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On Andrica's Conjecture, Cramér's Conjecture, gaps Between Primes and Jacobi Theta Functions III: A Simple Proof forAndrica's Conjecture

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1. NOTATION

We will use the notation of Armitage and Eberlein, [see 1, p. 103]:
k is a real number such that $0 < k < 1$;$k' = (1-k^2)^{1/2}$ is the complementary modulus,

$$(1) \qquad K = K(k) = \int_0^{2\pi} \frac{d\psi}{(1-k^2\sin^2\psi)}, \qquad K' = K(k').$$

$$(2) \qquad \tau = iK'/K, \qquad q = \exp(\pi i \tau).$$

2. PRELIMINARES

The Rosser's theorem [2] states that p_n is larger than $n\log n$. This can be improved by the following pair of bounds:

$$(3) \log n + \log\log n - 1 < \frac{p_n}{n} < \log n + \log\log n,$$

for $n \geq 6$.

3. THEOREM

THEOREM 1 (Andrica's Conjecture).*For* $n \in \mathbb{N}_{\geq 1}$, *then*

$$(3) \sqrt{p_{n+1}} - \sqrt{p_n} < 1.$$

Proof 1. Step 1. In previous paper[3, *On Andrica's Conjecture, gaps Between Primes and Jacobi Theta Functions*], Theorem 1, we discover that

$$(4) \quad \left(\frac{\theta_3-\theta_2}{\theta_2}\right)\sqrt{n\log n + n\log\log n - n} < \sqrt{p_{n+1}} - \sqrt{p_n} < \left(\frac{\theta_3-\theta_2}{\theta_2}\right)\sqrt{n\log n + n\log\log n},$$

for $k := \frac{p_n}{p_{n+1}}$ to be a k modulus.Ergo, dividindo ambos os membros of (14) by $\left(\frac{\theta_3-\theta_2}{\theta_2}\right)\sqrt{n\log n + n\log\log n}$, we have

$$\sqrt{\frac{n\log n + n\log\log n - n}{n\log n + n\log\log n}} < \left(\frac{\sqrt{p_{n+1}} - \sqrt{p_n}}{\sqrt{n\log n + n\log\log n}}\right)\left(\frac{\theta_2}{\theta_3-\theta_2}\right) < 1,$$

$$(5) \qquad \sqrt{1 - \frac{1}{\log n + \log\log n}} < \left(\frac{\sqrt{p_{n+1}} - \sqrt{p_n}}{\sqrt{n\log n + n\log\log n}}\right)\left(\frac{\theta_2}{\theta_3-\theta_2}\right) < 1,$$

But, by the Rosser's theorem, we find

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$$(6)\sqrt{(n+1)\log(n+1)+(n+1)\log\log(n+1)-(n+1)}-\sqrt{n\log n+n\log\log n-n}$$
$$<\sqrt{p_{n+1}}-\sqrt{p_n}$$
$$<\sqrt{(n+1)\log(n+1)+(n+1)\log\log(n+1)}-\sqrt{n\log n+n\log\log n}.$$

Dividing (6) by $\sqrt{n\log n+n\log\log n}$, we obtain

$$(7)\sqrt{\left(1+\frac{1}{n}\right)\frac{\log(n+1)+\log\log(n+1)-1}{\log n+\log\log n}}-\sqrt{1-\frac{1}{\log n+\log\log n}}<\frac{\sqrt{p_{n+1}}-\sqrt{p_n}}{\sqrt{n\log n+n\log\log n}}$$
$$<\sqrt{\left(1+\frac{1}{n}\right)\frac{\log(n+1)+\log\log(n+1)}{\log n+\log\log n}-1}.$$

Dividing (5) by (7), we encounter

$$(8)\frac{\sqrt{1-\frac{1}{\log n+\log\log n}}}{\sqrt{\left(1+\frac{1}{n}\right)\frac{\log(n+1)+\log\log(n+1)-1}{\log n+\log\log n}}-\sqrt{1-\frac{1}{\log n+\log\log n}}}<\frac{\theta_2}{\theta_3-\theta_2}$$
$$<\frac{1}{\sqrt{\left(1+\frac{1}{n}\right)\frac{\log(n+1)+\log\log(n+1)}{\log n+\log\log n}-1}}.$$

Substituting the left-hand side of (8) in (5), we have

$$(9)\qquad\frac{\sqrt{1-\frac{1}{\log n+\log\log n}}}{\sqrt{\left(1+\frac{1}{n}\right)\frac{\log(n+1)+\log\log(n+1)-1}{\log n+\log\log n}}-\sqrt{1-\frac{1}{\log n+\log\log n}}}\left(\frac{\sqrt{p_{n+1}}-\sqrt{p_n}}{\sqrt{n\log n+n\log\log n}}\right)$$
$$<\left(\frac{\sqrt{p_{n+1}}-\sqrt{p_n}}{\sqrt{n\log n+n\log\log n}}\right)\left(\frac{\theta_2}{\theta_3-\theta_2}\right)<1,$$

consequently,

$$(10)\frac{\sqrt{1-\frac{1}{\log n+\log\log n}}}{\sqrt{(n+1)[\log(n+1)+\log\log(n+1)-1]}-\sqrt{n(\log n+\log\log n-1)}}\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right)$$
$$<\left(\frac{\sqrt{p_{n+1}}-\sqrt{p_n}}{\sqrt{n\log n+n\log\log n}}\right)\left(\frac{\theta_2}{\theta_3-\theta_2}\right)<1.$$

Simplifying (10), we find

$$(11)\frac{\sqrt{1-\frac{1}{\log n+\log\log n}}}{\sqrt{(n+1)[\log(n+1)+\log\log(n+1)-1]}-\sqrt{n(\log n+\log\log n-1)}}\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right)<1$$
$$\Rightarrow\sqrt{p_{n+1}}-\sqrt{p_n}$$
$$<\frac{\sqrt{(n+1)[\log(n+1)+\log\log(n+1)-1]}-\sqrt{n(\log n+\log\log n-1)}}{\sqrt{\frac{\log n+\log\log n-1}{\log n+\log\log n}}}.$$

But, we observe that $\sqrt{\log(n+1)+\log\log(n+1)-1} \cong \sqrt{\log n+\log\log n-1}$;for example: 1) for $n=11$, then $\sqrt{\log(12)+\log\log(12)-1} = 1.54762454851\ldots$ and $\sqrt{\log(11)+\log\log(11)-1} = 1.50747691714\ldots$; 2) for $n=110$, then $\sqrt{\log(111)+\log\log(111)-1} = 2.2932767734\ldots$ and $\sqrt{\log(110)+\log\log(110)-1} = 2.29088303384\ldots$;3) for $n=1100$, then $\sqrt{\log(1101)+\log\log(1101)-1} = 2.81965456368\ldots$ and $\sqrt{\log(1100)+\log\log(1100)-1} = 2.81947041741\ldots$

Therefore, the equation (11) can be written as

$$(12)\sqrt{p_{n+1}}-\sqrt{p_n} \lesssim \frac{\sqrt{(n+1)(\log n+\log\log n-1)}-\sqrt{n(\log n+\log\log n-1)}}{\sqrt{\dfrac{\log n+\log\log n-1}{\log n+\log\log n}}}$$

$$=\frac{\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{\log n+\log\log n-1}}{\sqrt{\dfrac{\log n+\log\log n-1}{\log n+\log\log n}}}$$

$$=\frac{\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{\log n+\log\log n-1}}{\sqrt{\log n+\log\log n-1}}\sqrt{\log n+\log\log n}$$

$$=\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{\log n+\log\log n} < \sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{\log n}$$

$$< \sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n}.$$

On the other hand,for $f(n)=\sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n}$, the maximum occurs when n tends at infinity, and $f(n)\downarrow\frac{1}{\sqrt{2}}$;we have,

for$\lim_{n\to\infty}\sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n}=\frac{1}{\sqrt{2}}=0.707106781187\ldots$For $n\in\mathbb{N}_{\geq 6}$,because of condition for Rosser's theorem,then

$$(13)\sqrt{p_{n+1}}-\sqrt{p_n} < \sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n} \leq \lim_{n\to\infty}\sqrt{2}\left(\sqrt{(n+1)}-\sqrt{n}\right)\sqrt{n}$$
$$=0.707106781187\ldots<1.$$

Step 2. For $n=1$, then $\sqrt{p_2}-\sqrt{p_1}=\sqrt{3}-\sqrt{2}=0.317837245196\ldots<1$; for $n=2$, then $\sqrt{p_3}-\sqrt{p_2}=\sqrt{5}-\sqrt{3}=0.504017169931\ldots<1$; for $n=3$, then $\sqrt{p_4}-\sqrt{p_3}=\sqrt{7}-\sqrt{5}=0.409683333565\ldots<1$; for $n=4$, then $\sqrt{p_5}-\sqrt{p_4}=\sqrt{11}-\sqrt{7}=0.670873479291\ldots<1$; for $n=5$, then $\sqrt{p_6}-\sqrt{p_5}=\sqrt{13}-\sqrt{11}=0.267949192431\ldots<1.\square$

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[2] http://en.wikipedia.org/wiki/Prime_number_theorem, available in April 25, 2013.

[3] Prof. Dr. Raja Rama Gandhi and Guedes, Edigles, *On Andrica's Conjecture, gaps Between Primes and Jacobi Theta Functions,*

Table 1

The order of table: 1.[a] column: n; 2.[a] column: $\sqrt{p_{n+1}}-\sqrt{p_n}$; 3.[a] column: $\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{2n}$; 4.[a] column: $\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{2n}-\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right)$.

1	0.317837245196	0.585786437627	0.267949192431
2	0.504017169931	0.635674490392	0.13165732046
3	0.409683333565	0.656338798447	0.24665546488
4	0.670873479291	0.667701070844	−0.00317240844
5	0.288926485108	0.674898880549	0.385972395441
6	0.517554350154	0.679870015673	0.16231566552
7	0.235793317923	0.683510307647	0.447716989724
8	0.436932579772	0.686291501015	0.24935892124
9	0.589333283822	0.688485803641	0.09915251982
10	0.182599555695	0.69026135046	0.507661794765
11	0.514998167468	0.691727623168	0.1767294557
12	0.320361707134	0.692958984179	0.37259727704
13	0.154314286869	0.694007717489	0.53969343062
14	0.298216076099	0.694911658696	0.396695582597
15	0.424455288879	0.69569886461	0.27124357573
16	0.401035858588	0.696390581412	0.29535472282
17	0.129103928038	0.697003193363	0.567899265325
18	0.375103095966	0.697549538528	0.32244644256
19	0.240797001304	0.698039819092	0.457242817788
20	0.117853972141	0.698482244917	0.580628272776
21	0.344190671998	0.698883497306	0.3546928253
22	0.222239161828	0.699249068966	0.477009907137
23	0.323547552912	0.699583517091	0.37603596417
24	0.414876669739	0.699890654423	0.28501398468
25	0.201017819324	0.700173695313	0.499155875988
26	0.0990159439713	0.70043536869	0.601419424719
27	0.195188867696	0.700678006374	0.505489138677
28	0.096226076122	0.700903612773	0.604677536651
29	0.189839303824	0.701113920409	0.511274616585
30	0.63928185685	0.701310434503	0.06202857765
31	0.176095472675	0.701494469074	0.525398996399
32	0.25917676846	0.701667176365	0.442490407905
33	0.085126211832	0.701829570996	0.616703359164
34	0.416729493182	0.701982549917	0.28525305673
35	0.0816501117108	0.702126908986	0.620476797275
36	0.241758358697	0.702263356824	0.460504998127
37	0.237181248662	0.702392526447	0.465211277785
38	0.155702648516	0.702514985087	0.546812336571
39	0.230098454645	0.702631242525	0.472532787879
40	0.226141722293	0.702741758182	0.476600035888
41	0.0745358868141	0.702846947188	0.628311060374
42	0.366650914011	0.702947185597	0.33629627158
43	0.0721690283646	0.703042814876	0.630873786511
44	0.143224858168	0.703134145793	0.559909287625
45	0.0710671320477	0.703221461792	0.632154329745
46	0.419103066668	0.703305021932	0.28420195526
47	0.407345476734	0.703385063451	0.29603958671
48	0.133334650251	0.70346180402	0.570127153769
49	0.0662267771022	0.703535443718	0.637308666616
50	0.131591572052	0.703606166774	0.572014594722
51	0.195287311266	0.703674143105	0.508386831838
52	0.0645498625197	0.703739529681	0.639189667162
53	0.318804821495	0.703802471737	0.38499765024
54	0.188240024126	0.703863103844	0.515623079717

55	0.186055198345	0.703921550874	0.517866352529
56	0.18394472663	0.703977928859	0.52003320223
57	0.0608581662976	0.704032345758	0.643174179461
58	0.181239343939	0.704084902146	0.522845558207
59	0.119737637147	0.704135691838	0.584398054691
60	0.0595492270205	0.704184802444	0.644635575424
61	0.294638927363	0.704232315877	0.409593388514
62	0.404172699311	0.704278308806	0.30010560949
63	0.113776620613	0.70432285307	0.590546232457
64	0.0566139244057	0.704366016053	0.647752091647
65	0.11268780181	0.704407861024	0.591720059213
66	0.388911583895	0.704448447446	0.31553686355
67	0.164154352025	0.704487831256	0.540333479231
68	0.270376259511	0.704526065128	0.434149805617
69	0.0536056820722	0.704563198701	0.650957516629
70	0.106752535786	0.704599278799	0.597846743013
71	0.15900109344	0.704634349627	0.545633256186
72	0.209948739171	0.704668452948	0.494719713776
73	0.15596385516	0.704701628254	0.548737773094
74	0.154714418103	0.704733912918	0.550019494815
75	0.102463456849	0.704765342331	0.602301885482
76	0.152697132535	0.704795950033	0.552098817498
77	0.201775921855	0.704825767834	0.503049845979
78	0.100125549329	0.704854825922	0.604729276592
79	0.198764021656	0.704883152967	0.506119131311
80	0.245741074302	0.704910776214	0.459169701912
81	0.0487950382245	0.704937721571	0.656142683346
82	0.242254963343	0.704964013688	0.462709050345
83	0.0481125546581	0.704989676037	0.656877121379
84	0.143674793072	0.705014730977	0.561339937905
85	0.0952383400922	0.705039199822	0.609800859729
86	0.142054920568	0.705063102898	0.56300818233
87	0.187938226014	0.705086459605	0.51714823359
88	0.093352227152	0.705109288464	0.611757061312
89	0.0465242377661	0.705131607168	0.658607369402
90	0.0927479936243	0.705153432631	0.612405439006
91	0.275885843265	0.705174781023	0.429288937758
92	0.182007862474	0.705195667818	0.523187805343
93	0.0904433154464	0.705216107827	0.614772792381
94	0.179788097528	0.705236115234	0.525448017706
95	0.0893535883171	0.705255703629	0.615902115312
96	0.133366853351	0.705274886039	0.571908032688
97	0.264396075669	0.705293674955	0.440897599286
98	0.0437688310319	0.705312082363	0.661543251332
99	0.390213447167	0.705330119766	0.31511667259
100	0.128624427827	0.705347798209	0.576723370383
101	0.212816315359	0.705365128302	0.492548812944
102	0.126773592997	0.705382120241	0.578608527244
103	0.126099848343	0.705398783827	0.579298935483
104	0.0418854069439	0.705415128487	0.663529721544

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105	0.125218008231	0.705431163291	0.58021315506
106	0.207258580242	0.705446896967	0.498188316724
107	0.1235084446	0.705462337918	0.581953893318
108	0.122885177269	0.705477494239	0.58259231697
109	0.0408248432217	0.705492373727	0.664667530506
110	0.122068645247	0.7055069839	0.583438338653
111	0.12146681677	0.705521332003	0.584054515233
112	0.0806478904686	0.705535425024	0.624887534556
113	0.040225912501	0.705549269707	0.665323357206
114	0.240002764911	0.705562872557	0.465560107646
115	0.198264428183	0.705576239857	0.507311811674
116	0.0394668638676	0.705589377673	0.666122513805
117	0.0787500177419	0.705602291863	0.626852274121
118	0.117669994407	0.705614988089	0.587944993681
119	0.117130627625	0.705627471823	0.588496844197
120	0.038924958378	0.705639748355	0.666714789977
121	0.23232327778	0.705651822802	0.473328545021
122	0.0769801203697	0.705663700113	0.628683579743
123	0.115045028228	0.705675385076	0.590630356847
124	0.152610165445	0.705686882326	0.553076716881
125	0.189525733557	0.705698196352	0.516172462794
126	0.150649321641	0.705709331497	0.555060009856
127	0.187121444195	0.705720291971	0.518598847776
128	0.148762169841	0.705731081853	0.556968912011
129	0.111035215936	0.705741705093	0.594706489157
130	0.110581696774	0.705752165523	0.595170468749
131	0.0734719027416	0.705762466858	0.632290564116
132	0.146352871211	0.705772612699	0.559419741488
133	0.109253772305	0.705782606541	0.596528834235
134	0.0725954638722	0.705792451774	0.633196987901
135	0.144620799456	0.705802151688	0.561181352231
136	0.0720283011916	0.705811709478	0.633783408286
137	0.250642729295	0.705821128243	0.455178398948
138	0.177668148775	0.705830410997	0.528162262222
139	0.211736879669	0.705839560663	0.494102680993
140	0.0351364251405	0.705848580083	0.670712154943
141	0.175035831992	0.70585747202	0.530821640027
142	0.0348790117733	0.705866239156	0.670987227382
143	0.0696311135347	0.705874884101	0.636243770566
144	0.0347524086791	0.705883409391	0.671131000712
145	0.173136618144	0.705891817495	0.53275519935
146	0.2406670171	0.705900110811	0.465233093711
147	0.0683986035884	0.705908291674	0.637509688085
148	0.0341394428968	0.705916362356	0.671776919459
149	0.0681598636311	0.705924325069	0.637764461438
150	0.237324146785	0.705932181963	0.468608035179
151	0.06745836939	0.705939935136	0.638481565746
152	0.0336717568956	0.705947586627	0.672275829731
153	0.0672293074636	0.705955138423	0.638725830959
154	0.33389546908	0.705962592461	0.37206712338
155	0.0663358528219	0.705969950626	0.639634097804
156	0.132236236875	0.705977214756	0.573740977881
157	0.164488525808	0.705984386642	0.541495860834
158	0.130954421771	0.705991468031	0.57503704626

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159	0.065267570328	0.705998460625	0.640730890297
160	0.0976418065023	0.706005366083	0.60836355958
161	0.097332974008	0.706012186023	0.608679212015
162	0.225925530066	0.706018922025	0.480093391959
163	0.0642492908333	0.706025575629	0.641776284796
164	0.0961263144098	0.706032148338	0.609905833928
165	0.0958315970062	0.706038641618	0.610207044611
166	0.127321664212	0.706045056899	0.578723392687
167	0.0951543302995	0.70605139558	0.610897065281
168	0.189453540843	0.706057659024	0.516604118181
169	0.0629005771419	0.706063848562	0.64316327142
170	0.0941184733457	0.706069965495	0.61195149215
171	0.0313112183162	0.706076011094	0.674764792777
172	0.156098098663	0.706081986598	0.549983887934
173	0.0311286439717	0.706087893219	0.674959249247
174	0.093205562596	0.706093732143	0.612888169547
175	0.154746558831	0.706099504525	0.551352945695
176	0.0308606734913	0.706105211498	0.675244538007
177	0.15386479491	0.706110854167	0.552246059257
178	0.0306858239975	0.706116433612	0.675430609614
179	0.0918846747414	0.70612195089	0.614237276149
180	0.274117595125	0.706127407035	0.432009811909
181	0.0606060861584	0.706132803056	0.645526716898
182	0.0302613798056	0.706138139943	0.675876760137
183	0.0604398137253	0.706143418661	0.645703604936
184	0.0904534874729	0.706148640157	0.615695152685
185	0.0902077998622	0.706153805357	0.615946005495
186	0.119898323443	0.706158915165	0.586260591722
187	0.0896422260953	0.706163970469	0.616521744374
188	0.0894030725908	0.706168972135	0.616765899544
189	0.325795674552	0.706173921013	0.38037824646
190	0.0294627853245	0.706178817933	0.676716032609
191	0.14693211726	0.706183663711	0.55925154645
192	0.117091454577	0.706188459143	0.589097004565
193	0.145803290341	0.706193205009	0.560389914667
194	0.0871857990253	0.706197902074	0.619012103049
195	0.0869657244367	0.706202551087	0.619236826651
196	0.115614823985	0.706207152782	0.590592328797
197	0.172702632209	0.706211707877	0.533509075668
198	0.0573775504888	0.706216217077	0.648838666588
199	0.0858898150666	0.706220681072	0.620330866005
200	0.0856793858248	0.706225100537	0.620545714713