# The Analysis of Barcelo,Finazzi and Liberati applied to both Alcubierre and Natario Warp Drive Spacetimes: Horizons,Infinite Doppler Blueshifts and Quantum Instabilities($Natario \neq Alcubierre$)

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**Abstract**

Warp Drives are solutions of the Einstein Field Equations that allows superluminal travel within the framework of General Relativity. There are at the present moment two known solutions: The Alcubierre warp drive discovered in 1994 and the Natario warp drive discovered in 2001. However as stated by both Alcubierre and Natario themselves the warp drive violates all the known energy conditions because the stress energy momentum tensor is negative implying in a negative energy density. While from a classical point of view the negative energy is forbidden the Quantum Field Theory allows the existence of very small amounts of it being the Casimir effect a good example as stated by Alcubierre himself.The major drawback concerning negative energies for the warp drive is the huge amount of negative energy able to sustain the warp bubble.In order to perform an interstellar space travel to a "nearby" star at 20 light-years away with 3 potential habitable exo-planets(Gliese $667c$) at superluminal speeds in a reasonable amount of time a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor $10^{48}$ which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!! Some years ago Barcelo,Finazzi and Liberati published a work in which the composed mixed tensor $\langle T_{\mu}{}^{\nu} \rangle$ obtained from the negative energy density tensor $T_{\mu\nu}$ $\mu = 0, \nu = 0$ of the $1 + 1$ dimensional Alcubierre warp drive metric diverges when the velocity of the ship $vs$ exceeds the speed of light.(see pg 2 in [19]).We demonstrate in this work that in fact this do not happens and their results must be re-examined.We introduce here a shape function that defines the Natario warp drive spacetime as an excellent candidate to low the negative energy density requirements from $10^{48}$ to affordable levels.We also discuss Horizons and Doppler Blueshifts that affects the Alcubierre spacetime but not the Natario counterpart.

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# 1 Introduction

The Warp Drive as a solution of the Einstein Field Equations of General Relativity that allows superluminal travel appeared first in 1994 due to the work of Alcubierre.([1]) The warp drive as conceived by Alcubierre worked with an expansion of the spacetime behind an object and contraction of the spacetime in front.The departure point is being moved away from the object and the destination point is being moved closer to the object.The object do not moves at all[1].It remains at the rest inside the so called warp bubble but an external observer would see the object passing by him at superluminal speeds(pg 8 in [1])(pg 1 in [2]).

Later on in 2001 another warp drive appeared due to the work of Natario.([2]).This do not expands or contracts spacetime but deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [2]). Imagine the object being a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium whose walls do not expand or contract. An observer in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.

However the major drawback that affects the warp drive is the quest of large negative energy requirements enough to sustain the warp bubble.While from a classical point of view negative energy densities are forbidden the Quantum Field Theory allows the existence of very small quantities of such energies but unfortunately the warp drive requires immense amounts of it.Ford and Pfenning computed the negative energy density needed to maintain a warp bubble and they arrived at the conclusion that in order to sustain a stable configuration able to perform interstellar travel the amount of negative energy density is of about 10 times the mass of the Universe and they concluded that the warp drive is impossible.(see pg 10 in [3] and pg 78 in [5]).

Another drawback that affects the warp drive is the quest of the interstellar navigation:Interstellar space is not empty and from a real point of view a ship at superluminal speeds would impact asteroids,comets,interstellar space dust and photons of Cosmic Background Radiation(COBE).(see Appendices $K$,$L$,$M$ and $N$)

According to Clark,Hiscock and Larson a single collision between a ship at 200 times faster then light and a COBE photon would release an amount of energy equal to the photosphere of a star like the Sun.(see pg 11 in [9]).And how many photons of COBE we have per cubic centimeter of space??

These highly energetic collisions would pose a very serious threat to the astronauts as pointed out by McMonigal,Lewis and O'Byrne (see pg 10 in [10]).

Another problem:these highly energetic collisions would raise the temperature of the warp bubble reaching the Hawking temperature as pointed out by Barcelo,Finazzi and Liberati.(see pg 6 and 7 in [11]).At pg 9 they postulate that all future spaceships cannot bypass 99 percent of the light speed.

In section 3 we will see that these problems of interstellar navigation affects the Alcubierre warp drive but not the Natario one.

[1] do not violates Relativity

The last drawback raised against the warp drive is the fact that inside the warp bubble an astronaut cannot send signals with the speed of the light to control the front of the bubble because an Horizon(causally disconnected portion of spacetime)is established between the astronaut and the warp bubble.We discuss this in section 3 and in section 5 we discuss a possible way to overcome the Horizon problem using only General Relativity.

Some years ago Barcelo,Finazzi and Liberati published a work in which the composed mixed tensor $\langle T_{\mu}{}^{\nu}\rangle$ obtained from the negative energy density tensor $T_{\mu\nu}$ $\mu = 0, \nu = 0$ of the $1+1$ dimensional Alcubierre warp drive metric diverges when the velocity of the ship $vs$ exceeds the speed of light.(see pg 2 in [19]).We demonstrate in this work that in fact this do not happens and their results must be re-examined.

In this work we introduce a shape function that defines the Natario spacetime as an excellent candidate to lower the negative energy density requirements to arbitrary low levels.

We adopted the International System of Units where $G = 6,67 \times 10^{-11}$ $\frac{Newton \times meters^2}{kilograms^2}$ and $c = 3 \times 10^8$ $\frac{meters}{seconds}$ for negative energy density purposes and the Geometrized System of units in which $c = G = 1$ for geometrical purposes.

We consider here a Natario warp drive with a radius $R = 100$ meters a dimensionless thickness parameter with the values $@ = 50000$,$@ = 75000$ and $@ = 100000$ (and not $@ = 50000$ meters and not $@ = 75000$ meters and not $@ = 100000$ meters) moving with a speed 200 times faster than light implying in a $vs = 2 \times 10^2 \times 3 \times 10^8 = 6 \times 10^{10}$ and a $vs^2 = 3,6 \times 10^{21}$

We also adopt a warp factor as a dimensionless parameter in our Natario shape function with a value $WF = 200$.[2]Our warp factor matches eq 5 pg 3 in [19].

This work is organized as follows:

- Section 2)-Outlines the problems of the immense magnitude in negative energy density when a ship travels with a speed of 200 times faster than light. The negative energy density for such a speed is directly proportional to the factor $10^{48}$ which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!..

- Section 3)-The most important section in this work.According to Barcelo,Finazzi and Liberati the composed mixed tensor $\langle T_{\mu}{}^{\nu}\rangle$ obtained from the negative energy density tensor $T_{\mu\nu}$ $\mu = 0, \nu = 0$ of the two-dimensional Alcubierre warp drive metric diverges when the velocity of the ship $vs$ exceeds the speed of light.(see pg 2 in [19]).We demonstrate that in fact this do not happens and their results must be re-examined.

- Section 4)-We introduce a shape function that defines the Natario warp drive spacetime being this function an excellent candidate to lower the energy density requirements in the Natario warp drive to affordable levels completely obliterating the factor $10^{48}$ which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!..

- Section 5)-Outlines the possibility of how to overcome the Horizon problem from an original point of view of General Relativity.

[2]When reading Appendix $F$ remember that $WF >> 1$ and $vs >> 1$

Although this work was designed to be an independent and self-contained it can be regarded as a companion of our works [6],[7],[13] and [18]

## 2 The Problem of the Negative Energy in the Natario Warp Drive Spacetime-The Unphysical Nature of Warp Drive

The negative energy density for the Natario warp drive is given by(see pg 5 in [2])

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{1}$$

Converting from the Geometrized System of Units to the International System we should expect for the following expression:

$$\rho = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{rs}{2}n''(rs)\right)^2\sin^2\theta\right]. \tag{2}$$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for[3]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{3}$$

In the equatorial plane:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{4}$$

Note that in the above expressions the warp drive speed $vs$ appears raised to a power of 2. Considering our Natario warp drive moving with $vs = 200$ which means to say 200 times light speed in order to make a round trip from Earth to a nearby star at 20 light-years away in a reasonable amount of time(in months not in years) we would get in the expression of the negative energy the factor $c^2 = (3 \times 10^8)^2 = 9 \times 10^{16}$ being divided by $6,67 \times 10^{-11}$ giving $1,35 \times 10^{27}$ and this is multiplied by $(6 \times 10^{10})^2 = 36 \times 10^{20}$ coming from the term $vs = 200$ giving $1,35 \times 10^{27} \times 36 \times 10^{20} = 1,35 \times 10^{27} \times 3,6 \times 10^{21} = 4,86 \times 10^{48}$ !!!

A number with 48 zeros!!!Our Earth have a mass[4] of about $6 \times 10^{24}kg$

This term is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!or better:The amount of negative energy density needed to sustain a warp bubble at a speed of 200 times faster than light requires the magnitude of the masses of 1.000.000.000.000.000.000.000.000 planet Earths for both Alcubierre and Natario cases!!!!

And multiplying the mass of Earth by $c^2$ in order to get the total positive energy "stored" in the Earth according to the Einstein equation $E = mc^2$ we would find the value of $54 \times 10^{40} = 5,4 \times 10^{41} Joules$.

Earth have a positive energy of $10^{41} Joules$ and dividing this by the volume of the Earth(radius $R_{Earth} =$ 6300 km approximately) we would find the positive energy density of the Earth.Taking the cube of the Earth radius $(6300000m = 6,3 \times 10^6)^3 = 2,5 \times 10^{20}$ and dividing $5,4 \times 10^{41}$ by $(4/3)\pi R_{Earth}^3$ we would find the value of $4,77 \times 10^{20}\frac{Joules}{m^3}$. So Earth have a positive energy density of $4,77 \times 10^{20}\frac{Joules}{m^3}$ and we are talking about negative energy densities with a factor of $10^{48}$ for the warp drive while the quantum theory allows only microscopical amounts of negative energy density.

[3]see Appendix A

[4]see Wikipedia:The free Encyclopedia

So we would need to generate in order to maintain a warp drive with 200 times light speed the negative energy density equivalent to the positive energy density of $10^{28}$ Earths!!!!

A number with 28 zeros!!!.Unfortunately we must agree with the major part of the scientific community that says:"Warp Drive is impossible and unphysical!!"(eg Ford-Pfenning)(see pg 10 in [3] and pg 78 in [5]).

However looking better to the expression of the negative energy density in the equatorial plane of the Natario warp drive:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{5}$$

We can see that a very low derivative and hence its square can perhaps obliterate the huge factor of $10^{48}$ ameliorating the negative energy requirements to sustain the warp drive.

In section 4 we will introduce a shape function that defines the Natario warp drive spacetime and this function allows the reduction of the negative energy requirements from 10 times the mass of the Universe that would render the warp drive as impossible and unphysical to arbitrary low values completely obliterating the factor $10^{48}$ which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!...

## 3 The Analysis of Barcelo,Finazzi and Liberati applied to both Alcubierre and Natario Warp Drive Spacetimes: Horizons,Infinite Doppler Blueshifts and Quantum Instabilities($Natario \neq Alcubierre$).

In pg 2 of [19] Barcelo,Finazzi and Liberati claims that the stress energy momentum tensor (SEMT) in the $1+1$ dimensional Alcubierre warp drive diverges in the Horizon.[5].They use the result obtained by Hiscock(pg 2 in [4]).However the result obtained by Hiscock must be re-examined according to pg 10 and 11 in [18].

Still with pg 2 of [19] Barcelo,Finazzi and Liberati argues that it is not possible to create an Alcubierre warp drive due to this divergence in the Horizon.They compute the (SEMT) for a $1+1$ dimensional Alcubierre warp drive and shows that it increases exponentially in the Horizon implying in the fact that warp drives would become unstable at superluminal speeds.

However like the results obtained by Hiscock the results of Barcelo,Finazzi and Liberati for a $1+1$ dimensional Alcubierre warp drive must be re-examined.We will not examine their results here and we can demonstrate that in a $1+1$ dimensional Alcubierre warp drive the (SEMT) is zero wether the speed of the bubble $vs$ is superluminal or not or wether we take the value of the (SEMT) in the Horizon or not.

Now we are ready to demonstrate that the composed mixed tensor $\langle T_{\mu}{}^{\nu}\rangle$ obtained from the negative energy density tensor $T_{\mu\nu}$ $\mu = 0, \nu = 0$ of the $1+1$ dimensional Alcubierre warp drive metric do not diverges when the velocity of the ship $vs$ exceeds the speed of light and the results obtained by Hiscock needs to be re-examined or re-evaluated.(see pg 2 in [4]).

We will use the negative energy density equation.

The expressions for the negative energy density in the Alcubierre warp drive spacetime are given by:(see eq 8 pg 6 in [3],eq 19 pg 8 and pg 5 in [1],pg 4 eq 11 in [8] and pg 4 in [2])[6][7][8]:

- Negative Energy Density equation for the Alcubierre warp drive

$$\langle T^{\mu\nu}u_{\mu}u_{\nu}\rangle = \langle T^{00}\rangle = \frac{1}{8\pi}G^{00} = -\frac{1}{8\pi}\frac{v_s^2(t)[y^2+z^2]}{4r_s^2(t)}\left(\frac{df(r_s)}{dr_s}\right)^2 = -\frac{1}{32\pi}\frac{v_s^2(t)[y^2+z^2]}{r_s^2(t)}\left(\frac{df(r_s)}{dr_s}\right)^2 \tag{6}$$

According to eq 12 pg 4 in [8] the contravariant components of the negative energy density tensor are equivalent to its covariant counterparts.Then we have the following result:

$$T^{00} = T^{\mu\nu}\, u_{\mu}\, u_{\nu} = T_{00} = T_{\mu\nu}\, u^{\mu}\, u^{\nu} \dashrightarrow \mu = 0, \nu = 0 \tag{7}$$

[5]tend to infinity $+\infty$

[6]$f(rs)$ is the Alcubierre shape function.Equation written in the Geometrized System of Units $c = G = 1$

[7]Alcubierre defined $\rho = \sqrt{y^2+z^2}$ see pg 5 in [1] while other authors(eg Natario) uses $\rho = T_{\mu\nu}u^{\mu}u^{\nu}$ see pg 3 in [2].When reading works from different authors a conversion of notations must be made.This comment is meant for introductory readers

[8]in pg 4 eq 11 in [8] Lobo and Visser uses the $z$ as the axis of motion while Alcubierre,Natario and Ford-Pfenning uses the $x$ as the axis of motion.When reading Lobo-Visser equations a conversion to the $x$ axis must be made.This comment is also meant for introductory readers

So in resume we have:

$$T^{00} = T_{00} \tag{8}$$

Hiscock worked the Alcubierre warp drive equations in $1+1$ dimensions($x$,$t$) with $y^2+z^2=0$ and $dy+dz=0$.(see pg 3 and 4 in [4]).

But according to the equations of Alcubierre,Natario,Ford-Pfenning,Lobo and Visser a $1+1$ dimensional Alcubierre warp drive have a zero negative energy density given by:(see eq 8 pg 6 in [3],eq 19 pg 8 and pg 5 in [1],pg 4 eq 11 in [8] and pg 4 in [2])[9]

$$T^{00} = T_{00} = -\frac{1}{32\pi}\frac{v_s^2(t)[y^2+z^2]}{r_s^2(t)}\left(\frac{df(r_s)}{dr_s}\right)^2 = 0 \dashrightarrow [y^2+z^2]=0 \tag{9}$$

So we are left with:

$$T^{00} = T_{00} = 0 \tag{10}$$

Tensors with the same form of the composed mixed tensor $\langle T_\mu{}^\nu\rangle$ obtained from the negative energy density tensor $T_{\mu\nu}$ of the $1+1$ dimensional Alcubierre warp drive metric that Hiscock claims to diverge when the velocity of the ship $vs$ exceeds the speed of light(see pg 2 in [4]) are easily obtained applying the basic rules of Tensor Calculus as shown below:(see Carroll eqs 1.61 and 1.62 pg 24 in [17]).

$$\langle T_\mu{}^\nu\rangle = \langle T_{\mu\nu}g^{\nu\nu}\rangle = 0 \dashrightarrow \mu=0,\nu=0 \dashrightarrow \langle T_{00}g^{00}\rangle = 0 \dashrightarrow \langle T_{00}\rangle = 0 \tag{11}$$

$$\langle T_\mu{}^\nu\rangle = \langle T_{\mu\mu}g^{\mu\nu}\rangle = 0 \dashrightarrow \mu=0,\nu=1 \dashrightarrow \langle T_{00}g^{01}\rangle = 0 \dashrightarrow \langle T_{00}\rangle = 0 \tag{12}$$

$$\langle T^\mu{}_\nu\rangle = \langle T^{\mu\nu}g_{\nu\nu}\rangle = 0 \dashrightarrow \mu=0,\nu=0 \dashrightarrow \langle T^{00}g_{00}\rangle = 0 \dashrightarrow \langle T^{00}\rangle = 0 \tag{13}$$

$$\langle T^\mu{}_\nu\rangle = \langle T^{\mu\mu}g_{\mu\nu}\rangle = 0 \dashrightarrow \mu=0,\nu=1 \dashrightarrow \langle T^{00}g_{01}\rangle = 0 \dashrightarrow \langle T^{00}\rangle = 0 \tag{14}$$

Independently of the speed of the spaceship $vs$ exceeds the speed of light or not or independently of the Horizon mixed tensors of the forms $\langle T_\mu{}^\nu\rangle$ and $\langle T^\mu{}_\nu\rangle$ are impossible to diverge at least for the Alcubierre warp drive in $1+1$ dimensions because in this case the negative energy density is simply:zero!!!!

The results obtained by Hiscock for the divergence of the mixed tensor $\langle T_\mu{}^\nu\rangle$ needs to be re-examined or re-evaluated.(see pg 2 in [4])(see pg 10 and 11 in [18]).

In consequence of the fact pointed above the results obtained by Barcelo,Finazzi and Liberati for a $1+1$ dimensional Alcubierre warp drive needs also to be re-examined or re-evaluated because they based their results on the Hiscock result.(see pg 2 in [19])(see pg 10 and 11 in [18]).

We will now compute the Horizon for both Alcubierre and Natario warp drive spacetimes in $1+1$ dimensions.

[9] in pg 4 eq 11 in [8] Lobo and Visser uses the $z$ as the axis of motion while Alcubierre,Natario and Ford-Pfenning uses the $x$ as the axis of motion.When reading Lobo-Visser equations a conversion to the $x$ axis must be made.This comment is also meant for introductory readers

- 1)-Horizon for the Alcubierre warp drive spacetime in $1+1$ dimensions

The equation of the Alcubierre warp drive spacetime in $1+1$ dimensions is given by(see Appendix $H$):

$$ds^2 = [1 - vs^2 f(rs)^2]dt^2 + 2vsf(rs)dxdt - dx^2 \tag{15}$$

In the equation above $f(rs)$ is the Alcubierre shape function.Below is presented the equation of the Alcubierre shape function and its derivative square.(see eqs 6 and 7 pg 4 in [1] or top of pg 4 in [2])[10].

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{16}$$

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs - R)]}] \tag{17}$$

.

$$rs = \sqrt{(x - xs)^2 + y^2 + z^2} \tag{18}$$

According with Alcubierre any function $f(rs)$ that gives 1 inside the bubble and 0 outside the bubble while being $1 > f(rs) > 0$ in the Alcubierre warped region is a valid shape function for the Alcubierre warp drive.(see eqs 6 and 7 pg 4 in [1] or top of pg 4 in [2]).

In the Alcubierre shape function $xs$ is the center of the warp bubble where the ship resides. $R$ is the radius of the warp bubble and @ is the Alcubierre dimensionless parameter[11] related to the thickness.According to Alcubierre these can have arbitrary values.We outline here the fact that according to pg 4 in [1] the parameter @ can have arbitrary values.

$rs$ is the path of the so-called Eulerian observer that starts at the center of the bubble $xs$ and ends up outside the warp bubble. In our case due to Hiscock we consider the equatorial plane $(x \geq xs, y = 0, z = 0)$ and we have for $rs$ the following expression[12].

$$rs = \sqrt{(x - xs)^2} \tag{19}$$

$$rs = x - xs \tag{20}$$

An Horizon occurs every time an observer in the center of the warp bubble send a photon towards the front of the bubble.The photon will stop somewhere in the Alcubierre warped region never reaching the outermost layers of the Alcubierre warped region which are causally disconnected with respect to the observer in the center of the bubble.This means to say that the observer in the center of the bubble cannot signal or control the outermost regions of the warp bubble[13].

The motion of a photon being sent to the front of the bubble obeys the null-like geodesics $(ds^2 = 0)$

[10] $tanh[@(rs + R)] = 1$,$tanh(@R) = 1$ for the value of the Alcubierre thickness parameter $@ = 50000$

[11] see pg 6 in [18],pg 7 and 10 in [13]

[12] see Appendix A for the Natario warp drive

[13] at least using photons.

Inserting the condition of the null-like geodesics ($ds^2 = 0$) in the equation of the Alcubierre warp drive spacetime in $1 + 1$ dimensions we have:

$$0 = [1 - vs^2 f(rs)^2]dt^2 + 2vsf(rs)dxdt - dx^2 \tag{21}$$

Solving the quadratic form above for $\frac{dx}{dx}$ being the speed of the photon being sent to the front(or the rear) of the warp bubble we can see that we have two solutions:one for the photon sent towards the front of the ship($-$ sign) and another for the photon sent to the rear of the ship($+$ sign).See Appendix $D$

$$\frac{dx_{front}}{dt} = vsf(rs) - 1 \tag{22}$$

$$\frac{dx_{rear}}{dt} = vsf(rs) + 1 \tag{23}$$

We are interested in the result of the photon being sent to the front.

The Alcubierre shape function $f(rs)$ is defined as being 1 inside the warp bubble and 0 outside the warp bubble while being $1 > f(rs) > 0$ in the Alcubierre warped region according to eq 7 pg 4 in [1] or top of pg 4 in [2].

Expanding the quadratic term in eq 8 pg 4 in [1] and solving eq 8 for a null-like interval $ds^2 = 0$(Appendix $D$) we will have the following equation for the motion of the photon sent to the front(see pg 3 in [12] and pg 22 eqs 146 and 147 in [7])[14]:

$$\frac{dx}{dt} = vsf(rs) - 1 \tag{24}$$

Inside the Alcubierre warp bubble $f(rs) = 1$ and $vsf(rs) = vs$.Outside the warp bubble $f(rs) = 0$ and $vsf(rs) = 0$.

Somewhere inside the Alcubierre warped region when $f(rs)$ starts to decrease from 1 to 0 making the term $vsf(rs)$ decreases from $vs$ to 0 and assuming a continuous behavior then in a given point $vsf(rs) = 1$ and $\frac{dx}{dt} = 0$.The photon stops,A Horizon is established

The value of the Alcubierre shape function when or where the photon stops is given by(see Appendix $G$):

$$f(rs) = \frac{1}{vs} \tag{25}$$

Note that for superluminal speeds ($vs > 1$)($c = 1$) as higher the speed as close to zero $f(rs)$ becomes.So as higher the speed ($vs >>> 1$)($vs- > +\infty$) the point where the photon stops approaches zero[15] more and more keeping the major part of the layers of the Alcubierre warped region causally connected to the observer

$$\lim_{vs->+\infty}(\frac{1}{vs}) = 0 \tag{26}$$

[14]The coordinate frame for the Alcubierre warp drive as in [1] is the remote observer outside the ship

[15]value of $f(rs)$ outside the bubble

The exact position $rs$ of the so-called Eulerian observer where(or when) the photon stops in the Horizon of the $1+1$ Alcubierre warp drive spacetime is given by(see Appendix $G$):

$$rs = R + \frac{1}{@} arctanh(1 - \frac{2}{vs}) \tag{27}$$

As higher the thickness parameter @ becomes as thicker or thinner the warp bubble becomes too.(see pgs 8,10 and 11 in [13])

The term $(\frac{1}{@} \simeq 0)$ as long as $(@ \simeq +\infty)$ and the term $(arctanh(1 - \frac{2}{vs}) < 0)$ but $(arctanh(1 - \frac{2}{vs}) \simeq 0)$ as long as $(vs \simeq +\infty)$

So we can expect the following results:

$$\frac{1}{@} arctanh(1 - \frac{2}{vs}) < 0 \dashrightarrow \frac{1}{@} arctanh(1 - \frac{2}{vs}) \simeq 0 \tag{28}$$

Giving for the Horizon the result

$$rs = R + \frac{1}{@} arctanh(1 - \frac{2}{vs}) < R \dashrightarrow R + \frac{1}{@} arctanh(1 - \frac{2}{vs}) \simeq R \tag{29}$$

The photon never stops outside the bubble $(rs > R)$ and not over the bubble radius $(rs = R)$.It stops somewhere inside the Alcubierre warped region where $(1 > f(rs) > 0)$ because $(f(rs) = \frac{1}{vs})$ being $(0 < \frac{1}{vs} < 1)$ as long as $(vs >> 1)$.

Note that the (SEMT) for the $1+1$ Alcubierre warp drive spacetime is zero.So there are no concentrations of energy density or matter density in the front of the bubble or in the photon pathway which is empty and therefore there are nothing left to prevent the photon from reaching the Horizon.

Barcelo,Finazzi and Liberati claims that the Horizon is quantically unstable in the $1+1$ Alcubierre warp drive spacetime(see pg 2 in [19]) and they are correct.

However this quantum unstability is not due a divergence in the (SEMT) which is zero.This unstability is due to an Infinite Doppler Blueshift a photon suffers when reaching the Horizon.This was first spotted by Natario.(see pg 8 in [2]).

We are ready now to examine the topic of the Infinite Doppler Blueshifts in the Horizon suffered by photons sent towards the front of the Warp Bubble.(See pg 6 and 8 in [2])(see also Appendix $I$).

The unit vector $n$ defined below represents the direction of the corresponding light ray from the point of view of the Eulerian observer outside the warp bubble.(See pg 6 in [2])

$$n = \frac{dx}{dx} - vsf(rs) \tag{30}$$

In the Horizon $(\frac{dx}{dx} = 0)$ and $(vsf(rs) = 1)$
Then we have:

$$n = \frac{dx}{dx} - vsf(rs) = 0 - 1 = -1 \tag{31}$$

The equation of the observed energy is given by(See pg 8 in [2]):

$$E_0 = E(1 + n.vsf(rs)) \tag{32}$$

So when a photon reaches the Horizon we have the following conditions: $n = -1$ and $vsf(rs) = 1$.Inserting these values in the equation of the energy(pg 8 in [2]) we have:

$$E = \frac{E_0}{(1 + n.vsf(rs))} = \frac{E_0}{(1 + -1.1)} = \frac{E_0}{0} \tag{33}$$

And then we have the Infinite Doppler Blueshift in the Horizon as mentioned by Natario in pg 8 in [2].

According to pg 9 in [9] and pg 8 in [2] photons still outside the warp bubble but approaching the warp bubble from the front appears with the frequency highly Doppler Blueshifted when seen by the observer in the center of the bubble.

If the warp bubble moves with a speed $vs$ then a source of photons(a star) is being seen by the observer in the center of the bubble with a relative approximation speed $vs$

Computing the Blueshift using the classical Doppler-Fizeau formula for a photon approaching the ship from the front we have:[16]

$$f = f_0 \frac{c + va}{c - vb} \tag{34}$$

The terms above are:

- 1)-$f$ is the photon frequency seen by an observer
- 2)-$f_0$ is the original frequency of the emitted photon
- 3)-$c$ is the light speed.in our case $c = 1$
- 4)-$va$ is the relative speed of the light source approaching the observer.In our case is $vs$
- 5)-$vb$ in the relative speed of the light source moving away from the observer. In our case because the photon is coming to the observer $vb = 0$

Rewriting the Doppler-Fizeau expression for an incoming photon approaching the warp bubble from the front we should expect for(see eq 26 pg 9 in [9],see also pg 8 in [2]):

$$f = f_0(1 + vs) \tag{35}$$

$$f = f_0(1 + X) \tag{36}$$

Energy $E$ is Planck Constsnt $\hbar$ multiplied by frequency so for the energy we would have:

$$E = E_0(1 + vs) \tag{37}$$

[16]Remember that the warp drive do not obey Lorentz transformations so the classical formula can be applied to get the results of [2] and [9]

Note that as larger is $vs > 1$ as large is the Blueshift and this is a serious obstacle that compromises the physical feasibility of the warp drive according with Clark,Hiscock and Larson.See pg 11 in [9].Incoming Blueshifted photons are hazardous for any crew inside the warp bubble.According to pg 11 in [9] COBE photons are Blueshifted in the front of the warp bubble to energies equivalent to a solar photosphere for a speed of 200 times faster then light.These highly energetic photons would pose a very serious threat to the astronauts as pointed out by McMonigal,Lewis and O'Byrne (see pg 10 in [10]).

- 1)-in this point we agree with Clark,Hiscock,Larson,McMonigal,Lewis and O'Byrne (pg 11 in [9])(pg 10 in [10]) when they say the Alcubierre warp drive in a $1 + 1$ spacetime is impossible.

Outside the bubble the photon is seen with a relative approximation speed $vs$ while in the interior of the bubble the relative approximation speed is 0 (the case of an observer $B$ in front of the observer $A$ but stationary with respect to $A$ sending to him a photon with a frequency $f_0$ and an energy $E_0$.Both observers $A$ and $B$ are in the neighborhoods of each other and inside the bubble.) then a photon coming from outside approaching the bubble with a relative approximation speed $vs$ arrives at the Alcubierre warped region with an energy $E_0$ according to the Doppler-Fizeau formula multiplied by a Doppler Blueshift of $1 + vs$ being the incoming energy $E = E_0(1+vs)$ then the photon crosses the Alcubierre warped region and finally arrives at the region inside the warp bubble where the relative approximation speed is 0 so $E = E_0$.Note that $E_0(1 + X) >> E_0$ which means to say that the incoming photon passed from a state of high energy $E_0(1 + X)$ to a state of low energy $E_0$.

At this point one would ask:what happened with the remaining energy??

This excess of energy is "released" when the photon crosses the warped region rendering this region unstable and generating the Hawking temperatures that are another big problem that compromises the physical stability of the warp drive(see pgs 6 and 7 in [11])).

So Barcelo,Finazzi and Liberati are correct when they say that the Alcubierre warp drive in $1+1$ spacetime is impossible due to the quantum instabilities that occurs in the warped region and in the Horizon(see pg 2 in [19]) but these quantum instabilities are due to the Doppler Blueshifts and not due to the divergences in the (SEMT) which is zero in this case.

- 1)-in this point we agree with Barcelo,Finazzi and Liberati (pg 2 in [19]) when they say the Alcubierre warp drive in a $1 + 1$ spacetime is impossible.

- 2)-Horizon for the Natario warp drive spacetime in $1+1$ dimensions

The equation of the Natario warp drive spacetime is given by(see Appendices $B$ and $C$):

$$ds^2 = [1 - (X^{rs})^2 - (X^{\theta})^2]dt^2 + 2[X^{rs}drs + X^{\theta}rsd\theta]dt - drs^2 - rs^2d\theta^2 \tag{38}$$

The expressions for $X^{rs}$ and $X^{\theta}$ in the equation above are given by:(see pg 5 in [2])

$$X^{rs} = -2v_s n(rs)\cos\theta \tag{39}$$

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{40}$$

$$X^{\theta} = v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{41}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{42}$$

We are interested in the two-dimensional $1+1$ version of the Natario warp drive in the dimensions $rs$ and $t$ (motion over the $x - axis$ only with $\theta = 0$ $cos\theta = 1$ and $\sin\theta = 0$.see also Appendix $A$) given by:

$$ds^2 = [1 - (X^{rs})^2]dt^2 + 2X^{rs}drsdt - drs^2 \tag{43}$$

With $X^{rs}$ being given by:

$$X^{rs} = 2v_s n(rs) \tag{44}$$

According to Natario(pg 5 in [2]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive.

A Natario warp drive valid shape function can be given by:

$$n(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{45}$$

Its derivative square is :

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{46}$$

The shape function above gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region(see pg 5 in [2]).

Note that the Alcubierre shape function is being used to define its Natario shape function counterpart. The term $WF$ in the Natario shape function is dimensionless too:it is the warp factor that will squeeze the region where the derivatives of the Natario shape function are different than 0.See eq 5 pg 3 in [19].

For the Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

An Horizon occurs every time an observer in the center of the warp bubble send a photon towards the front of the bubble.The photon will stop somewhere in the Natario warped region never reaching the outermost layers of the Natario warped region which are causally disconnected with respect to the observer in the center of the bubble.This means to say that the observer in the center of the bubble cannot signal or control the outermost regions of the warp bubble[17].

The motion of a photon being sent to the front of the bubble obeys the null-like geodesics ($ds^2 = 0$)

Inserting the condition of the null-like geodesics ($ds^2 = 0$) in the equation of the Natario warp drive spacetime in $1+1$ dimensions we have:

$$ds^2 = 0 \dashrightarrow [1 - (X^{rs})^2]dt^2 + 2X^{rs}drsdt - drs^2 = 0 \tag{47}$$

Solving the quadratic form above for $\frac{dx}{dx}$ being the speed of the photon being sent to the front(or the rear) of the warp bubble we can see that we have two solutions:one for the photon sent towards the front of the ship($-$ sign) and another for the photon sent to the rear of the ship($+$ sign).See Appendix $E$

$$\frac{drs_{front}}{dt} = X^{rs} - 1 \tag{48}$$

$$\frac{drs_{rear}}{dt} = X^{rs} + 1 \tag{49}$$

$$X^{rs} = 2v_s n(rs) \tag{50}$$

We are interested in the result of the photon being sent to the front.

The Natario shape function $n(rs)$ is defined as being 0 inside the warp bubble and $\frac{1}{2}$ outside the warp bubble while being $\frac{1}{2} > n(rs) > 0$ in the Natario warped region according to pg 5 in [2].

Solving the Natario warp drive equation in a $1+1$ spacetime for a null-like interval $ds^2 = 0$(Appendix $E$) we will have the following equation for the motion of the photon sent to the front:

$$\frac{dx}{dt} = 2vsn(rs) - 1 \tag{51}$$

Inside the Natario warp bubble $n(rs) = 0$ and $2vsn(rs) = 0$.
Outside the Natario warp bubble $n(rs) = \frac{1}{2}$ and $2vsn(rs) = vs$.

Somewhere inside the Natario warped region when $n(rs)$ starts to increase from 0 to $\frac{1}{2}$ making the term $2vsn(rs)$ increases from 0 to $vs$ and assuming a continuous behavior then in a given point $2vsn(rs) = 1$ and $\frac{dx}{dt} = 0$.The photon stops,A Horizon is established

The value of the Natario shape function when or where the photon stops is given by(see Appendix $F$):

$$n(rs) = \frac{1}{2vs} \tag{52}$$

[17] at least using photons.

Note that for superluminal speeds $(vs > 1)(c = 1)$ as higher the speed as close to zero $n(rs)$ becomes.So as higher the speed $(vs >>> 1)(vs- > +\infty)$ the point where the photon stops. approaches zero[18][19]

$$\lim_{vs->+\infty}(\frac{1}{2vs}) = 0 \tag{53}$$

The exact position $rs$ of the so-called Eulerian observer where(or when) the photon stops in the Horizon of the $1+1$ Natario warp drive spacetime is given by(see Appendix $F$):

$$rs = R + \frac{1}{@}arctanh(1 - 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}}) \tag{54}$$

As higher the thickness parameter @ becomes and also as higher the warp factor $WF$ becomes as thicker or thinner the warp bubble becomes too.(see pgs 8,10 and 11 in [13])

The term $(\frac{1}{@} \simeq 0)$ as long as $(@ \simeq +\infty)$ and the terms $(\sqrt[WF]{\frac{1}{vs}} \simeq 0)$ and $(\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}} \simeq 1)$ giving $(2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}} \simeq 2)$ as long as $(vs \simeq +\infty)$

But $(\sqrt[WF]{\frac{1}{vs}} > 0)$[20] and $(1 - \sqrt[WF]{\frac{1}{vs}} \simeq 1)$ although we will always have the condition $(1 - \sqrt[WF]{\frac{1}{vs}} < 1)$ implying in $(\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}} < 1)$ and in consequence $(2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}} < 2)$ however we can clearly see that $(2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}} > 1)$ as long as $vs >> 1$ superluminal.

From the conclusions given above we can see that $(1 - 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}} < 0)$

So we can expect the following results concerning the position of the Eulerian observer in the Horizon of the $1+1$ Natario warp drive spacetime:

$$\frac{1}{@} \simeq 0 \tag{55}$$

$$arctanh(1 - 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}}) < 0 \tag{56}$$

$$\frac{1}{@}arctanh(1 - 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}}) << 0 \dashrightarrow \frac{1}{@}arctanh(1 - 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}}) \simeq 0 \tag{57}$$

$$rs = R + \frac{1}{@}arctanh(1 - 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}}) < R \dashrightarrow rs = R + \frac{1}{@}arctanh(1 - 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}}) \simeq R \tag{58}$$

[18] value of $n(rs)$ inside the bubble

[19] although $(vs- > +\infty)$ we will always have the condition $(vs < +\infty)$ then $vs$ will never reach $(\infty)$ so $(\frac{1}{2vs} - > 0)$ but will never reaches zero and the condition $(\frac{1}{2} > \frac{1}{2vs} > 0)$ will always remains valid keeping the Horizon inside the Natario warped region

[20] again $(vs- > +\infty)$ and $(vs < +\infty)$ then $vs$ will never reach $(\infty)$

The photon never stops outside the bubble ($rs > R$) and not over the bubble radius ($rs = R$).It stops somewhere inside the Natario warped region where ($\frac{1}{2} > n(rs) > 0$) because ($n(rs) = \frac{1}{2vs}$) being ($0 < \frac{1}{2vs} < 1$) as long as ($vs >> 1$).

- )-For a while we will neglect the geometrical distribution of energy in the (SEMT) of the $1+1$ dimensional Natario warp drive spacetime

We are ready now to examine the topic of the Infinite Doppler Blueshifts in the Horizon suffered by photons sent towards the front of the Warp Bubble.(See pg 6 and 8 in [2])(see also Appendix $J$).

The unit vector $n$ defined below represents the direction of the corresponding light ray from the point of view of the Eulerian observer outside the warp bubble.(See pg 6 in [2])

$$n = \frac{dx}{dx} - 2vsn(rs) \tag{59}$$

In the Horizon ($\frac{dx}{dx} = 0$) and ($2vsn(rs) = 1$)
Then we have:

$$n = \frac{dx}{dx} - 2vsn(rs) = 0 - 1 = -1 \tag{60}$$

The equation of the observed energy is given by(See pg 8 in [2]):

$$E_0 = E(1 + n.vsf(rs)) \tag{61}$$

So when a photon reaches the Horizon we have the following conditions: $n = -1$ and $2vsn(rs) = 1$.Inserting these values in the equation of the energy(pg 8 in [2]) we have:

$$E = \frac{E_0}{(1 + n.vsf(rs))} = \frac{E_0}{(1 + -1.1)} = \frac{E_0}{0} \tag{62}$$

And then we have the Infinite Doppler Blueshift in the Horizon as mentioned by Natario in pg 8 in [2].

- )-Considering now the geometrical distribution of energy in the (SEMT) of the $1+1$ dimensional Natario warp drive spacetime compared to its Alcubierre counterpart:

The infinite Doppler Blueshift happens in the Alcubierre warp drive but not in the Natario one.This means to say that Alcubierre warp drive is physically impossible to be achieved but the Natario warp drive is perfectly physically possible to be achieved.

Consider again the negative energy density distribution (SEMT) in the Alcubierre warp drive spacetime(see eq 8 pg 6 in [3])[21][22]:

$$\langle T^{\mu\nu}u_{\mu}u_{\nu}\rangle = \langle T^{00}\rangle = \frac{1}{8\pi}G^{00} = -\frac{1}{8\pi}\frac{v_s^2(t)[y^2+z^2]}{4r_s^2(t)}\left(\frac{df(r_s)}{dr_s}\right)^2, \tag{63}$$

And considering again the negative energy density distribution (SEMT) in the Natario warp drive spacetime(see pg 5 in [2])[23][24]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs)+\frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{64}$$

In pg 6 in [2] a warp drive with a x-axis only (a $1+1$ dimensional warp drive)is considered.In this case for the Alcubierre warp drive $[y^2+z^2]=0$

$$\langle T^{\mu\nu}u_{\mu}u_{\nu}\rangle = \langle T^{00}\rangle = \frac{1}{8\pi}G^{00} = -\frac{1}{8\pi}\frac{v_s^2(t)[y^2+z^2]}{4r_s^2(t)}\left(\frac{df(r_s)}{dr_s}\right)^2, = 0!!!! \tag{65}$$

And the negative energy density is zero!!! but the Natario energy density is not zero and given by:.

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2\right] \tag{66}$$

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1-f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{67}$$

Note that in front of the ship in the Alcubierre case the spacetime is empty and there is nothing to deflect the photon before the Horizon point but in the Natario case there exists negative energy density in the front of the ship.This negative energy density deflects the photon before the point of the Horizon.

The amount of negative energy density in the Horizon can be computed inserting the value of the $rs$ in the Horizon into the (SEMT) equation

$$rs = R + \frac{1}{@}arctanh(1 - 2\sqrt[WF]{1-\sqrt[WF]{\frac{1}{vs}}}) \tag{68}$$

And the negative energy density that exists in the Natario warped region before the Horizon point will deflect the photon before the Horizon point.

So while Infinite Doppler Blueshifts generates instabilities in the Alcubierre warp drive these instabilities never happens in the Natario warp drive because the photon never reaches the Horizon.The photon is deflected by the negative energy in front of the ship.

[21] $f(rs)$ is the Alcubierre shape function.Equation written in the Geometrized System of Units $c=G=1$
[22] Equation written in Cartezian Coordinates
[23] $n(rs)$ is the Natario shape function.Equation written the Geometrized System of Units $c=G=1$
[24] Equation written in Cartezian Coordinates.See Appendix $A$

Still according with Natario in pg 7 before section 5.2 in [14] negative energy density means a negative mass density and a negative mass density generates a repulsive gravitational field. This repulsive gravitational field in front of the ship in the Natario warp drive spacetime protects the ship from impacts with the interstellar matter and also protects the ship from COBE photons highly Doppler Blueshifted approaching the ship from the front.So in the Natario warp drive the problem of the Hawking temperature will never happens.

Adapted from the negative energy in Wikipedia:The free Encyclopedia:

"if we have a small object with equal inertial and passive gravitational masses falling in the gravitational field of an object with negative active gravitational mass (a small mass dropped above a negative-mass planet, say), then the acceleration of the small object is proportional to the negative active gravitational mass creating a negative gravitational field and the small object would actually accelerate away from the negative-mass object rather than towards it."

The Natario warp drive as a solution of the Einstein Field Equations of General Relativity that allows faster than light motion is the first valid candidate for interstellar space travel.

In order to terminate this section we would like to outline the following statements:

- 1)-The objections raised by Clark,Hiscock,Larson,McMonigal,Lewis,O'Byrne,Barcelo,Finazzi and Liberati (pg 11 in [9])(pg 10 in [10])(pg 2 in [19])(pgs 6 and 7 in [11])) are not valid for the Natario warp drive spacetime.The negative energy density in front of the ship according to Natario in pg 7 before section 5.2 in [14] have a repulsive behavior that deflects all the hazardous interstellar matter and Doppler Blueshifted photons.
- 2)-So they cannot say the warp drive is impossible.
- 3)-At least they can say the Alcubierre warp drive is impossible because in front of the ship there exists nothing to deflect Doppler Blueshifted photons or incoming hazardous interstellar matter.
- 4)-At pg 9 in [11]) Barcelo,Finazzi and Liberati postulate that all future spaceships cannot bypass 99 percent of the light speed,This is true for the Alcubierre warp drive but not for the Natario one.
- 5)-When we study warp drives a clear difference between Natario and Alcubierre must be made or in short:$(Natario \neq Alcubierre)$.

# 4 Reducing the Negative Energy Density Requirements in the Natario Warp Drive in a $1+1$ Dimensional Spacetime

Now we are ready to demonstrate how the negative energy density requirements can be greatly reduced for the Natario warp drive in a $1+1$ dimensional spacetime:

We already know the form of the equation of the Natario warp drive in a $1+1$ dimensional spacetime:

$$ds^2 = [1 - (X^{rs})^2]dt^2 + 2X^{rs}drsdt - drs^2 \tag{69}$$

$$X^{rs} = 2v_s n(rs) \tag{70}$$

According to Natario(pg 5 in [2]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive.

A Natario warp drive valid shape function can be given by:

$$n(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{71}$$

Its derivative square is :

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{72}$$

The shape function above gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region(see pg 5 in [2]).

Note that the Alcubierre shape function is being used to define its Natario shape function counterpart. The term $WF$ in the Natario shape function is dimensionless too:it is the warp factor that will squeeze the region where the derivatives of the Natario shape function are different than 0.See eq 5 pg 3 in [19].

For the Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

We consider here a Natario warp drive with a radius $R = 100$ meters a dimensionless thickness parameter with the values $@ = 50000$ $@ = 750000$ and $@ = 100000$.We also consider a warp factor with a value $WF = 200$.

We provide numerical plots of the values of the derivative squares of both Alcubierre and Natario shape functions for two warp drives in $1+1$ dimensions.

- Numerical plot with $@ = 50000$

| $rs$ | $f(rs)$ | $n(rs)$ | $f'(rs)^2$ | $n'(rs)^2$ |
|---|---|---|---|---|
| $9,99970000000E+001$ | 1 | 0 | $2,650396620740E-251$ | 0 |
| $9,99980000000E+001$ | 1 | 0 | $1,915169647489E-164$ | 0 |
| $9,99990000000E+001$ | 1 | 0 | $1,383896564748E-077$ | 0 |
| $1,00000000000E+002$ | $0,5$ | $0,5$ | $6,250000000000E+008$ | $3,872591914849E-103$ |
| $1,00001000000E+002$ | 0 | $0,5$ | $1,383896486082E-077$ | 0 |
| $1,00002000000E+002$ | 0 | $0,5$ | $1,915169538624E-164$ | 0 |
| $1,00003000000E+002$ | 0 | $0,5$ | $2,650396470082E-251$ | 0 |

- Numerical plot with $@ = 75000$

| $rs$ | $f(rs)$ | $n(rs)$ | $f'(rs)^2$ | $n'(rs)^2$ |
|---|---|---|---|---|
| $9,99980000000E+001$ | 1 | 0 | $5,963392481410E-251$ | 0 |
| $9,99990000000E+001$ | 1 | 0 | $1,158345097767E-120$ | 0 |
| $1,00000000000E+002$ | $0,5$ | $0,5$ | $1,406250000000E+009$ | $8,713331808411E-103$ |
| $1,00001000000E+002$ | 0 | $0,5$ | $1,158344999000E-120$ | 0 |
| $1,00002000000E+002$ | 0 | $0,5$ | $5,963391972940E-251$ | 0 |

- Numerical plot with $@ = 100000$

| $rs$ | $f(rs)$ | $n(rs)$ | $f'(rs)^2$ | $n'(rs)^2$ |
|---|---|---|---|---|
| $9,99990000000E+001$ | 1 | 0 | $7,660678807684E-164$ | 0 |
| $1,00000000000E+002$ | $0,5$ | $0,5$ | $2,500000000000E+009$ | $1,549036765940E-102$ |
| $1,00001000000E+002$ | 0 | $0,5$ | $7,660677936765E-164$ | 0 |

Examining the plot above we can see that both warp drives wether in Alcubierre or Natario case have two warped regions:

- 1)-The warped region where $1 > f(rs) > 0$ or $0 < n(rs) < \frac{1}{2}$ according with both Alcubierre and Natario requirements.-This warped region is known as the Geometrized warped region.
- 2)-The warped region where the derivative squares of both Alcubierre and Natario shape functions are not zero resulting in a non-null negative energy density and in a non-flat spacetime.-This warped region is known as the Energized warped region

Note that as higher the thickness parameter @ becomes as thicker or thinner the limits or the boundaries of the Energized warped region(the region where the derivatives of $f(rs)$ are not zero)becomes too.

Note that the warp drives with thickness parameter $@ = 100000$ have an Energized warped region with a width (a thickness) smaller than its $@ = 50000$ and $@ = 75000$ counterparts.

Note that the Geometrized warped region is distributed in a thin layer over the neighborhoods of the bubble radius $rs \simeq R$ in both Alcubierre and Natario cases but the Energized warped region extends itself over regions inside and outside the bubble $(rs < R)$ and $(rs > R)$ in the Alcubierre case. A warp drive with a thickness parameter $@ = 1000000000000000000000$ would have the limits of the Energized warped region infinitely close to each other resulting in an infinitely small thickness.

Note that in the Natario case the Energized warped region is distributed in a very thin layer over the neighborhoods of the bubble radius starting in a region very close to the bubble radius
$(rs <= R, rs \cong R, R - rs \cong 0)$ and terminating also in a region very close to the bubble radius
$(rs >= R, rs \cong R, rs - R \cong 0)$

Calling the point where the Energized warped region begins as point $a$ and the point where the Energized warped region ends as point $b$ then the point $b$ lies almost infinitely closed to the point $a$ resulting in an Energized warped region of almost infinite small thickness.

$(rs <= R, rs \cong R, R - rs \cong 0, R - rs = a, a \cong 0)$
$(rs >= R, rs \cong R, rs - R \cong 0, rs - R = b, b \cong 0)$
$(a \cong 0, b \cong 0, b >= a, b \cong a, b - a \cong 0)$

Why this behavior that contains both the Geometrized and Energized warped regions in the same boundary limits $a$ and $b$ occurs only with the Natario case and not also with its Alcubierre counterpart??Why in the Natario case the Energized warped region coincides with the Geometrized warped region??

Starting with the square of the derivative of the Natario shape function:

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{73}$$

Inside the bubble $f(rs) = 1$ and $[1 - f(rs)^{WF}]^{2(WF-1)} = 0$ resulting in a $n'(rs)^2 = 0$.This is the reason why the Natario shape function dont have derivatives inside the bubble.

Outside the bubble $f(rs) = 0$ and $[f(rs)^{2(WF-1)}] = 0$ resulting also in a $n'(rs)^2 = 0$.This is the reason why the Natario shape function dont have derivatives outside the bubble.

In the Geometrized warped region for the Alcubierre warp drive $1 > f(rs) > 0$.In this region the derivatives of the Natario shape function do not vanish because if $f(rs) < 1$ then $f(rs)^{WF} << 1$ resulting in an $[1 - f(rs)^{WF}]^{2(WF-1)} << 1$ but $[1 - f(rs)^{WF}]^{2(WF-1)} > 0$.Also if $f(rs) < 1$ then $[f(rs)^{2(WF-1)}] << 1$ too but $[f(rs)^{2(WF-1)}] > 0$

Note that if $[1 - f(rs)^{WF}]^{2(WF-1)} << 1$ and $[f(rs)^{2(WF-1)}] << 1$ their product
$[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}] <<<< 1$

Note that inside the Alcubierre Geometrized warped region $1 > f(rs) > 0$ when $f(rs)$ approaches 1 $n'(rs)^2$ approaches 0 due to the factor $[1 - f(rs)^{WF}]^{2(WF-1)}$ and when $f(rs)$ approaches 0 $n'(rs)^2$ approaches 0 again due to the factor $[f(rs)^{2(WF-1)}]$ .

The maximum value for $n'(rs)^2 \cong 10^{-103}$ occurs in the middle of the Alcubierre Geometrized warped region where $f(rs) = 0,5$.Note that this value for the square of the derivative of the Natario shape function can obliterate the factor $10^{48}$ resulting in an extremely low level of negative energy density.

Due to the fact that the values of $n'(rs)^2$ grows from 0 to $10^{-103}$ and decreases to 0 again we choose the Natario warp drive as the best candidate to low the negative energy density requirements completely obliterating the factor $10^{48}$ which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!.

Back again to the negative energy density in the Natario warp drive[25]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{74}$$

The total energy needed to sustain the Natario warp bubble is obtained by integrating the negative energy density $\rho$ over the volume of the Natario warped region(The region where the derivatives of the Natario shape function are not null)(points $a$ and $b$ the beginning and the end of the Natario Energized warped region).
Since we are in the equatorial plane then only the term in $rs$ accounts for and the total energy integral can be given by:

$$E = \int(\rho)drs = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\int(3(n'(rs))^2)drs = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\int((n'(rs))^2)drs \tag{75}$$

Above we placed the constant terms $c$,$G$ and $vs^2$ outside the integral. But the integral now becomes:

$$\int((n'(rs))^2)drs = \int([\frac{1}{4}]WF^4[1-f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2)drs \tag{76}$$

$$\int((n'(rs))^2)drs = \frac{1}{4}WF^4\int([1-f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2)drs \tag{77}$$

Since $WF$ ia also a constant.Then the total energy integral for the Natario warp drive is given by:

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\int((n'(rs))^2)drs = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{1}{4}WF^4\int([1-f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2)drs \tag{78}$$

Unfortunately integrals of this form do not have known primitives and also the integration methods to compute the integral are not known.In order to compute the total energy needed to sustain the Natario warp bubble we must employ the numerical integration by the Trapezoidal Rule.(see pg 13 eqa 27 in [13])[26]

$$\int_a^b f(x)dx = (b-a)\frac{f(b)+f(a)}{2} \cong 0 \dashrightarrow b-a \cong 0 \dashrightarrow b \cong a \dashrightarrow f(b) \cong f(a) \tag{79}$$

The total energy integral needed to sustain the Natario warp bubble now becomes:

$$E = \int_a^b \rho drs = \int_a^b -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3n'(rs))^2\right]drs = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\int_a^b\left[n'(rs)^2\right]drs \tag{80}$$

$$\int_a^b\left[(n'(rs)^2\right]drs = (b-a)\frac{n'(b)^2+n'(a)^2}{2} \tag{81}$$

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}(b-a)\frac{n'(b)^2+n'(a)^2}{2} \simeq 0 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \tag{82}$$

[25]written in the International System of units
[26]see Wikipedia the Free Encyclopedia

The beginning of the region where the derivatives of the shape function ceases to be zero is the beginning of the Natario Energized warped (point $a$).The end of the region where the derivatives of the shape function still are not zero is the end of the Natario Energized warped region (point $b$)

If the difference between the points $a$ and $b$ is close to zero due to the fact that the points $a$ and $b$ are infinitely close to each other then this is enough to obliterate the value of $10^{48}$ or any other value.$(b \simeq a)(b - a \simeq 0)$

However there exists an error margin in the integration by the Trapezoidal Rule.The correct procedure is to decompose the area to be integrated in $n$ slices and integrate numerically still by the Trapezoidal Rule separately each slice and in the end we sum the result of all the slices integrated.This method is known as the Composite Trapezoidal Rule.(see pg 14 in [13])

As higher the number $n$ of slices is as more precise the result of the integral by the Composite Trapezoidal Rule is.(see pgs 14 in [13]).Remember that a warp bubble wether in the Alcubierre or Natario cases with a radius of 100 meters moving at 200 times light speed have the total amount of negative energy equal to the product of $10^{48}$ by the integral of the square derivatives of the shape function in the region between the point $b$(end of the Energized warped region) and the point $a$(beginning of the Energized warped region).If we want to integrate from the point $a$ to point $b$ using the Composite Trapezoidal Rule reducing the error margin we must divide the region between $a$ and $b$ in $n$ slices and integrate separately each slice also by the Trapezoidal Rule and in the end we sum the result of all the integrations.As higher the number $n$ of slices as accurate the integration becomes.Following pg 14 in [13] if we want to divide the region between $a$ and $b$ in $n$ slices each slice have a width given by:

$$h = \frac{b-a}{n} \tag{83}$$

And the final integration is the sum of the integration of all the slices by the Trapezoidal Rule given by:

$$\int_a^b n'(rs)^2 drs = \int_a^{a+h} n'(rs)^2 drs + \int_{a+h}^{a+2h} n'(rs)^2 drs + ....... + \int_{a+(n-2)h}^{a+(n-1)h} n'(rs)^2 drs + \int_{a+(n-1)h}^{b} n'(rs)^2 drs \tag{84}$$

Note that each slice above have a width $h$.

Writing the integral using sums we get the following expressions(pg 14 in [13]):

$$\int_a^b n'(rs)^2 drs = \frac{1}{2n}(b-a)[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \cong 0 \dashrightarrow b - a \cong 0 \dashrightarrow b \cong a \dashrightarrow n >> 1 \tag{85}$$

$$\int_a^b n'(rs)^2 drs = \frac{b-a}{2n}[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \cong 0 \dashrightarrow b - a \cong 0 \dashrightarrow b \cong a \dashrightarrow n >> 1 \tag{86}$$

Inserting these expressions in the integral of the negative energy density we get:

$$E = \int_a^b \rho drs = \int_a^b -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3n'(rs))^2\right] drs = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\int_a^b \left[n'(rs)^2\right] drs \tag{87}$$

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{1}{2n}(b-a)[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \tag{88}$$

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{b-a}{2n}[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \tag{89}$$

Look again to the equations of the total negative energy needed to sustain a Natario warp bubble:

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{1}{2n}(b-a)[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \tag{90}$$

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{b-a}{2n}[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \tag{91}$$

Now note an interesting thing:

- 1)-The number of slices raises the accuracy of the integration method and we have the factor $\frac{c^2}{G} \times \frac{vs^2}{8\pi}$ generating the huge factor $10^{48}$ for a bubble speed $vs = 200$ times light speed constraining the negative energy densities in the Natario warp bubble.
- 2)-How about to make the numerical integration by the Trapezoidal Rule using $10^{48}$ slices??.How about to make $n = 10^{48}$ ???

If $n = 10^{48}$ then $n$ in the denominator of the fraction $\frac{b-a}{2n}$ will completely obliterate the factor $\frac{c^2}{G} \times \frac{vs^2}{8\pi}$ in the equations of the total energy integral lowering the negative energy density requirements for the Natario warp bubble.

If we want a rigorous integration of the Natario negative energy density warped region we must divide this region in $10^{48}$ slices each one with a width $h$ of:

$$h = \frac{b-a}{n} = \frac{b-a}{10^{48}} \tag{92}$$

Inserting these values in the total energy integral equations we should expect for:

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{b-a}{2\times 10^{48}}[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \tag{93}$$

We choose to divide the Natario negative energy warped region in $10^{48}$ slices due to the factor $\frac{c^2}{G}\frac{v_s^2}{8\pi}$ which is $10^{48}$ for a bubble speed $vs = 200$ times faster than light .In the equation above keeping the bubble radius $R = 100$ meters and the thickness parameter $@ = 5000$ and the warp factor $WF = 200$ all constants and since $c$ and $G$ are constants if we use a different bubble velocity $vs$ higher than 200 times faster than light giving a factor $\frac{c^2}{G}\frac{v_s^2}{8\pi} > 10^{48}$ then the number of slices $n$ needed to integrate accurately by the Trapezoidal Rule the energy density in the Natario warp bubble must be equal to this new factor in order to reduce the total energy integral.

# 5 A Causally Connected Superluminal Natario Warp Drive Spacetime using Micro Warp Bubbles

In 2002 Gauthier,Gravel and Melanson appeared with the idea of the micro warp bubbles.([15],[16])

According to them,microscopical particle-sized warp bubbles may have formed spontaneously immediately after the Big Bang and these warp bubbles could be used to transmit information at superluminal speeds.These micro warp bubbles may exist today.(see abs of [16])

A micro warp bubble with a radius of $10^{-10}$ meters could be used to transport an elementary particle like the electron whose Compton wavelength is $2.43 \times 10^{-12}$ meters at a superluminal speed.These micro warp bubbles may have formed when the Universe had an age between the Planck time and the time we assume that Inflation started.(see pg 306 of [15])

Following the ideas of Gauthier,Gravel and Melanson ([15],[16]) a micro warp bubble can send information or particles at superluminal speeds.(abs of [16],pg 306 in [15]).Since the infinite Doppler Blueshift affect the Alcubierre warp drive but not the Natario one and a superluminal micro warp bubble can only exists without Infinite Doppler Blueshifts[27] we consider in this section only the Natario warp drive spacetime.

The idea of Gauthier,Gravel and Melanson ([15],[16]) to send information at superluminal speeds using micro warp bubbles is very interesting and as a matter of fact shows to us how to solve the Horizon problem. Imagine that we are inside a large superluminal warp bubble and we want to send information to the front.Photons sent from inside the bubble to the front would stop in the Horizon but we also know that incoming photons from outside would reach the bubble.[28] The external observer outside the bubble have all the bubble causally connected while the internal observer is causally connected to the point before the Horizon.Then the external observer can create the bubble while the internal observer cannot. This was also outlined by Everett-Roman in pg 3 in [12]. Unless we find a way to overcome the Horizon problem. We inside the large warp bubble could create and send one of these micro warp bubbles to the front of the large warp bubble but with a superluminal speed $vs2$ larger than the large bubble speed $X = 2vsn(rs)$.Then $vs2 >> X$ or $vs2 >> 2vsn(rs)$ and this would allow ourselves to keep all the warp bubble causally connected from inside overcoming the Horizon problem without the need of the "tachyonic" matter.

- 1)- Superluminal micro warp bubble sent towards the front of the large superluminal warp bubble $vs2 = \frac{dx}{dt} > X - 1 > vs - 1 \dashrightarrow X = 2vsn(rs)$

From above it easy to see that a micro warp bubble with a superluminal speed $vs2$ maintains a large superluminal warp bubble with speed $vs$ causally connected from inside if $vs2 > vs$

[27] assuming a continuous growth of the warp bubble speed $vs$ from zero to a superluminal speed at a given time the speed will be equal to $c$ and the Infinite Doppler Blueshift crashes the bubble.The Alcubierre warp drive can only exists for $vs < c$ so it cannot sustain a micro warp bubble able to shelter particles or information at superluminal speeds

[28] true for the Alcubierre warp drive but not for the Natario one because the negative energy density in the front with repulsive gravitational behavior would deflect all the photons sent from inside and outside the bubble effectively shielding the Horizon from the photon avoiding the catastrophical Infinite Doppler Blueshift

From the point of view of the astronaut inside the large warp bubble he is the internal observer with respect to the large warp bubble but he is the external observer from the point of view of the micro warp bubble so he keeps all the light-cone of the micro warp bubble causally connected to him so he can use it to send superluminal signals to the large warp bubble from inside.(Everett-Roman in pg 3 in [12]).

Gauthier,Gravel and Melanson developed the concept of the micro warp bubble but the idea is at least 5 years younger.The first time micro warp bubbles were mentioned appeared in the work of Ford-Pfenning pg 10 and 11 in [3].

According with Natario in pg 7 before section 5.2 in [14] negative energy density means a negative mass density and a negative mass generates a repulsive gravitational field that deflect photons or positive mass-density particles from the interstellar medium or particles sent to the bubble walls by the astronaut inside the bubble.

However while the negative mass deflects the positive mass or photons[29] a negative mass always attracts another negative mass so the astronaut cannot send positive particles or photons to the large warp bubble but by sending micro warp bubbles these also possesses negative masses that will be attracted by the negative mass of the large warp bubble effectively being a useful way to send signals.

[29] by negative gravitational bending of light

# 6 Conclusion

In this work we demonstrated that the composed mixed tensor $\langle T_{\mu}{}^{\nu} \rangle$ obtained from the negative energy density tensor $T_{\mu\nu}$ $\mu = 0, \nu = 0$ of the $1+1$ dimensional Alcubierre warp drive metric do not diverges when the velocity of the ship $vs$ exceeds the speed of light and the results obtained by Barcelo,Finazzi and Liberati needs to be re-examined or re.evaluated.(see pg 2 in [19]).

We also introduced a shape functions for the Natario warp drive spacetime that is very efficient lowering the negative energy densities in the Natario warp drive to affordable levels.

Also we demonstrated that the objections raised by Clark,Hiscock,Larson,McMonigal,Lewis,O'Byrne,Barcelo, Finazzi and Liberati in the introduction of this work and valid for the Alcubierre warp drive making it physically impossible independently from arbitrary lower levels of negative energy do not affect the Natario warp drive which is perfectly possible to be achieved.This was the main reason behind our interest in the derivatives of a particular form of the shape function for the Natario warp drive spacetime.

The Natario warp drive once created can survive against all the obstacles pointed as physical impossibilities that rules out the warp drive as a dynamical spacetime.(see Appendices $K$,$L$,$M$ and $N$)

Lastly and in order to terminate this work:There exists another problem not covered here:the fact that we still dont know how to generate the negative energy density and negative mass and above everything else we dont know how to generate the shape function that distorts the spacetime geometry creating the warp drive effect.So unfortunately all the discussions about warp drives are still under the domain of the mathematical conjectures.

However we are confident to affirm that the Natario warp drive will survive the passage of the Century $XXI$ and will arrive to the Future.The Natario warp drive as the first Human candidate for faster than light interstellar space travel will arrive to the the Century $XXIV$ on-board the future starships up there in the middle of the stars helping the human race to give his first steps in the exploration of our Galaxy

Live Long And Prosper

# 7 Appendix A:The Natario Warp Drive Negative Energy Density in Cartezian Coordinates

The negative energy density according to Natario is given by(see pg 5 in [2])[30]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{94}$$

In the bottom of pg 4 in [2] Natario defined the x-axis as the polar axis.In the top of page 5 we can see that $x = rs\cos(\theta)$ implying in $\cos(\theta) = \frac{x}{rs}$ and in $\sin(\theta) = \frac{y}{rs}$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{95}$$

Considering motion in the equatorial plane of the Natario warp bubble (x-axis only) then $[y^2 + z^2] = 0$ and $rs^2 = [(x - xs)^2]$ and making $xs = 0$ the center of the bubble as the origin of the coordinate frame for the motion of the Eulerian observer then $rs^2 = x^2$ because in the equatorial plane $y = z = 0$.

Rewriting the Natario negative energy density in cartezian coordinates in the equatorial plane we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{96}$$

[30] $n(rs)$ is the Natario shape function.Equation written in the Geometrized System of Units $c = G = 1$

# 8 Appendix B:Mathematical Demonstration of the Natario Warp Drive Equation using the Natario Vector $nX$ for a constant speed $vs$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [2])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{97}$$

where $X^i$ is the so-called shift vector.This shift vector is the responsible for the warp drive behavior defined as follows(pg 2 in [2]):

$$X^i = X, Y, Z \curvearrowright i = 1, 2, 3 \tag{98}$$

The warp drive spacetime is completely generated by the Natario vector $nX$(pg 2 in [2])

$$nX = X^i \frac{\partial}{\partial x^i} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}, \tag{99}$$

Defined using the canonical basis of the Hodge Star in spherical coordinates as follows(pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \tag{100}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \tag{101}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \tag{102}$$

Redefining the Natario vector $nX$ as being the rate-of-strain tensor of fluid mechanics as shown below(pg 5 in [2]):

$$nX = X^r \mathbf{e}_r + X^\theta \mathbf{e}_\theta + X^\varphi \mathbf{e}_\varphi \tag{103}$$

$$nX = X^r dr + X^\theta r d\theta + X^\varphi r\sin\theta d\varphi \tag{104}$$

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{105}$$

$$X^i = r, \theta, \varphi \curvearrowright i = 1, 2, 3 \tag{106}$$

We are interested only in the coordinates $r$ and $\theta$ according to pg 5 in [2])

$$ds^2 = dt^2 - (dr - X^r dt)^2 - (rd\theta - X^\theta dt)^2 \tag{107}$$

$$(dr - X^r dt)^2 = dr^2 - 2X^r dr dt + (X^r)^2 dt^2 \tag{108}$$

$$(rd\theta - X^{\theta}dt)^2 = r^2 d\theta^2 - 2X^{\theta} r d\theta dt + (X^{\theta})^2 dt^2 \tag{109}$$

$$ds^2 = dt^2 - (X^r)^2 dt^2 - (X^{\theta})^2 dt^2 + 2X^r dr dt + 2X^{\theta} r d\theta dt - dr^2 - r^2 d\theta^2 \tag{110}$$

$$ds^2 = [1 - (X^r)^2 - (X^{\theta})^2] dt^2 + 2[X^r dr + X^{\theta} r d\theta] dt - dr^2 - r^2 d\theta^2 \tag{111}$$

making $r = rs$ we have the Natario warp drive equation:

$$ds^2 = [1 - (X^{rs})^2 - (X^{\theta})^2] dt^2 + 2[X^{rs} drs + X^{\theta} rs d\theta] dt - drs^2 - rs^2 d\theta^2 \tag{112}$$

According with the Natario definition for the warp drive using the following statement(pg 4 in [2]):any Natario vector $nX$ generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of $rs$ defined by Natario as the interior of the warp bubble and $nX = -vs(t)dx$ or $nX = vs(t)dx$ with $X = vs$ for a large value of $rs$ defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble

The expressions for $X^{rs}$ and $X^{\theta}$ are given by:(see pg 5 in [2])

$$nX \sim -2v_s n(rs) \cos\theta \mathbf{e}_{rs} + v_s(2n(rs) + (rs)n'(rs)) \sin\theta \mathbf{e}_{\theta} \tag{113}$$

$$nX \sim 2v_s n(rs) \cos\theta \mathbf{e}_{rs} - v_s(2n(rs) + (rs)n'(rs)) \sin\theta \mathbf{e}_{\theta} \tag{114}$$

$$nX \sim -2v_s n(rs) \cos\theta drs + v_s(2n(rs) + (rs)n'(rs)) \sin\theta rs d\theta \tag{115}$$

$$nX \sim 2v_s n(rs) \cos\theta drs - v_s(2n(rs) + (rs)n'(rs)) \sin\theta rs d\theta \tag{116}$$

But we already know that the Natario vector $nX$ is defined by(pg 2 in [2]):

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{117}$$

Hence we should expect for:

$$X^{rs} = -2v_s n(rs) \cos\theta \tag{118}$$

$$X^{rs} = 2v_s n(rs) \cos\theta \tag{119}$$

$$X^{\theta} = v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{120}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{121}$$

We are interested in the two-dimensional $1+1$ version of the Natario warp drive in the dimensions $rs$ and $t$ (motion over the $x - axis$ only with $\theta = 0$ $cos\theta = 1$ and $\sin\theta = 0$) given by:

$$ds^2 = [1 - (X^{rs})^2] dt^2 + 2X^{rs} drs dt - drs^2 \tag{122}$$

$$X^{rs} = 2v_s n(rs) \tag{123}$$

# 9 Appendix C:Differential Forms,Hodge Star and the Mathematical Demonstration of the Natario Vectors $nX = -vsdx$ and $nX = vsdx$ for a constant speed $vs$

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector $nX$

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{124}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{125}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{126}$$

From above we get the following results

$$dr \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{127}$$

$$rd\theta \sim r\sin\theta (d\varphi \wedge dr) \tag{128}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{129}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(pg 8 in [22]):

$$*dr = r^2 \sin\theta (d\theta \wedge d\varphi) \tag{130}$$

$$*rd\theta = r\sin\theta (d\varphi \wedge dr) \tag{131}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{132}$$

Back again to the Natario equivalence between spherical and cartezian coordinates(pg 5 in [2]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2 \sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2} r^2 \sin^2\theta d\varphi\right) \tag{133}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{134}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta rd\theta \tag{135}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{136}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi\wedge dr)] \tag{137}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] - [r\sin^2\theta(d\varphi\wedge dr)] \tag{138}$$

We know that the following expression holds true(see pg 9 in [21]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{139}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] + [r\sin^2\theta(dr\wedge d\varphi)] \tag{140}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [2]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{141}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{142}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{143}$$

According to pg 10 in [21] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{144}$$

Because and according to pg 10 in [21]:

$$d(\alpha+\beta) = d\alpha + d\beta \tag{145}$$

$$d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{146}$$

$$d(dx) = d(dy) = d(dz) = 0 \tag{147}$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta\wedge dd\varphi = 2 sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{148}$$

$$*[d(r^2)d\varphi] = 2rdr\wedge d\varphi + r^2\wedge dd\varphi = 2r(dr\wedge d\varphi) \tag{149}$$

And then we derived again the Natario result of pg 5 in [2]

$$r^2\sin\theta\cos\theta(d\theta\wedge d\varphi) + r\sin^2\theta(dr\wedge d\varphi) \tag{150}$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [2] except that we replaced $\frac{1}{2}$ by the function $f(r)$ :

$$*d[f(r)r^2\sin^2\theta d\varphi] \tag{151}$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \tag{152}$$

$$f(r)r^2 2sin\theta\cos\theta(d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{153}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{154}$$

Comparing the above expressions with the Natario definitions of pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta)\wedge(r\sin\theta d\varphi) \sim r^2\sin\theta(d\theta\wedge d\varphi) \tag{155}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi)\wedge dr \sim r\sin\theta(d\varphi\wedge dr) \sim -r\sin\theta(dr\wedge d\varphi) \tag{156}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr\wedge(rd\theta) \sim r(dr\wedge d\theta) \tag{157}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(d\theta\wedge d\varphi)] + 2f(r)\ sin\theta[r\sin\theta(dr\wedge d\varphi)] + f'(r)r\sin\theta[r\sin\theta(dr\wedge d\varphi)] \tag{158}$$

$$2f(r)\ cos\theta e_r - 2f(r)\ sin\theta e_\theta - rf'(r)\sin\theta e_\theta \tag{159}$$

$$*d[f(r)r^2\sin^2\theta d\varphi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{160}$$

Defining the Natario Vector as in pg 5 in [2] with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{161}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{162}$$

We can get finally the latest expressions for the Natario Vector $nX$ also shown in pg 5 in [2]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{163}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{164}$$

With our pedagogical approaches

$$nX = 2vs(t)f(r)\ cos\theta dr - vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{165}$$

$$nX = -2vs(t)f(r)\ cos\theta dr + vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{166}$$

# 10 Appendix D:Solution of the Quadratic Form $ds^2 = 0$ for the Horizon problem in the Alcubierre Warp Drive in a $1+1$ Spacetime

The equation of the Alcubierre warp drive spacetime in $1+1$ dimensions is given by:(see Appendix $H$)

$$ds^2 = [1 - (vsf(rs))^2]dt^2 + 2vsf(rs)dxdt - dx^2 \tag{167}$$

Inserting the condition of the null-like geodesics ($ds^2 = 0$) in the equation of the Alcubierre warp drive spacetime in $1+1$ dimensions we have:

$$ds^2 = 0 \dashrightarrow [1 - (vsf(rs))^2]dt^2 + 2vsf(rs)dxdt - dx^2 = 0 \tag{168}$$

$$[1 - (vsf(rs))^2] + 2vsf(rs)\frac{dx}{dt} - (\frac{dx}{dt})^2 = 0 \tag{169}$$

$$U = \frac{dx}{dt} \tag{170}$$

$$[1 - (vsf(rs))^2] + 2vsf(rs)U - U^2 = 0 \tag{171}$$

$$U^2 - 2vsf(rs)U - [1 - (vsf(rs))^2] = 0 \tag{172}$$

$$U = \frac{2vsf(rs) \pm \sqrt{4(vsf(rs))^2 + 4[1 - (vsf(rs))^2]}}{2} \tag{173}$$

$$U = \frac{2vsf(rs) \pm 2}{2} \tag{174}$$

$$U = \frac{dx}{dt} = vsf(rs) \pm 1 \tag{175}$$

From above we can see that we have two solutions:one for the photon sent towards the front of the ship($-$ sign) and another for the photon sent to the rear of the ship($+$ sign).

$$U_{front} = \frac{dx_{front}}{dt} = vsf(rs) - 1 \tag{176}$$

$$U_{rear} = \frac{dx_{rear}}{dt} = vsf(rs) + 1 \tag{177}$$

# 11 Appendix E:Solution of the Quadratic Form $ds^2 = 0$ for the Horizon problem in the Natario Warp Drive in a $1 + 1$ Spacetime

The equation of the Natario warp drive spacetime in $1 + 1$ dimensions is given by:(see Appendix $B$)

$$ds^2 = [1 - (X^{rs})^2]dt^2 + 2X^{rs}drsdt - drs^2 \tag{178}$$

Inserting the condition of the null-like geodesics ($ds^2 = 0$) in the equation of the Natario warp drive spacetime in $1 + 1$ dimensions we have:

$$ds^2 = 0 \dashrightarrow [1 - (X^{rs})^2]dt^2 + 2X^{rs}drsdt - drs^2 = 0 \tag{179}$$

$$[1 - (X^{rs})^2] + 2X^{rs}\frac{drs}{dt} - (\frac{drs}{dt})^2 = 0 \tag{180}$$

$$U = \frac{drs}{dt} \tag{181}$$

$$[1 - (X^{rs})^2] + 2X^{rs}U - U^2 = 0 \tag{182}$$

$$U^2 - 2X^{rs}U - [1 - (X^{rs})^2] = 0 \tag{183}$$

$$U = \frac{2X^{rs} \pm \sqrt{4(X^{rs})^2 + 4[1 - (X^{rs})^2]}}{2} \tag{184}$$

$$U = \frac{2X^{rs} \pm 2}{2} \tag{185}$$

$$U = \frac{drs}{dt} = X^{rs} \pm 1 \tag{186}$$

From above we can see that we have two solutions:one for the photon sent towards the front of the ship($-$ sign) and another for the photon sent to the rear of the ship($+$ sign).

$$U_{front} = \frac{drs_{front}}{dt} = X^{rs} - 1 \tag{187}$$

$$U_{rear} = \frac{drs_{rear}}{dt} = X^{rs} + 1 \tag{188}$$

$$X^{rs} = 2v_s n(rs) \tag{189}$$

## 12 Appendix F:Mathematical Demonstration of the Horizon Equation in the Natario Warp Drive for a $1+1$ Spacetime

An Horizon occurs every time we get:(see Appendix $E$)[31]

$$U_{front} = \frac{drs_{front}}{dt} = X^{rs} - 1 = 2v_s n(rs) - 1 = 0 \tag{190}$$

The value of the $n(rs)$ in the Horizon is given by:

$$2v_s n(rs) - 1 = 0 \dashrightarrow 2v_s n(rs) = 1 \tag{191}$$

$$n(rs) = \frac{1}{2vs} = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{192}$$

$$\frac{1}{vs} = [1 - f(rs)^{WF}]^{WF} \tag{193}$$

$$\sqrt[WF]{\frac{1}{vs}} = 1 - f(rs)^{WF} \tag{194}$$

$$f(rs)^{WF} = 1 - \sqrt[WF]{\frac{1}{vs}} \tag{195}$$

$$f(rs) = \sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}} = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{196}$$

$$1 - tanh[@(rs - R)] = 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}} \tag{197}$$

$$tanh[@(rs - R)] = 1 - 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}} \tag{198}$$

$$@(rs - R) = arctanh(1 - 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}}) \tag{199}$$

$$rs - R = \frac{1}{@}arctanh(1 - 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}}) \tag{200}$$

$$rs = R + \frac{1}{@}arctanh(1 - 2\sqrt[WF]{1 - \sqrt[WF]{\frac{1}{vs}}}) \tag{201}$$

[31] When reading this appendix remember that $WF >> 1$ and $vs >> 1$

# 13 Appendix G:Mathematical Demonstration of the Horizon Equation in the Alcubierre Warp Drive for a $1+1$ Spacetime

An Horizon occurs every time we get:(see Appendix $D$)

$$\frac{dx_{front}}{dt} = vsf(rs) - 1 = 0 \tag{202}$$

The value of the $f(rs)$ in the Horizon is given by:

$$vsf(rs) - 1 = 0 \dashrightarrow f(rs) = \frac{1}{vs} = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{203}$$

$$\frac{2}{vs} = 1 - tanh[@(rs - R)] \tag{204}$$

$$tanh[@(rs - R)] = 1 - \frac{2}{vs} \tag{205}$$

$$@(rs - R) = arctanh(1 - \frac{2}{vs}) \tag{206}$$

$$rs - R = \frac{1}{@}arctanh(1 - \frac{2}{vs}) \tag{207}$$

$$rs = R + \frac{1}{@}arctanh(1 - \frac{2}{vs}) \tag{208}$$

# 14 Appendix H:Mathematical Demonstration of the Alcubierre Warp Drive Equation for a $1 + 1$ Spacetime

The Alcubierre warp drive equation in four-dimensions using signature $(+, -, -, -)$[32] is given by:(see eq 8 pg 4 in [1],eq 1 pg 3 in [3] and eq 1 pg 3 in [4])

$$ds^2 = dt^2 - [dx - vsf(rs)dt]^2 - dy^2 - dz^2 \tag{209}$$

Hiscock uses the equation of the $1 + 1$ dimensional version of the Alcubierre warp drive metric where $[y^2 + z^2] = 0$ implying in $[dy + dz] = 0$ (see pg 3 and 4 in [4]) given by:

$$ds^2 = dt^2 - [dx - vsf(rs)dt]^2 \tag{210}$$

Expanding the squared term we have:

$$[dx - vsf(rs)dt]^2 = dx^2 - 2vsf(rs)dxdt + vs^2f(rs)^2dt^2 \tag{211}$$

Inserting these expanded terms in the main equation we get:

$$ds^2 = dt^2 - [dx^2 - 2vsf(rs)dxdt + vs^2f(rs)^2dt^2] \tag{212}$$

$$ds^2 = dt^2 - dx^2 + 2vsf(rs)dxdt - vs^2f(rs)^2dt^2] \tag{213}$$

Rearranging the common algebraic terms we get the $1 + 1$ dimensional Alcubierre warp drive equation in its expanded form in signature $(+, -, -, -)$ as given below:(see eq 7 pg 4 in [4])

$$ds^2 = [1 - vs^2f(rs)^2]dt^2 + 2vsf(rs)dxdt - dx^2 \tag{214}$$

[32] Alcubierre,Ford-Pfenning and Hiscock used the signature $(-, +, +, +)$.This comment is meant for introductory readers

## 15 Appendix I:Infinite Doppler Blueshift in the Horizon for the Alcubierre Warp Drive in a $1+1$ Spacetime

Defining the following term $X$ as being:

$$X = vsf(rs) \tag{215}$$

We know that in the Horizon the value of $f(rs)$ is given by(see Appendix $G$):

$$f(rs) = \frac{1}{vs} \tag{216}$$

Then we have for $X$:

$$X = vsf(rs) = 1 \tag{217}$$

Also for the Horizon we have:

$$\frac{dx}{dt} = 0 \tag{218}$$

The photon stops!!!
From Appendix $D$ we know that:

- 1)- photon sent towards the front of the Warp Bubble $\frac{dx}{dt} = X - 1 = vsf(rs) - 1$
- 2)- photon sent towards the rear of the Warp Bubble $\frac{dx}{dt} = X + 1 = vsf(rs) + 1$

Or even better:

- 1)- photon sent towards the front of the Warp Bubble $\frac{dx}{dt} - X = \frac{dx}{dt} - vsf(rs) = -1$
- 2)- photon sent towards the rear of the Warp Bubble $\frac{dx}{dt} + X = \frac{dx}{dt} + vsf(rs) = +1$

Then we can easily see that(pg 6 in [2]):

$$\parallel \frac{dx}{dt} - X \parallel = 1 \tag{219}$$

$$\parallel \frac{dx}{dt} - vsf(rs) \parallel = 1 \tag{220}$$

Defining now the term $n$ in function of $X$ we get the following values for $n$:

- 1)-photon sent to the front of the bubble

$$n = \frac{dx}{dt} - X = \frac{dx}{dt} - vsf(rs) = -1 \dashrightarrow vsf(rs) = 1 \dashrightarrow \frac{dx}{dt} = 0 \tag{221}$$

- 1)-photon sent to the rear of the bubble

$$n = \frac{dx}{dt} + X = \frac{dx}{dt} + vsf(rs) = +1 \dashrightarrow vsf(rs) = 1 \dashrightarrow \frac{dx}{dt} = 0 \tag{222}$$

We are interested in the results for the photon sent to the front of the bubble.

We are ready now to examine the topic of the Infinite Doppler Blueshifts in the Horizon suffered by photons sent towards the front of the Warp Bubble.See pg 6 and 8 in [2].

The unit vector $n$ defined below represents the direction of the corresponding light ray from the point of view of the Eulerian observer outside the warp bubble.(See pg 6 in [2])

$$n = \frac{dx}{dx} - X = \frac{dx}{dx} - vsf(rs) \tag{223}$$

In the Horizon ($\frac{dx}{dx} = 0$) and ($X = vsf(rs) = 1$)
Then we have:

$$n = \frac{dx}{dx} - X = \frac{dx}{dx} - vsf(rs) = 0 - 1 = -1 \tag{224}$$

The equation of the observed energy is given by(See pg 8 in [2]):

$$E_0 = E(1 + n.X) \tag{225}$$

So when a photon reaches the Horizon we have the following conditions: $n = -1$ and $X = vsf(rs) = 1$.Inserting these values in the equation of the energy(pg 8 in [2]) we have:

$$E = \frac{E_0}{(1 + n.X)} = \frac{E_0}{(1 + -1.1)} = \frac{E_0}{0} \tag{226}$$

And then we have the Infinite Doppler Blueshift in the Horizon as mentioned by Natario in pg 8 in [2].

# 16 Appendix J:Infinite Doppler Blueshift in the Horizon for the Natario Warp Drive in a $1+1$ Spacetime

Defining the following term $X$ as being:

$$X = 2vsn(rs) \tag{227}$$

We know that in the Horizon the value of $n(rs)$ is given by(see Appendix $F$):

$$n(rs) = \frac{1}{2vs} \tag{228}$$

Then we have for $X$:

$$X = 2vsn(rs) = 1 \tag{229}$$

Also for the Horizon we have:

$$\frac{dx}{dt} = 0 \tag{230}$$

The photon stops!!!
From Appendix $E$ we know that:

- 1)- photon sent towards the front of the Warp Bubble $\frac{dx}{dt} = X - 1 = 2vsn(rs) - 1$
- 2)- photon sent towards the rear of the Warp Bubble $\frac{dx}{dt} = X + 1 = 2vsn(rs) + 1$

Or even better:

- 1)- photon sent towards the front of the Warp Bubble $\frac{dx}{dt} - X = \frac{dx}{dt} - 2vsn(rs) = -1$
- 2)- photon sent towards the rear of the Warp Bubble $\frac{dx}{dt} + X = \frac{dx}{dt} + 2vsn(rs) = +1$

Then we can easily see that(pg 6 in [2]).:

$$\parallel \frac{dx}{dt} - X \parallel = 1 \tag{231}$$

$$\parallel \frac{dx}{dt} - 2vsn(rs) \parallel = 1 \tag{232}$$

Defining now the term $n$ in function of $X$ we get the following values for $n$:

- 1)-photon sent to the front of the bubble

$$n = \frac{dx}{dt} - X = \frac{dx}{dt} - 2vsn(rs) = -1 \dashrightarrow 2vsn(rs) = 1 \dashrightarrow \frac{dx}{dt} = 0 \tag{233}$$

- 1)-photon sent to the rear of the bubble

$$n = \frac{dx}{dt} + X = \frac{dx}{dt} + 2vsn(rs) = +1 \dashrightarrow 2vsn(rs) = 1 \dashrightarrow \frac{dx}{dt} = 0 \tag{234}$$

We are interested in the results for the photon sent to the front of the bubble.

We are ready now to examine the topic of the Infinite Doppler Blueshifts in the Horizon suffered by photons sent towards the front of the Warp Bubble.See pg 6 and 8 in [2].

The unit vector $n$ defined below represents the direction of the corresponding light ray from the point of view of the Eulerian observer outside the warp bubble.(See pg 6 in [2]).

$$n = \frac{dx}{dx} - X = \frac{dx}{dx} - 2vsn(rs) \tag{235}$$

In the Horizon ($\frac{dx}{dx} = 0$) and ($X = 2vsn(rs) = 1$)
Then we have:

$$n = \frac{dx}{dx} - X = \frac{dx}{dx} - 2vsn(rs) = 0 - 1 = -1 \tag{236}$$

The equation of the observed energy is given by(See pg 8 in [2]):

$$E_0 = E(1 + n.X) \tag{237}$$

So when a photon reaches the Horizon we have the following conditions: $n = -1$ and $X = 2vsn(rs) = 1$.Inserting these values in the equation of the energy(pg 8 in [2]) we have:

$$E = \frac{E_0}{(1 + n.X)} = \frac{E_0}{(1 + -1.1)} = \frac{E_0}{0} \tag{238}$$

And then we have the Infinite Doppler Blueshift in the Horizon as mentioned by Natario in pg 8 in [2].

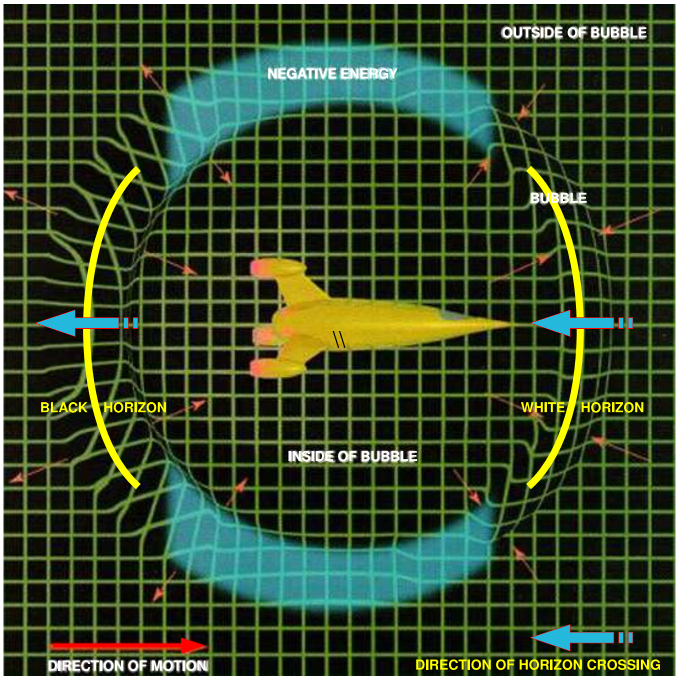


Figure 1: Artistic representation of the Energy Density distribution in the Alcubierre Warp Drive . (Source:fig 2 pg 4 in Barcelo,Finazzi and Liberati [11])

# 17 Appendix K:Artistic Presentation of the Energy Density distribution in the Alcubierre Warp Drive

Above is being presented the artistic graphical presentation of the energy density for the Alcubierre warp drive (SEMT) given by the following expressions(pg 4 in [2]):

$$\rho = -\frac{1}{32\pi} v_s^2 \left[f'(r_s)\right]^2 [\frac{y^2+z^2}{r_s^2}] \tag{239}$$

$$\rho = -\frac{1}{32\pi} v_s^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2+z^2}{r_s^2}] \tag{240}$$

Note that the negative energy density is located on a toroidal region above and below the ship and perpendicular to the direction of motion due to the term $y^2+z^2 \neq 0$.The front of the ship is "empty" space where the contraction of spacetime in front occurs. There are no negative energy densities in front of the ship in the Alcubierre warp drive

In front of the ship in the $x-axis$ only (a $1+1$ dimensional Alcubierre warp drive spacetime) we have $y^2+z^2=0$ and then the Alcubierre energy density will have the following value:

$$\rho = -\frac{1}{32\pi} v_s^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2+z^2}{r_s^2}] = 0!!!! \tag{241}$$

Note that an observer stationary inside an Alcubierre warp drive that moves with a superluminal speed $vs > 1$ with respect to the rest of the Universe[33] will suffer from the Horizons and Doppler blueshift problems because there are no negative energies in front of the ship to "deflect" photons.This is the reason why the Hawking temperatures that compromises the physical stability of the Alcubierre warp drive(see pgs 6 and 7 in [11]) cannot be avoided .

So Barcelo,Finazzi and Liberati are right when they say the Alcubierre warp drive in a $1 + 1$ spacetime is quantically unstable but this is not due to a divergence in the (SEMT) which is zero but it is due to impacts with the hazardous interstellar matter (particles of space dust,Doppler Blueshifted photons molecules of gas etc) since there is nothing in front of the ship to protect the crew.

The blue arrow in front of the ship pointing towards the ship in fig 1 represents the hazardous interstellar matter entering in the bubble and this will compromise the integrity of the ship

Although Natario and Alcubierre warp drive spacetimes are members of the same family of the Einstein Field Equations of General Relativity,due to the different geometrical distribution of the negative energy densities ,one(Natario) will assume a different behavior when compared to the other(Alcubierre)(see Appendices $L$,$M$ and $N$).

[33] Observer at rest inside the bubble $X = 0$ while outside the bubble $X = vs$

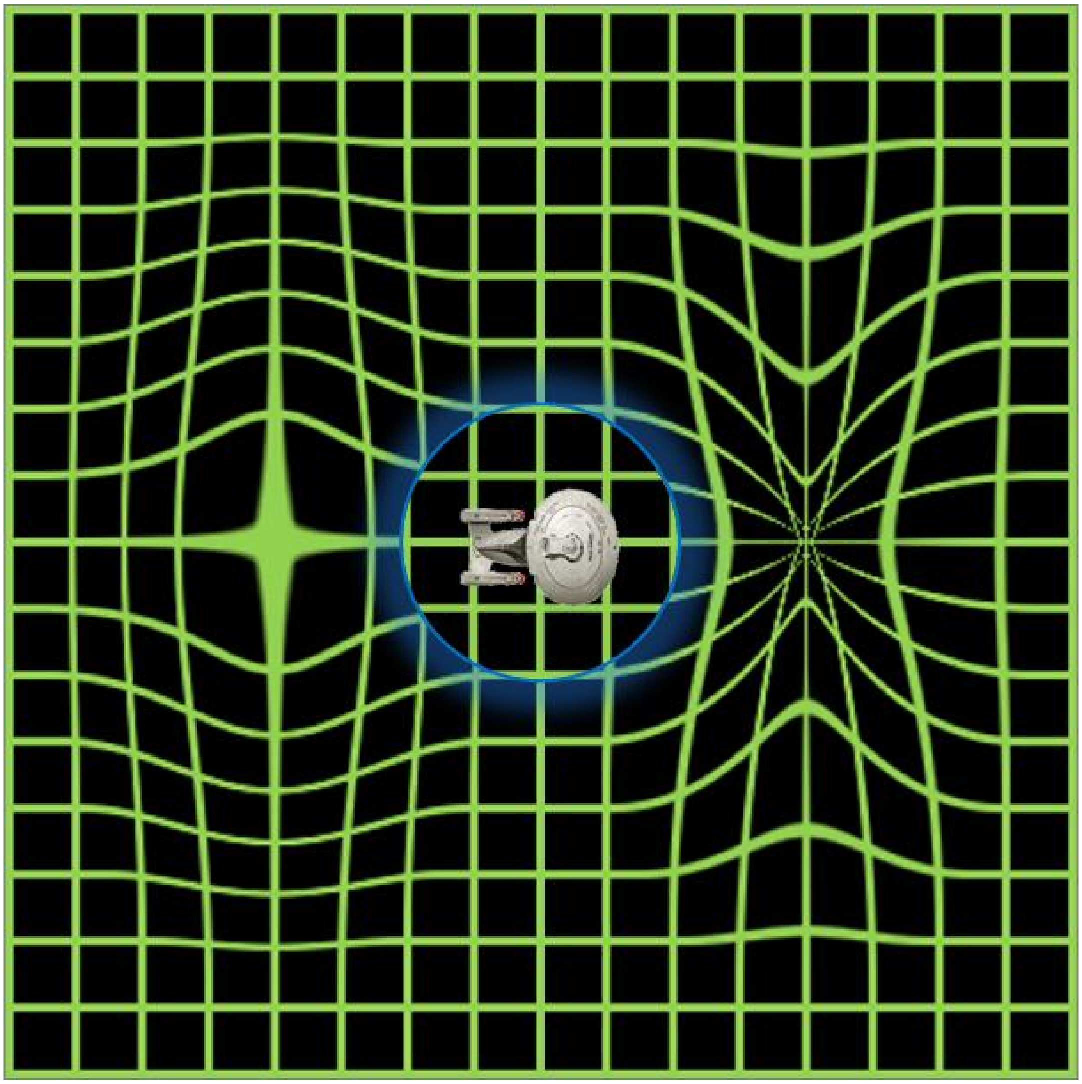

Figure 2: Artistic representation of the Natario Warp Bubble .(Source:Internet)

# 18 Appendix L:Artistic Presentation of the Natario Warp Bubble

According to the Natario definition for the warp drive using the following statement(pg 4 in [2]):

- 1)-Any Natario vector $nX$ generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of $rs$ defined by Natario as the interior of the bubble and $nX = -vs(t)dx$ or $nX = vs(t)dx$ with $X = vs$ for a large value of $rs$ defined by Natario as the exterior of the bubble with $vs(t)$ being the speed of the bubble(pg 5 in [2]).

  The blue region is the Natario warped region(bubble walls) where the negative energy is located.

  Note that the negative energy surrounds entirely the "spaceship" inside the bubble deflecting incoming hazardous interstellar objects protecting the crew.(see Appendix $M$).

A given Natario vector $nX$ generates a Natario warp drive Spacetime if and only if satisfies these conditions stated below:

- 1)-A Natario vector $nX$ being $nX = 0$ for a small value of $rs$(interior of the bubble)
- 2)-A Natario vector $nX = -Xdx$ or $nX = Xdx$ for a large value of $rs$(exterior of the bubble)
- 3)-A shift vector $X$ depicting the speed of the bubble being $X = 0$(interior of the bubble) while $X = vs$ seen by distant observers(exterior of the bubble).

The Natario vector $nX$ is given by:

$$nX = -v_s(t)d\left[n(rs)rs^2\sin^2\theta d\varphi\right] \sim -2v_s n(rs)\cos\theta drs + v_s(2n(rs) + rsn'(rs))rs\sin\theta d\theta \tag{242}$$

This holds true if we set for the Natario vector $nX$ a continuous Natario shape function being $n(rs) = \frac{1}{2}$ for large $rs$(outside the bubble) and $n(rs) = 0$ for small $rs$(inside the bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the bubble(pg 5 in [2])

The Natario vector $nX = -vs(t)dx = 0$ vanishes inside the bubble because inside the bubble there are no motion at all because $dx = 0$ or $n(rs) = 0$ or $X = 0$ while being $nX = -vs(t)dx \neq 0$ not vanishing outside the bubble because $n(rs)$ do not vanish.Then an external observer would see the bubble passing by him with a speed defined by the shift vector $X = -vs(t)$ or $X = vs(t)$.

The "spaceship" above lies in the interior of the bubble at the rest $X = vs = 0$ but the observer outside the bubble sees the "spaceship" passing by him with a speed $X = vs$.

See also pgs 14,15 and 16 in [20] and pgs 7,8 and 9 in [2] for more graphical presentations of the Natario warp bubble

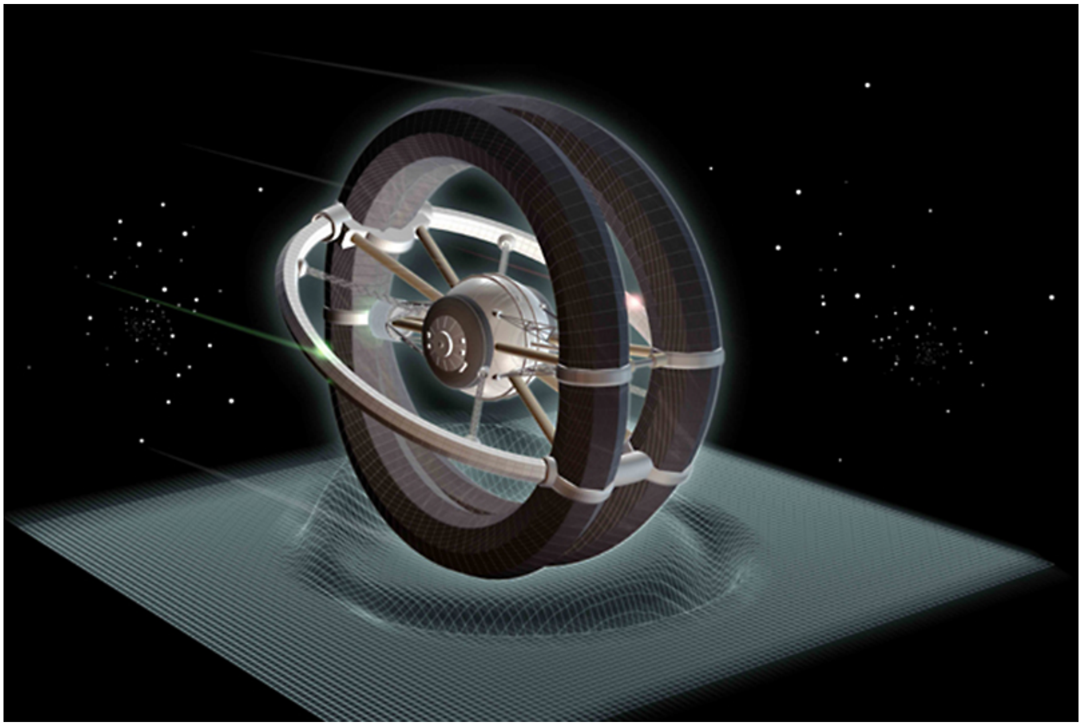

Figure 3: Artistic representation of the Natario Warp Drive .Note the Alcubierre Expansion of the Normal Volume Elements below.(Source:Internet)

# 19 Appendix M:Artistic Presentation of the Natario Warp Drive

Note that according to the geometry of the Natario warp drive the spacetime contraction in one direction(radial) is balanced by the spacetime expansion in the remaining direction(perpendicular).

Remember also that the expansion of the normal volume elements in the Natario warp drive is given by the following expressions(pg 5 in [2]). :

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s n'(r)\cos\theta \tag{243}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = v_s n'(r)\cos\theta; \tag{244}$$

$$K_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi} + \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} = v_s n'(r)\cos\theta \tag{245}$$

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{246}$$

If we expand the radial direction the perpendicular direction contracts to keep the expansion of the normal volume elements equal to zero.

This figure is a pedagogical example of the graphical presentarion of the Natario warp drive.

The "bars" in the figure were included to illustrate how the expansion in one direction can be counterbalanced by the contraction in the other directions.These "bars" keeps the expansion of the normal volume elements in the Natario warp drive equal to zero.

Note also that the graphical presentation of the Alcubierre warp drive expansion of the normal volume elements according to fig 1 pg 10 in [1] is also included

Note also that the energy density in the Natario Warp Drive being given by the following expressions(pg 5 in [2]):

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[ 3(n'(r))^2 \cos^2\theta + \left( n'(r) + \frac{r}{2} n''(r) \right)^2 \sin^2\theta \right]. \tag{247}$$

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[ 3(\frac{dn(r)}{dr})^2 \cos^2\theta + \left( \frac{dn(r)}{dr} + \frac{r}{2} \frac{d^2 n(r)}{dr^2} \right)^2 \sin^2\theta \right]. \tag{248}$$

Is being distributed around all the space involving the ship(above the ship $\sin\theta = 1$ and $\cos\theta = 0$ while in front of the ship $\sin\theta = 0$ and $\cos\theta = 1$).The negative energy in front of the ship "deflect" photons so these will not reach the Horizon and the Natario warp drive will not suffer from Doppler blueshifts.The illustrated "bars" are the obstacles that deflects photons or incoming particles from outside the bubble never allowing these to reach the interior of the bubble(see Appendix $N$)

- )-Energy directly above the ship($y - axis$)

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[ \left( \frac{dn(r)}{dr} + \frac{r}{2} \frac{d^2 n(r)}{dr^2} \right)^2 \sin^2\theta \right]. \tag{249}$$

- )-Energy directly in front of the ship($x - axis$)

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[ 3(\frac{dn(r)}{dr})^2 \cos^2\theta \right]. \tag{250}$$

Note that as fast as the ship goes by,then the negative energy density requirements grows proportionally to the square of the bubble speed $vs$.

For a bubble speed 200 times faster than light then ($vs = 6 \times 10^{10}$) and this affects the negative energy requirements by the factor $vs^2$ being ($vs^2 = 3,6 \times 10^{21}$) [34]

This raises the amount of negative energy to enormous levels making this a major concern when studying warp drive spacetimes.

The derivatives of the Natario shape function $n(rs)$ must be very low in order to obliterate the factor $vs^2$.

[34] the mass of the Earth according to Wikipedia is $M_{\oplus} = 5,9722 \times 10^{24}$ kilograms.In tons this reaches $10^{21}$ tons,exactly the factor $vs^2$ for 200 times light speed

Note also that even in a $1+1$ dimensional spacetime the Natario warp drive retains the zero expansion behavior:

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s n'(r)\cos\theta \tag{251}$$

$$K_{\theta\theta} = \frac{X^r}{r} = v_s n'(r)\cos\theta; \tag{252}$$

$$K_{\varphi\varphi} = \frac{X^r}{r} = v_s n'(r)\cos\theta \tag{253}$$

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{254}$$

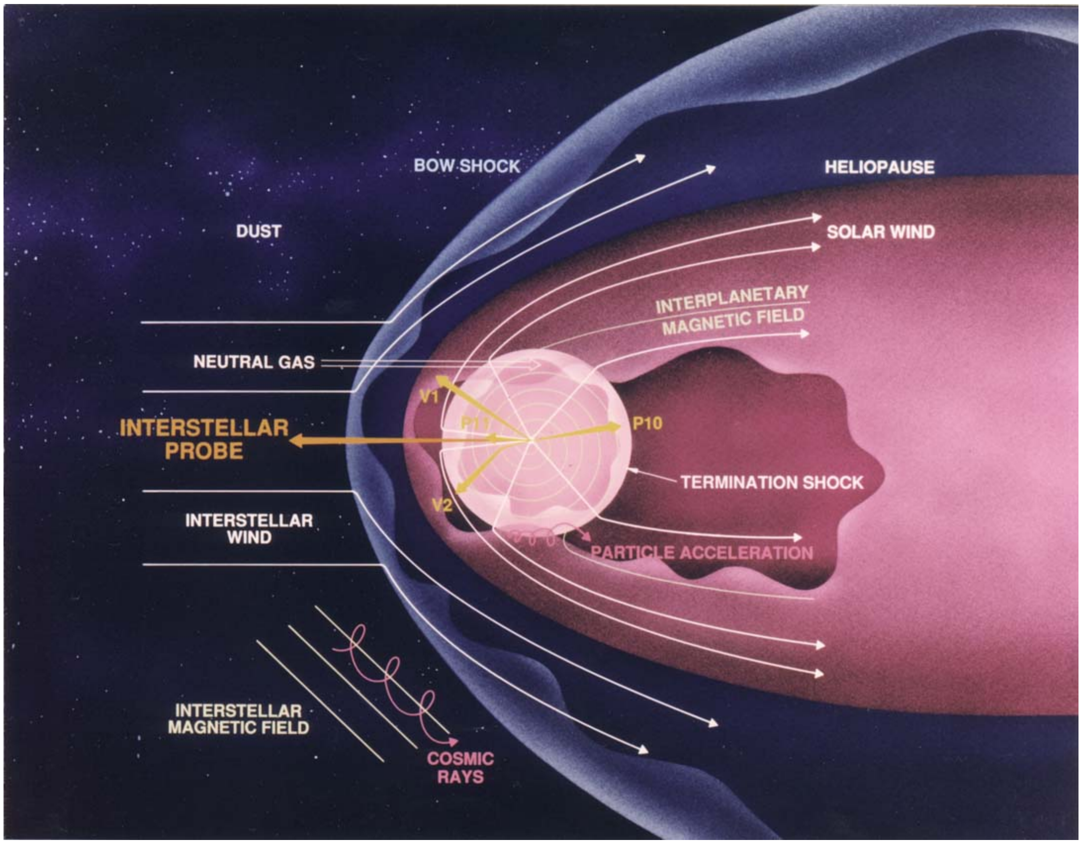


Figure 4: Artistic representation of a Natario Warp Drive in a real superluminal space travel .Note the negative energy in front of the ship deflecting incoming hazardous interstellar matter.(Source:Internet)

## 20 Appendix N:Artistic Presentation of a Natario Warp Drive In A Real Faster Than Light Interstellar Spaceflight

According to Clark,Hiscock and Larson a single collision between a ship at 200 times faster then light and a COBE photon would release an amount of energy equal to the photosphere of a star like the Sun.(see pg 11 in [9]).And how many photons of COBE we have per cubic centimeter of space??

These highly energetic collisions would pose a very serious threat to the astronauts as pointed out by McMonigal,Lewis and O'Byrne (see pg 10 in [10]).

Another problem:these highly energetic collisions would raise the temperature of the warp bubble reaching the Hawking temperature as pointed out by Barcelo,Finazzi and Liberati.(see pg 6 and 7 in [11]).At pg 9 they postulate that all future spaceships cannot bypass 99 percent of the light speed.

The statements pointed above are correct for the Alcubierre warp drive but not for the Natario one.Remember from Appendices $A$ and $M$ that the negative energy in front of the ship is not zero even for the $1+1$ dimensional Natario warp drive.

Still according with Natario in pg 7 before section 5.2 in [14] negative energy density means a negative mass density and a negative mass density generates a repulsive gravitational field. This repulsive gravitational field in front of the ship in the Natario warp drive spacetime protects the ship from impacts with the interstellar matter and also protects the ship from COBE photons highly Doppler Blueshifted approaching the ship from the front.So in the Natario warp drive the problem of the Hawking temperature will never happens.

Adapted from the negative energy in Wikipedia:The free Encyclopedia:

"if we have a small object with equal inertial and passive gravitational masses falling in the gravitational field of an object with negative active gravitational mass (a small mass dropped above a negative-mass planet, say), then the acceleration of the small object is proportional to the negative active gravitational mass creating a negative gravitational field and the small object would actually accelerate away from the negative-mass object rather than towards it." The Natario warp drive as a solution of the Einstein Field Equations of General Relativity that allows faster than light motion is the first valid candidate for interstellar space travel.

In order to terminate we would like to outline the following statements:

- 1)-The objections raised by Clark,Hiscock,Larson,McMonigal,Lewis,O'Byrne,Barcelo,Finazzi and Liberati (pg 11 in [9])(pg 10 in [10])(pg 2 in [19])(pgs 6 and 7 in [11])) are not valid for the Natario warp drive spacetime.The negative energy density in front of the ship according to Natario in pg 7 before section 5.2 in [14] have a repulsive behavior that deflects all the hazardous interstellar matter and Doppler Blueshifted photons.

- 2)-So they cannot say the warp drive is impossible.

- 3)-At least they can say the Alcubierre warp drive is impossible because in front of the ship there exists nothing to deflect Doppler Blueshifted photons or incoming hazardous interstellar matter.

- 4)-At pg 9 in [11]) Barcelo,Finazzi and Liberati postulate that all future spaceships cannot bypass 99 percent of the light speed,This is true for the Alcubierre warp drive but not for the Natario one.

- 5)-When we study warp drives a clear difference between Natario and Alcubierre must be made or in short:($Natario \neq Alcubierre$).

## 21 Epilogue

- "The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C.Clarke[35]
- "The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein[36][37]

## 22 Remarks

Reference [15] was online at the time we picked it up for our records.It ceased to be online but we can provide a copy in PDF Acrobat reader of this reference for those interested.

Reference [16] we only have access to the abstract.

Reference [20] can be obtained from the web pages of Professor Jose Natario at Instituto Superior Tecnico Lisboa Portugal as long as the site remains on-line. [38] [39]

We performed all the numerical calculus of our simulations for both Alcubierre and Natario warp drive spacetimes using Microsoft Excel[40].We can provide our Excel files to those interested and although Excel is a licensed program there exists another program that can read Excel files available in the Internet as a freeware for those that perhaps may want to examine our files:the OpenOffice[41] at http://www.openoffice.org

[35]special thanks to Maria Matreno from Residencia de Estudantes Universitas Lisboa Portugal for providing the Second Law Of Arthur C.Clarke

[36]"Ideas And Opinions" Einstein compilation, ISBN $0-517-88440-2$, on page 226."Principles of Research" ([Ideas and Opinions],pp.224-227), described as "Address delivered in celebration of Max Planck's sixtieth birthday (1918) before the Physical Society in Berlin"

[37]appears also in the Eric Baird book Relativity in Curved Spacetime ISBN $978-0-9557068-0-6$

[38]http://www.math.ist.utl.pt/ jnatar/

[39]http://www.math.ist.utl.pt/ jnatar/seminario.pdf

[40]Copyright(R) by Microsoft Corporation

[41]Copyright(R) by Oracle Corporation

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