Quantum Field Theory as Manifestation of Fractal Geometry

Ervin Goldfain

Photonics CoE, Welch Allyn Inc., Skaneateles Falls, NY 13153, USA

Abstract

We discuss two theoretical arguments strongly suggesting that the continuum limit of Quantum Field Theory (QFT) leads to fractal geometry. The first argument stems from the Path Integral formulation of QFT, whereas the second one is an inevitable consequence of the Renormalization Group (RG).

1. QFT as Critical Behavior in Statistical Physics

A basic task in perturbative QFT is to compute the time-ordered *n*-point Green function, i.e. [1]

$$\langle 0 | T\{\varphi(x_1)\varphi(x_2)...\varphi(x_n)\} | 0 \rangle = \frac{\int D\varphi\,\varphi(x_1)\varphi(x_2)...\varphi(x_n)\, e^{iS}}{\int D\varphi\, e^{iS}} \qquad (1)$$

Performing the rotation to Euclidean space $e^{iS} = e^{-S_E}$ and taking the above integral to run over all configurations that vanish as the Euclidean time goes to infinity ($t_E = \pm\infty$), leads to the conclusion that (1) is formally identical to the correlation function of a classical statistical system. A natural question is then: *What kind of statistical system is able to duplicate the properties of a QFT described by (1)?*

In order to compute (1), it is convenient to discretize the four-dimensional Euclidean space using, for example, a four-dimensional lattice with constant spacing δ. Under the assumption that the number of lattice sites is finite, the path integral of (1) becomes well defined and the question posed above amounts to taking the *continuum limit* $\delta \to 0$ at the end of calculations.

To fix ideas, consider the two-point Green function for a massive field theory defined on a four-dimensional spacetime with Euclidean metric δ_μ^ν

$$\langle \varphi(x)\varphi(0) \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{\exp(ipx)}{p^2 + m^2} \tag{2}$$

with $|p|^2 = p_\mu p^\mu$ and $px = p_\mu x^\mu$. Calculations are considerably simplified if $m|x| >> 1$, in which case (2) becomes

$$\langle \varphi(x)\varphi(0) \rangle \sim \frac{1}{|x|^2} \exp(-m|x|) \tag{3}$$

Expressing the spacetime separation as $|x| = n\delta$ and assuming $n >> 1$, leads to

$$\langle \varphi(x)\varphi(0) \rangle \sim \exp(-n\delta m) \tag{4}$$

By analogy with statistical physics, the behavior of

$$\langle \varphi(x)\varphi(0) \rangle \sim \exp(-\frac{n}{\xi}) \tag{5}$$

determines the dimensional correlation length ξ. Comparing (4) and (5) yields

$$\xi = \frac{1}{\delta m} \tag{6}$$

It is immediately apparent that the continuum limit $\delta \to 0$ of the massive theory ($m \neq 0$) implies a divergent correlation length, that is, $\xi \to \infty$. This conclusion shows that QFT models phenomena that are strikingly similar with the ones describing critical behavior in classical statistical physics. Since all statistical phenomena near criticality are scale-free and lay on a fractal foundation [2], it is clear that *the continuum limit of QFT necessarily leads to fractal geometry*.

2. RG and the Onset of Self-similarity in QFT

As it is known, the RG studies the evolution of dynamical systems scale-by-scale as they approach criticality [2]. It does so by defining a map between the observation scale (μ) and the distance ($x = |\mu - \mu_c|$) from the critical point. The universal utility of the RG is based on the existence of *self-similarity of all observables* as $x \to 0$.

To illustrate this point, consider a generic model whose fields are evenly distributed on the discrete lattice of points. The behavior of the Lagrangian $L(x)$ in the RG formalism is given by the following set of transformations [2]

$$x' = \sigma(x) \tag{7}$$

$$L(x) = h(x) + \frac{1}{\Delta} L[\sigma(x)] \tag{8}$$

Here, Δ is a constant describing the rescaling of the Lagrangian upon shifting the scale to the critical value ($\mu \to \mu_c$), the function $\sigma(x)$ is called the *flow map* and

$$L(x) = L(\mu) - L(\mu_c) \tag{9}$$

such that $L(x)=0$ at the critical point. The function $h(x)$ represents the non-singular part of $L(x)$. Assuming that both $L(x)$ and $\varphi(x)$ are differentiable, the critical points are defined as the set of values at which $L(x)$ becomes singular, that is, when $\frac{dL}{dx}\to\infty$. Then, the formal solution of (8) can be presented as the recursive sequence

$$f_0(x)=h(x) \tag{10}$$

$$f_{n+1}(x)= f_0(x)+\frac{1}{\Delta} f_n\left|\sigma(x)\right|, \quad n=0,1,2.... \tag{11}$$

where

$$f_n(x)=\sum_{i=0}^{n}\frac{1}{\Delta^i}h\,[\sigma^{(i)}(x)] \tag{12}$$

Here, the superscripts (n) denote composition, that is,

$$\sigma^{(2)}=\sigma\left[\sigma(x)\right], \sigma^{(3)}=\sigma\left[\sigma^{(2)}(x)\right]... \tag{13}$$

The renormalized Lagrangian assumes the form

$$L(x)=\lim_{n\to\infty} f_n(x) \tag{14}$$

The above relation indicates that *all copies of the Lagrangian specified by the iteration index* n *become self-similar in the limit* $n\to\infty$. Furthermore, if x designates a generic coupling constant ($x=g(\mu)$) whose critical value occurs at $g_c=g(\mu_c)$, the Lagrangian

$$L(g) = \sum_{n=0}^{\infty} \frac{1}{\Delta^n} h\left[\sigma^{(n)}(g)\right] \tag{15}$$

may be shown to become singular at $g = g_c$. In the neighborhood of $g = g_c$ ( ) obeys a power law that is typical for the onset of fractal behavior, namely:

$$L(g) = (const).(g - g_c)^m \tag{16}$$

where m stands for the critical exponent.

3. Conclusion

This brief analysis points out that QFT is a manifestation of fractal geometry. As we have repeatedly shown over the years, exploiting the fractal underpinnings of QFT and RG may lead to a satisfactory resolution of the many puzzles associated with the Standard Model of particle physics [3].

References

[1] M. Maggiore, "*A Modern Introduction to Quantum Field Theory*", Oxford University Press, 2006.

[2] D. Sornette, "*Critical Phenomena in Natural Sciences: Chaos, Fractals, Self-organization and Disorder*", Springer Series on Complex Systems, 2006.

[3] http://scholar.google.com/citations?user=xWO3weoAAAAJ&hl=en