

TWIN PRIMES

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ABSTRACT

This paper, which is a revision/expansion of the author's earlier paper published in an international mathematics journal in 2003, approaches the twin primes problem from a few different perspectives.

MSC: 11-XX (Number Theory)

INTRODUCTION

Many believe that the twin primes are infinite. In fact, twin primes pairs could easily be found among the integers. There is evidently no region of the natural number system so remote that it lies beyond the largest twin primes pair. It is even possible to forecast the approximate number of twin primes pairs found in any region of the natural number system.

The occurrence of twin primes pairs is evidently unpredictable or random. This means that the chance of 2 numbers x and $x + 2$ being prime (twin primes) is somewhat similar to the chance of getting heads on 2 successive tosses of a coin. If 2 successive tosses of a coin are independent, the chance of success of obtaining heads for the 2 successive tosses of the coin is the product of the chances of success of obtaining a head for each toss of the coin. As each coin has probability $\frac{1}{2}$ of coming up heads with a toss, 2 coins would have probability $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ of coming up a pair of heads with a toss.

The prime number theorem, which had been proven, states that if n is a large number, and we select a number x at random between 0 and n , the chance that x is prime would be approximately $1/\log n$, the larger n is, the better would be the approximation given by $1/\log n$ to the proportion of primes in the numbers up to n . Like 2 coins coming up heads, the chance that both x and $x + 2$ are prime (twin primes) would be approximately $1/(\log n)^2$. That is, there would be approximately $n/(\log n)^2$ twin primes pairs between 0 and n . As n goes to infinity, this fraction approaches infinity. This represents a quantitative version of the twin primes conjecture.

As $x + 2$ being prime depends on the fact that x is already prime, we should modify the estimate $n/(\log n)^2$ to $(1.32032\dots)n/(\log n)^2$.

The following is a comparison between the twin primes predicted by the above formula and the twin primes found, where the agreement is evidently very good:-

| INTERVAL | TWIN PRIMES | |
|--|------------------|--------------|
| | <u>PREDICTED</u> | <u>FOUND</u> |
| 100,000,000 - 100,150,000 | 584 | 601 |
| 1,000,000,000 - 1,000,150,000 | 461 | 466 |
| 10,000,000,000 - 10,000,150,000 | 374 | 389 |
| 100,000,000,000 - 100,000,150,000 | 309 | 276 |
| 1,000,000,000,000 - 1,000,000,150,000 | 259 | 276 |
| 10,000,000,000,000 - 10,000,000,150,000 | 221 | 208 |
| 100,000,000,000,000 - 100,000,000,150,000 | 191 | 186 |
| 1,000,000,000,000,000 - 1,000,000,000,150,000 | 166 | 161 |

All this represents numerical evidence that the twin primes are infinite as we could find more twin primes pairs whenever we look for them. But the proof is lacking.

Keywords: building-blocks; symmetry; infinity; composite numbers; unique factorisation; prime factors

Theorem:- The list of the twin primes pairs is infinite.

Proof:-

Lemma 1: According to the precepts of fractal geometry and group theory, symmetry is a very important, intrinsic part of nature. There is symmetry all around us and within us. There is evident symmetry in human bodies, the structures of viruses and bacteria, polymers and ceramic materials, the permutations of numbers, the universe and many others, even the movements of

prices in financial markets, the growths of populations, the sound of music, the flow of blood through our circulatory system, the behavior of people en masse, etc.. In other words, regularity, pattern, order, uniformity or symmetry is evident everywhere.

The reasoning here makes use of a very important idea in fractal geometry and group theory, namely, symmetry.

A prime number is an integer which is divisible only by 1 and itself, e.g., 2, 3, 7, 19, etc.. A twin primes pair are 2 primes which differ from one another by 2, e.g., 5 & 7, 11 & 13, 17 & 19, and, 29 & 31, etc.. A composite number or non-prime is a product of primes or prime factors, e.g., the composite number 15 is the product of 2 primes, 3 and 5 ($15 = 3 \times 5$), and 231 is the product of 3 primes, 3, 7 and 11 ($231 = 3 \times 7 \times 11$), etc.. The integers or whole numbers are either primes or composites and are infinite.

The primes, which Euclid had proven to be infinite, are the atoms or building-blocks of the infinite integers or whole numbers, which comprise of the infinite list of the odd numbers that are all either primes or products of primes (i.e., composites), and, the infinite list of the even numbers that are all products of primes (i.e., composites, with the exception of 2 which is a prime, e.g., $6 = 2 \times 3$, $8 = 2 \times 2 \times 2$ and $10 = 2 \times 5$, etc.). The infinite list of the integers or whole numbers may be classified as an infinite group, with various symmetries, subgroups and infinite elements, hidden within it. These various symmetries, subgroups and infinite elements, within this infinite group may be classified as follows:-

- (1) Subgroup A: Infinite consecutive primes such as 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, etc. to infinity, separated by 2 integers (twin primes), 4 integers, 6 integers, 8 integers, 10 integers, etc., which, incidentally, except for 2, are all odd numbers; this splitting up of the subgroup into infinite elements is shown below:
 - (i) Element A1: Infinite list of all the primes pairs separated by 2 integers (twin primes) (Example: 17 & 19)
 - (ii) Element A2: Infinite list of all the primes pairs separated by 4 integers/1 odd composite - single composite (Example: 79 & 83 separated by 81)
 - (iii) Element A3: Infinite list of all the primes pairs separated by 6 integers/2 consecutive odd composites - twin composites (Example: 47 & 53 separated by 49 & 51)
 - (iv) Element A4: Infinite list of all the primes pairs separated by 8 integers/3 consecutive odd composites - "triple" composites (Example: 359 & 367 separated by 361, 363 & 365)
 - (v) Element A5: Infinite list of all the primes pairs separated by 10 integers/4 consecutive odd composites - "four-ple" composites (Example: 709 & 719 separated by 711, 713, 715 & 717)

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(2) Subgroup B: Infinite consecutive odd composites such as 9, 15, 21, 25, 27, 33, 35, 39, 45, 49, etc. to infinity, of “various sizes” sandwiched between 2 primes; this splitting up of the subgroup into infinite elements is shown below:

- (i) Element B1: Infinite list of all “1 odd composite sandwiched between 2 primes - single composite” (Example: 9 sandwiched between the primes 7 & 11)
- (ii) Element B2: Infinite list of all “2 consecutive odd composites sandwiched between 2 primes - twin composites” (Example: 253 & 255 sandwiched between the primes 251 & 257)
- (iii) Element B3: Infinite list of all “3 consecutive odd composites sandwiched between 2 primes - “triple” composites” (Example: 685, 687 & 689 sandwiched between the primes 683 & 691)
- (iv) Element B4: Infinite list of all “4 consecutive odd composites sandwiched between 2 primes - “four-ple” composites” (Example: 2,769, 2,771, 2,773 & 2,775 sandwiched between the primes 2,767 & 2,777)
- (v) Element B5: Infinite list of all “5 consecutive odd composites sandwiched between 2 primes - “five-ple” composites” (Example: 19,291, 19,293, 19,295, 19,297 & 19,299 sandwiched between the primes 19,289 & 19,301)

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(3) Subgroup C: Infinite consecutive odd composites separated by 4 integers and 6 integers respectively; this splitting up of the subgroup into the 2 infinite elements is shown below:

- (i) Element C1: Infinite list of all “2 consecutive odd composites separated by 4 integers/1 prime” (Example: 209 & 213 separated by the prime 211)
- (ii) Element C2: Infinite list of all “2 consecutive odd composites separated by 6 integers/2 primes” (Example: 279 & 285 separated by the twin primes 281 & 283)

(4) Subgroup D: Infinite single primes and twin primes separating 2 consecutive odd composites; this splitting up of the subgroup into the 2 infinite elements is shown below:

- (i) Element D1: Infinite list of all the single primes separating 2 consecutive odd composites (Example: 23 separating the 2 consecutive odd composites 21 & 25)
- (ii) Element D2: Infinite list of all the twin primes separating 2 consecutive odd composites (Example: 11 & 13 separating the 2 consecutive odd composites 9 & 15)

- (5) Subgroup E: Infinite consecutive even composites such as 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, etc. to infinity, all separated by only 2 integers; this subgroup may be classified as a single infinite element There is always 1 even number between a twin primes pair, which is separated by 2 integers, a prime and a composite which are separated by 2 integers, and, 2 composites which are separated by 2 integers. That is, the even numbers are always found in Subgroup A, Subgroup B, Subgroup C and Subgroup D above always evenly spaced out in consecutive order by 2 integers.

There is an evident symmetry in the above-mentioned infinite group, which would be broken if any of the elements within were to be finite. There are close interlinks between all the various infinite elements in all the five subgroups above, e.g., the infinity of the list of all the primes pairs each separated by 6 integers (Element A3/Subgroup A) implies the infinity of the list of all the “2 consecutive odd composites sandwiched between 2 primes - twin composites” (Element B2/Subgroup B) and vice versa, the infinity of the list of all the primes pairs each separated by 2 integers (twin primes) (Element A1/Subgroup A) implies the infinity of the list of all the “2 consecutive odd composites separated by 6 integers/2 primes” (Element C2/Subgroup C) and vice versa, the infinity of the list of all the infinite elements (A1, A2, A3, etc. to infinity) in Subgroup A above, which represents the infinity of the list of the primes which Euclid had in fact proven, implies the infinity of the list of all the “2 consecutive odd composites separated by 4 integers/1 prime” (Element C1/Subgroup C) and vice versa, the infinity of the list of all the “2 consecutive odd composites separated by 6 integers/2 primes” (Element C2/Subgroup C) implies the infinity of the list of all the twin primes separating 2 consecutive odd composites (Element D2/Subgroup D) and vice versa, the infinity of all the lists of all the infinite elements (A1, A2, A3, B1, B2, B3, C1 & C2, D1 & D2) in Subgroup A, Subgroup B, Subgroup C and Subgroup D above implies the infinity of the list of the consecutive even composites, i.e., 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, etc. to infinity (Subgroup E), which we know to be true in any case, and vice versa, etc..

Subgroup A and Subgroup B above are practically “mirror” images of one another - they represent the viewing of the primes and the composites from 2 variant angles - the infinitude, or, finiteness of either implies the infinitude, or, finiteness of the other; the same applies to both Subgroup C and Subgroup D above. It is similar to the following way of viewing a glass which is partially filled: this glass could be described as “half full” or “half empty” if it is half filled, “three-quarter full” or “one-quarter empty” if it is three-quarter filled, or, “one-quarter full” or “three-quarter empty” if it is one-quarter filled, etc..

It is evident that the infinitude, or, finiteness of any one of the above-mentioned elements would imply the infinitude, or, finiteness of the other element that is interlinked with it and vice versa. All these infinite elements are evidently entangled together and complementary, being all the infinite building-blocks of the infinite integers or whole numbers. The infinity of the list of the integers or whole numbers, the primes included, in fact implies that all these various elements within it are infinite, and, vice versa, since all these various elements are closely interlinked and could not do without each other.

Therefore, the breaking of the evident intrinsic symmetry of this whole infinite group, i.e., the infinite list of the integers or whole numbers, due to the finiteness of any of the elements within it, could not be possible.

We pose a very important question: Besides questioning whether the infinite list of all the primes pairs separated by 2 integers (twin primes) is really infinite, should we not also be questioning whether the following are really infinite?:

- (a) Infinite lists of all the primes pairs separated respectively by 4 integers, 6 integers, 8 integers, 10 integers and sequentially larger integers (as in Subgroup A above).
- (b) Infinite lists of all the respective consecutive odd composites of “various sizes” sandwiched between 2 primes (as in Subgroup B above).
- (c) The 2 infinite lists with respectively “2 consecutive odd composites separated by 4 integers/1 prime” and “2 consecutive odd composites separated by 6 integers/2 primes” (as in Subgroup C above).
- (d) The 2 infinite lists of respective single primes and twin primes separating 2 consecutive odd composites (as in Subgroup D above).
- (e) Infinite list of the consecutive even composites all separated by only 2 integers (as in Subgroup E above).

Could there possibly be any symmetry-breaking in the above-mentioned infinite group whence one or more of the elements within it would be finite? In particular, could there be a possibility for the symmetry of this infinite group to be broken due to the finiteness of Element A1 (i.e., the finiteness of the twin primes) within it? Since the above-mentioned group, i.e., the list of the integers or whole numbers, is infinite, it is indeed not possible for all of these elements to be finite. And, there is no evident reason to account for why any of these elements, especially Element A1, i.e., the list of primes separated by 2 integers, or, twin primes, should be finite. In fact, all these infinite elements are like the slabs of various sizes in a building. They are all necessary for the construction of the infinite building known as the “infinite list of the integers or whole numbers” and should thus all be infinite, wherein the symmetry of the infinite group, i.e., the infinite list of the integers or whole numbers, would be preserved.

Therefore, by Lemma 1, all the elements in Subgroup A, Subgroup B, Subgroup C, Subgroup D and Subgroup E above are infinite.

Lemma 2: The Fundamental Theorem of Arithmetic or Unique Factorisation Theorem states that there is only one possible combination of primes which will multiply together to produce any particular composite number, e.g., the only combination of primes which will produce the composite number 2,079 is: $3 \times 3 \times 3 \times 7 \times 11$. In the same manner, the following composite numbers are also uniquely factorised:

- (1) $63 = 3 \times 3 \times 7$ (only)
- (2) $153 = 3 \times 3 \times 17$ (only)

$$(3) 1,021,020 = 2 \times 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \text{ (only)}$$

In other words, every positive whole number which is not prime (i.e., every positive whole number which is composite) can be broken up into prime factors, and, this can happen in only 1 way:

$$c = \prod_{p \text{ prime}} p \text{ (in only 1 way)}$$

The 10 consecutive twin primes 3 & 5 to 107 & 109, e.g., give rise to the following 10 composite numbers which can be factorised in only 1 way, i.e., can be factorised only by the respective twin primes:

- (1) $15 = 3 \times 5$ (only)
- (2) $35 = 5 \times 7$ (only)
- (3) $143 = 11 \times 13$ (only)
- (4) $323 = 17 \times 19$ (only)
- (5) $899 = 29 \times 31$ (only)
- (6) $1,763 = 41 \times 43$ (only)
- (7) $3,599 = 59 \times 61$ (only)
- (8) $5,183 = 71 \times 73$ (only)
- (9) $10,403 = 101 \times 103$ (only)
- (10) $11,663 = 107 \times 109$ (only)
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As the composite numbers are infinite, this implies that there is an infinitude of twin primes acting as prime factors for an infinitude of composite numbers in only 1 way as the twin primes are indispensable, i.e., necessary, as prime factors for the formation of an infinite number of composite numbers which can only be formed through the product of twin primes in only 1 way - the twin primes can never be substituted as prime factors of these composite numbers by other primes.

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