \frac{d}{dx} f^2(x)x dx} = \frac{1}{2} & \text{coeff. of } \eta^{1,3} = (\delta^2)' \\ a_2 &= \textstyle\int_{-\infty}^{+\infty} f_*(x)x^2 dx = 0 & \text{coeff. of } \eta^{2,3} = \delta^{''} \end{aligned} \tag{32}$$

we have:

$$\delta(x)\delta'(x) = \frac{1}{2}\eta^{1,3} + R\left(\eta^{3,3}\right) \tag{33}$$

Once again, there is a second way to get the same result. By taking twice the derivative of both sides of the (24), and rearranging the terms we get:

$$\delta(x)\delta'(x) = -\frac{1}{3}u(x)\delta^{''}(x) + \frac{1}{6}\delta^{''}(x) + R\left(\eta^{3,3}\right) \tag{34}$$

We see easily that, taking into account the (29), the (33) and the (34) are in perfect agreement.

Example 6.4: Evaluate the following product:

$$sign^2(x)\delta(x) \tag{35}$$

We use the proposition above. We have:

$$sign^2(x)\delta(x) = \lim_{n\to\infty} n\,(2f^{(-1)}(nx) - 1)^2 f(nx) \to q = 1 \tag{36}$$

which is actually the sum of three products one of which is trivial. We have:

$$sign^2(x)\delta(x) = \frac{1}{3}\delta(x) + R\left(\eta^{1,1}\right) \tag{37}$$

compare with [3] §1.1 ex. iii and with [5].

References

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