# Scale-Invariant Embeddings in a Riemannian Spacetime

PRELIMINARY DRAFT

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October 26, 2013

**Abstract**

A framework for calculations in a semi-*Riemann*ian space with the typical metric connection and curvature expressions is developed, with an emphasis on deriving them from an embedding function as a more fundamental object than the metric tensor.

The scale-invariant and 'linearizing' logarithmic nature of an 'infinitesimal embedding' of a tangent space into its neighbourhood is observed, and a composition scheme of spacetime scenarios from 'outer' non-invariant and 'inner' scale-invariant embeddings is briefly outlined.

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# 1 Framework

## 1.1 General Conventions

Tensors are written in index notation.
The *Einstein* summation convention is always active, unless noted otherwise.

More unusually, contraction indices may be doubled when unambiguous, like for example in ${\Gamma^a}_{\gamma\gamma} \cdot g^{\gamma\gamma}$, since at least one of the involved tensors is symmetric in that index pair.
Multiple indices of same variance are sometimes used to denote symmetries between all indices involved, as in ${\Gamma^a}_{bb,b}$, and might even be contracted only in part, as in ${\Gamma^a}_{\gamma\gamma,\gamma} \cdot g^{\gamma\gamma}$, as long as the operation is unambiguous.
Contraction might span over equations, $\left( A_{\delta\delta} = B_{\delta\delta} \right) g^{\delta\delta}$.

Index instances are printed in bold, like $(T_{\boldsymbol{t}}, T_{\boldsymbol{x}}, T_{\boldsymbol{y}}, T_{\boldsymbol{z}})$, when in $T_a$, $a \in \{\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$.

As usual, a comma before an index $(,\delta)$ might be used as a short form for the partial derivative ($\partial_\delta$ or $\frac{\partial}{\partial x_\delta}$), and a semicolon $(;\delta)$ for the covariant derivative $(\nabla_\delta)$. The partial derivative by an index instance may leave out the comma, $T_{\boldsymbol{t}} := T_{,\boldsymbol{t}} = \partial_{\boldsymbol{t}} T$.

A dot $(\cdot)$ denotes a product, but in index notation not a 'dot product', since tensor contractions are already signified by index notation, so ${A^a}_b \cdot {B^b}_c = {B^b}_c \cdot {A^a}_b$.

The term 'pro-symmetric' might be used in contrast to 'anti-symmetric'.

In matrices, zero elements may be left blank or replaced by a dot $(\cdot)$, see the null matrix,

$$0_{ab} := \begin{bmatrix} \cdot & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{bmatrix} .$$

The *Minkowski* metric is chosen with signature $(-,+,+,+)$, that is,

$$\eta_{ab} := \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} .$$

## 1.2 Embedding Functions and the *Jacobi* Matrix

Be there a vector-valued function $x^a \mapsto y^\mu = J^\mu(x^a)$, which converts 'inner' coordinates $x^a$ to 'outer' coordinates $y^\mu$ and can be seen as an embedding of an 'inner' space into an 'outer' space.[1]

The ***Jacobi* matrix** is the matrix of first partial derivatives of the embedding function, if the latter is explicitly given, *Jacobi* matrix

$$J^\mu{}_a := J^\mu{}_{,a} = \partial_a J^\mu .$$

In the special case that the embedding function is linear in all dimensions, the *Jacobi* matrix is constant and the embedding function is determined by the linear transformation

$$y^\mu = J^\mu(x^a) \stackrel{\star}{=} J^\mu{}_a \cdot x^a .$$

If the embedding function is not explicitly available, the *Jacobi* matrix itself is what to start with. In any case, it shall be the most 'fundamental tensor' within this framework, instead of the metric tensor.

The *Jacobi* matrix of a concatenation of embedding functions is the product of the *Jacobi* matrices of those functions, so the concatenation of two embeddings $A \cdot B$, and their inverse, $B^{-1} \cdot A^{-1}$, is

$$J^\mu{}_a := \overset{A}{J}{}^\mu{}_\gamma \cdot \overset{B}{J}{}^\gamma{}_a \qquad \Leftrightarrow \qquad J_\mu{}^a := \overset{B}{J}{}_\mu{}^\gamma \cdot \overset{A}{J}{}_\gamma{}^a ,$$

and with three and more consecutive embeddings with linearly increasing complexity,

$$J^\mu{}_a := \overset{A}{J}{}^\mu{}_\beta \cdot \overset{B}{J}{}^\beta{}_\gamma \cdot \overset{C}{J}{}^\gamma{}_a \qquad \Leftrightarrow \qquad J_\mu{}^a := \overset{C}{J}{}_\mu{}^\beta \cdot \overset{B}{J}{}_\beta{}^\gamma \cdot \overset{A}{J}{}_\gamma{}^a , \qquad \text{etc} \cdots$$

Note that in the case of modeling a *Minkowski* spacetime, the signature of the metric is not contained in the *Jacobi* matrix, but introduced from outside, as shown below, 1.3. infinitesimal

Consider an **infinitesimal embedding** from one point to its neighbour, or a map transition function of an atlas of infinitely many maps, one for each point on the manifold.

In this limit, the *Jacobi* matrix, and its inverse, are infinitesimally equal to the identity matrix,

$$J^\mu{}_a \stackrel{\star}{\to} \delta^\mu{}_a , \qquad J_\mu{}^a \stackrel{\star}{\to} \delta_\mu{}^a ,$$

but their derivatives need not vanish.

[1]similar [2, (9.3-4)], without explicitly naming the entity of derivatives

## 1.3 Metric Tensors

The first index of the *Jacobi* matrix, $\mu$, lives in the outer space. To lower it, Outer metric

$$J_{\mu a} := \overset{o}{g}_{\mu\nu} \cdot J^{\nu}{}_{a} ,$$

or raise it again, an **outer metric tensor** $\overset{o}{g}_{\mu\nu}$ must be given. In the simplest case it is a flat *Euclid*ean, $\overset{o}{g}_{\mu\nu} = \delta_{\mu\nu}$, or a *Minkowski* metric, $\overset{o}{g}_{\mu\nu} = \eta_{\mu\nu}$. But even when lowered, the first index still lives in the outer space and can't be contracted with indices from the inner space.

To '**transport**' a contravariant vector from the outer space, $y^{\mu}$, to one in the inner space, $x^{a}$, the *Jacobi* matrix itself, together with the outer metric is used,

$$x_{a} := \overset{o}{g}_{\mu\nu} \cdot J^{\mu}{}_{a} \cdot y^{\nu} .$$

In this way the first index of the *Jacobi* matrix can be transported and lowered, which results in the **(overall) metric tensor**[2] which is symmetric in its indices, as long as the outer metric is symmetric,[3] Overall metric

$$g_{ab} := \overset{o}{g}_{\mu\nu} \cdot J^{\mu}{}_{a} \cdot J^{\nu}{}_{b} , \qquad \text{or} \qquad g_{aa} = \overset{o}{g}_{\mu\mu} \cdot J^{\mu}{}_{a} \cdot J^{\mu}{}_{a} .$$

In the limit of an **infinitesimal embedding**, where $J^{\mu}{}_{a} \xrightarrow{\star} \delta^{\mu}{}_{a}$, the overall metric tensor reduces to the outer metric tensor,

$$g_{aa} \xrightarrow{\star} \overset{o}{g}_{aa} , \qquad g^{aa} \xrightarrow{\star} \overset{o}{g}{}^{aa} , \tag{1}$$

but again, its derivatives need not vanish.

The metric tensor, as obtained from the *Jacobi* matrix with arbitrary mixed symmetry, is always pro-symmetric[4], and thus has lost some information.

In a 1-dimensional example, the single component of the metric is the square of the single component of the *Jacobi*an, $g = j^2$, so that recovery of the original sign is ambiguous.

In 4D, the metric has only 10 different components, where the *Jacobi* matrix had 16, so even more information is lost. Especially the orthogonal part of the *Jacobi* matrix is invisible to the metric, so the metric is not susceptible to rotations any more.

Thus it is preferable to solve for the *Jacobi* matrix in the first place rather than for the metric tensor, since the latter can be exactly formed from the former, but not the other way around.

[2] similar to [2, (9.5)], but generalized to a non-flat outer metric tensor.
[3] Now the summation convention is extended to doubling indices
[4] introducing 'pro-symmetric' in contrast to 'anti-symmetric'

## 1.4 The Functional determinant

The **functional determinant**, sometimes called the '*Jacobi*an', is the determinant of the *Jacobi* matrix, and often expressed as a square root from the determinant of the metric tensor, Functional determinant

$$\det\left(J^{\mu}{}_{a}\right) = \sqrt{\left|\det\left(g_{ab}\right)\right|} = \sqrt{|g|}\,.$$

The functional determinant of a concatenated embedding is the product of the separate *Jacobi* determinants,

$$\det\left(J^{\mu}{}_{a}\right) \;=\; \det\bigl(\overset{A}{J}{}^{\mu}{}_{a}\bigr)\cdot\det\bigl(\overset{B}{J}{}^{\mu}{}_{a}\bigr)\cdot\det\bigl(\overset{C}{J}{}^{\mu}{}_{a}\bigr)\cdots\,.$$

## 1.5 A '*Hesse* Stack'

The **second partial derivatives** of the embedding function form slices *Hesse* matrix stack

$$J^{\mu}{}_{ab} \;:=\; J^{\mu}{}_{a,b} \;=\; \partial_b\, J^{\mu}{}_{a} \;=\; \partial_b\partial_a\, J^{\mu}\,,$$

of a stack of ***Hesse* matrices**, which are symmetric in $(a,b)$,
since partial derivatives commute,

$$J^{\mu}{}_{ab} \;=\; J^{\mu}{}_{ba}\,,$$

and shall be called here **'*Hesse* (matrix) stack'**.

Since partial derivatives commute, each additional covariant index, which is appended by expanding the *Hesse* stack through a partial derivation, has also a symmetry interchangeable with the former covariant indices,

$$\partial_a J^{\mu}{}_{aa} \;=\; J^{\mu}{}_{aaa}\,, \quad \partial_a J^{\mu}{}_{aaa} \;=\; J^{\mu}{}_{aaaa}\,, \qquad \text{etc}\ldots$$

The combined *Hesse* stack from a concatenation is formed by the *Leibniz* rule
from the separate *Hesse* stacks together with the separate *Jacobi* matrices, like

$$J^{\mu}{}_{ad} \;=\; \overset{A}{J}{}^{\mu}{}_{\alpha\delta}\cdot\overset{B}{J}{}^{\alpha}{}_{a}\cdot\overset{B}{J}{}^{\delta}{}_{d} + \overset{A}{J}{}^{\mu}{}_{\alpha}\cdot\overset{B}{J}{}^{\alpha}{}_{ad}\,, \tag{2}$$

and for more consecutive embeddings with the number of matrix multiplications increasing in second order, for example with 3 embeddings

$$J^{\mu}{}_{ad} \;=\; \overset{A}{J}{}^{\mu}{}_{\nu\gamma}\cdot\overset{B}{J}{}^{\nu}{}_{\alpha}\cdot\overset{C}{J}{}^{\alpha}{}_{a}\cdot\overset{B}{J}{}^{\gamma}{}_{\delta}\cdot\overset{C}{J}{}^{\delta}{}_{d} + \overset{A}{J}{}^{\mu}{}_{\nu}\cdot\overset{B}{J}{}^{\nu}{}_{\alpha\delta}\cdot\overset{C}{J}{}^{\alpha}{}_{a}\cdot\overset{C}{J}{}^{\delta}{}_{d} + \overset{A}{J}{}^{\mu}{}_{\nu}\cdot\overset{B}{J}{}^{\nu}{}_{\gamma}\cdot\overset{C}{J}{}^{\gamma}{}_{ad}\,,$$

which is obviously a rather impractical calculation.

## 1.6 Metric Connections

### From the Metric ...

The usual deduction[5] of the metric-compatible affine connection, or metric connection, follows from the definition that the covariant derivative of the metric tensor vanishes identically,

$$g_{ab;c} \;=\; g_{ab,c} - g_{\alpha b} \cdot \Gamma^{\alpha}{}_{ac} - g_{a\beta} \cdot \Gamma^{\beta}{}_{bc} \;:=\; 0\,,$$

so that

$$g_{ab,c} \;=\; \Gamma_{abc} + \Gamma_{bac}\,, \tag{3}$$

and since scalar addition commutes, the affine connection being the metric connection requires that again $g_{ab}$ must be symmetric in $(a, b)$.

Permuting

$$\begin{aligned} g_{ac,b} &= \Gamma_{acb} + \Gamma_{cab}\,, \\ g_{cb,a} &= \Gamma_{cba} + \Gamma_{bca}\,, \end{aligned}$$

and assuming that the Gamma symbol is symmetric in its last two indices, $\Gamma_{abc} = \Gamma_{acb}$, we find a chain

$$\Gamma_{abc} \;=\; g_{ab,c} - \Gamma_{bac}\,, \qquad \Gamma_{bac} \;=\; g_{cb,a} - \Gamma_{cba}\,, \qquad \Gamma_{cab} \;=\; g_{ac,b} - \Gamma_{abc}\,,$$

so that

$$\begin{aligned} \Gamma_{abc} &= g_{ab,c} + g_{ac,b} - g_{bc,a} - \Gamma_{abc} \quad \Rightarrow \\ 2\,\Gamma_{abc} &= g_{ab,c} + g_{ac,b} - g_{bc,a} \quad \Rightarrow \\ \Gamma_{abc} &= \tfrac{1}{2}\big(\, g_{ab,c} + g_{ac,b} - g_{bc,a} \,\big)\,. \end{aligned} \tag{4}$$

With the opposite assumption, that the Gamma symbol be anti-symmetric in its last two indices, no such equation can be found.

### ... to the *Hesse* tensor

Alternatively, by substituting

$$g_{ab,c} \;=\; \big(J^{\mu}{}_{a} \cdot J^{\nu}{}_{b} \cdot \overset{o}{g}_{\mu\nu}\big)_{,c} \;=\; \big(J^{\mu}{}_{ac} \cdot J^{\nu}{}_{b} + J^{\mu}{}_{a} \cdot J^{\nu}{}_{bc}\big) \cdot \overset{o}{g}_{\mu\nu} + J^{\mu}{}_{a} \cdot J^{\nu}{}_{b} \cdot \overset{o}{g}_{\mu\nu,c}$$

[5] here similar to [2, (11.15-16)]

into (4), the ***Christoffel* symbol of the first kind** can be rewritten in terms of the *Jacobi* matrix, the *Hesse* tensor and the outer metric

$$2\,\Gamma_{abc} = \overset{o}{g}_{\mu\nu}\cdot\left(J^\mu{}_a\cdot J^\nu{}_{bc} + J^\mu{}_b\cdot J^\nu{}_{ac} + J^\mu{}_a\cdot J^\nu{}_{cb} + J^\mu{}_c\cdot J^\nu{}_{ab} - J^\mu{}_b\cdot J^\nu{}_{ca} - J^\mu{}_c\cdot J^\nu{}_{ba}\right)$$
$$+\; J^\mu{}_a\cdot J^\nu{}_b\cdot \overset{o}{g}_{\mu\nu,c} + J^\mu{}_a\cdot J^\nu{}_c\cdot \overset{o}{g}_{\mu\nu,b} - J^\mu{}_b\cdot J^\nu{}_c\cdot \overset{o}{g}_{\mu\nu,a}\,,$$

equivalent to

$$\begin{aligned}\Gamma_{abc} &= \overset{o}{g}_{\mu\nu}\cdot J^\mu{}_a\cdot J^\nu{}_{bc} \quad + \quad \tfrac{1}{2}\left( J^\mu{}_a\cdot J^\nu{}_b\cdot \overset{o}{g}_{\mu\nu,c} + J^\mu{}_a\cdot J^\nu{}_c\cdot \overset{o}{g}_{\mu\nu,b} - J^\mu{}_b\cdot J^\nu{}_c\cdot \overset{o}{g}_{\mu\nu,a}\right)\\ &= \overset{o}{g}_{\mu\nu}\cdot J^\mu{}_a\cdot J^\nu{}_{bc} \quad + \quad J^\mu{}_a\cdot \overset{o}{\Gamma}_{\mu\nu\gamma}\cdot J^\nu{}_b\cdot J^\gamma{}_c\,,\end{aligned}$$

where the symmetry in $(b, c)$ can again be established from the corresponding symmetry of the *Hesse* tensor, and written in multi-index notation, *Christoffel*, 1st kind

$$\Gamma_{abb} = \overset{o}{g}_{\mu\mu}\cdot J^\mu{}_a\cdot J^\mu{}_{bb} \quad + \quad J^\mu{}_a\cdot \overset{o}{\Gamma}_{\mu\gamma\gamma}\cdot J^\gamma{}_b\cdot J^\gamma{}_b\,. \tag{5}$$

Notice how the outer connection is transported in all three indices into the inner space.

In case the outer metric is constant, $\overset{o}{\Gamma}_{abc} \overset{\star}{=} 0$, this reduces to

$$\Gamma_{abb} \overset{\star}{=} \overset{o}{g}_{\mu\mu}\cdot J^\mu{}_a\cdot J^\mu{}_{bb}\,,$$

so expressing the metric connection from the embedding might be somewhat simpler than deducing it from the metric tensor's derivatives, (4).

**Theorem.** *The* Christoffel *symbols are symmetric in the last two indices but the first one, $\Gamma^a{}_{bb}$, as much as the metric tensor is symmetric ($g_{aa}$, $g^{aa}$) or partial derivatives commute.*[6]

The ***Christoffel* symbol of the second kind**,

$$\Gamma^a{}_{bb} := g^{aa}\cdot\Gamma_{abb}\,, \tag{6}$$

can in a similar way be expressed from the inverse *Jacobi* matrix and the *Hesse* tensor[7], so that calculation of an inner metric connection is even independent from the outer metric, *Christoffel*, 2nd kind

$$\overset{i}{\Gamma}{}^a{}_{bb} = J_\mu{}^a\cdot J^\mu{}_{bb}\,, \tag{7}$$

and in the limit of a flat inner embedding ($J_\mu{}^a \overset{\star}{\to} \delta_\mu{}^a$) becomes identical to the *Hesse* stack,

$$\overset{i}{\Gamma}{}^a{}_{bb} \overset{\star}{\to} J^\mu{}_{bb}\,.$$

[6]Note that both the symmetries of $g_{aa}$ and $\Gamma_{abb}$ come naturally from the model. It shall be suspected that adding any additional concept to loosen restrictions rather than to constrain a system, might be a counterproductive endeavour and even the latter a questionable procedure.

[7]similar [2, (11.14)] without explicitly naming the entity of second derivatives

The *Christoffel* 2nd of a combined embedding, $\overset{A}{J}{}^{\mu}{}_{\gamma} \cdot \overset{B}{J}{}^{\gamma}{}_{a}$, follows from (7) with (2),

$$\begin{aligned}
\Gamma^a{}_{bd} &= \overset{B}{J}{}_{\alpha}{}^{a} \cdot \overset{A}{J}{}_{\mu}{}^{\alpha} \cdot J^{\mu}{}_{bd} \\
&= \overset{B}{J}{}_{\gamma}{}^{a} \cdot \overset{A}{J}{}_{\mu}{}^{\gamma} \cdot \left( \overset{A}{J}{}^{\mu}{}_{\beta\delta} \cdot \overset{B}{J}{}^{\beta}{}_{b} \cdot \overset{B}{J}{}^{\delta}{}_{d} + \overset{A}{J}{}^{\mu}{}_{\nu} \cdot \overset{B}{J}{}^{\nu}{}_{bd} \right) \\
&= \overset{B}{J}{}_{\alpha}{}^{a} \cdot \overset{A}{J}{}_{\mu}{}^{\alpha} \cdot \overset{A}{J}{}^{\mu}{}_{\beta\delta} \cdot \overset{B}{J}{}^{\beta}{}_{b} \cdot \overset{B}{J}{}^{\delta}{}_{d} + \overset{B}{J}{}_{\alpha}{}^{a} \cdot \overset{A}{J}{}_{\mu}{}^{\alpha} \cdot \overset{A}{J}{}^{\mu}{}_{\gamma} \cdot \overset{B}{J}{}^{\gamma}{}_{bd} \\
&= \overset{B}{J}{}_{\alpha}{}^{a} \cdot \overset{A}{\Gamma}{}^{\alpha}{}_{\beta\delta} \cdot \overset{B}{J}{}^{\beta}{}_{b} \cdot \overset{B}{J}{}^{\delta}{}_{d} + \overset{B}{\Gamma}{}^{a}{}_{bd} \,.
\end{aligned}$$

Since $\overset{A}{J}{}^{\mu}{}_{\gamma}$ does not appear any more, we can take $A$ as an outer embedding, leaving only $B$ for the inner,

$$\Gamma^a{}_{bd} = J_{\alpha}{}^{a} \cdot \overset{o}{\Gamma}{}^{\alpha}{}_{\beta\delta} \cdot J^{\beta}{}_{b} \cdot J^{\delta}{}_{d} + \overset{i}{\Gamma}{}^{a}{}_{bd} \,,$$

so the overall connection is again the sum of the inner connection with the outer connection transported into the inner space in all three indices, similar to (5),

$$\Gamma^a{}_{bb} = \overset{i}{\Gamma}{}^{a}{}_{bb} + J_{\mu}{}^{a} \cdot \overset{o}{\Gamma}{}^{\mu}{}_{\gamma\gamma} \cdot J^{\gamma}{}_{b} \cdot J^{\gamma}{}_{b} \,,$$

and in the limit of a flat inner embedding ($J^{\nu}{}_{b} \overset{\star}{\to} \delta^{\nu}{}_{b}$, $J_{\mu}{}^{a} \overset{\star}{\to} \delta_{\mu}{}^{a}$),
becomes simply the sum of the outer and inner connection,

$$\Gamma^a{}_{bb} \overset{\star}{\to} \overset{i}{\Gamma}{}^{a}{}_{bb} + \overset{o}{\Gamma}{}^{a}{}_{bb} \,, \tag{8}$$

and with a constant outer metric ($\overset{o}{g}_{ab,c} \overset{\star}{=} 0$) reduces to the inner connection,

$$\Gamma^a{}_{bb} \overset{\star}{\to} \overset{i}{\Gamma}{}^{a}{}_{bb} \,.$$

A more historical notation of the *Christoffel* symbols is

$$\left[ {b\ c \atop a} \right] = \Gamma_{abc} \,, \qquad \left\{ {b\ c \atop a} \right\} = \Gamma^a{}_{bc} \,,$$

explicitly meaning the metric connection, whereas the gamma symbol $\Gamma_{abc}$, $\Gamma^a{}_{bc}$ more generally denotes just an affine (*Levi-Civita*) connection, and might be declared in the context to be the metric-compatible connection, as is the case in the present text.

## 1.7 Connection Products

We examine of the 6-ranked outer product of the connection, $\Gamma_{abc}\,\Gamma^d{}_{ef}$ or $\Gamma^a{}_{bc}\,\Gamma^d{}_{ef}$,
the 4-ranked first contractions.

Of the singly inter-contracted products of the *Christoffel* symbols of the first and second kind, let us call the contraction between first indices the '**connection product of the first kind**', Connection product, 1st kind

$$\Gamma_{\gamma ab} \cdot \Gamma^{\gamma}{}_{cd} \;=\; \Gamma_{\gamma cd} \cdot \Gamma^{\gamma}{}_{ab} \,.$$

Reversely expressing the *Hesse* stack from the *Christoffel* 1st by (6) and (7),

$$J^{\mu}{}_{bc} \;=\; \overset{o}{g}{}^{\mu\mu} \cdot J_{\mu}{}^{a} \cdot \Gamma_{abc} \,,$$

this product is equivalent to the most symmetric single-contracted outer product of the *Hesse* stack with itself,

$$\begin{aligned} J^{\mu}{}_{ab} \cdot \overset{o}{g}{}_{\mu\mu} \cdot J^{\mu}{}_{cd} \;&=\; \Gamma_{\gamma ab} \cdot J_{\nu}{}^{\gamma} \cdot \overset{o}{g}{}^{\nu\mu} \cdot \overset{o}{g}{}_{\mu\mu} \cdot \overset{o}{g}{}^{\mu\nu} \cdot J_{\nu}{}^{\beta} \cdot \Gamma_{\beta cd} \\ &=\; \Gamma_{\gamma ab} \cdot g^{\gamma\gamma} \cdot \Gamma_{\gamma cd} \;=\; \Gamma^{\gamma}{}_{ab} \cdot g_{\gamma\gamma} \cdot \Gamma^{\gamma}{}_{cd} \\ &=\; \Gamma_{\gamma ab} \cdot \Gamma^{\gamma}{}_{cd} \;=\; \Gamma^{\gamma}{}_{ab} \cdot \Gamma_{\gamma cd} \,, \end{aligned}$$

which contains the metric as a factor.

Of 6 possible **contractions** are only 2 different, where one occurs twice,

$$\left(\Gamma_{\gamma ab}\, \Gamma^{\gamma}{}_{\delta\delta} \;=\; \Gamma_{\gamma\delta\delta}\, \Gamma^{\gamma}{}_{ab}\right) g^{\delta\delta} \,,$$

and another one 4-fold,

$$\Gamma_{\gamma aa}\, \Gamma^{\gamma}{}_{bb}\, g^{ab} \,,$$

which are now free from the metric factor.

There are two total **scalar contractions**,

$$\Gamma_{\gamma aa}\, \Gamma^{\gamma}{}_{bb}\, g^{ab}\, g^{ab} \qquad \text{and} \qquad \Gamma_{\gamma aa}\, \Gamma^{\gamma}{}_{bb}\, g^{aa}\, g^{bb} \,,$$

which again contain the inverse metric as a factor.

Let us call the 'daisy chain' contraction between a last index of a *Christoffel* 1st or 2nd with the first index of a *Christoffel* 2nd the '**connection product of the second kind**', Connection product, 2nd kind

$$\Gamma_{ab\gamma}\, \Gamma^{\gamma}{}_{cd} \qquad \text{or} \qquad \Gamma^{a}{}_{b\gamma}\, \Gamma^{\gamma}{}_{cd} \,,$$

the first containing the metric factor and the latter being metric-free.

Contract in the back,

$$\Gamma_{abb}\, \Gamma^{b}{}_{\delta\delta}\, g^{\delta\delta} \qquad \text{or} \qquad \Gamma^{a}{}_{bb}\, \Gamma^{b}{}_{\delta\delta}\, g^{\delta\delta} \,,$$

or cross-wise,

$$\Gamma_{a\gamma\gamma}\, \Gamma^{\gamma}{}_{bb}\, g^{\gamma b} \qquad \text{or} \qquad \Gamma^{a}{}_{\gamma\gamma}\, \Gamma^{\gamma}{}_{bb}\, g^{\gamma b} \,,$$

with, respectively, the former now being metric-free
and the latter containing the inverse metric factor.

Again there are two total scalar contractions,

$$\Gamma^{\alpha}{}_{\alpha\alpha}\,\Gamma^{\alpha}{}_{\delta\delta}\,g^{\delta\delta} \qquad \text{and} \qquad \Gamma^{\alpha}{}_{\gamma\gamma}\,\Gamma^{\gamma}{}_{\alpha\alpha}\,g^{\gamma\alpha}\,,$$

also containing the inverse metric factor.

## 1.8 Partial Connection Derivatives

Partial derivation is in general not interchangeable with raising and lowering of any index,

$$\left(g_{aa}\cdot\Gamma^{a}{}_{bb}\right)_{,d} \neq g_{aa}\cdot\left(\Gamma^{a}{}_{bb,d}\right),$$

indeed, expressing from each other

$$\begin{aligned}
\Gamma_{abc,d} &= \left(g_{a\gamma}\cdot\Gamma^{\gamma}{}_{bc}\right)_{,d} \\
&= g_{a\gamma}\cdot\Gamma^{\gamma}{}_{bc,d} + g_{a\gamma,d}\cdot\Gamma^{\gamma}{}_{bc} \\
&= g_{a\gamma}\cdot\Gamma^{\gamma}{}_{bc,d} + \left(\Gamma_{a\gamma d}+\Gamma_{\gamma ad}\right)\cdot\Gamma^{\gamma}{}_{bc} \\
&= g_{a\gamma}\cdot\Gamma^{\gamma}{}_{bc,d} + \Gamma_{a\gamma d}\cdot\Gamma^{\gamma}{}_{bc} + \Gamma_{\gamma ad}\cdot\Gamma^{\gamma}{}_{bc} \\
&= g_{a\gamma}\cdot\Gamma^{\gamma}{}_{bc,d} + \Gamma_{ad\gamma}\cdot\Gamma^{\gamma}{}_{bc} + \Gamma_{\gamma bc}\cdot\Gamma^{\gamma}{}_{ad}\,,
\end{aligned}$$

gives

$$g_{a\gamma}\cdot\Gamma^{\gamma}{}_{bc,d} = \Gamma_{abc,d} - \Gamma_{ad\gamma}\cdot\Gamma^{\gamma}{}_{bc} - \Gamma_{\gamma bc}\cdot\Gamma^{\gamma}{}_{ad}\,. \tag{9}$$

Partial *Christoffel* 1st

Expressing the overall partial derivative of ***Christoffel* 1st** from the separate inner and outer connections,

$$\begin{aligned}
\Gamma_{abc,d} = \overset{i}{\Gamma}_{abc,d} &+ J^{\mu}{}_{ad}\cdot\overset{o}{\Gamma}_{\mu\beta\gamma}\cdot J^{\beta}{}_{b}\cdot J^{\gamma}{}_{c} \\
&+ J^{\mu}{}_{a}\cdot\overset{o}{\Gamma}_{\mu\beta\gamma,\delta}\cdot J^{\beta}{}_{b}\cdot J^{\gamma}{}_{c}\cdot J^{\delta}{}_{d} \\
&+ J^{\mu}{}_{a}\cdot\overset{o}{\Gamma}_{\mu\beta\gamma}\cdot J^{\beta}{}_{bd}\cdot J^{\gamma}{}_{c} \\
&+ J^{\mu}{}_{a}\cdot\overset{o}{\Gamma}_{\mu\beta\gamma}\cdot J^{\beta}{}_{b}\cdot J^{\gamma}{}_{cd}\,,
\end{aligned}$$

becomes more sensible in the limit of a flat *Jacobi* matrix, $J^{\nu}{}_{b} \xrightarrow{\star} \delta^{\nu}{}_{b}$, $J_{\mu}{}^{a} \xrightarrow{\star} \delta_{\mu}{}^{a}$, where also $J^{a}{}_{bb} \xrightarrow{\star} \overset{i}{\Gamma}{}^{a}{}_{bb}$, so that partial derivation is still not additive with respect to inner and outer connection, but may be expressed from the gamma symbols alone,

$$\Gamma_{abc,d} \xrightarrow{\star} \overset{o}{\Gamma}_{abc,d} + \overset{i}{\Gamma}_{abc,d} + \overset{o}{\Gamma}_{\gamma bc}\cdot\overset{i}{\Gamma}{}^{\gamma}{}_{ad} + \overset{o}{\Gamma}_{ac\gamma}\cdot\overset{i}{\Gamma}{}^{\gamma}{}_{bd} + \overset{o}{\Gamma}_{ab\gamma}\cdot\overset{i}{\Gamma}{}^{\gamma}{}_{cd}\,, \tag{10}$$

Expressing the overall partial derivative of ***Christoffel* 2nd** from the separate inner and outer connections, Partial *Christoffel* 2nd

$$\begin{aligned}\Gamma^a{}_{bc,d} \;=\; \overset{i}{\Gamma}{}^a{}_{bc,d} &+ J_\mu{}^a{}_{,d} \cdot \overset{o}{\Gamma}{}^\mu{}_{\beta\gamma} \cdot J^\beta{}_b \cdot J^\gamma{}_c \\ &+ J_\mu{}^a \cdot \overset{o}{\Gamma}{}^\mu{}_{\beta\gamma,\delta} \cdot J^\beta{}_b \cdot J^\gamma{}_c \cdot J^\delta{}_d \\ &+ J_\mu{}^a \cdot \overset{o}{\Gamma}{}^\mu{}_{\beta\gamma} \cdot J^\beta{}_{bd} \cdot J^\gamma{}_c \\ &+ J_\mu{}^a \cdot \overset{o}{\Gamma}{}^\mu{}_{\beta\gamma} \cdot J^\beta{}_b \cdot J^\gamma{}_{cd}\,,\end{aligned}$$

and in the limit of a flat *Jacobi* matrix, $J^\mu{}_a \xrightarrow{\star} \delta^\mu{}_a$, $J_\mu{}^a \xrightarrow{\star} \delta_\mu{}^a$,
where also $J^a{}_{bb} \xrightarrow{\star} \overset{i}{\Gamma}{}^a{}_{bb}$, so that partial derivation is still not additive with respect to inner and outer connection, but may be expressed from the gamma symbols alone,

$$\Gamma^a{}_{bc,d} \xrightarrow{\star} \overset{o}{\Gamma}{}^a{}_{bc,d} + \overset{i}{\Gamma}{}^a{}_{bc,d} \overset{?}{-} \overset{i}{\Gamma}{}^a{}_{d\gamma} \cdot \overset{o}{\Gamma}{}^\gamma{}_{bc} + \overset{o}{\Gamma}{}^a{}_{c\gamma} \cdot \overset{i}{\Gamma}{}^\gamma{}_{bd} + \overset{o}{\Gamma}{}^a{}_{b\gamma} \cdot \overset{i}{\Gamma}{}^\gamma{}_{cd}\,, \tag{11}$$

TODO: Assuming $J_\mu{}^a{}_{,d} \xrightarrow{\star} -\Gamma^a{}_{d\mu}$, still missing a proof!

## 1.9 Covariant Connection Derivatives

Expressing the covariant derivative of ***Christoffel* 1st**
from the partial derivative of *Christoffel* 1st, Covar *Christoffel* 1st

$$\Gamma_{abc;d} \;=\; \Gamma_{abc,d} - \Gamma_{\gamma bc} \cdot \Gamma^\gamma{}_{ad} - \Gamma_{a\gamma c} \cdot \Gamma^\gamma{}_{bd} - \Gamma_{ab\gamma} \cdot \Gamma^\gamma{}_{cd}\,. \tag{12}$$

Expressing the covariant derivatives of ***Christoffel* 2nd**
from the partial derivative of *Christoffel* 2nd, Covar *Christoffel* 2nd

$$\Gamma^a{}_{bc;d} \;=\; \Gamma^a{}_{bc,d} + \Gamma^a{}_{d\gamma} \cdot \Gamma^\gamma{}_{bc} - \Gamma^a{}_{\gamma c} \cdot \Gamma^\gamma{}_{bd} - \Gamma^a{}_{b\gamma} \cdot \Gamma^\gamma{}_{cd}\,. \tag{13}$$

**Corollary.** *Swapping* (13) *in* $(c,d)$ *yields in the difference*

$$\left(\Gamma^a{}_{bc;d} - \Gamma^a{}_{bd;c}\right) \;=\; \left(\Gamma^a{}_{bc,d} - \Gamma^a{}_{bd,c}\right) - 2\left(\Gamma^a{}_{c\gamma} \cdot \Gamma^\gamma{}_{bd} - \Gamma^a{}_{d\gamma} \cdot \Gamma^\gamma{}_{bc}\right). \tag{14}$$

Since the covariant derivative of the metric tensor per definiton vanishes identically, $g_{aa;d} = 0$, covariant derivation is interchangeable with raising and lowering of any index,

$$\left(g_{aa}\Gamma^a{}_{bb}\right)_{;d} \;=\; g_{aa;d} \cdot \Gamma^a{}_{bb} + g_{aa} \cdot \left(\Gamma^a{}_{bb;d}\right).$$

**Theorem.** *Covariant derivation of the* Christoffel *symbols is interchangeable with raising and lowering of any index,*

$$\left(g_{aa}\Gamma^a{}_{bb}\right)_{;d} \;=\; g_{aa}\left(\Gamma^a{}_{bb;d}\right). \tag{15}$$

## 1.10 Curvature Tensors

While mixed second partial derivatives commute, $A^a{}_{,c,d} = A^a{}_{,d,c}$,
mixed second covariant derivatives do not commute in general, $(A^a{}_{;c})_{;d} \neq (A^a{}_{;d})_{;c}$.

With the first derivative of a contravariant vector $A^a$,

$$A^a{}_{;c} \;=\; A^a{}_{,c} \;+\; A^b\,\Gamma^a{}_{bc}\,,$$

the twice covariant derivative can be expressed as

$$\begin{aligned}(A^a{}_{;c})_{;d} \;&=\; (A^a{}_{,c})_{;d} \;+\; (A^b\,\Gamma^a{}_{bc})_{;d}\\ &=\; (A^a{}_{,c})_{;d} \;+\; A^b{}_{;d}\,\Gamma^a{}_{bc} \;+\; A^b\cdot(\Gamma^a{}_{bc})_{;d}\,,\end{aligned}$$

with the covariant derivative of the partial derivative,

$$(A^a{}_{,c})_{;d} \;=\; A^a{}_{,c,d} \;+\; A^b{}_{,c}\,\Gamma^a{}_{bd} \;-\; A^a{}_{,b}\,\Gamma^b{}_{cd}\,,$$

and the covariant derivative of the connection (13),

$$(\Gamma^a{}_{bc})_{;d} \;=\; \Gamma^a{}_{bc,d} \;+\; \Gamma^a{}_{d\gamma}\,\Gamma^\gamma{}_{bc} \;-\; \Gamma^a{}_{\gamma c}\,\Gamma^\gamma{}_{bd} \;-\; \Gamma^a{}_{b\gamma}\,\Gamma^\gamma{}_{cd}\,.$$

Leaving out terms symmetric in $(c, d)$,

$$(A^a{}_{;c})_{;d} \;\stackrel{\star}{=}\; A^b{}_{,c}\,\Gamma^a{}_{bd} \;+\; A^b{}_{,d}\,\Gamma^a{}_{bc} \;+\; A^b\,\Gamma^a{}_{bc,d} \;+\; A^b\,\Gamma^a{}_{d\gamma}\,\Gamma^\gamma{}_{bc}\,, \tag{16}$$

and expressing the commutator of covariant derivation as the difference
of (16) with itself when $(c, d)$ swapped,

$$\begin{aligned}&(A^a{}_{;c})_{;d} \;-\; (A^a{}_{;d})_{;c}\\ &\quad=\; A^b\cdot\big(\Gamma^a{}_{bc,d} - \Gamma^a{}_{bd,c} \;+\; \Gamma^a{}_{d\gamma}\,\Gamma^\gamma{}_{bc} - \Gamma^a{}_{c\gamma}\,\Gamma^\gamma{}_{bd}\big)\\ &\quad=\; -\,A^b\,R^a{}_{bcd}\,,\end{aligned}$$

the ***Riemann* curvature tensor** is introduced[8]

*Riemann* tensor

$$R^a{}_{bcd} \;=\; \big(\,\Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d}\,\big) \;+\; \big(\,\Gamma^a{}_{c\gamma}\,\Gamma^\gamma{}_{bd} - \Gamma^a{}_{d\gamma}\,\Gamma^\gamma{}_{bc}\,\big)\,, \tag{17}$$

from which follows anti-symmetry in $(c, d)$.

Contracting the first two indices without the metric, we get the doubly covariant
*Ricci* tensor, which does not contain the metric,

*Ricci* tensor

$$R_{bd} \;=\; \big(\,\Gamma^\gamma{}_{bd,\gamma} - \Gamma^\gamma{}_{\gamma b,d}\,\big) \;+\; \big(\,\Gamma^\alpha{}_{\alpha\gamma}\,\Gamma^\gamma{}_{bd} - \Gamma^\alpha{}_{b\gamma}\,\Gamma^\gamma{}_{d\alpha}\,\big)\,,$$

In the following contractions the inverse inner metric is involved, that is,
the *Ricci* tensor in mixed variance,

$$R^a{}_b \;=\; \big(\,\Gamma^a{}_{\delta\delta,b} - \Gamma^a{}_{b\delta,\delta}\,\big)\,g^{\delta\delta} \;+\; \big(\,\Gamma^a{}_{b\gamma}\,\Gamma^\gamma{}_{\delta\delta} - \Gamma^a{}_{\delta\gamma}\,\Gamma^\gamma{}_{\delta b}\,\big)\,g^{\delta\delta}\,, \tag{18}$$

and the fully contracted *Ricci* scalar,

*Ricci* scalar

$$R \;=\; \big(\,\Gamma^\alpha{}_{\delta\delta,\alpha} - \Gamma^\alpha{}_{\alpha\delta,\delta}\,\big)\,g^{\delta\delta} \;+\; \big(\,\Gamma^\alpha{}_{\alpha\gamma}\,\Gamma^\gamma{}_{\delta\delta} - \Gamma^\alpha{}_{\delta\gamma}\,\Gamma^\gamma{}_{\delta\alpha}\,\big)\,g^{\delta\delta}\,.$$

[8] similar [3, 6.5] or [2, (18.5-8)], see also [1, §12]

**Corollary.** *Expressing the* Riemann *tensor with covariant derivatives instead of the partials,*

$$\begin{aligned} R^{a}{}_{bcd} &= \Gamma^{a}{}_{bd;c} + \Gamma^{a}{}_{b\gamma}\,\Gamma^{\gamma}{}_{cd} + \Gamma^{a}{}_{d\gamma}\,\Gamma^{\gamma}{}_{bc} - \Gamma^{a}{}_{c\gamma}\,\Gamma^{\gamma}{}_{bd} \\ &\quad - \Gamma^{a}{}_{bc;d} - \Gamma^{a}{}_{b\gamma}\,\Gamma^{\gamma}{}_{cd} - \Gamma^{a}{}_{c\gamma}\,\Gamma^{\gamma}{}_{bd} + \Gamma^{a}{}_{d\gamma}\,\Gamma^{\gamma}{}_{bc} \\ &\quad + \Gamma^{a}{}_{c\gamma}\,\Gamma^{\gamma}{}_{bd} - \Gamma^{a}{}_{d\gamma}\,\Gamma^{\gamma}{}_{bc} \\ &= \big(\Gamma^{a}{}_{bd;c} - \Gamma^{a}{}_{bc;d}\big) - \big(\Gamma^{a}{}_{c\gamma}\,\Gamma^{\gamma}{}_{bd} - \Gamma^{a}{}_{d\gamma}\,\Gamma^{\gamma}{}_{bc}\big)\,, \end{aligned}$$

*the result looks similar to* (17)*, but with the product terms inverted.*

**Corollary.** *Expressing the* Riemann *tensor 'halfway' with one covariant derivative,*

$$\begin{aligned} R^{a}{}_{bcd} &= \Gamma^{a}{}_{bd,c} - \Gamma^{a}{}_{bc;d} - \Gamma^{a}{}_{b\gamma}\,\Gamma^{\gamma}{}_{cd} - \Gamma^{a}{}_{c\gamma}\,\Gamma^{\gamma}{}_{bd} + \Gamma^{a}{}_{d\gamma}\,\Gamma^{\gamma}{}_{bc} + \Gamma^{a}{}_{c\gamma}\,\Gamma^{\gamma}{}_{bd} - \Gamma^{a}{}_{d\gamma}\,\Gamma^{\gamma}{}_{bc} \\ &= \Gamma^{a}{}_{bd,c} - \Gamma^{a}{}_{bc;d} - \Gamma^{a}{}_{b\gamma}\,\Gamma^{\gamma}{}_{cd}\,, \end{aligned}$$

*we find*

$$\Gamma^{a}{}_{bc;d} + R^{a}{}_{bcd} \;=\; \Gamma^{a}{}_{bd,c} - \Gamma^{a}{}_{b\gamma}\,\Gamma^{\gamma}{}_{cd}\,. \tag{19}$$

# 2 An Infinitesimal Toolkit

In the former section, we investigated properties of and interactions between an outer finite, and an inner flat, or infinitesimal, embedding. In this section, we discover special properties of and interactions between infinitesimal embeddings themselves.

## 2.1 Matrix Exponential and Logarithm

The exponential of a square matrix is well-defined[9], so that any real square matrix $\boldsymbol{L}$ maps to another real square matrix $\boldsymbol{E}$ with a positive-definite determinant,

$$\boldsymbol{E} = \exp(\boldsymbol{L}) := \boldsymbol{I} + \sum_{k=1}^{\infty} \frac{1}{k!}\cdot \boldsymbol{L}^{k}\,, \qquad \text{with} \qquad \det(\boldsymbol{E}) > 0\,,$$

where $\boldsymbol{I}$ is the identity matrix.

The determinant of the exponential matrix is the exponential of the trace of the original matrix,

$$\det(\boldsymbol{E}) \;=\; \exp\big(\operatorname{tr}\big(\boldsymbol{L}\big)\big)\,.$$

[9] see [4, 5.6, p.350]

Inversely, the logarithm of a positive-definite real square matrix is a well-defined real square matrix[10], with the necessary condition, that the determinant of the original matrix is positive, $\det(\boldsymbol{E}) > 0$, and the sufficient condition, that all eigenvalues are positive,

$$\boldsymbol{L} = \log(\boldsymbol{E}) \;:=\; \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \cdot (\boldsymbol{E} - \boldsymbol{I})^k \,, \qquad \text{as long as} \qquad \det(\boldsymbol{E}) \in ]0,2[\,,$$

where the above series converges perfectly in case the original matrix is infinitesimally near identity, $\boldsymbol{E} \overset{\star}{\to} \boldsymbol{I}$.

The trace of the log matrix is the logarithm of the determinant of the original matrix,

$$\operatorname{tr}\big(\boldsymbol{L}\big) \;=\; \log\big(\det(\boldsymbol{E})\big)\,.$$

To '**normalize**' an exponential matrix, so that its determinant becomes one, $\det(\hat{\boldsymbol{E}}) = 1$, it may be 'squeezed' in an isotropic way,

$$\hat{\boldsymbol{E}} \;=\; \frac{\boldsymbol{E}}{\det(\boldsymbol{E})}\,, \quad \text{for example} \quad \hat{E}^{\mu}{}_{a} = \frac{E^{\mu}{}_{a}}{\det(E^{\mu}{}_{a})}\,.$$

The corresponding operation in the logarithmic domain is then to zero out the trace of a logarithmic matrix,

logarithmic matrix normalization

$$\hat{\boldsymbol{L}} \;=\; \boldsymbol{L} - \frac{\operatorname{tr}(\boldsymbol{L})}{\operatorname{tr}(\boldsymbol{I})}\,\boldsymbol{I}\,, \quad \text{for example} \quad \hat{L}^{a}{}_{b} \;=\; L^{a}{}_{b} - \frac{L^{\gamma}{}_{\gamma}}{\delta^{\gamma}{}_{\gamma}}\,\delta^{a}{}_{b}\,, \tag{20}$$

or for a mixed-variant matrix in a 4D space,

$$\hat{L}^{a}{}_{b} \;=\; L^{a}{}_{b} - \tfrac{1}{4}\,L^{\gamma}{}_{\gamma}\,\delta^{a}{}_{b}\,.$$

We could express matrix **multiplication through addition** of matrix logarithms,

$$\exp(\boldsymbol{A}) \cdot \exp(\boldsymbol{B}) \;=\; \exp(\boldsymbol{A} + \boldsymbol{B})\,,$$

but while addition of matrices always commutes, $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$,
multiplication of matrices, $\exp(\boldsymbol{A}) \cdot \exp(\boldsymbol{B}) \neq \exp(\boldsymbol{B}) \cdot \exp(\boldsymbol{A})$,
does not commute in general.

We could define a 'mean matrix multiplication',

$$\exp(\boldsymbol{A} + \boldsymbol{B}) \;:=\; \lim_{n\to\infty} \prod^{n} \exp(\boldsymbol{A})^{1/n} \exp(\boldsymbol{B})^{1/n} \;=\; \lim_{n\to\infty} \prod^{n} \exp(\boldsymbol{B})^{1/n} \exp(\boldsymbol{A})^{1/n}\,.$$

which is still questionable when we deal with finite values.

But in infinitesimal steps $\partial\boldsymbol{A}$, $\partial\boldsymbol{B}$, where all involved matrices are infinitesimally near identity, $\partial\boldsymbol{A} \overset{\star}{\to} \boldsymbol{I}$, $\partial\boldsymbol{B} \overset{\star}{\to} \boldsymbol{I}$, matrix multiplication commutes in the limit,

$$\exp(\partial\boldsymbol{A} + \partial\boldsymbol{B}) \;=\; \exp(\partial\boldsymbol{A}) \cdot \exp(\partial\boldsymbol{B}) \;=\; \exp(\partial\boldsymbol{B}) \cdot \exp(\partial\boldsymbol{A})\,.$$

[10] see [5, 8.8]

**Theorem.** *Matrix multiplication commutes in the infinitesimal limit and can then be replaced by the exponential of the sum of matrix logarithms.*

**Concrete solutions** for exponentiation and logarithm of matrices may be composed of some 'standard solutions'.

standard solutions

The exponential of the null matrix is the identity matrix, for example in 4D,

$$\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \xrightarrow{\exp} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} .$$

The logarithm of a diagonal matrix is also a diagonal matrix, and operations are simply element-wise,

$$\begin{bmatrix} t & & & \\ & x & & \\ & & y & \\ & & & z \end{bmatrix} \xrightarrow{\exp} \begin{bmatrix} e^t & & & \\ & e^x & & \\ & & e^y & \\ & & & e^z \end{bmatrix} .$$

The exponential of a pro-symmetric submatrix is another pro-symmetric submatrix, containing the hyperbolic sine and cosine functions,

$$\begin{bmatrix} \cdot & \phi \\ \phi & \cdot \end{bmatrix} \xrightarrow{\exp} \begin{bmatrix} \cosh(\phi) & \sinh(\phi) \\ \sinh(\phi) & \cosh(\phi) \end{bmatrix} .$$

The exponential of an anti-symmetric submatrix is an orthogonal submatrix, containing the trigonometric sine and cosine functions,

$$\begin{bmatrix} \cdot & -\phi \\ \phi & \cdot \end{bmatrix} \xrightarrow{\exp} \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} .$$

## 2.2 Infinitesimal Embeddings

In the infinitesimal limit, we define a logarithm of the *Jacobi* matrix,

*Jacobi* logarithm

$$\Gamma^\mu{}_a \;:=\; \log(J^\mu{}_a) \;\to\; 0^\mu{}_a \,,$$

so that the *Jacobi* matrix, and its inverse, can both be expressed as exponentials of that logarithm,

$$J^\mu{}_a \;=\; \exp(\,\Gamma^\mu{}_a) \;\to\; \delta^\mu{}_a \,, \qquad J_\mu{}^a \;=\; \exp(-\Gamma^\mu{}_a) \;\to\; \delta_\mu{}^a \,,$$

and the *Jacobi* logarithm of the concatenation of infinitesimal embedding functions is the sum of their *Jacobi* logarithms,

$$\Gamma^{\mu}{}_{a} = \overset{A}{\Gamma}{}^{\mu}{}_{a} + \overset{B}{\Gamma}{}^{\mu}{}_{a} + \overset{C}{\Gamma}{}^{\mu}{}_{a} \; \cdots \, . \tag{21}$$

The partial derivative of the *Jacobi* logarithm is exactly the *Christoffel* 2nd,

*Christoffel*, 2nd kind

$$\partial_c \, \Gamma^{a}{}_{b} = \Gamma^{a}{}_{b,c} = \Gamma^{a}{}_{bc} \, ,$$

that's why the 'Γ' symbol has also been used for the *Jacobi* logarithm in the first place.

The *Christoffel* 2nd of the concatenation of infinitesimal embedding functions is the sum of their *Christoffel* 2nds,

$$\Gamma^{a}{}_{bc} = \overset{A}{\Gamma}{}^{a}{}_{bc} + \overset{B}{\Gamma}{}^{a}{}_{bc} + \overset{C}{\Gamma}{}^{a}{}_{bc} \; \cdots \, . \tag{22}$$

When the *Hesse* tensor is expressed back from the *Jacobi* logarithm and its derivative, the introduction of the exponential can be observed,

$$J^{\mu}{}_{ab} = J^{\mu}{}_{a,b} = \exp(\,\Gamma^{\mu}{}_{a})_{,b} = \exp(\,\Gamma^{\mu}{}_{\gamma}) \cdot \Gamma^{\gamma}{}_{a,b} = J^{\mu}{}_{\gamma} \cdot \Gamma^{\gamma}{}_{ab} \, .$$

Expressing *Christoffel* 2 from (7) shows how the exponential gets eliminated,

$$J_{\mu}{}^{a} \cdot J^{\mu}{}_{bc} = \exp(-\Gamma^{a}{}_{\beta}) \cdot \exp(\,\Gamma^{\beta}{}_{\gamma}) \cdot \Gamma^{\gamma}{}_{bc} = \Gamma^{a}{}_{bc} \, ,$$

thus *Christoffel* 2nd is a scale-invariant expression of the derivative of the *Jacobi* matrix, and identical with the derivative of the *Jacobi* logarithm.

The logarithm of the functional determinant is the trace of the *Jacobi* logarithm,

Logarithmic functional determinant

$$\log(\,\det(J^{\mu}{}_{a})) = \operatorname{tr}(\,\Gamma^{\mu}{}_{a}) = \Gamma^{\mu}{}_{\mu} \, ,$$

and the partial derivative of the logarithmic functional determinant is the contraction of *Christoffel* 2nd with its first index,

$$\partial_d \, \log(\,\det(J^{\mu}{}_{a})) = \frac{\partial_d \, \det(J^{\mu}{}_{a})}{\det(J^{\mu}{}_{a})} = \Gamma^{\mu}{}_{\mu,d} = \Gamma^{\mu}{}_{\mu d} \, .$$

**Corollary.** *The logarithmic functional determinant of the sum of infinitesimal embeddings is the sum of the separate logarithmic functional determinants,*

$$\log(\,\det(J^{\mu}{}_{a})) = \overset{A}{\Gamma}{}^{\mu}{}_{\mu} + \overset{B}{\Gamma}{}^{\mu}{}_{\mu} + \overset{C}{\Gamma}{}^{\mu}{}_{\mu} \cdots \, ,$$

*and the logarithmic functional determinant derivatives of a concatenated embedding is then the sum of the separate logarithmic functional determinant derivatives,*

$$\partial_d \log(\,\det(J^{\mu}{}_{a})) = \overset{A}{\Gamma}{}^{\mu}{}_{\mu d} + \overset{B}{\Gamma}{}^{\mu}{}_{\mu d} + \overset{C}{\Gamma}{}^{\mu}{}_{\mu d} \cdots \, .$$

**Corollary.** *To 'normalize' the logarithmic* Jacobi *matrix, so that its trace vanishes,* $\hat{\Gamma}^{\gamma}{}_{\gamma} := 0$*, the trace is zeroed out by subtraction, as in* (20)*,*

$$\hat{\Gamma}^{a}{}_{b} = \Gamma^{a}{}_{b} - \frac{\Gamma^{\gamma}{}_{\gamma}}{\delta^{\gamma}{}_{\gamma}}\,\delta^{a}{}_{b}\,,$$

*and especially in a 4D space,*

$$\hat{\Gamma}^{a}{}_{b} \stackrel{\star}{=} \Gamma^{a}{}_{b} - \tfrac{1}{4}\,\Gamma^{\gamma}{}_{\gamma}\,\delta^{a}{}_{b}\,.$$

## 2.3 Scale Invariance

Expressing the laws of nature from derivatives is essential for a near field picture. But the derivatives of functions are still dependend on the 'magnitude' of the functions, unless when viewed in a logarithmic local context.

To compare a 1D example with the general case, let there be an outer positive real function $f > 0$ the chain of exponentiation with an inner function $g \in \mathbb{R}$,

$$f(x) = e^{g(x)} \quad \sim \quad J^{\mu}{}_{a}\,,$$

then the derivative contains the outer function as a factor,

$$f'(x) = \frac{\partial}{\partial x} f(x) = g'(x) \cdot \exp(g(x)) = g'(x) \cdot f(x) \qquad \sim \quad J^{\mu}{}_{ab}\,,$$

which gets eliminated in the quotient

$$g'(x) = \frac{f'(x)}{f(x)} = \frac{\partial}{\partial x} \ln f(x) \quad \sim \quad \Gamma^{a}{}_{bc} = J^{\mu}{}_{ab} \cdot \left(J^{\mu}{}_{a}\right)^{-1},$$

that is the essence of this 'scale invariance'.

Canonically, the logarithmic *Jacobi* matrix is the constant identity,

$$\frac{f(x)}{f(x)} = 1 \quad \sim \quad \Gamma^{a}{}_{b} = J^{\mu}{}_{a} \cdot \left(J^{\mu}{}_{a}\right)^{-1} = \delta^{a}{}_{b}\,,$$

remembering that it is her derivatives which in general do not vanish.

The flat *Euclid*ean and *Minkowski* metrics are both scale-invariant and products and partial derivatives of scale-invariant entities are themselves scale-invariant entities. Contraction of mutually contravariant indices leaves scale invariance untouched, but raising and lowering of indices and contraction with the metric breaks scale invariance.

## 2.4 A '*Jacobi* weight'

To quantify the degree of non-scale-invariance of an expression, be the exponent with which the *Jacobi* matrix is multiplied in called the '*Jacobi* weight'. Within expressions and equations, only terms of same weight may be added or equated.

Then the weight of any scale-invariant entity is 0, the weight of the *Jacobi* matrix itself is 1, the weight of the metric tensor is twice that, and the weights of inverses are negative.

The *Jacobi* weights of the relevant entities within this framework are summarized as follows:

*Jacobi* weights

**Basic entities**

| Entity | Weight | similar | Text |
|---|---|---|---|
| $g_{ab}$, $\Gamma_{abc}$ | $+2$ | $f(x)^2, f'(x)f(x)$ | 'containing the metric' |
| $J^{\mu}{}_{b}$, $J^{\mu}{}_{bc}$ | $+1$ | $f'(x), f(x)$ | |
| $\delta^a{}_b$, $\eta^a{}_b$, $\Gamma^a{}_b$, $\Gamma^a{}_{bc}$ | $0$ | $1,\ f'(x)f(x)^{-1}$ | 'free from the metric', 'scale-invariant' |
| $J_{\mu}{}^{b}$ | $-1$ | $f(x)^{-1}$ | |
| $g^{ab}$ | $-2$ | $f(x)^{-2}$ | 'containing the inverse metric' |

**Connection products**

| Entity | Weight | similar | Text |
|---|---|---|---|
| $\Gamma_{\gamma ab}\Gamma^{\gamma}{}_{cd}$, $\Gamma_{ab\gamma}\Gamma^{\gamma}{}_{cd}$ | $+2$ | $f'(x)^2$ | 'containing the metric' |
| $\Gamma^a{}_{b\gamma}\Gamma^{\gamma}{}_{cd}$, $\Gamma_{ab\gamma}\Gamma^{\gamma}{}_{\delta\delta}\, g^{\delta\delta}$ | $0$ | $f'(x)^2f(x)^{-2}$ | 'free from the metric', 'scale-invariant' |
| $\Gamma^a{}_{b\gamma}\Gamma^{\gamma}{}_{\delta\delta}\, g^{\delta\delta}$ | $-2$ | $f'(x)^2f(x)^{-4}$ | 'containing the inverse metric' |

**Connection derivatives**

| Entity | Weight | similar | Text |
|---|---|---|---|
| $\Gamma_{abc,d}$ | $+2$ | $(f'(x)f(x))'$ | 'containing the metric' |
| $\Gamma_{ab\delta,\delta}\, g^{\delta\delta}$ | $0$ | $(f'(x)f(x))'\, f(x)^{-2}$ | 'free from the metric', 'scale-invariant' |
| $\Gamma^a{}_{bc,d}$ | $0$ | $(f'(x)/f(x))'$ | 'free from the metric', 'scale-invariant' |
| $\Gamma^a{}_{b\delta,\delta}\, g^{\delta\delta}$ | $-2$ | $(f'(x)/f(x))'\, f(x)^{-2}$ | 'containing the inverse metric' |

**Curvature tensors**

| Entity | Weight | Text |
|---|---|---|
| $R_{abcd}$ | $+2$ | ‘containing the metric’ |
| $R^{a}{}_{bcd}$, $R_{ab}$ | $0$ | ‘free from the metric’, ‘scale-invariant’ |
| $R^{a}{}_{b}$, $R$ | $-2$ | ‘containing the metric’ |

## 2.5 A Spacetime Test Bench Scheme

In the preceding section we found, that the composition of an infinitesimal embedding from separate infinitesimal embeddings yields desirable simplifications,

- infinitesimal embeddings do not contribute to the overall metric tensor (1),
- the *Jacobi* logarithms are additive (21),
- the *Christoffel* 2nd are additive (22).

If not all embeddings can be made infinitesimal, then by moving any (possibly composed) non-infinitesimal embedding to the place of an ‘outer’ embedding, the ‘inner’ embedding is again free to be built from infinitesimal components only. This way most benefits are retained,

- the overall metric tensor equals the outer metric tensor (1),
- the inner *Jacobi* logarithms are still additive (21),
- additivity of the inner *Christoffel* 2nd (22)
  even extends to additivity with the outer connection (8),

while in this case combining the outer and inner second derivatives, (10), (11),
is still somewhat more complicated.

# 3 Conclusion

It has been shown, that the deduction not only of the metric tensor, but also of the metric connection from the *Jacobi* matrix of an embedding function is a particularly simple and clear operation.

It has also been shown, that a linearization of the metric connection is possible by taking the logarithm of an infinitesimal embedding viewed as a strictly local phenomenon and expressing its derivatives through exactly the *Christoffel* symbol of the second kind, where a notion of ‘scale-invariance’ becomes manifest.

So the connection field can be represented as if simply ‘pasted’ onto distorted spacetime, and separated from the distortion which it itself is actually causing.

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