FEATURES OF THE HILBERT BOOK MODEL

Colophon

Written by Ir J.A.J. van Leunen
The subject of this book is a new model of fundamental physics.

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Preface

I started the Hilbert Book Model during my studies in physics in the sixties on the Technical University of Eindhoven (TUE).

In the first two years the lectures concerned only classical physics. In the third year quantum physics was introduced. I had great difficulty in understanding why the methodology of doing physics changed drastically. So I went to the teacher, which was an old nearly retired and very wise professor and asked him:

"Why is quantum mechanics done so differently from classical mechanics?".

His answer was short. He stated":

"The reason is that quantum mechanics is based on the superposition principle".

I quickly realized that this was part of the methodology and could not be the reason of the difference in methodology. So I went back and told him my concern. He told me that he could not give me a better answer and if I wanted a useful answer I should research that myself. So, I first went to the library, but the university was quite new and its library only contained rather old second hand books, which they got as a gift from other institutions. Next I went to the city's book shops. I finally found a booklet from P. Mittelstaedt: (Philosophische Probleme der modernen Physik, BI Hochschultaschenbücher, Band 50, 1963) that contained a chapter on quantum logic.

Small particles appear to obey a kind of logic that differs from classical logic. As a result their dynamic behavior differs from the behavior of larger objects. I concluded that this produced the answer that I was looking for.

I searched further and encountered papers from Garret Birkhoff and John von Neumann that explained the correspondence between

quantum logic and separable Hilbert spaces. That produced a more conclusive answer to my question.

The lectures also told me that observables were related to eigenvalues of Hermitian operators. These eigenvalues are real numbers. However, it was clearly visible that nature has a 3+1D structure. So I tried to solve that discrepancy as well. After a few days of puzzling I discovered a number system that had this 3+1D structure and I called them compound numbers. I went back to my professor and asked him why such compound numbers were not used in physics. Again he could not give a reasonable answer.

When I asked the same question to a much younger assistant professor he told me that these numbers were discovered more than a century earlier by William Rowan Hamilton when he was walking with his wife over a bridge in Dublin. He was so glad about his discovery that he carved the formula that treats the multiplication of these numbers into the sidewall of the bridge. The inscription has faded away, but it is now molded in bronze and fixed to the same wall. The numbers are known as quaternions. So, I went to the library and searched for papers on quaternions.

In those years C. Piron wrote his papers on the number systems that can be used by Hilbert spaces. That information completed my insight in this subject. I finalized my physics study with an internal paper on quaternionic Hilbert spaces.

The university was specialized in applied physics and not in theoretical physics. This did not stimulate me to proceed with the subject. Next, I went into a career in industry where I used my knowledge of physics in helping to analyze intensified imaging and in assisting with the design of night vision equipment and X-ray image intensifiers. That put me with my nose on the notion of quanta.

The output window of image intensifiers did not show radiation. Instead they showed clouds of impinging quanta. In those times I had not much opportunity to deliberate on that fact. However, after my retirement I started to rethink the matter. That was the instant that the Hilbert Book Model project was started.

In 2009 I started the Hilbert Book Model project. The HBM is a very simple model of a tiny part of physics that is completely deduced. For that reason it is strictly based on a solid foundation. For that foundation I choose the lattice structure of traditional quantum logic. The lattice structure of this logic system is isomorphic to the lattice structure of the sub-spaces of a Hilbert space.

Since neither the logic system nor the Hilbert space can represent dynamics, a full dynamics model is based on an ordered sequence of such static sub-models. This sequence shows great similarity with the set of pages of a book. This has led to the name "Hilbert Book Model"

Thus, in a few words: The Hilbert Book Model tries to explain the existence of quanta. It does that by starting from traditional quantum logic.

You will find the model to be in many aspects controversial and non-conventional. That is why the author took great efforts in order to keep the model self-consistent.

Its main purpose is to get insight into the possibilities of the physical toolkit.

Each time that I read this book I encounter small and sometimes big inconsistencies. When I see them I repair them. Due to my sloppy nature there must still be a lot of them left. I apologize to the reader

for this inconvenience. I do not consider myself a good and precise mathematician and I consider myself as a horrible physicist. The Great Creator must be a lot better. For a better manuscript you better invite Him. He constructed this structure.

If a mathematical theory is self-consistent, then there is a realistic chance that nature somewhere somehow uses it.

If that theory is compatible with traditional quantum logic, then there is a much larger *chance that nature will use it.*

This drives my intuition.

HvL

Ir J.A.J. van Leunen

PHYSICS OF THE HILBERT BOOK MODEL

ACKNOWLEDGEMENTS

I thank my wife Albertine, who tolerated me to work days and nights on a subject that can only be fully comprehended by experts in this field. For several years she had to share me with my text processor. She stimulated me to bring this project to a feasible temporary end, because this project is in fact a never ending story.

I also have to thank my friends and discussion partners that listened to my lengthy deliberations on this non society chitchat suitable subject and patiently tolerated that my insights changed regularly.

DETAILS

The Hilbert Book Model is the result of a still ongoing research project.

That project started in 2009.

This version added sections on the progression factor, progression interval, the Higgs phenomenon and the covariant derivative.

The continuing status of the project can be followed at
http://www.e-physics.eu

The author's e-print site is:
http://vixra.org/author/j_a_j_van_leunen.
This book is accompanied by a slide show at
http://vixra.org/abs/1202.0033

The nice thing about laws of physics is that they repeat themselves. Otherwise they would not be noticed. The task of physicists is to notice the repetition.

Contents

1. Introduction

I present you my personal view on the lower part of the hierarchy of objects that occur in nature. Only fields and elementary particles are treated in some detail. Composite particle objects are treated in a more general way. Cosmology is touched.

For the greater part, the model is deduced. For that reason the model is founded on a solid and well accepted foundation. That foundation is traditional quantum logic.

The model does not aim at experimental verification of its results, but it uses experimentally verified results of physics as a guidance. The model uses mathematical tools for extending the foundation. In some cases "new" mathematics is applied.

The paper is founded on three starting points:

- A sub-model in the form of traditional quantum logic that represents a static status quo.
- A correlation vehicle that establishes cohesion between subsequent members of a sequence of such static sub-models.
- The cosmological principle.

Further it uses a small set of hypotheses. It turns out that the cosmological principle is already a corollary of the first two points.

The correlation vehicle must provide sufficient cohesion between the subsequent members of the sequence. The cohesion must not be too stiff otherwise no dynamics will take place.

The cosmological principle means that at large scales, universe looks the same for whomever and wherever you are. One of the consequences is that at larger scales universe possesses no preferred directions. It is ***quasi-isotropic*** (on average isotropic).

The mathematical concepts are treated in more detail in the second part, which is called Q-formulæ

The HBM refines quantum logic to Hilbert Logic. A Hilbert logic system resembles a separable Hilbert space much closer than quantum logic does. Together with quantum logic this refined logical system represents a new hierarchy that introduces nature's building blocks and their constituents.

The paper explains[1] all features of fundamental physics that are encountered in the discussed hierarchy which ranges from propositions about physical objects until elementary particles and their composites. Amongst them are the cosmological principle, the existence of quantum physics, the existence of a maximum speed of information transfer, the existence of physical fields, the origin of curvature, the origin of inertia, the dynamics of gravity, the existence of elementary particles, the existence of generations of elementary particles, the existence of the Pauli principle and the history of the universe.

On the other hand the current HBM does not explore further than composites that are constructed from elementary particles. It only touches some aspects of cosmology.

New mathematics is involved in the dynamic generation of potential functions.

[1] Or it indicates a possible explanation

2 The Book Model

The name of the research project that treats this investigation is due to the main starting point on which this study is based.

The Hilbert Book Model (HBM) is based on the assumption that universe steps with universe wide progression steps and that the static status quo of each of these steps can be described by an infinite dimensional separable Hilbert space and its Gelfand triple. With other words a dynamic model will consist of an ordered sequence of these static sub-models.

The sequence of the static models show similarity with the sequence of pages in a book. That is why the name "Hilbert Book Model" is selected for the project.

The progression step size defines an ultra-high frequency, which represents the ***basic carrier frequency*** for transport of information.

3 General remarks

3.1 Completely deduced model

The Hilbert Book Model is completely deduced. It is based on a solid foundation, which is extended by using trustworthy mathematical tools. We want to understand the physics of the developed model. For that reason the first priority of the HBM is to understand how this model works and it is not considered her primary task to verify whether nature behaves that way. This is compensated by pursuing a strong degree of self-consistence of the model. At the same time the knowledge of how nature works is a guide in the development of the model.

For example the HBM uses proper time instead of coordinate time. Proper time is a Lorentz invariant measure of time. The corresponding clock ticks at the location of the observed item. Our common notion of time is coordinate time. The coordinate time clock ticks at the location of the observer. The HBM adds to this fact that all proper time clocks are synchronized. The HBM does not bother about the fact that in general proper time cannot practicably be measured.

Further, the model includes lower level objects that cannot be observed as individuals. Only as groups these objects become noticeable behavior.

The result is that the HBM introduces its own methodology that often deviates considerably from the methodology of contemporary physics. The advantage is that this approach enables the researcher to dive deeper into the undercrofts of physics than is possible with conventional methodology.

As a consequence the HBM must be reluctant in comparing these methodologies and in using similar names. Confusions in discussion groups about these items have shown that great care is necessary. Otherwise, the author can easily be accused from stealing ideas from other theories that are not meant to be included in the HBM model.

This again will make it difficult to design measurements. Measuring methods are designed for measuring physical phenomena that are common in contemporary physics. This is best assured when is sought for phenomena that are similar between the model and contemporary physics. This action contradicts the caution not to use similar terms and concepts. This is the main reason why the HBM does not make experimental verification to its first priority.

On the other hand, also contemporary physics contains items that cannot be measured. For example color charge is an item that cannot

(yet) be measured. Due to color confinement, quarks have never been detected as separate objects.

As indicated above, proper time is a concept that also exists in contemporary physics, but in general it cannot be measured. Contemporary physics uses the field concept, but except for the cases that the fields are raised by properties of separate particles contemporary physics does not bother what causes the field.

3.2 Generators, spread and descriptors.

The HBM allows very pictorial representations of its fundamental concepts. Let me give you a small preview.

In the model, generators produce coherent groups of discrete objects that are spread over an embedding continuum. The density distribution and the current density distribution of these coherent groups are continuous functions that describe and categorize these groups.

Depending on a suitable Green's function, the distributions of discrete objects also correspond to potential functions. Depending on the way in which the potential is generated, the potential function corresponds to a local curvature of the embedding space. This can be comprehended when the groups are generated dynamically in a rate of one element per progression step.

During its very short existence the element transmits a wave front[2] that slightly folds and thus curves the embedding space. The wave front keeps floating away with light speed from its previous source. It represents a trace of the existence of the element. This trace survives the element when that element is long gone. These traces can be observed without affecting the emitter.

For each coherent group, the elements are generated at a rate of one element per progression step. With other words the wave fronts

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[2] For anisotropic elements the message is transmitted by an anisotropic wave.

form ultra-high frequency waves that move with light speed away from their source. However, each wave front is emitted at a slightly different location. Already at a small distance it appears as if they originate from the same center location. The coherent group forms a building block. These waves together constitute the potential function(s) of this building block.

The elements act as step stones and together they form a micro-path for the corresponding group. This micro-movement can be considered as a combination of a quasi-oscillation and a quasi-rotation. Indirectly, the generator influences space curvature. The descriptors only describe the influence of the potentials on the local space curvature. The ultra-high frequency wave cannot be observed. Only its averaged effect is observable. The resulting potential is an integral and therefore a rather static effect. Modulations of this wave that are due to oscillations of the emitter can be observed. These modulation waves possess a much lower frequency than the ultra-high frequency carrier wave has.

The element generator can be described by the convolution of a sharp continuous function and a low scale spread function that blurs the continuous function. In this way, the spreading part can be seen as the activator of local space curvature, while the derivative of the sharp part defines a local metric that can be considered as the descriptor of the local curvature. The two parts must be in concordance. In this way two kinds of descriptors of local curvature exist. The first is the density distribution that describes the spread of the discrete objects. It corresponds to a potential function. The second descriptor is the local metric. Since these functions act on different scales, they can usually be treated separately.

The ***origin of the local curvature*** is the dynamic stochastic process that produces the low scale spread of the discrete objects. As

described above these objects transmit waves that curve the local space. The HBM suggests the combination of a Poisson process that is coupled to a binomial process, where the attenuation of the binomial process is implemented by a 3D spread function[3]. The stochastic generator process will generate according to a standard plan. In principle, at each location where it is active the generator produces locally the same kind of patterns. In undisturbed (natal) format, these patterns may only differ in their symmetry properties. However, these patterns cause space curvature. The local curvature is generated by the considered group and by neighboring groups. Due to an existing uniform move of the building block and due to the variance in space curvature, the center location of the pattern may become displaced. Both effects disturb the natal state of the distributions that are generated by the generating process. Since the patterns are generated with a single element per progression step, the generation poses a large chance to not generate the target natal shape but instead a distorted shape that in addition is spread over the path that the center location decides to follow. The produced distribution can still be described by a continuous function, but that function will differ from the continuous function that describes the undisturbed natal state. So the generation process is characterized by two functions. The first one represents the characteristics of the local generation process. It describes the natal state of the intended distribution. It is more a prospector than a descriptor. The second one describes the actually produced distribution that is distorted by the local space curvature and spread out by the movement of the center location. Further the generation of the distribution may not be completely finished, because not enough elements were generated since the generation of the pattern was started. The generated element only lives during the current progression step. In the next step a newly generated element replaces

[3] See: The enumeration process.

the previous object. At any instant the generated distribution consists of only one element. Thus for its most part the distribution can be considered as a set of virtual elements that lived in the past or will live in the future. The virtual distribution together with its current non-virtual element represents a pattern. The local curvature is partly caused by the pattern itself, but for another part it is caused by neighbor patterns.

The previous description of the natal generation can be imagined visually. At a rate of one element per progression instant the generator produces step stones that are used by the generated building block. The step stones are located randomly in a coherent region of 3D space. The building block walks along these step stones. As a consequence even at rest the building block follows a stochastic micro-path. Any movement of the building block as a whole, will be superposed on the micro-path. At every arrival at a step stone, the building block transmits its presence via a wave front that slightly folds and thus curves the embedding continuum. These wave fronts and the transmitted content constitute the potentials of the building block.

Nobody said that the undercrofts of physics behave in a simple way!

3.3 Coupling and events

The HBM introduces the notion of ***coupling*** of fields. It also means that non-coupled fields exists. Coupling is described by a ***coupling equation***, which is a special kind of ***differential continuity equation***[4].

Coupling takes place between ***stochastic fields***. Stochastic fields describe ***density distributions*** and ***current density distributions*** of

[4] See: Coupling

lower order objects. The distributions are generated by a local generation process that in each progression step produces ONE lower order object per stochastic field.

Coupling is implemented by messages that are transmitted in the embedding continuum by the active elements of the distribution via wave fronts[5] that slightly fold and thus curve this continuum. Together these waves constitute the potentials that are raised by the distribution. It is sensible to presume that the element generator reacts on the potentials that are active in that location.

When the particle is annihilated, the coupling stops. This also means that no further wave fronts are generated that contribute to the potential. However, the existing wave fronts keep flowing away from their original source. They keep extending their reach with light speed. With these wave fronts the potential flees away.

In order to keep the considered group coherent, an inbound or outbound micro-move must on average be followed by a move in a reverse direction. This must hold separately in each spatial dimension. Thus in each spatial dimension a kind of quasi oscillation takes place. The synchronization of this quasi oscillation may differ per dimension. In a similar way a quasi-rotation can exist. A certain kind of coupling of fields may be based on induced synchronization of these quasi oscillations and quasi-rotations.

Coupling becomes complicated when it involves coupling dependencies that live in different dimensions. Such cases can no longer be solved by separating the problem per dimension. It also means that the problem is inherently quaternionic and cannot be

[5] For anisotropic elements the message is transmitted by an anisotropic wave.

solved by simple complex number based technology. This occurs in the coupling equation of elementary particles where two quaternionic functions are coupled that belong to different discrete symmetry sets. Dirac has solved this problem by applying spinors and Dirac matrices. The HBM solves this with quaternionic methodology. The HBM applies special indices that identify symmetry flavors.

The wave fronts that constitute the potentials of the building blocks are non-coupled fields. When the source oscillates then these ultra-high frequency carrier waves get modulated. The corresponding modulation frequency is much lower than the carrier frequency. Photons are examples of these modulating waves.

3.4 Wave particle duality

A point-like object can hop along a stochastically distributed set of step stones that together form a micro-path. The step stones form a coherent distribution that can be described by a continuous object density distribution. Via a properly selected Green's function the step stone distribution can also be converted into a potential function. Each suitable Green's function corresponds to a corresponding potential function.

A direct conversion from density distribution to a potential function is also possible and also uses a dedicated Green's function.

These higher level objects are different views of the same thing. Let us call it a building block.

Both the density distribution and the potential function have a Fourier transform and can be considered as a wave package. Problem with this view is the fact that the step stones only are used in a single progression instant. So most of the time the step stones are virtual. This becomes less relevant when the step stone distribution is generated according to a given plan. In that case the plan represents the building block.

The most impressing view is raised by the fact that at every arrival at a step stone the building block emits a wave front that contains information about its presence and about its properties. The flow of these wave fronts are controlled by the Huygens principle. Together the wave fronts constitute an ultra-high frequency carrier wave that cannot be observed directly, but that can show observable lower frequency modulations and that shows its influence in the form of the potentials of the building block.

Now we have a higher level object that at the same time is a point-like particle and will act as a wave package and an ultra-high frequency carrier wave

This idea is exploited by the Hilbert Book Model.

4 The logic model

In this chapter the basic phenomena of physics will be deduced from its logical foundation. The HBM choses traditional quantum logic as its most basic foundation. In 1936, this foundation was suggested by Garret Birkhoff and John von Neumann[6].

4.1 Static status quo

4.1.1 Quantum logic

The most basic level of objects in nature is formed by the propositions that can be made about the objects that occur in nature. The relations between these propositions appear to be restricted by the axioms of traditional quantum logic. This set of related propositions can only describe a static status quo. The axioms that specify quantum logic are specified in Q-FORMULÆ 2.

In mathematical terminology the propositions whose relations are described by traditional quantum logic form a lattice. More particular, they form an orthomodular lattice that contains a countable infinite set of atomic (=mutually independent) propositions. Within the same quantum logic system multiple versions of sets of these mutually independent atoms exist. In this phase of the model the content of the propositions is totally unimportant. As a consequence these

[6] http://en.wikipedia.org/wiki/John_von_Neumann#Quantum_logics & Stanford Encyclopedia of Philosophy, *Quantum Logic and Probability Theory*, http://plato.stanford.edu/entries/qt-quantlog/

atoms form principally an unordered set[7]. Only the interrelations between the propositions count.

Traditional quantum logic shows narrow similarity with classical logic, however the modular law, which is one of the about 25 axioms that define the classical logic, is weakened in quantum logic. This is the cause of the fact that the structure of quantum logic is significantly more complicated than the structure of classical logic.

4.1.2 Hilbert logic

The set of propositions of traditional quantum logic is lattice isomorphic with the set of closed subspaces of a separable Hilbert space. However still significant differences exist between this logic system and the Hilbert space. This gap can be closed by a small refinement of the quantum logic system.

Step 1: Define ***linear propositions*** (also called ***Hilbert propositions***) as quantum logical propositions that are characterized by a number valued strength or relevance. This number is taken from a division ring.

Step 2: Require that linear combinations of Hilbert propositions also belong to the logic system.

Step 3: Introduce the notion of ***relational relevance*** between two linear propositions. This measure has properties that are similar to the inner product of Hilbert space vectors.

Step 4: Close the subsets of the new logic system with respect to this relational coupling measure.

The relational relevance measure can have values that are taken from a suitable division ring. The resulting logic system will be called ***Hilbert logic***.

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[7] This fact will prove to be the underpinning of the cosmologic principle.

The Hilbert logic is lattice isomorphic as well topological isomorphic with the corresponding Hilbert space.

The definition of Hilbert logic is specified in Q-FORMULÆ 3.

In this correspondence, Hilbert propositions are the equivalents of Hilbert vectors. General quantum logic propositions are the equivalents of (closed) subspaces of a Hilbert space.

The measure of the relational relevance between two Hilbert propositions is the equivalent of the inner product between two Hilbert vectors.

Due to this similarity the Hilbert logic will also feature operators[8].

In a Hilbert logic, linear operators can be defined that have atomic Hilbert propositions as their eigen-propositions. Their eigenspace is countable.

In a Hilbert logic system the ***superposition principle*** holds. A linear combination of Hilbert proposition is again a Hilbert proposition.

[8] The Hilbert logic does not feature dynamic operators.

5 Dynamic model

A dynamic model can be constructed from an ordered sequence of the above static sub-models. Care must be taken to keep sufficient coherence between subsequent static models. Otherwise, the model just represents dynamical chaos. However, some deviation must be tolerated, because otherwise, nothing dynamical will happen in this new dynamic model. The cohesion is established by a suitable correlation vehicle.

5.1 Correlation vehicle

The correlation vehicle supports and guards the coherence of the dynamics of the model. The correlation vehicle uses a toolkit consisting of an enumerator generator, an embedding continuum and a continuous function that maps the enumerators onto the continuum. The function is a continuous function of both the sequence number of the sub-models and the enumerators that are attached to a member of the selected set of atomic propositions. The enumeration is artificial and is not allowed to structurally add extra characteristics or functionality to the attached proposition. For example, if the enumeration takes the form of a coordinate system, then this coordinate system cannot have a unique origin and it is not allowed to structurally introduce preferred directions. These restrictions lead to an affine space. The avoidance of preferred directions produces problems in multidimensional coordinate systems. As a consequence, in case of a multidimensional coordinate system the correlation vehicle must use a smooth touch. This means, that at very small scales the coordinate system must get blurred. This means that the guarantee for coherence between subsequent sub-models cannot be made super hard. Instead coherence is reached with an acceptable tolerance. In any case a super hard coherence is unwanted.

The correlation vehicle also takes care of the perseverance of the emitted potential. For that reason it uses the Huygens principle. At every progression step the existing ultra-high frequency waves are re-emitted from locations at the wave fronts.

6 Isomorphic model

The natural form of the enumeration system can be derived from the lattice isomorphic companion of the quantum logic sub-model. Or it can be derived via a corresponding Hilbert logic system. Here we follow the historical development that was initialized by Birkhoff and von Neumann.

In the third decade of the twentieth century Garret Birkhoff and John von Neumann[9] were able to prove that for the set of propositions in the traditional quantum logic model a mathematical lattice isomorphic model exists in the form of the set of the closed subspaces of an infinite dimensional separable Hilbert space. The Hilbert space is a linear vector space that features an inner vector product. It offers a mathematical environment that is far better suited for the formulation of physical laws than what the purely logic model can provide.

Some decades later Constantin Piron[10] proved that the only number systems that can be used to construct the inner products of the Hilbert vectors must be division rings. Later Solèr's theorem formulated this discovery more precisely. The only suitable division rings are the real numbers, the complex numbers and the quaternions[11]. Quaternions can be seen as combinations of a real scalar and a 3D (real) vector. The number system of the quaternions represent a

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[9] http://en.wikipedia.org/wiki/John_von_Neumann#Quantum_logics & Stanford Encyclopedia of Philosophy, *Quantum Logic and Probability Theory*, http://plato.stanford.edu/entries/qt-quantlog/

[10] C. Piron 1964; _Axiomatique quantique_

[11] Bi-quaternions have complex coordinate values and do not form a division ring.

1+3D coordinate system. It can be shown that the eigenvalues of normal operators must also be taken from the same division ring.

Since the set of real numbers is multiple times contained in the set of complex numbers and the set of complex numbers is multiple times contained in the set of quaternions, the most extensive isomorphic model is contained in an infinite dimensional quaternionic separable Hilbert space. For our final model we will choose the quaternionic Hilbert space, but first we study what the real Hilbert space model and the complex Hilbert space model provide. What can be done by using a quaternionic Hilbert space can also be done in a real or complex Hilbert space by adding extra structure[12].

It appears that a cross product of two quaternionic Hilbert spaces no longer equals a quaternionic Hilbert space[13]. The HBM does not use such cross products.

The set of closed subspaces of the Hilbert space represents the set of propositions that forms the static quantum logic system. Like the sets of mutually independent atoms in the quantum logic system, multiple sets of orthonormal base vectors exist in the Hilbert space. The base vectors do not form an ordered set. However, a so called normal operator will have a set of eigenvectors that form a complete orthonormal base. The corresponding eigenvalues may provide a means for enumeration and thus for ordering these base vectors. An arbitrary normal operator will in general not fit the purpose of providing an affine eigenspace Usually the eigenvalues of a normal operator introduce a unique n origin and in the case of a multidimensional eigenspace the eigenspace may structurally contain preferred directions. Still, suitable enumeration operators exist. Several things can already be said about the eigenspace of the wanted enumeration operator. Its eigenspace is countable. It has no unique origin. It does

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[12] http://math.ucr.edu/home/baez/rch.pdf

[13] The result is an abstraction to a real Hilbert space.

not show preferred directions. Its eigenvalues can be embedded in an appropriate reference continuum.

As part of its corresponding Gelfand triple[14] a selected separable Hilbert space forms a sandwich that features uncountable orthonormal bases and (compact) normal operators with eigenspaces that form a continuum. A reference continuum can be taken as the eigenspace of the corresponding enumeration operator that resides in the Gelfand triple of this reference Hilbert space.

Together with the pure quantum logic model, we now have a dual model that is significantly better suited for use with calculable mathematics. Both models represent a static status quo.

The Hilbert space model suits as part of the toolkit that is used by the correlation vehicle.

As a consequence***, an ordered sequence of infinite dimensional quaternionic separable Hilbert spaces forms the isomorphic model of the dynamic logical model***.

6.1 Hierarchy

The refinement of quantum logic to Hilbert logic also can deliver an enumeration system. However, the fact that the selected separable Hilbert space offers a reference continuum via its Gelfand triple make the Hilbert space more suitable for implementing the Hilbert Book Model.

The two logic systems feature a hierarchy that is replicated in the Hilbert space. Quantum logic propositions can be represented by closed sub-spaces of the Hilbert space. Atomic Hilbert propositions can be represented by base vectors of the Hilbert space. The base vectors that span a closed sub-space belong to that sub-space. This

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[14] See http://vixra.org/abs/1210.0111 for more details on the Hilbert space and the Gelfand triple. See the paragraph on the Gelfand triple.

situation becomes interesting when the base vectors are eigenvectors. In that case the corresponding eigenvalues can be used to enumerate the eigenvectors of the Hilbert space operator and the corresponding eigen atoms of the Hilbert logic operator.

A similar hierarchy can be found when a coherent set of lower order objects forms a building block. Here the lower order objects correspond to atomic Hilbert propositions and to corresponding Hilbert base vectors. The building block corresponds to the quantum logical proposition and to the corresponding closed Hilbert subspace.

6.2 Correspondences

Several correspondences exist between the sub models:

Quantum logic	**Hilbert space**		**Hilbert logic**
Propositions: a, b	Subspaces a,b	Vectors: $\|a\rangle, \|b\rangle$	Hilbert propositions: a, b
atoms c, d		Base vectors: $\|c\rangle, \|d\rangle$	atoms c, d
Relational complexity: $C_{omplexity}(a \cap b)$	Relational complexity: $C_{omplexity}(a \cap b)$	Inner product: $\langle a\|b\rangle$	Relational coupling measure
Inclusion: $(a \cup b)$	Inclusion: $(a \cup b)$	Linear combination: $\alpha\|a\rangle + \beta\|b\rangle$	Linear combination: $\alpha a + \beta b$
For atoms c_i: $\bigcup_i c_i$	Subspace $\left\{\sum_i \alpha_i \|c_i\rangle\right\}_{\forall \alpha_i}$		Subset $\left\{\sum_i \alpha_i c_i\right\}$

The distribution

$$a(i) \equiv \{\langle a|c_i\rangle\}_{\forall_i}$$

has no proper definition in quantum logic. It can be interpreted via the Hilbert logic and Hilbert space sub-models.

6.3 Affine space

The set of mutually independent atomic propositions is represented by an orthonormal set of base vectors in Hilbert space. Both sets span the whole of the corresponding structure. An arbitrary orthonormal base is not an ordered set. It has no start and no end. It is comparable to an affine space. However, all or a part of these base vectors can be enumerated for example with rational quaternions. Enumeration introduces an artificial origin and may introduce artificially preferred directions. Thus, in general enumeration will apply to a part of the affine space. As is shown in the last paragraph this enumeration process defines a corresponding normal operator.

The installation of the correlation vehicle requests the introduction of enumerators. The enumeration may introduce an ordering. In that case the attachment of the numerical values of the enumerators to the Hilbert base vectors defines a corresponding operator. It must be remembered that the selection of the enumerators and therefore the corresponding ordering is kind of artificial. The eigenspace of the enumeration operator has ***no unique origin*** and is has ***no natural preferred directions.*** Thus it has no natural axes. It can only indicate the distance between two or more locations. It will be shown that for multidimensional rational enumerators the distance is not precise. In that case the enumeration represents a blurred coordinate system. Both in the Hilbert space and in its Gelfand triple, the enumeration can be represented by a normal enumeration operator.

6.4 Continuity

The task of the correlation vehicle is to arrange sufficient coherence between subsequent members of the sequence. Coherence translates to a moderate form of continuity.

6.4.1 Arranging dynamics

Embedding the enumerators in a continuum highlights the interspacing between the enumerators. Having a sequence of static submodels is no guarantee that anything happens in the dynamic model. A fixed (everywhere equal) interspacing will effectively lame any dynamics. A more effective dynamics can be arranged by playing with the sizes of the interspacing in a stochastic way. This is the task of ***an enumerator generator.***

6.4.2 Establishing coherence

The coherence between subsequent static models can be established by embedding each of the countable sets in an appropriate continuum. For example the whole Hilbert space can be embedded in its Gelfand triple. The enumerators of the base vectors of the separable Hilbert space or of a subspace can also be embedded in a corresponding continuum. In the reference Hilbert space that continuum is formed by the values of the enumerators that enumerate a corresponding orthonormal base of the Gelfand triple[15]. For subsequent Hilbert spaces a new appropriate embedding continuum will be used, but that continuum may be curved.

Next a correlation vehicle is established by introducing a continuous allocation function that controls the coherence between subsequent members of the sequence of static models. It does that by creating a moderate relocation in the countable set of the enumerators that act in the separable Hilbert space by mapping them to the embedding continuum. The relocation is controlled by a stochastic process. In fact the differential of the allocation function is used to specify the small scale working space for this stochastic process[16]. The

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[15] See Gelfand triple

[16] The differential defines a local metric.

allocation function also takes care of the persistence of the embedding continuum.

The equivalence of this action for the logic model is that the enumerators of the atomic propositions are embedded in a continuum that is used by an appropriate correlation vehicle.

The allocation function uses a combination of progression and the enumerator id as its parameter value. The value of the progression might be included in the value of the id. Apart from their relation via the allocation function, the enumerators and the embedding continuum are mutually independent[17]. For the selected correlation vehicle it is useful to use numbers as the value of the enumerators. The type of the numbers will be taken equal to the number type that is used for specifying the inner product of the corresponding Hilbert space and Gelfand triple.

The danger is then that in general a direct relation between the value of the enumerator of the Hilbert base vectors and the embedding continuum is suggested. An exception is formed by the selected reference Hilbert space. So, for later Hilbert spaces a warning is at its place. Without the allocation function there is no relation between the value of the enumerators and corresponding values in the embedding continuum that is formed by the Gelfand triple. However, there is a well-defined relation between the images[18] produced by the allocation function and the selected embedding continuum[19]. The relation between the members of a countable set and the members of a continuum raises a serious one-to-many problem. That problem can

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[17] This is not the case for the reference Hilbert space in the sequence. There a direct (close) relation exists.

[18] Later these images will be called Qpatches

[19] Later the nature of this embedding continuum will be revealed. In later Hilbert spaces the embedding continuum is constituted by potentials.

easily be resolved for real Hilbert spaces and complex Hilbert spaces, but it requires a special solution for quaternionic Hilbert spaces. That solution is treated below.

Together with the selected embedding continuum and the Hilbert base enumeration set the allocation function defines the ***evolution*** of the model.

6.4.3 Structure of the correlation vehicle

At every progression step the correlation vehicle regenerates the eigenspaces of the non-conserved operators[20]. This regeneration runs at an ultra-high frequency. That frequency is set by the progression step size.

An important part of the functionality of the correlation vehicle is implemented by the allocation function. This function is the convolution of a continuous part and a local blur. The local blur is implemented by the combination of a Poisson process and a binomial process. The binomial process is implemented by a 3D spread function.

The derivative of the continuous part of the allocation function defines a local metric.

Another part of the functionality of the correlation vehicle concerns the regeneration of the embedding continuums. This regeneration is governed by Huygens principle. It is implemented by wave fronts that flow with the constant maximum speed of information transfer.

Later we will see that the correlation vehicle is restricted by color confinement.

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[20] These operators reside in Hilbert logic, in the corresponding Hilbert space and in the corresponding Gelfand triple.

7 Hilbert spaces

Sets of subsets of Hilbert spaces represent quantum logical systems and associated Hilbert logic systems. Closed subspaces of the Hilbert space represent quantum logical propositions and Hilbert space vectors represent Hilbert propositions.

The Hilbert space is a static hull. A normal operator with a countable ordered set of eigenvalues can be used as a reference operator. This operator will be used for enumeration purposes. These enumerators will be used as parameters for the functions that implement the correlation mechanism.

Each Hilbert space corresponds to a Gelfand triple. That space features operators which have a continuum as it eigenspace. Also in this space a normal operator with an ordered set of eigenvalues can be used as a reference operator.

The reference operators are static objects.

Several normal operators in the Gelfand triple will be used to deliver target values for functions that implement the correlation mechanism. These operators are dynamic objects. They will be re-created at every progression step.

7.1 Real Hilbert space model

When a real separable Hilbert space is used to represent the static quantum logic, then it is sensible to use a countable set of real numbers for the enumeration. A possible selection is formed by the natural numbers. Within the real numbers the natural numbers have a fixed interspacing. Since the rational number system has the same cardinality as the natural number system, the rational numbers can

also be used as enumerators. In that case it is sensible to specify ***a (fixed) smallest rational number*** as the enumeration step size. In this way the notion of interspacing is preserved and can the allocation function do its scaling task[21]. In the realm of the real Hilbert space model, the continuum that embeds the enumerators is formed by the real numbers. The values of the enumerators of the Hilbert base vectors are used as parameters for the allocation function. The value that is produced by the allocation function determines the target location for the corresponding enumerator in the target embedding continuum. The target embedding continuum is taken from an operator that resides in the Gelfand triple. The interspacing freedom is used in order to introduce dynamics in which something happens.

In fact what we do is defining an enumeration operator that has the enumeration numbers as its eigenvalues. The corresponding eigenvectors of this operator are the target of the enumerator.

With respect to the logic model, what we do is enumerate a previously unordered set of atomic propositions that together span the quantum logic system and next we embed the numerators in an appropriate continuum. The correlation vehicle takes care of the cohesion between subsequent quantum logical systems.

While the progression step is kept fixed, the (otherwise fixed) space step might scale with progression.

Instead of using a fixed smallest rational number as the enumeration step size and a map into a reference continuum we could also have chosen for a model in which the rational numbered step size varies with the index of the enumerator.

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[21] Later, in the quaternionic Hilbert space model, this freedom is used to introduce space curvature and it is used for resolving the one to many problem.

7.2 Gelfand triple

The Gelfand triple of a real separable Hilbert space can be understood via the enumeration model of the real separable Hilbert space. This enumeration is obtained by taking the set of eigenvectors of a normal operator that has rational numbers as its eigenvalues. Let the smallest enumeration value of the rational enumerators approach zero. Even when zero is reached, then still the set of enumerators is countable. Now add all limits of converging rows of rational enumerators to the enumeration set. When appropriate also add a corresponding eigenvector. After this operation the enumeration set has become a continuum and has the same cardinality as the set of the real numbers. This operation converts the Hilbert space into its Gelfand triple and it converts the normal operator in a new operator that has the real numbers as its eigenspace. It means that the orthonormal base of the Gelfand triple that is formed by the eigenvectors of the new normal operator has the cardinality of the real numbers. It also means that linear operators in this Gelfand triple have eigenspaces that are continuums and have the cardinality of the real numbers[22]. The same reasoning holds for complex number based Hilbert spaces and quaternionic Hilbert spaces and their respective Gelfand triples.

7.3 Complex Hilbert space model

When a complex separable Hilbert space is used to represent quantum logic, then it is sensible to use rational complex numbers for the enumeration. Again a smallest enumeration step size is introduced. However, the imaginary fixed enumeration step size may differ from the real fixed enumeration step size. The otherwise fixed imaginary enumeration step may be scaled as a function of progression. In the complex Hilbert space model, the continuum that embeds

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[22] This story also applies to the complex and the quaternionic Hilbert spaces and their Gelfand triples.

the enumerators of the Hilbert base vectors is formed by the system of the complex numbers. This continuum belongs as eigenspace to the enumerator operator that resides in the Gelfand triple. It is sensible to let the real part of the Hilbert base enumerators represent progression. The same will happen to the real axis of the embedding continuum. On the real axis of the embedding continuum the interspacing can be kept fixed. Instead, it is possible to let the allocation function control the interspacing in the imaginary axis of the embedding continuum. The values of the rational complex enumerators are used as parameters for the allocation function. The complex value of the allocation function determines the target location for the corresponding target value in the continuum. The allocation function establishes the necessary coherence between the subsequent Hilbert spaces in the sequence. The difference with the real Hilbert space model is, that now the progression is included into the values of the enumerators. The result of these choices is that the whole model steps with (very small, say practically infinitesimal) fixed progression steps.

In the model that uses complex Hilbert spaces, the enumeration operator has rational complex numbers as its eigenvalues. In the complex Hilbert space model, the fixed enumeration real step size and the fixed enumeration imaginary step size define ***a maximum speed***. The fixed imaginary step size may scale as a function of progression. The same will then happen with the maximum speed, defined as space step divided by progression step. However, if information steps one step per progression step, then the information transfer speed will be constant. Progression plays the role of proper time. Now define a new concept that takes the length of the complex path step as the step value. Call this concept the coordinate time step. Define a new speed as the space step divided by the coordinate time step. This new ***maximum speed is a model constant***. Proper time is

the time that ticks in the reference frame of the observed item. Coordinate time is the time that ticks in the reference frame of the observer[23]. Coordinate time is our conventional notion of time.

Again the eigenvectors of the (complex enumeration) operator are the targets of the enumerator whose value corresponds to the complex eigenvalue.

In the complex Hilbert space model the squared modulus of the quantum state function represents the probability of finding the location of the corresponding particle at the position that is defined by the parameter of this function.

If we ignore the case of negative progression, then the complex Hilbert model exist in two forms, one in which the interspacing appears to expand and one in which the interspacing decreases with progression[24].

7.4 Quaternionic Hilbert space model

When a quaternionic separable Hilbert space is used to model the static quantum logic, then it is sensible to use rational quaternions for the enumeration. Again the fixed enumeration step sizes are applied for the real part of the enumerators and again the real parts of the enumerators represent progression. The reference continuum that embeds the enumerators is formed by the number system of the quaternions. The scaling allocation function of the complex Hilbert space translates into an isotropic scaling function in the quaternionic Hilbert space. However, we may instead use a full 3D allocation function that incorporates the isotropic scaling function. This new

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[23] In fact coordinate time is a mixture of progression and space. See paragraph on spacetime metric.

[24] The situation that expands from the point of view of the countable enumeration set, will contract from the point of view of the embedding continuum of enumerators.

allocation function may act differently in different spatial dimensions. However, when this happens at very large scales, then it conflicts with the cosmological principle. At those scales the allocation function must be quasi isotropic. The allocation function is not allowed to create preferred directions.

Now the enumeration operator of the Hilbert space has rational quaternions as its eigenvalues. The relation between eigenvalues, eigenvectors and enumerators is the same as in the case of the complex Hilbert space. Again the whole model steps with fixed progression steps.

In the quaternionic Hilbert space model the real part of the quantum state function represents the probability of finding the location of the corresponding particle at the position that is defined by the parameter of this function. It corresponds to a density distribution of the locations where the corresponding building block can/could be found.

7.4.1 Curvature and fundamental fuzziness

The spatially fixed interspacing that is used with complex Hilbert spaces poses problems with quaternionic Hilbert spaces. Any regular spatial interspacing pattern will introduce preferred directions. Preferred directions are not observed in nature[25] and the model must not create them. A solution is formed by the ***randomization of the interspacing***. Thus instead of a fixed imaginary interspacing we get an average interspacing. This problem does not play on the real axis. On the real axis we can still use a fixed interspacing. The result is an ***average maximum speed***. This speed is measured as space step per coordinate time step, where the coordinate time step is given by the

[25] Preffered directions are in conflict with the cosmological principle.

length of the 1+3D quaternionic path step. Further, the actual location of the enumerators in the embedding continuum will be determined by the combination of a sharp continuous allocation function (SCAF) $\wp$ and a stochastic spatial spread function (SSSF) $\mathcal{S}$ that specifies the local blur. The form factor of the blur may differ in each direction and is set by the differential of the sharp allocation function $\wp$. The total effect is given by the convolution $\mathcal{P} = \wp . \mathcal{S}$ of the sharp allocation function $\wp$ and spread function $\mathcal{S}$. The result is a blurred allocation function $\mathcal{P}$. The result of $\mathcal{S}$ alone is described by a quaternionic probability amplitude function (QPAD). This is a descriptor. It describes the planned distribution of a set of discrete objects that will be generated in a sequence. The result of $\mathcal{P}$ is the actual local QPAD. In the quaternionic Hilbert space model it conforms to the quaternionic quantum state function. It is a close equivalent of the ell known wave function.

The requirement that the cosmological principle must be obeyed is the cause[26] of a fundamental fuzziness of the quaternionic Hilbert model. It is the reason of existence of quantum physics.

An important observation is that the blur mainly occurs locally. The blur has a very limited extent. On the other hand, due to the emission of potential generating wave fronts, the blur corresponds to a potential function that has an unlimited extent, but its influence decreases with distance.

At larger distances the freedom that is tolerated by the allocation function causes ***curvature of observed space***. However, as explained before, at very large scales the allocation function must be quasi isotropic[27]. The local curvature is described by the differential of the sharp part of the allocation function.

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[26] Another cause is the requirement that coherence between subsequent progression steps must not be too stiff.

[27] Quasi-isotropic = on average isoropic.

The continuous part of the allocation function defines the current target embedding continuum. In fact it determines the eigenspace of a corresponding operator that resides in the Gelfand triple. Apart from the exceptional case of the reference Hilbert space, the selection of this operator poses a problem. The HBM selects the superposition of all gravitational potentials as the proper choice for subsequent Hilbert spaces.

This picture only tells that space curvature might exist. It does not describe the origin of space curvature. For a more detailed explanation of the origin of space curvature, please see the paragraph on the enumeration process.

7.4.2 Discrete symmetry sets

Quaternionic number systems exist in 16 versions (sign flavors[28]) that differ in their discrete symmetry sets. The same holds for sets of rational quaternionic enumerators and for continuous quaternionic functions. Four members of the set represent isotropic expansion or isotropic contraction of the imaginary interspacing. At large scales two of them are symmetric functions of progression. The other two are at large scales anti-symmetric functions of progression. We will take the symmetrical member that expands with positive progression as the ***reference rational quaternionic enumerator set***. Each member of the set corresponds with a quaternionic Hilbert space model. Thus apart from a reference continuum we now have a reference rational quaternionic enumerator set. Both reference sets meet at the reference Hilbert space. Even at the instance of the reference Hilbert space, the allocation function must be a continuous function of progression.

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[28] See paragraph on Qpattern coupling

When the real parts are ignored, then eight sign flavors result. These eight flavors are discerned by their "color" and their handedness. Besides of color, we use special indices in order to mark the sign flavors.

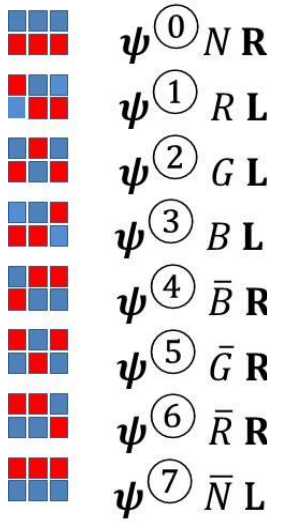

Figure 1: Sign flavors

Eight sign flavors
(discrete symmetries)
Colors N, R, G, B, $\bar{R}$, $\bar{G}$, $\bar{B}$, W
Right or Left handedness **R,L**

Within a coherent set of enumerators or in the images of such a set that are produced by the allocation function all objects possess the same sign flavor.

A similar split in quaternionic sign flavors as exists with quaternionic number systems occurs with continuous quaternionic functions. In the picture they are listed as $\psi^{⓪} \ldots \psi^{⑦}$.
In the picture the color *N* and the continuous function version $\psi^{⓪}$ represent the reference sign flavor.

For each discrete symmetry set of their parameter space, the function values of the continuous quaternionic distribution exist in 16 versions that differ in their discrete symmetry set. Within the target domain of the continuous quaternionic distribution the symmetry set will stay constant.

7.4.3 Generations and Qpatterns

Depending on its characteristics, the local generator of enumerators can generate a certain distribution of randomized enumerators. A Poisson generator combined by a binomial process that is implemented by a suitable 3D isotropic spread function can implement a suitable distribution. The planned distribution is described by a local QPAD. The local QPAD corresponds to the characteristics of the generator, but depending on its starting condition the stochastic generator can generate different distributions. Thus, different distributions may correspond to a single QPAD. The QPAD is a continuous quaternionic function that describes in its real part the density of the elements of the described distribution. In its imaginary part the QPAD describes the associated current density distribution.

If generators with different characteristics exist, then several generations[29] of local QPAD's exist.

HYPOTHESIS 1: *For a selected generation the following holds:*

[29] See the later paragraph on generations

Apart from the adaptation of the form factor that is determined by the local curvature and apart from the discrete symmetry set of the QPAD, the natal QPAD's are everywhere in the model the same.

Therefore we will call the distribution of objects that is described by this basic form of the selected QPAD generation a ***Qpattern***. For each generation, *QPAD's exist in 16 versions that differ in their* discrete symmetry set. Each Qpattern has a weighted center location, which is called ***Qpatch***.

At each progression step, all generators produce only a single element of the distribution. ***This means that each Hilbert space contains only one element of the Qpattern.*** That element is called ***Qtarget***.

7.4.4 Microstate

A Qpattern corresponds with the statistic mechanical notion of a microstate. A microstate of a gas is defined as a set of numbers which specify in which cell each atom is located, that is, a number labeling the atom, an index for the cell in which atom s is located and a label for the microstate[30].

7.5 Optimal ordering

In the Hilbert space it is possible to select a base that has optimal ordering for the eigenvalues of a normal operator. Optimally ordered means that these sections are uniformly distributed and that stochastic properties of these sections are the same. In the Hilbert logic system a similar selection is possible for the set of mutually independent

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[30] http://www.intechopen.com/books/theoretical-concepts-of-quantum-mechanics/quantum-mechanical-ensembles-and-the-h-theorem

atomic propositions. There the atoms are enumerated by the same set of rational quaternionic values.

For the Hilbert spaces it means that in the Gelfand triple a corresponding operator exist whose eigen space maps onto the well-ordered eigenspace of the operator that resides in the Hilbert space.

We will call these operators "reference operators".

7.6 The reference Hilbert space

The reference Hilbert space is taken as the member of the sequence of Hilbert spaces at the progression instance where the allocation function is a symmetric function of progression that expands in directions that depart from the progression value of the reference Hilbert space.

At large and medium scales the reference member of the sequence of quaternionic Hilbert spaces is supposed to have a quasi-uniform[31] distribution of the enumerators in the embedding continuum. This is realized by requiring that the eigenspace of the enumeration operator that acts in the Gelfand triple of the zero progression value Hilbert space represents the reference embedding continuum.

At this instance of progression, the target embedding continuum is ***flat***. For the reference Hilbert space the isotropic scaling function is symmetric at zero progression value. Thus for the reference Hilbert space at the reference progression instance the distribution of the enumerators will realize a ***densest packaging[32] of the target images***.

For all subsequent Hilbert spaces the embedding continuum will be taken from the superposition of potentials that are initiated in earlier Hilbert spaces.

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[31] quasi-uniform = on average uniform.

[32] The densest packaging will also be realized locally when the geometry generates black regions.

The (reference) Hilbert space together with its Gelfand triple and the reference operators in both structures form a static block that reappears in all later members of the sequence.

7.7 The embedding continuum

For the reference Hilbert space the embedding continuum is taken from a flat normal location operator that resides in its Gelfand triple. That continuum is the virginal reference continuum.

For subsequent Hilbert spaces the embedding continuum for fermions is formed by the superposition of all potentials that are generated by objects that lived in previous Hilbert spaces. The result is a curved version of the virginal reference continuum. The curvature is caused by the mechanism that emits the ultra-high frequency waves that constitute the potentials.

Bosons use an embedding continuum that is formed by the potentials that are emitted locally in previous Hilbert spaces.

The correlation vehicle takes care of the persistence of the potentials.

7.8 The cosmological principle revisited

The enumeration process attaches an artificial content to each of the members in the unordered set of atomic propositions. The unrestricted enumeration with rational quaternions generates an artificial origin and it generates artificial preferred directions that are not present in the original set of atomic propositions. The correlation vehicle is not allowed to attach this extra functionality to the original propositions. However, the vehicle must still perform its task to establish cohesion between subsequent sub-models. One measure is to turn the enumeration space into an affine space or to restrict the enumeration to a closed subset of a larger affine space. An affine space has no origin. The next measure is to randomize the enumeration process sufficiently such that an acceptable degree of cohesion is reached and at the same time a quasi-isotropy of this affine space is established. This measure requires the freedom of some interspacing, which is obtained by assigning a lowest rational number. In principle, a lowest rational number can be chosen for the real part and a

different smallest base number can be chosen for the imaginary part. This choice defines a basic notion of speed. The resulting (imaginary) space is on average isotropic. The randomization results in a local blur of the continuous function that regulates the enumeration process.

The result of these measures is that roughly the cosmologic principle is installed. Thus, in fact ***the cosmological principle is a corollary of the other two starting points***.

However, according to this model, apart from the low scale randomization, the universe would be quite well ordered. After a myriad of progression steps this medium to large scale ordering is significantly disturbed.

Looking away[33] from any point in universe is in fact looking back in proper time. Looking as far as is physically possible will open the view at a reference member of the Hilbert Book Model. This reference member represents a densest and well-ordered packaging. This will result in a uniform background at the horizon of the universe.

The well-known microwave background radiation is not fully uniform and is expelled by members that are close to the densest packaged member.

[33] Looking away = receiving messages from other objects.

8 The HBM picture

In the advance of quantum physics two views on quantum physics existed. This manuscript adds two extra pictures.

8.1 The Schrödinger picture

The Schrödinger picture describes a dynamic implementation in Hilbert space in which the quantum states carry the time dependence. The operators are static[34].

8.2 The Heisenberg picture

The Heisenberg picture describes a dynamic implementation in Hilbert space in which the operators (represented by matrices) carry the time dependence. The quantum states are static[35].

8.3 The Hilbert Book Model picture

In the HBM picture an ordered sequence of Hilbert spaces and their corresponding Gelfand triples are used. Each of these spaces represent a static status quo.

In the HBM the whole Hilbert space carries the ***proper time*** dependence. Both the enumeration operator and the patterns that represent the quantum state functions depend on the progression parameter. However, for the enumerator operator only the real part of the eigenvalue is affected. Other operators describe the target images of these enumerators. These target images form the Qtargets. For each Qpattern the Hilbert space contains only the actual element, the current Qtarget. Thus if only a single Hilbert space is considered, then the Qpatterns cannot be recognized. The virtual elements are not actually present in any member of the sequence of Hilbert spaces. The

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[34] http://en.wikipedia.org/wiki/Schr%C3%B6dinger_picture

[35] http://en.wikipedia.org/wiki/Heisenberg_picture

virtual elements can only exist as place holders. However, the potentials of Qpatterns act as traces of the existing and passed Qpatterns and the corresponding wave fronts form traces of the Qtargets. They affect the embedding continuum that is formed by the potentials of particles that existed in the past.

The correlation vehicle ensures the cohesion between subsequent Hilbert spaces and takes care of the persistence of the emitted potentials. In order to achieve this the correlation vehicle uses at each progression step the Huygens principle[36].

The potentials survive the extinction of the sources that created them. If they do not compensate each other, then they exist forever. ***Gravitation potentials do not compensate each other. This fact renders the HBM into a never ending story.***

8.4 The operational picture

In the operational picture only a single Hilbert space and its Gelfand triple are used. An operator that resides in the Hilbert space acts as the reference operator. It has an equivalent in the Gelfand triple and the eigenspaces of these operators map onto each other in an orderly fashion. Together with the Hilbert space and Gelfand triple these reference operators represent the static part of the model[37].

The eigenvalues of the reference operators represent the progression value in their real part.

In the Hilbert space and in its Gelfand triple the correlation vehicle supports the existence of progression dependent operators. This concerns a stochastically operating operator in the Hilbert space and

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[36] If the potentials are emitted in two dimensions, then the situation is more complicated.

[37] An exception holds for the real parts of the eigenvalues. They represent progression.

for each potential type a compact normal operator that installs the temporal behavior of these potentials.

The correlation vehicle uses the eigenspaces of the reference operators as its parameter spaces. It uses eigenspaces of other operators as its target space. As a consequence these target operators depend on progression.

This picture comes close to the Heisenberg picture, but it does not keep states static.

8.5 Discussion

Obviously the Hilbert Book Model selects the HBM picture. According to the feel of the author this picture offers the cleanest view. In this picture the difference between virtual and actual elements of a building block can be clearly explained.

The Hilbert space and Gelfand triple hulls together with the reference operators form the static part of both the HBM picture and the operational picture. In the HBM picture this static part is represented by the reference Hilbert space, its Gelfand triple and the reference operators. There is one small exception to this static behavior: the eigenvalues of the reference operators represent the progression value in their real parts.

Not all of the eigenvectors of the Hilbert space reference operator are constantly in use. Annihilation and (re)creation events regulate this usage. Virtual elements of building blocks are not used. Only the Qtarget is used, which is an actual element.

The models only use a huge subspace of the Hilbert space(s). Enumeration is considered to be an artificial action and the enumerators must be seen as to be embedded in an affine space.

The correlation vehicle controls all aspects of dynamics. It does that both in Hilbert space and in the Gelfand triple. Since the Hilbert space and the Gelfand triple are static hulls, the correlation vehicle

controls a selected set of operators that reside in these spaces. The tools of the correlation vehicle are the allocation function (in the Hilbert space) and the Huygens principle (in the Gelfand triple). Its actions are coordinated.

8.6 Quantum state function

In contemporary physics the "quantum state function" is used in its complex format. There it is a complex probability amplitude distribution (CPAD). It is also called "wave function". The squared modulus of the quantum state function is interpreted as the probability to be able to detect the corresponding building block at the location that is specified by the parameter of the wave function. The complex phase of the wave function can be freely selected. This freedom is used in gauge transformations.

In quaternionic quantum physics the quaternionic quantum state function is defined as a continuous quaternionic function. Its real part equals the squared modulus of the complex quantum state function and has the same interpretation. This part has no complex phase. In fact the real part can be interpreted as an object density distribution, where the objects are the locations where the corresponding building block can be detected. The imaginary part of the quaternionic quantum state function can be interpreted as the associated current density distribution. In fact it registers the displacement of the described building block since its last location. At every progression instant the building block gets a new location.

The displacement is the some of the displacement that is due to the movement of the building block as a whole and the displacement that is caused by the stochastic spatial spread. This last category of displacements cause the walk of the building block along a random micro-path.

In quaternionic quantum physics the gauge transformation reduces to a mathematical trick.

The characterization of the quaternionic quantum state function as a quaternionic probability amplitude distribution (QPAD) is used in analogy to the characterization of the complex quantum state function as a complex probability amplitude distribution (CPAD), but care must be taken because the interpretations of the CPAD and the QPAD slightly differ.

9 Fields

Field theory exists independent of what it describes. It describes fields varying from fluid dynamics, via electromagnetism to gravitation. You can describe scalar fields and vector fields separately or combined in a quaternionic field. Apart from that tensor fields exist.

Fields can be seen as variations (modulations) of an embedding continuum such as photons and gluons. Other types of fields can be seen as representing the distribution of the density of discrete objects and the corresponding current densities. Fields can also represent the potentials of these distributions of discrete objects. Examples of this last category are gravitation fields and electrostatic fields. The type of the potential is set by its Green's function. All these fields have many similarities and some differences. Only in case of density distributions and corresponding potentials the fields describe the same objects, which form the discrete distribution that underlies these fields. The elements of the distributions are treated as anonymous objects. However, it is also possible to enumerate them and allow each individual object to possess a series of properties. The elements can also share properties. These properties will characterize the distribution and the corresponding fields.

10 The enumeration process

It is not yet clear how Qpatterns will be shaped. This information can be derived from the requirements that are set for the correlation vehicle. We will start with a suggestion for the enumeration process that for this vehicle will lead to the wanted functionality.

HYPOTHESIS 2: *At small scales the enumeration process is governed by a Poisson process. The lateral spread that goes together with the low scale randomization of the interspacing plays the role of a binomial process. The combination of a Poisson process and a binomial process is again a Poisson process, but locally it has a lower efficiency than the original Poisson process. The binomial distribution is implemented by a continuous 3D spread function.*

As an ***example***, we consider the special situation that this combination produces a 3D normal distribution. For a large number of enumerator generations the resulting Poisson distribution resembles a Gaussian distribution[38]. If the generated enumerators are considered as charge carriers, then the corresponding potential has the shape of an Error function divided by r. Already at a short distance from its center location the potential function starts decreasing with distance r as a $1/r$ function[39].

10.1 Gravity and electrostatics

Potentials depend on the Green's function that is used to convert the corresponding density distribution into a potential function. Apart from their Green's function, gravity and electrostatics can be treated by similar equations.

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[38] http://en.wikipedia.org/wiki/Poisson's_equation#Potential_of_a_Gaussian_charge_density

[39] http://farside.ph.utexas.edu/teaching/em/lectures/node28.html through node31

Description	***Gravity***	***Electrostatics***
Field	$\boldsymbol{g} = -\boldsymbol{\nabla}\,\varphi$	$\boldsymbol{E} = -\boldsymbol{\nabla}\,\varphi$
Force	$\boldsymbol{F} = m\boldsymbol{g}$	$\boldsymbol{F} = Q\boldsymbol{E}$
Gauss law	$\langle \boldsymbol{\nabla}, \mathrm{g} \rangle = -4\pi G\rho$	$\langle \boldsymbol{\nabla}, \mathrm{E} \rangle = \frac{Q}{\varepsilon}$
Poisson law $\Delta\varphi = \langle\, \boldsymbol{\nabla}, \boldsymbol{\nabla}\varphi \rangle$	$\Delta\varphi = 4\pi G\rho$	$\Delta\varphi = -\frac{Q}{\varepsilon}$
Greens function	$\frac{-\rho(\boldsymbol{r}')}{\lvert\boldsymbol{r} - \boldsymbol{r}'\rvert}$	$\frac{Q}{\lvert\boldsymbol{r} - \boldsymbol{r}'\rvert}$
Single charge potential	$\varphi = -\frac{4\pi Gm}{\lvert\boldsymbol{r}\rvert}$	$\varphi = \frac{Q}{4\pi\varepsilon\lvert\boldsymbol{r}\rvert}$
Single charge field	$g = -\frac{4\pi Gm}{\lvert\boldsymbol{r}\rvert^2}\boldsymbol{r}$	$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon\lvert\boldsymbol{r}\rvert^2}\boldsymbol{r}$
Two charge force	$\boldsymbol{F} = -\frac{4\pi Gm_1m_2}{\lvert\boldsymbol{r}\rvert^3}\boldsymbol{r}$	$\boldsymbol{F} = \frac{Q_1Q_2}{4\pi\varepsilon\lvert\boldsymbol{r}\rvert^3}\boldsymbol{r}$
Mode	**attracting**	**repelling**

The table shows that the Greens functions of both fields differ in sign. For the gravitation potential the Green's function is charged with the local "charge" density $\rho(\boldsymbol{r}')$. For the electrostatic potential the Green's function is charged with a (constant) electric charge Q.

The Yukawa potential[40] uses a short range Green's function:

$$\frac{-\rho(\boldsymbol{r}')}{\lvert\boldsymbol{r} - \boldsymbol{r}'\rvert}\exp(-\mu\lvert\boldsymbol{r} - \boldsymbol{r}'\rvert) \tag{1}$$

[40] http://en.wikipedia.org/wiki/Yukawa_potential

In this example we use the gravitational Green's function.

Since the items are carriers with charge ρ_i, the density distribution$\rho_f(\mathbf{r})$ correspond to a potential $\varphi(\boldsymbol{r})$. Every item contributes a term $\varphi_i(\boldsymbol{r}-\boldsymbol{r_i}) = \frac{-\rho_i}{|\boldsymbol{r}-\boldsymbol{r_i}|}$

$$\varphi(\boldsymbol{r}) = \sum_i \varphi_i(\boldsymbol{r}-\boldsymbol{r_i}) = \sum_i \frac{-\rho_i}{|\boldsymbol{r}-\boldsymbol{r_i}|}$$

Example: If there is a static spherically symmetric Gaussian charge density

$$\rho_g(r) = \frac{\rho_c}{\sigma^3\sqrt{2\pi}^3}\exp(\frac{-r^2}{2\sigma^2})$$

where ρ_c is the total charge, then the solution $\varphi(r)$ of Poisson's equation,

$$\nabla^2\varphi = \rho_g$$

is given by

$$\varphi(r) = \frac{\rho_c}{4\pi\varepsilon r}\mathrm{erf}\left(\frac{r}{\sqrt{2}\sigma}\right) = \frac{-1}{4\pi\varepsilon}\int \frac{\rho_g(\boldsymbol{r'})}{|\boldsymbol{r}-\boldsymbol{r'}|}d^3\boldsymbol{r'}$$

where $erf(x)$ is the error function.

Note that, for r much greater than σ, the erf function approaches unity and the potential $\varphi\ (r)$ approaches the point charge potential

$$\varphi(r) \approx \frac{-\rho_c}{4\pi\varepsilon r}$$

as one would expect. Furthermore the erf function approaches 1 extremely quickly as its argument increases; in practice for $r > 3\sigma$ the relative error is smaller than one part in a thousand.

10.1.1 Bertrand's theorem

Now we remember Bertrand's theorem.[41] :

Bertrand's theorem states that only two types of central force potentials produce stable, closed orbits:

(1) an inverse-square central force such as the gravitational or electrostatic potential

$$V(r) = \frac{-k}{r} \tag{1}$$

and

(2) the radial harmonic oscillator potential

$$V(r) = \tfrac{1}{2}\, k\, r^2 \tag{2}$$

According to this investigation it becomes acceptable to assume that the undisturbed shape of the Qpatterns can be characterized by something that comes close to a 3D Gaussian distributions. Since such a distribution produces the correct shape of the gravitation potential***, the underlying mechanism would explain the origin of curvature***.

[41] http://en.wikipedia.org/wiki/Bertrand's_theorem.

10.2 The internal dynamics of Qpatterns

A Qpattern is generated in a rate of one element per progression step. A corresponding allocation operator that resides in the Hilbert space will reflect these Qtargets in its eigenspace.

During each progression step an increment is added to the static potential function. This is performed by transmitting a message to the environment of the Qtarget. The Qtarget is the element, which is currently active. Depending on the discrete symmetric difference with the embedding continuum to which the building block couples, the wave is either spherical or anisotropic, or better said, it is isotropic in less than three dimensions. For full 3D isotropic coupling Qtargets[42] the message is sent in the form of a 3D tsunami-like spherical wave ront . The wave folds the embedding continuum. This is the mechanism, which is used in order to transport the message. By repeating that message for every Qtarget a constant stream of messages is produced that together form a wave pattern that oscillates with ultra-high frequency[43]. If the Qpattern does not move, then at some distance the situation looks as if a "sine" wave is transmitted from a single source. If the Qpattern oscillates, then that ultra-high temporal frequency wave gets a lower temporal frequency amplitude and a phase modulation.

The geometry of the emitted wave fronts may depend on the type of the potential and on the symmetry properties of the emitting Qtarget.

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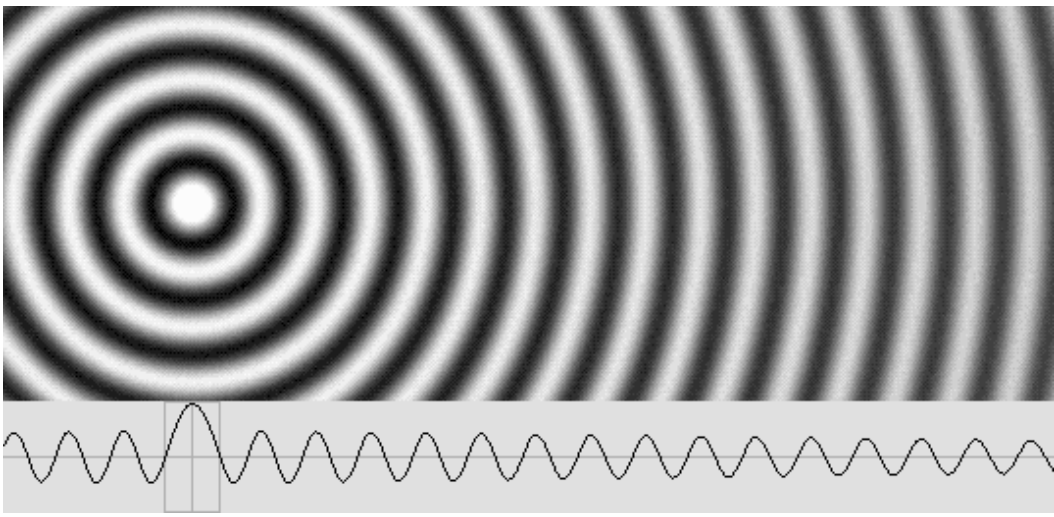
[42] See Discrete symmetry sets.

[43] That frequency is determined by the progression step size.



The waves curve the embedded continuum. The effect on local curvature diminishes with distance from the Qtargets. This effect is described by the corresponding potential function[44].

The sharp continuous part of the allocation function registers the effect on the embedding continuum and stores this data for the creation of the next version of the embedding continuum. A corresponding operator that resides in the Gelfand triple will reflect the embedding continuum in its eigen space. The correlation vehicle applies the Huygens principle for recreating the embedding continuum at every progression step.

[44] See: Waves that spread information.

10.3 Qpatterns

10.3.1 Natal and actual Qpatterns

The Qpattern is a dynamic building block. Qpatterns extend over many progression steps. A Qtarget lasts only during a single progression step.

A Qpattern is a coherent collection of objects that are distributed in space by a stochastic process. It means that each Qpattern is created differently. This coherent distribution can be described by two density distributions. The first one is a scalar function that describes the distribution of the density of the spatial locations. The second one describes the corresponding current density distribution. It administers the displacement since the last element generation. The two descriptions combine in a single Quaternionic Probability Amplitude Distribution (QPAD). The QPAD is a continuous quaternionic function. According to the hypothesis, Qpatterns of a given generation have a QPAD with a fixed natal shape.

The distribution of discrete objects corresponds to several potential functions. For each suitable Green's function a corresponding potential function exists. In this way the scalar density distribution correspond to a set of scalar potential functions and the current density distribution corresponds to a set of 3D vector potential functions.

A direct conversion from density distribution to a potential function uses a dedicated Green's function. Each suitable Green's function gives a corresponding potential function. The reverse conversion is only possible when the design plan of the Qpattern is known.

Each natal Qpattern corresponds to a plan. Not all enumerations that are required for generating the planned Qpattern must be used during the life of the actual Qpattern. Per progression step the generator creates only a single member of the Qpattern and that member

is replaced in the next step by another member. At every instant of progression, Qpatterns contain one actual member and for the rest it consists of virtual members. The actual member is a location where an event can happen. This actual element is called Qtarget. That event may be the annihilation of the Qpattern. After that the generation of new elements stops. In any case at every progression instant at the location of the Qtarget, small contributions to the potentials of the Qpattern are generated by the current Qtarget.

Each realization of a Qpattern corresponds to a micro-path that runs along step stones. The Qpatch may move and/or oscillate. The actual distribution of Qtargets spreads along the actual path of the building block. This actual path differs from the planned micro-path. The contributions to the potentials are transmitted by Qtargets at the halts along the actual path.

The natal Qpattern can be described by a temporal function that produces a stochastic spatial location at every subsequent progression interval.

Since the collection is generated in a rate of one element per progression step, the contributions to the potential functions are also generated in that rate and at the locations of the Qtargets, which form the current actual element. It is shown above that the potential functions are generated with the help of wave fronts[45] that with light speed move away from the locations of the elements that generated them.

These wave fronts are emitted with a fixed ultra-high frequency. In the HBM no higher frequency exists.

Only if the Qpattern stays fixed at a single location in an non-curved part of the embedding continuum, then that location will see the generation of a virtual Qpattern that takes a shape that approaches

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[45] The isotropy of the wave front depends on the isotropy of the emitting Qtarget.

the planned target distribution. It will take a huge number of progression steps to reach that condition.

A moving Qpattern will be spread along the path of the corresponding building block.

A move of the building block may affect the life of the realizable part of the Qpattern[46].

10.3.2 Micro-paths

Qpatterns are representatives of nature's building blocks. They are coherent collections of lower order objects that each can be considered as a location where the building block can be. These objects are generated in a rate of one element per progression step. The situation can be interpreted as if the building block hops from step stone to step stone. These micro-movements form a micro-path in the form of a random string. At each arrival at a step stone the building block emits a message. That emission contributes to the potentials of the building block. The emission does not affect the natal Qpattern. However, it may affect the actual Qpattern.

In order to stay at the same position, a step in a given direction will on average be followed by a step in the reverse direction. Otherwise the average location will move away or the pattern will implode or explode. This means that the particle moves along a micro-path and this path is characterized by quasi-oscillations. Similarly the micro-path may show quasi-rotations.

10.3.3 Qpattern history

A Qpattern can be created and it can be annihilated. If a Qpattern is annihilated, then the generator stops producing new elements. Thus, also the generation of new potential waves will stop. However,

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[46] http://en.wikipedia.org/wiki/Particle_decay

existing potential waves will keep proceeding. The last generated wave closes a train of previous waves. This edge moves away with light speed. A previously rather "static" potential will be replaced by a dynamic phenomenon. The annihilation frees the identifier of the Qpattern and makes it available for reuse. In this way the identifiers of the Qpatterns refer to their virgin equivalents that were born in the reference Hilbert space.

We will define "looking away" as receiving messages from distant objects. Looking away is looking back in proper time. Looking back as far as is possible is looking back at the virginal state of the historic Qpattern. Looking as far away as is possible is looking at the virginal state. In this way a Qpattern can be coupled both to its past and to its distant background. On the other side this means that the transmitted potential waves from this virgin state reach the current local Qpattern.

The superposition of all transmitted potentials that were emitted in the past and that contribute via superposition to the local potential results in huge background potential that acts as a (curved) embedding continuum (for fermions). This effect installs inertia[47].

10.3.4 Fourier transform

A QPAD that has the form of a QPAD of a Gaussian distribution has a Fourier transform that also has the form of a QPAD of a Gaussian distribution. However, the characteristics of the distributions will differ.

The QPAD of a coupled Qpattern is compact in configuration space and wide spread in canonical conjugated space.

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[47] See inertia

The Fourier transform of a Qpattern is its characteristic function[48]. It is a quaternionic function.

10.4 Qtargets

In fact the actual elements, called Qtargets, are represented by ***three*** different rational quaternions. These rational quaternions define locations or displacements relative to an embedding continuum. That continuum might be curved.

1. The real part of the *first quaternion* represents progression. Its imaginary part acts as the identifier of the element. For each Qtarget, the first quaternion plays the role of the corresponding parameter. This also holds at zero progression value. The Qtargets walk through a path as a function of progression.
2. The imaginary part of the *second quaternion* defines the location of the Qtarget in its current embedding continuum. Its real part specifies the local density. It also acts as the relevance factor of the corresponding Hilbert proposition.
3. The imaginary part of the *third quaternion* defines the displacement . The discrete symmetry set of this quaternion determines the "charge" of the Qtarget. The effective charge is set by the difference between the discrete symmetry set of the Qtarget and the discrete symmetry set of the embedding continuum. Apart from the discrete symmetry set this third quaternion

[48] http://en.wikipedia.org/wiki/Characteristic_function_(probability_theory)

> contains no new information. It contains the displacement of the previous Qtarget to the current Qtarget.

The planned and the actual distribution can be described by a charged carrier density distribution and a corresponding current density distribution. Via appropriate Green's functions these density distributions correspond to a scalar potential and a corresponding vector potential. The potentials reflect the transmittance of the existence and the discrete properties of the Qtarget via ultra-high frequency information carrier waves.

Since Qtargets are elements of Qpatterns and their identifier is also Qtarget of a Qpattern that existed at zero progression value, the two patterns are connected as well.

10.5 New mathematics

The idea that wave fronts[49] implement the contribution that Green's functions add to the potential functions, represents new mathematics. This is quite clear for the gravitational potential. The emitted wave folds and thus curves the embedding continuum. In this way curvature can be explained.

It is less clear for other potentials. Especially the encoding of electric charge information in the emitted information is not yet properly established. This encoding uses the difference in discrete symmetry between the Qtarget and the embedding continuum[50].

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[49] For anisotropic Qpatterns the message is transmitted by an anisotropic wave.

[50] See elementary particle properties

10.5.1 Waves that spread information

A Qtarget exists during a single progression step. Even when they belong to the same Qpattern will subsequent Qtargets be generated at different locations. If the Qtarget is generated, then in the embedding continuum the Qtarget corresponds to a tsunami-like wave front that has its source at the location of the Qtarget. After the disappearance of the Qtarget the wave front keeps spreading out. The wave fronts that belong to preceding Qtargets and the wave fronts that belong to other Qpatterns will interfere with that wave front. If the Qpatch is stationary, then at sufficient distance it will look as if the waves are generated by a single source. The train of emitted wave fronts will resemble an ultra-high frequency oscillating wave. The amplitude of this oscillating wave decreases with distance from the source. For isotropic spherical waves, this is the reason of the contribution of the term $\frac{Q_i}{|\boldsymbol{r}-\boldsymbol{r_i}|}$ to the static potential integral.

Example: Generation process with one element per progression instant. Here we use the electrostatic Green's function.

- Poisson process coupled to a binomial process
  - Binomial process implemented by a 3D spread function
  - Produces a 3D distribution
- Which approaches a 3D Gaussian distribution
  - $\rho_f(r) = \frac{Q}{\sigma^3\sqrt{2\pi}^3}\exp\left(\frac{-r^2}{2\sigma^2}\right)$
- This corresponds to a ***scalar potential*** of the form

$$\varphi(r) = \frac{Q}{4\pi\varepsilon r}\operatorname{erf}\left(\frac{r}{\sqrt{2}\sigma}\right) = \frac{1}{4\pi\varepsilon}\int \frac{\rho_f(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3\boldsymbol{r}' \approx \frac{Q}{4\pi\varepsilon r}(r \gg \sigma)$$

- And a ***vector potential*** of the form

$$\frac{\mathbf{Q}}{4\pi\varepsilon r}(r \gg \sigma)$$

  - Charge **Q** represents the discrete symmetry set difference between the carrier and the embedding continuum.

If an event occurs, then the generator stops generating Qtargets for this Qpattern in the configuration space. However the wave fronts that have been started will proceed spreading over the embedding configuration space.

When the local generator stops generating then no new wave fronts will be formed. The last wave front and foregoing wave fronts proceed spreading with light speed.

The fact that the wave fronts keep spreading is a consequence of the characteristics of the correlation vehicle, which is implemented by the enumerator generating mechanism. That mechanism also regenerates the embedding continuum for use in the next progression step. The mechanism uses the Huygens principle in order to establish persistence of the floating wave fronts.

The scalar potential functions and vector potential functions that correspond to the charge and current density distributions reflect the transmission of the information that is transmitted by the Qtargets.

The potential functions reveal the existence and the properties of the Qpattern. The potentials can be observed without affecting the Qpattern.

10.5.2 Waves that shrink space

The tsunami-like wave fronts appear to shrink space. The local shrinkage diminishes when the distance from the source increases. As a consequence, for 3D spherical information carrier waves, the influence diminishes as 1/r. Also this fact is a consequence of the actions of the correlation vehicle.

All quaternionic quantum state functions are fields (they are quaternionic probability amplitude distributions) that extend over a limited region of the embedding space. Their potentials extend over a part of universe that falls within the information horizon of the corresponding particles. When a particle annihilates, then the information about its existence keeps spreading. However, no new information is generated. The potential functions act as traces of Qpatterns.

The tsunami-like wave front that spreads this information appears to shrink the space where it passes. However its influence diminishes with distance. For spherical waves the influence diminishes with distance r as 1/r.

As long as a particle lives, it keeps sending these tsunami-like wave fronts. This might be the way that gravitation/ space curvature is implemented.

10.5.3 Information carrier waves

Information carrier wave fronts are emitted by Qtargets. The corresponding building block emits these wave fronts at an ultra-high frequency that is set by the progression step size. The wave fronts move with “light speed”. This speed is the highest possible speed that can be achieved for information transmission. Even when the Qtargets belong to the same Qpattern will subsequent Qtargets emit their information carrier wave fronts from different locations.

The spread of information carrier waves is governed by the Huygens principle. The correlation vehicle uses this principle in order to retransmit the waves at every progression step. This holds for transmission in odd numbers of dimensions. For transmission in two dimensions the situation is more complicated.

Information carrier waves pass unblocked through the embedding continuum. These waves are only influenced by other information carrier waves. Information carrier waves interfere. In that case, the information that they carry combines into a new information set.

If the emitting building block moves, then the sources of the emitted wave fronts move as well.

If the emitting building block oscillates, then the information carrier wave gets an amplitude and/or phase modulation. The frequency

of that modulation will be much lower than the ultra-high frequency of the carrier.

A train of emitted carrier wave fronts constitute a potential field. The interrelation is set by an appropriate Green's function.

10.5.4 Spreading electric charge information

The Qtarget also contains information about the electric charge of the corresponding particle. The process of spreading that information corresponds to the way that gravitational information is transmitted. In this case not the existence and local density, but the charge is transmitted. The charge is determined by the discrete symmetry of the Qtarget in comparison to the discrete symmetry of the embedding continuum. Only the symmetries of the imaginary parts that encode displacement are relevant.

10.5.5 Huygens principle

The correlation vehicle applies the Huygens principle. It means that in every progression step, every location on a wave front can be seen as a source of a new wave. The Huygens principle acts differently for waves that operate in different numbers of dimensions[51]. The Green's function differs accordingly. For odd dimensions the mechanism works in the commonly understood way.

The Huygens principle acts on ultra-high frequency waves that transmit the information that is contained in potential fields. The corresponding wave fronts proceed with light speed.

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[51] An interesting discussion is given at: http://www.math-pages.com/home/kmath242/kmath242.htm

10.6 Quasi oscillations and quasi rotations

In order to keep the distribution on average coherent in each dimension, any step in positive direction must be followed by a step in negative direction. With other words a kind of quasi oscillation takes place. This oscillation can be synchronous to a reference or it can be asynchronous. This (a)synchrony may differ per dimension. In a similar way a quasi-rotation can exist.

A special kind of coupling/interaction between fields can be the result of these induced quasi oscillations and or quasi rotations, where distant sources of oscillating potentials induce this coupling with local oscillations.

10.7 Distant Qtargets

The Qtargets of distant Qpatterns also send messages that encode their presence in tsunami-like wave fronts. These waves contribute to a huge local potential. This effect represents the origin of inertia[52]. Together the potentials of ***all*** Qpatterns constitute a local potential that can act as an embedding continuum.

It is a bit strange that electrostatic potential plays no role in this effect.

In this respect http://en.wikipedia.org/wiki/Common_integrals_in_quantum_field_theory may show interesting.

10.8 Spurious elements

Qtargets need not be generated in coherent distributions as is the case with Qpatterns. Coherent distributions correspond to potential functions that are constructed dynamically in a large series of steps. In extreme cases the distribution consists of a single element that pops up and disappears in a single progression step. During its existence the element still produces a tsunami-like signal in the form of a

[52] See Inertia

wave front[53] that travels in the embedding continuum. Again this wave front causes a local curvature. In large numbers these spurious elements may cause a noticeable effect.

10.9 The tasks of the correlation vehicle

The primary task of the element generator is the generation of Qtargets that are part of Qpatterns. After the generation and vanishing of the Qtarget the correlation vehicle takes care of the transmission of the information about the generation incident over the embedding continuum in which the Qtarget was produced. This is done in the form of the described tsunami-like wave fronts. This is the second task of the correlation vehicle. When the generator stops generating Qtargets for the current Qpattern, then it does not transmit new information but the correlation mechanism keeps supporting the existing flow of information. This means that a third task of the correlation mechanism is the care for the survival of the embedding continuum when the Qtargets vanish.

The transmission of incident information causes space curvature. The sharp part of the allocation function ***describes*** the strength of the local space curvature. It does this via its differential which specifies a local metric.

Apart from describing the curvature, the correlation mechanism also recreates at every progression step the corresponding embedding continuum. For that purpose it uses the Huygens principle.

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[53] For anisotropic Qtargets the message is transmitted by an anisotropic wave.

Geometric model

11 Geometrics

Geometrics enters the model as soon as numerical enumerators are applied. These enumerators are taken from the eigenspaces of operators. This can happen in Hilbert logic and in the Hilbert space and its Gelfand triple.

The geometric model applies the quaternionic Hilbert space model. From now on the complex Hilbert space model and the real Hilbert space model are considered to be abstractions of the quaternionic model. It means that the special features of the quaternionic model bubble down to the complex and real models. For example both lower dimensional enumeration spaces will show blur at small enumeration scales. Further, both models will show a simulation of the discrete symmetry sets that quaternionic systems and functions possess. This can be achieved with spinors and Dirac matrices or with the combination of Clifford algebras, Grassmann algebras and Jordan algebras[54].

The real and complex models suit in situations where multidimensional phenomena can be decoupled from the dimensions in which they appear.

At large scales the model can properly be described by the complex Hilbert space model. After a sufficient number of progression steps, at very large scales the quaternionic model is quasi isotropic.

We will place the reference Hilbert space at zero progression value. This reference Hilbert space can be a subspace of a much larger Hilbert space. However, in the reference Hilbert subspace a state of densest packaging must reside.

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[54] See: http://math.ucr.edu/home/baez/rch.pdf

Quaternionic numbers exist in 16 discrete symmetry sets. When used as enumerators, half of this set corresponds with ***negative progression and will not be used in this geometric model.***

As a consequence we will call the Hilbert space at zero progression value the ***start*** of the model.

This model does not start with a Big Bang. Instead it starts in a state that is characterized by densest packaging of the Qpatches. This reference sub-model is well-ordered.

12 Distribitions of quaternions

12.1 Continuous quaternionic distributions

Quaternionic distributions consist of a real scalar distribution and an imaginary 3D vector distribution.

It is the sum of a symmetric distribution and an asymmetric distribution.

The complex Fourier transform of a symmetric (complex) function is a cosine transform. It is a real function.

The complex Fourier transform of an anti-symmetric (complex) function is a sine transform. It is an imaginary function.

This cannot directly be translated to quaternionic functions. The simplest solution is to consider the symmetric parts and asymmetric parts separately. An asymmetric quaternionic function is always anisotropic. A symmetric function can be isotropic.

As shown before the continuous quaternionic distributions can be interpreted as descriptors of the density distribution of a coherent distribution of discrete objects. However the potential functions that can be derived from coherent distributions of discrete objects are also quaternionic functions. In the HBM these associated potentials can be considered to be generated dynamically.

12.2 RQE's

In principle the base vectors of the Hilbert space can be enumerated by members of a countable affine space. Here we concentrate on a ***huge subsp***ace in which the base vectors are enumerated by rational quaternions. The huge subspace is covered by a large number of ***small dedicated subspaces*** that all are identified by a ***Qpatch region***.

The ordering and the corresponding origin of space become relevant when an observer object considers one or more observed objects. The real parts of the enumerators define progression. In conventional physics progression conforms to proper time. In the HBM all proper time clocks are synchronized. As a consequence according to our model, the equivalent of proper time steps with a fixed step.

RQE stands for ***Rational Quaternionic Enumerator***. This lowest geometrical level is formed by the enumerators of a selected base of a selected member of the sequence of Hilbert spaces. The selected base vectors represent atoms of the Hilbert logic system. In this level, the embedding continuum plays a secondary role. The sequence number corresponds with the progression value in the real part of the value of the RQE. In principle the enumerators enumerate a previously unordered set.

The dedicated subspaces are spanned by eigenvectors whose eigenvalues form the elements of ***Qpatterns***. Qpatterns are identified by a ***Qpatch***, which is the weighted center and by a ***Qtarget***, which is the currently actual element. All other elements of the Qpattern and all other vectors of the dedicated subspace are virtual. Virtual means: "reserved, but currently not in use".

Two types of RQE's exist.

- The first type of RQE plays the role the parameter that via the continuous part of the allocation function determine the origins of planned Qpatterns. We will call these RQE's "***parameter RQE***".
- The second type of RQE is defined relative to these origin RQE's. We will call these RQE's "***relative RQE***"

The relative RQE's are targets of the stochastic function that defines the relative locations of the elements of the natal Qpattern. The relative RQE's can be considered to be the target values of the separate stochastic part of the allocation function. They define a natal Qpattern.

The actual Qtarget is the image produced by the total allocation function of the parameter RQE. The total allocation function is the ***convolution*** of the continuous part of the allocation function and the stochastic part of the allocation function. It maps a parameter RQE onto a selected embedding continuum. For the reference Hilbert space its Gelfand triple delivers the reference continuum as embedding continuum. For later Hilbert spaces the role of the embedding continuum is taken over by the superposition of one type of the potentials. That potential type is the gravitation potential.

12.2.1 Reference Hilbert space

A zero value of the real part of an RQE indicates its role in the reference Hilbert space. In the reference Hilbert space the parameter RQE's are well ordered and embedded in a reference continuum that is taken from the eigenspace of a reference operator that resides in the Gelfand triple of that reference Hilbert space.

The considered huge subspace of the selected reference member of the sequence of Hilbert spaces represents a state of densest packaging of the parameter RQE's. This means that in this subspace of the selected Hilbert space a normal ***allocation operator*** exists whose discrete and countable eigenspace has eigenvalues that are parameter RQE's, while in the Gelfand triple of this Hilbert space an allocation operator exists whose continuous eigenspace embeds the values of these parameter RQE's in a well ordered and relative dense way. The relative density is limited by a lowest size of rational quaternions.

Due to this restriction the parameter RQE-space is not afflicted with splits and ramifications[55].

Thus, both the parameter RQE's and the reference continuum are taken from the eigenspace of a corresponding normal allocation operator. These operators will be called reference operators.

In the reference Hilbert space the continuous part of the allocation function is a unity map. The Qpatches in the reference Hilbert space are linear combinations of a coherent set of relative RQE's that together with the parameter RQE of that set correspond to eigenvectors, which together span the dedicated subspace. This dedicated subspace corresponds to a building block.

In the reference Hilbert space the notion of an absolute RQE makes sense. It is the sum of a parameter RQE and a relative RQE. The Qpatch is the average value of all absolute RQE's that belong to the building block.

12.2.2 Later Hilbert spaces

In each Hilbert space the planned Qpattern has its own local origin. In later Hilbert spaces the embedding continuum is no longer

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[55] http://en.wikipedia.org/wiki/Quaternion_algebra#Quaternion_algebras_over_the_rational_numbers

flat as it is in the reference Hilbert space. Also the parameter RQE may have another location (has another imaginary value) than it had in the reference Hilbert space. With other words the parameter RQE's may move.

Still, the actual Qpatch is the average value of all target RQE's that belong to the corresponding building block. The continuous part of the allocation function images the current parameter RQE on a temporary target. This temporary target is taken as the parameter of the stochastic part of the allocation function. This second part produces the Qtarget as a location in the selected embedding continuum.

Here the selected embedding continuum is formed by superposed potentials and is represented by the eigenspace of a dedicated operator that resides in the Gelfand triple. The corresponding potential is a special type. It is the gravitation potential.

Relative RQE's act as target vales for elements of actual Qpatterns. They are target values for a corresponding parameter RQE of the complete allocation function. The Qpatch of the actual building block will become the expectation value of the Qtargets. Thus, at higher progression values, it no longer corresponds to the average value of the undistorted absolute RQE's that characterize the natal Qpattern.

In general, Qtargets are locations in a curved space. Only in the reference Hilbert space, that space is flat.

HYPOTHESIS 3: *At the start of the life of the considered huge subspace the HBM used only one discrete symmetry set for its lowest level of geometrical objects. This discrete symmetry set is the same set that characterizes the reference continuum. This situation stays throughout the history of the model. This set corresponds with the set*

of eigenvalues of an RQE allocation operator that resides in the reference quaternionic Hilbert space model.

For each building block, in the reference Hilbert space one of the relative RQE's becomes the actual element and will be called Qtarget. In each subsequent Hilbert space another relative RQE will be selected whose image becomes the Qtarget. The selection of the relative RQE occurs via a random process.

In subsequent Hilbert spaces a new eigenvalue of the reference allocation operator becomes the parameter RQE of the new Qtarget of the building block. This goes together with the selection of a new relative RQE. The relative RQE will differ in a random way from the original relative RQE. Thus Qtargets are for a part a continuous functions ($\wp$) of the corresponding parameter RQE's and for another part the function result is blurred by a random generator function ($\mathcal{S}$). The convolution ($\mathcal{P}$) of the continuous function and the random generator function ($\mathcal{S}$) determines the location of the current Qtarget.

$$\mathcal{P} = \wp \circ \mathcal{S} \tag{1}$$

($\mathcal{S}$) stands for stochastic spatial spread function. The assignment of the value of the random function ($\mathcal{S}$) occurs according to a given ***plan***. The natal (undisturbed) result of ($\mathcal{S}$) is a natal Qpattern that is described by a quaternionic probability amplitude distribution (QPAD) ψ. A significant difference may exist between the planned building block and the actually realized building block.

12.3 Potentials

Relative RQE's are the (relative) identifiers of the elements of a Qpattern. Parameter RQE's are parameters of Qtargets. Qpatterns exist during a series of subsequent Hilbert spaces. They represent nature's building blocks. The absolute RQE's reside in the reference

Hilbert space, which occurred in the past. The real part of the RQE's reflect the current progression value. The parameter RQE's reside in each of the subsequent Hilbert spaces. Qpatches are linear combinations of the values of elements of a Qpattern. They represent the expectation values of the Qtargets. The elements of the Qpatterns correspond to base vectors of dedicated Hilbert subspaces. The Qtargets emit contributions to the potentials of the Qpatterns.

Potentials depend on their Green's function. Apart from that, two kinds of potentials exist: scalar potentials and vector potentials. Potentials of the same type superpose. The potentials that possess sufficient reach may together add up to huge local potentials[56]. Locally the superposition of scalar potentials constitute a curved continuum that can be used to embed localizable objects. This continuum installs inertia for the embedded Qpatterns.

For all continuous quaternionic functions and for each discrete symmetry set of its parameter space, the function exists in 16 different discrete symmetry sets for its function values. In the HBM the discrete symmetry set of the parameter RQE's is fixed. The quaternionic potentials are continuous functions. Their superpositions constitute embedding continuums. This means that for vector potentials also 16 different embedding continuums exist.

Also the allocation function exists in 16 different discrete symmetry sets for its function values. The sharp continuous part of the allocation function describes an embedding continuum. The allocation function keeps its discrete symmetry set throughout its life.

Discrete symmetry sets do not influence the scalar potentials that are connected to object density distributions. Thus the superposition of these scalar potentials constitutes a special embedding continuum.

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[56] See Inertia

This continuum characterizes the ***Palestra***. It is described by the gravitation potential field. This does not say that in the realm of the Palestra no other potentials play their role.

12.4 Palestra

The second geometric level is a curved space, called Palestra. As ingredients, it consists of an embedding continuum, the embedded Qtarget set and a sharp continuous quaternionic allocation function. The local curvature is defined via the differential of the continuous (sharp) quaternionic allocation function. The parameter space of the allocation function embeds the parameter RQE-set. Thus since the parameter RQE-set is countable, the Palestra contains a countable set of images of the sharp allocation function. We have called these images "local origins" of Qpatterns. The Qpatches represent the expectation values of the corresponding Qtarget values. The allocation function exists in 16 versions. The version determines the discrete symmetry set of the Qpattern and of the corresponding Qtargets.

The allocation function may include an isotropic scaling function. The differential of the allocation function defines an infinitesimal quaternionic step. In physical terms the length of this step is the infinitesimal coordinate time interval. The differential is a linear combination of sixteen partial derivatives. It defines a quaternionic metric[57]. The enumeration process adds a coordinate system. The selection of the coordinate system is arbitrary. The origin and the axes of this coordinate system only become relevant when the distance between locations must be handled. The origin is taken at the location of the current observer. The underlying space is an affine space. It does not have a unique origin. We only consider an enumerated compartment of the affine space.

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[57] See the paragraph on the spacetime metric.

12.5 Qpatch regions

The third level of geometrical objects consists of a countable set of space patches that occupy the Palestra. We already called them ***Qpatch regions***. Qpatches are expectation values of the Qtarget images of the parameter RQE's that house in the first geometric object level. The set of parameter RQE's is used for the part of the allocation function that produces the local Qpattern origins. Apart from the rational quaternionic value of the corresponding local origin, the discrete symmetry set of that origin will be shared by all elements of the corresponding Qpattern. The curvature of the second level space relates to the density distribution of the local origins of the Qpatterns and to the total energy of the corresponding Qpattern. The Qpatches represent the weighted centers of the locations of the ***regions***[58] where next level objects can be detected. The name Qpatch stands for space patches with a quaternionic value. The charge of the Qpatches can be named Qsymm, Qsymm stands for discrete symmetry set of a quaternion. However, we already established that the value of the enumerator is also contained in the property set that forms the Qsymm charge.

The enumeration problems that come with the quaternionic Hilbert space model indicate that the Qpatches are in fact centers of a fuzzy environment that houses the potential locations where the actual parameter RQE images (the Qtargets) can be found. The subsequent Qtargets form a micro-path.

12.6 QPAD's and Qtargets

The fuzziness in the sampling of the enumerators and their images in the embedding continuum is described by a ***quaternionic probability amplitude distribution*** (QPAD). The squared modulus of the ***complex probability amplitude distribution*** (CPAD) represents the

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[58] Not the exact locations.

probability that an image of a parameter RQE will be detected on the exact location that is specified by the value of the target of the blurred allocation function. In the QPAD this location probability is represented by the real part of the QPAD. The imaginary part describes a corresponding displacement probability. The real part is an object density distribution and the imaginary part is the associated current density distribution. The real part is a scalar function and the imaginary part is a 3D vector function.

Both a CPAD and a QPAD can describe a Qpattern. A QPAD gives a more complete description.

A natal Qpattern is generated in a rate of one element per progressions step. Thus the generator function ($\mathcal{S}$) is a stochastic function of progression. Its anchor point is the image by the continuous part ($\wp$) of the allocation function ($\mathcal{P}$) of the selected parameter RQE. Its target domain is an embedding continuum. The natal Qtarget is one of the function values. Usually, the actual Qtarget is displaced with respect to the natal Qtarget.

A natal Qpattern is generated via a fixed statistical plan and is not disturbed by space curvature or a moving local origin. Since a Qpattern is generated by a stochastic process, the same natal QPAD can correspond to different natal Qpatterns. The ***QPAD's that describe natal Qpatterns have a flat target space in the form of a quaternionic continuum.***

This natal QPAD describes the planned blur (ψ) to the image of the sharp allocation function ($\wp$). The blurred allocation function ($\mathcal{P}$) is formed by the ***convolution*** of the sharp allocation function ($\wp$) with stochastic generator function ($\mathcal{S}$). The results of this generator function are described by the natal QPAD (ψ) that on its turn describes the natal Qpattern. In this way the local form of the actually

realized QPAD describes a deformed Qpattern. The adaptation concerns the form factor and the gradual displacement of the deformed QPAD. The form factor may differ in each direction. It is determined by the local differential ($d\wp$) of the sharp allocation function ($\wp$).

The image of a parameter RQE that is produced by the *blurred allocation function* ($\mathcal{P}$) is a ***Qtarget***. ***Qtargets only live during a single progression step***. Qtargets mark the location where (higher level) objects may be detected. In this way QPAD's exist in two types. The natal QPAD type describes the undisturbed natal Qpattern. It describes a fixed plan. The second QPAD type describes the potential Qtargets that at a rate of one element per progression step are or will be[59] ***locally*** generated by the blurred allocation function. That is why this second QPAD type is also called an actual local QPAD.

The natal Qpattern can also be described by a function ($\mathcal{S}$) that produces a stochastic spatial location at every subsequent progression interval. That natal Qpattern describes a natal micro-path.

The fact that Qtargets only exist during a single progression step means that on the instant of an event the generation of the Qpattern might stop or might proceed in a different mode. Only if the Qpattern stays untouched, a rather complete Qpattern will be generated at that location. When the Qpatch moves, then the corresponding actual Qpattern smears out. With other words the natal QPAD is a plan rather than reality.

An event means that a Qpattern stops being generated or is generated in a different mode. Being generated means that it is coupled to an embedding continuum. The generator will create a relatively small pattern in that continuum. Coupling means that the generated Qpattern is coupled via its Qpatch to a mirror Qpattern that houses

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[59] Adding to the QPAD Qtargets that still have to be generated can be considered as an odd decision.

in the embedding continuum. This is reflected in the coupling equation[60].

The parameter space of the blurred allocation function ($\mathcal{P}$) is a flat quaternionic continuum. The parameter RQE's form points in that continuum.

Local QPAD's are quaternionic distributions that contain a scalar density distribution in their real part that describes a density distribution of ***potential*** Qtargets. Further they contain a 3D vector function in their imaginary part that describes the associated current density distribution of these potential Qtargets.

Continuous quaternionic distributions exist in sixteen different discrete spatial symmetry sets. However, the QPAD's inherit the discrete symmetry of their connected sharp allocation function. The Qpatterns may mingle and then the QPAD's will superpose. However the spatial extent of Qpatterns is quite moderate. In contrast, the potentials of their Qtargets reach very far. Quite probably these potentials will superpose. Together the potentials of distant building blocks form a background potential. Depending on the Green's functions, the local QPAD's correspond to several types of quaternionic potential functions. These quaternionic potential functions combine a scalar potential and a vector potential.

The QPAD's are continuous functions. The objects that are described by these distributions form coherent countable discrete sets.

A Qtarget is an actually existing object. A Qpattern is a mostly virtual object. A natal Qpattern conforms to a plan. A QPAD may describe a Qpattern. In that case it describes a mostly virtual object. A natal QPAD describes a plan.

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[60] See coupling equation.

12.6.1 Inner products of QPAD's and their Qpatches

(this section needs editing)

Each Qpattern is a representative of a Hilbert subspace and indirectly the Qpattern represents a quantum logic proposition. The corresponding Qpatch is represented by a linear combination of Hilbert base vectors and is represented by a Hilbert proposition. These base vectors are eigenvectors of the location operator. The coefficients are determined by the values of the real part of the QPAD. The Qpatch vector may represent the QPAD.

Two QPAD's a and b have an inner product defined by

$$\langle a|b\rangle = \int_V a\ b\ dV \tag{1}$$

Since the Fourier transform $\mathcal{F}$ preserves inner products, the Parseval equation holds for the inner product:

$$\langle a|b\rangle = \langle \mathcal{F}a|\mathcal{F}b\rangle = \langle \tilde{a}|\tilde{b}\rangle = \int_{\tilde{V}} \tilde{a}\ \tilde{b}\ d\tilde{V} \tag{2}$$

QPAD's have a norm

$$|a| = \sqrt{\langle a|a\rangle} \tag{3}$$

12.7 Blurred allocation functions

The blurred allocation function $\mathcal{P}$ has a flat parameter space that is formed by rational quaternions. It is the convolution of the sharp allocation function $\wp$ with a stochastic spatial spread function $\mathcal{S}$ that generates a blur that is represented by a planned natal Qpattern and is described by QPAD ψ. The sharp allocation function $\wp$ has a flat parameter space that is formed by real quaternions. ψ has rational quaternionic parameters.

$$\mathcal{P} = \wp \circ \mathcal{S} \tag{1}$$

$\wp$ describes the long range variation and ψ describes the short range variation. Due to this separation it is possible to describe the effect of the convolution on the actual local QPAD as a deformed natal QPAD that on its turn describes a natal Qpattern, where the

form factor is controlled by the differential $d\wp$ of the sharp allocation function. The sharp part of the allocation function specifies the current embedding continuum. In fact this function defines the eigenspace of a corresponding operator that resides in the Gelfand triple of the current Hilbert space.

The planned Qpattern is the result of a Poisson process that is coupled to a binomial process, while the binomial process is implemented by a 3D spread function. This second part $\mathcal{S}$ of the allocation function $\mathcal{P}$ influences the local curvature. The differential $d\wp$ of the first part $\wp$defines a quaternionic ***metric*** that describes the ***local spatial curvature***. This means that the two parts must be in concordance with each other.

Fourier transforms cannot be defined properly for functions with a curved parameter space, however, the blurred allocation function $\mathcal{P}$ has a well-defined Fourier transform $\tilde{\mathcal{P}}$, which is the product of the Fourier transform $\widetilde{\wp}$ of the sharp allocation function and the Fourier transform $\tilde{\mathcal{S}}$ of the stochastic spatial spread function $\mathcal{S}$.

$$\tilde{\mathcal{P}} = \widetilde{\wp} \times \tilde{\mathcal{S}} \tag{2}$$

This corresponds to a Fourier transform $\tilde{\psi}$ of the actual local QPAD ψ .The Fourier transform pairs and the corresponding canonical conjugated parameter spaces form a double-hierarchy model.

The Fourier transform $\tilde{\mathcal{P}}$ of the blurred allocation function $\mathcal{P}$ equals the product of the Fourier transform $\widetilde{\wp}$ of the sharp allocation function $\wp$ and the Fourier transform $\tilde{\mathcal{S}}$ of the generator function $\mathcal{S}$.

16 blurred allocation functions exist that together cover all Qpatches. One of the 16 blurred allocation functions acts as reference. The corresponding sharp allocation function and thus the corresponding actual QPAD ψ have the same discrete symmetry set as the lowest level space.

The fact that the blur ψ mainly has a local effect makes it possible to treat $\wp$ and ψ seperately[61].

12.8 Local QPAD's and their superpositions

The model uses Qpatterns in order to implement the fuzziness of the local interspacing. After adaptation of the form factor to the differential of the sharp allocation function a local QPAD is generated. The non-deformed natal QPAD describes a natal Qpattern. Each Qpattern possess a private inertial reference frame[62].

The superposition of neighboring deformed local QPAD's, eventually including neighboring (deformed) descriptors of the higher generations of the Qpatterns, forms a new QPAD. Each of the 16 blurred allocation functions may correspond to such QPAD superpositions.

Each of the natal Qpatterns extends over a restricted part of the embedding continuum. The probability amplitude of the elements of these Qpatterns quickly diminishes with the distance from their center point[63].

The gravitation potential of a Qpattern extends over the whole embedding continuum. As a consequence superpositions of such potentials may cover the whole embedding continuum.

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[61] ψ concerns quantum physics. $\wp$ concerns general relativaty.

[62] See the paragraph on inertial reference frames.

[63] See the paragraph on the enumeration process.

12.9 Generations

Photons and gluons correspond to a special kind of fields. They differ in temporal frequency from the fields that constitute the potentials of particles. They can be interpreted as amplitude modulations of the potential generating fields. Two photon types and six[64] gluon types exist[65].

For fermions, three generations of Qpatterns exist that have non-zero extension and that differ in their basic form factor. This paper does not explain these generations.

The generator of enumerators is for a part a random number generator. That part is implemented by a Poisson process and a subsequent binomial process. Generations correspond to different characteristics of the enumerator generator.

All generated Qpatterns may differ in their quasi-oscillations and quasi-rotations.

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[64] In the Standard Model gluons appear as eight superpositions of the six base gluons.

[65] Bertrand's theorem indicates that under some conditions, photons and gluons might be described as radial harmonic oscillators.

13 Coupling

According to the coupling equation, coupling may occur because the two QPAD's that constitute the coupling take the same location. Several reasons can be given for this coupling. The strongest reason is that the Qpattern generator produces two patterns that subsequently are coupled.

Other reasons are:

Coupling between Qpatterns can be achieved by coupling to each other's potential functions.

- Coupling may occur between the local Qpattern and the potentials of very distant Qpatterns. This kind of coupling causes inertia. These coupling products appear to be fermions.
- Coupling may occur between the local Qpattern and the potentials of locally situated Qpatterns. These coupling products appear to be bosons.

The fermion coupling uses the gravitation potential, which is a scalar potential. On itself this does not enforce a discrete symmetry. (Suggestion: That symmetry can be enforced/induced by involving the discrete symmetry of the parameter space and/or the discrete symmetry of the virgin Qpattern).

Coupling can also occur via induced quasi oscillations and or induced quasi rotations. These quasi-oscillations and quasi-rotations occur in the micro-paths of the Qpatterns. Because they differ in their discrete symmetry they may take part in a local oscillation where an outbound move is followed by an inbound move and vice versa[66].

For fermions coupling also occurs with the parameter RQE and with the historic Qpattern that belongs to this RQE.

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[66] See: Coupling Qpatterns.

13.1 Background potential

We use the ideas of Denis Sciama[67][68][69].

The superposition of all real parts of potentials of distant Qpatterns that emit potential contributions in the form of spherical waves produces a uniform background potential. At a somewhat larger distance r these individual scalar potentials diminish in their amplitude as $1/r$. However, the number of involved Qpatterns increases with the covered volume. Further, on average the distribution of the Qpatterns is isotropic and uniform. The result is a huge (real) local potential Φ

$$\Phi = -\int_V \frac{\bar{\rho}_0}{r} dV = -\bar{\rho}_0 \int_V \frac{dV}{r} = 2\pi R^2 \bar{\rho}_0 \tag{1}$$

$$\bar{\rho} = \bar{\rho}_0; \bar{\boldsymbol{\rho}} = \mathbf{0} \tag{2}$$

Apart from its dependence on the average value of $\bar{\rho}_0$, Φ is a huge constant. Sciama relates Φ to the gravitational constant G.

$$G = (-c^2) \,/\, \Phi \tag{3}$$

If a local Qpattern moves relative to the universe with a uniform speed $\boldsymbol{v}$, then a vector potential $\boldsymbol{A}$ is generated.

$$\boldsymbol{A} = -\int_V \frac{\boldsymbol{v}\,\bar{\rho}_0}{c\,r} dV \tag{4}$$

Both $\bar{\rho}_0$ and **v** are independent of *r*. The product $\boldsymbol{v}\,\bar{\rho}_0$ represents a current. Together with the constant *c* they can be taken out of the integral. Thus

$$\boldsymbol{A} = \Phi \frac{\boldsymbol{v}}{c} \tag{5}$$

Field theory learns:

$$\mathfrak{E} = -\boldsymbol{\nabla}\boldsymbol{\Phi} - \frac{\mathbf{1}}{\boldsymbol{c}} \cdot \dot{\boldsymbol{A}} \tag{6}$$

If we exclude the first term because it is negligible small, we get:

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[67] http://arxiv.org/abs/physics/0609026v4.pdf

[68] http://www.adsabs.harvard.edu/abs/1953MNRAS.113...34S

[69] http://rmp.aps.org/abstract/RMP/v36/i1/p463_1

$$\mathfrak{E} = -\frac{\Phi}{c^2}\,\dot{\boldsymbol{v}} = G\,\dot{\boldsymbol{v}} \tag{7}$$

The fields Φ and $\boldsymbol{A}$ together form a quaternionic potential. However, this time the fields Φ and $\boldsymbol{A}$ do not represent the potential of a Qpattern.

13.2 Interpretation

As soon as an acceleration of a local Qpattern occurs, an extra component $\dot{\boldsymbol{A}}$ of field $\mathfrak{E}$ appears that corresponds to acceleration $\dot{\boldsymbol{v}}$.[70]

In our setting the component $\boldsymbol{\nabla}\Phi$ of the field $\mathfrak{E}$ is negligible. With respect to this component the items compensate each other's influence. This means that if the influenced subject moves with uniform speed $\boldsymbol{v}$, then $\mathfrak{E} \approx 0$. However, a vector potential $\boldsymbol{A}$ is present due to the movement of the considered local Qpattern. Any acceleration of the considered local item goes together with an extra non-zero $\mathfrak{E}$ field. In this way the universe of particles causes inertia in the form of a force that acts upon the scalar potential of the accelerating item.

The amplitude of Φ says something about the number of coupled Qpatterns of the selected generation that exist in universe. If it is constant and the average interspacing grows with progression, then the universe dilutes with increasing progression. Also the volume of the reference continuum over which the integration must be done will increase with progression. The total energy of these coupled Qpatterns that is contained in universe equals:

$$E_{total} = \sqrt{\int_V \left|\frac{\bar{\rho}_0}{r}\right|^2 dV} \tag{1}$$

[70] See: Inertia from the coupling equation.

The background potential Φ is the superposition of the contributions of waves that are emitted by distant particles. The emission occurred with ultra-high frequency. This is the highest frequency that exists in the HBM. The background potential constitutes an embedding continuum.

The enumerator generator uses the background potential as the embedding continuum for its embedded products. The allocation function describes this embedding continuum and takes care of its permanence.

Fields that oscillate with a lower frequency, such as photons, are generated by oscillating sources and can be considered as amplitude modulations of the ultra-high frequency (potential) field.

13.3 Isotropic vector potential

The scalar background potential may be accompanied by a similar background vector potential that is caused by the fact that the considered volume that was investigated in order to calculate the scalar background potential is enveloped by a surface that delivers a non-zero surface integral. The isotropic background potential corresponds to an isotropic scaling factor. This factor was already introduced in the first phases of the model.

13.4 Quantum fluid dynamics

13.4.1 Quaternionic nabla

The quaternionic nabla stands for

$$\nabla \stackrel{\text{def}}{=} \left\{\frac{\partial}{\partial\tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} \tag{1}$$

$$\psi \stackrel{\text{def}}{=} \psi_0 + \boldsymbol{\psi} \tag{2}$$

Here τ stands for the progression parameter.

$$\phi = \nabla\psi \tag{3a}$$

$$\phi_0 = \nabla_0\psi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle \tag{3b}$$

$$\boldsymbol{\phi} = \nabla_0\boldsymbol{\psi} + \boldsymbol{\nabla}\psi_0 + \boldsymbol{\nabla} \times \boldsymbol{\psi} \tag{3c}$$

Is the differential equation for continuous quaternionic distributions. Rearranging shows:

$$\nabla\psi = \phi \tag{4}$$

This is the differential continuity equation. It holds for QPAD's

13.4.2 The differential and integral continuity equations

Let us approach the balance equation from the integral variety of the balance equation. Balance equation is another name for continuity equation.

We replace ψ by ρ, ψ_0 by ρ_0 and $\boldsymbol{\psi}$ by $\boldsymbol{\rho} = \rho_0\boldsymbol{v}/c$.

$$\rho \stackrel{\text{def}}{=} \rho_0 + \boldsymbol{\rho} \tag{1}$$

When ρ_0 is interpreted as a charge density distribution, then the conservation of the corresponding charge[71] is given by the continuity equation:

$$\text{Total change within } V = \text{flow into } V + \text{production inside } V \tag{2}$$

In formula this means:

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[71] Also see Noether's laws: http://en.wikipedia.org/wiki/Noether%27s_theorem

$$\frac{d}{d\tau}\int_V \rho_0\, dV = \oint_S \hat{\boldsymbol{n}}\rho_0 \frac{\boldsymbol{v}}{c}\, dS + \int_V s_0\, dV \tag{3}$$

$$\int_V \nabla_0\rho_0\, dV = \int_V \langle \boldsymbol{\nabla}, \boldsymbol{\rho}\rangle\, dV + \int_V s_0\, dV \tag{4}$$

The conversion from formula (2) to formula (3) uses the Gauss theorem[72]. Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface S, $\boldsymbol{v}(\tau, \boldsymbol{q})$ is the velocity at which the charge density $\rho_0(\tau, \boldsymbol{q})$ enters volume V and s_0 is the source density inside V. In the above formula $\boldsymbol{\rho}$ stands for

$$\boldsymbol{\rho} = \rho_0 \boldsymbol{v}/c \tag{4b}$$

It is the flux (flow per unit area and unit time) of ρ_0 .

The combination of $\rho_0(q)$ and $\boldsymbol{\rho}(q)$ is a quaternionic skew field $\rho(q)$ and can be seen as a probability amplitude distribution (QPAD). ρ is a function of q.

$$q \stackrel{\text{def}}{=} q_0 + \boldsymbol{q};\ q_0 = \tau \tag{5}$$

$\rho(q)\rho^*(q)$ can be seen as an overall probability density distribution of the presence of the carrier of the charge. $\rho_0(q)$ is a charge density distribution. $\boldsymbol{\rho}(q)$ is the current density distribution.

This results in the law of charge conservation:

$$\begin{aligned} s_0(q) &= \nabla_0\rho_0(q) \mp \langle \boldsymbol{\nabla}, \left(\rho_0(q)\boldsymbol{v}(q) + \boldsymbol{\nabla} \times \boldsymbol{a}(q)\right)\rangle \\ &= \nabla_0\rho_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) + \boldsymbol{A}(q)\rangle \\ &= \nabla_0\rho_0(q) \mp \langle \boldsymbol{v}(q), \boldsymbol{\nabla}\rho_0(q)\rangle \\ &\qquad \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(q)\rangle\, \rho_0(q) \\ &\qquad \mp \langle \boldsymbol{\nabla}, \boldsymbol{A}(q)\rangle \end{aligned} \tag{6}$$

The blue colored ± indicates quaternionic sign selection through conjugation of the field $\rho(q)$. The field $\boldsymbol{a}(q)$ is an arbitrary differentiable vector function.

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[72] http://en.wikipedia.org/wiki/Divergence_theorem

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{a}(q) \rangle = 0 \tag{7}$$

$\boldsymbol{A}(q) \stackrel{\text{def}}{=} \boldsymbol{\nabla} \times \boldsymbol{a}(q)$ is always divergence free. In the following we will neglect $\boldsymbol{A}(q)$.

Equation (6) represents a balance equation for charge density. What this charge actually is, will be left in the middle. It can be one of the properties of the carrier or it can represent the full ensemble of the properties of the carrier.

Up to this point the investigation only treats the real part of the full equation. The full continuity equation runs:

$$\begin{aligned} s(q) = \nabla\rho(q) &= s_0(q) + \boldsymbol{s}(q) \\ &= \nabla_0\rho_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle \pm \nabla_0 \boldsymbol{\rho}(\tau, \boldsymbol{q}) \\ &\quad + \boldsymbol{\nabla}\rho_0(\tau, \boldsymbol{q}) \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{\rho}(\tau, \boldsymbol{q}) \right) \\ &= \nabla_0\rho_0(\tau, \boldsymbol{q}) \mp \langle \boldsymbol{v}(q), \boldsymbol{\nabla}\rho_0(q) \rangle \\ &\quad \mp \langle \boldsymbol{\nabla}, \boldsymbol{vq} \rangle \rho_0(q) \\ &\quad \pm \nabla_0 \boldsymbol{v}(q) + \nabla_0\rho_0(q) + \boldsymbol{\nabla}\rho_0(q) \\ &\quad \pm \big(\pm (\rho_0(q)\, \boldsymbol{\nabla} \times \boldsymbol{v}(q) - \boldsymbol{v}(q) \\ &\quad \times \boldsymbol{\nabla}\rho_0(q) \big) \end{aligned} \tag{8}$$

$$s_0(q) = 2\nabla_0\rho_0(q) \mp \langle \boldsymbol{v}(q), \boldsymbol{\nabla}\rho_0(q) \rangle \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(q) \rangle \rho_0(q) \tag{9}$$

$$\begin{aligned} \boldsymbol{s}(q) &= \pm \nabla_0 \boldsymbol{v}(q) \pm \boldsymbol{\nabla}\rho_0(q) \\ &\quad \pm \left(\pm \big(\rho_0(q)\, \boldsymbol{\nabla} \times \boldsymbol{v}(q) - \boldsymbol{v}(q) \times \boldsymbol{\nabla}\rho_0(q) \big) \right) \end{aligned} \tag{10}$$

The red sign selection indicates a change of handedness by changing the sign of one of the imaginary base vectors. Conjugation also causes a switch of handedness. It changes the sign of all three imaginary base vectors.

In its simplest form the full continuity equation runs:

$$s(q) = \nabla\rho(q) \tag{11}$$

Thus the full continuity equation specifies a quaternionic distribution s as a flat differential $\nabla\rho$.

When we go back to the integral balance equation, then holds for the imaginary parts:

(12)

$$\frac{d}{d\tau}\int_V \boldsymbol{\rho}\, dV = -\oint_S \hat{\boldsymbol{n}}\rho_0\, dS - \oint_S \hat{\boldsymbol{n}} \times \boldsymbol{\rho}\, dS + \int_V \boldsymbol{s}\, dV$$

$$\int_V \nabla_0\, \boldsymbol{\rho}\, dV = -\int_V \boldsymbol{\nabla}\rho_0\, dV - \int_V \boldsymbol{\nabla} \times \boldsymbol{\rho}\, dV + \int_V \boldsymbol{s}\, dV \tag{13}$$

For the full integral equation holds:

$$\frac{d}{d\tau}\int_V \rho\, dV + \oint_S \hat{\boldsymbol{n}}\rho\, dS = \int_V s\, dV \tag{14}$$

$$\int_V \nabla\, \rho\, dV = \int_V s\, dV \tag{15}$$

Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface S, $\boldsymbol{v}(q)$ is the velocity at which the charge density $\rho_0(q)$ enters volume V and s_0 is the source density inside V. In the above formula ρ stands for

$$\rho = \rho_0 + \boldsymbol{\rho} = \rho_0 + \frac{\rho_0 \boldsymbol{v}}{c} \tag{16}$$

It is the flux (flow per unit of area and per unit of progression) of ρ_0 . τ stands for progression (not coordinate time).

13.5 The coupling equation

The coupling equation is a special form of the continuity equation. ψ is a normalized quaternionic distribution.

$$\langle\psi|\psi\rangle = \int_V |\psi|^2 \, dV = 1 \tag{1}$$

$$\nabla\psi = \phi \tag{2}$$

We also normalize ϕ by dividing a by a real factor m

$$\phi = m\,\varphi \tag{3}$$

$$\langle\varphi|\varphi\rangle = \int_V |\varphi|^2 \, dV = 1 \tag{4}$$

This results in the coupling equation, which holds for coupled field pairs $\{\psi, \varphi\}$

$$\langle\phi|\phi\rangle = \int_V |\phi|^2 \, dV = m^2 \tag{5}$$

$$\langle\nabla\psi|\nabla\psi\rangle = \int_V |\nabla\psi|^2 \, dV = m^2 \tag{6}$$

This equation does not depend on φ, thus it also holds for composites. The coupling equation reads:

$$\nabla\psi = m\,\varphi \tag{7}$$

The quaternionic format of the Dirac equation for the electron is a special form of the coupling equation.

$$\nabla\psi = m\,\psi^* \tag{8}$$

The coupling equation appears to hold for elementary particles and simple composite particles. For anti-particles hold.

$$(\nabla\psi)^* = m\,\varphi^* \tag{9}$$

Due to the fact that the parameter space is not conjugated, equation (9) differs from equation (7).

The quaternionic format of the Dirac equation for the positron is a special form of the coupling equation for anti-particles.

$$(\nabla\psi)^* = m\,\psi \tag{11}$$

13.6 Energy

This makes $|\phi|$ to the distribution of ***the local energy*** and m to the ***total energy*** of the quantum state function. The coupling equation can be split in a real equation and an imaginary equation.

$$\nabla_0\psi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle = m\,\varphi_0 \tag{1}$$

$$\nabla_0\boldsymbol{\psi} + \boldsymbol{\nabla}\psi_0 + \boldsymbol{\nabla} \times \boldsymbol{\psi} = m\,\boldsymbol{\varphi} \tag{2}$$

Bold characters indicate imaginary quaternionic distributions and operators. Zero subscripts indicate real distributions and operators.

The quantum state function of a particle moving with uniform speed $\boldsymbol{v}$ is given by

$$\psi = \chi + \chi_0\,\boldsymbol{v} \tag{3}$$

$$\chi_0 = \psi_0 \tag{4}$$

Here χ stands for quantum state function of the particle at rest.

We introduce new symbols. In order to indicate the difference with Maxwell's equations we use Gotic capitals:

$$\mathfrak{E} = \nabla_0\boldsymbol{\psi} + \boldsymbol{\nabla}\psi_0 \tag{5}$$

$$\mathfrak{B} = \boldsymbol{\nabla} \times \boldsymbol{\psi} \tag{6}$$

The local field energy E is given by:

$$E = |\phi| = \sqrt{\phi_0\phi_0 + \langle \boldsymbol{\phi}, \boldsymbol{\phi} \rangle} = \sqrt{\phi_0\phi_0 + \langle \mathfrak{E}, \mathfrak{E} \rangle + \langle \mathfrak{B}, \mathfrak{B} \rangle + 2\langle \mathfrak{E}, \mathfrak{B} \rangle} \tag{7}$$

The total energy is given by the volume integral

$$E_{total} = \sqrt{\int_V |\phi|^2\, dV} \tag{8}$$

In a static situation the local energy E reduces to

$$E_{static} = \sqrt{\langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle^2 + \langle \mathfrak{E}, \mathfrak{E} \rangle + \langle \mathfrak{B}, \mathfrak{B} \rangle} \tag{9}$$

13.6.1 Fourier transform

In a region of little or no space curvature the Fourier transform of the local QPAD can be taken.

$$\nabla\psi = \phi = m\,\varphi \tag{1}$$

$$\mathcal{M}\tilde{\psi} = \tilde{\phi} = m\,\tilde{\varphi} \tag{2}$$
$$\langle\tilde{\psi}|\mathcal{M}\tilde{\psi}\rangle = m\,\langle\tilde{\psi}|\tilde{\varphi}\rangle \tag{3}$$
$$\mathcal{M} = \mathcal{M}_0 + \boldsymbol{M} \tag{4}$$
$$\mathcal{M}_0\tilde{\psi}_0 - \langle\boldsymbol{M},\tilde{\boldsymbol{\psi}}\rangle = m\,\tilde{\varphi}_0 \tag{5}$$
$$\mathcal{M}_0\boldsymbol{\psi} + \boldsymbol{M}\tilde{\psi}_0 + \boldsymbol{M}\times\tilde{\boldsymbol{\psi}} = m\,\tilde{\boldsymbol{\varphi}} \tag{6}$$
$$\int_{\tilde{V}}\tilde{\phi}^2\,d\tilde{V} = \int_{\tilde{V}}\left(\widetilde{\mathcal{M}\psi}\right)^2\,d\tilde{V} = m^2 \tag{7}$$

In general $|\tilde{\psi}\rangle$ is not an eigenfunction of operator $\mathcal{M}$. That is only true when $|\tilde{\psi}\rangle$ and $|\tilde{\varphi}\rangle$ are equal. For elementary particles they are equal apart from their difference in discrete symmetry.

14 Elementary particles

Elementary particles are constituted by the coupling of two QPAD's that belong to the same generation. One of the QPAD's is the quantum state function of the particle. The other QPAD can be interpreted to implement inertia. Apart from their sign flavors these constituting QPAD's form the same quaternionic distribution. However, the sign flavor may differ and their progression must have the same direction. It means that the density distribution is the same, but the signs of the flows of the concerned objects differ between the two distributions. The second QPAD only simulates a Qpattern. It represents the coupling of the quantum state function to the embedding continuum, which is used in constructing the potentials of the particle. Coupling of elementary particles is governed by a special coupling equation

The quantum state function is a mostly virtual distribution. Only one element is actual. The second QPAD is completely virtual.

The coupling uses pairs $\{\psi^x, \psi^y\}$ of two sign flavors of the same basic Qpattern and its corresponding QPAD, which is indicated by $\psi^{⓪}$. The special coupling equation runs:

$$\nabla\psi^x = m\,\psi^y \tag{1}$$

Corresponding anti-particles obey

$$(\nabla\psi^x)^* = m\,(\psi^y)^* \tag{2}$$

As claimed above, coupling (also) occurs by embedding the message waves in the potential(s) of other particles.

In this specification the form of the quaternionic Dirac equations play a significant, but at the same time a very peculiar role. The fact that ψ^x and ψ^y must be equal apart from a discrete symmetry difference is very strange and it is highly improbable that this strong relation is constituted by accident. On the other hand it is known that the step stones couple to the embedding continuum. Two different types

of this embedding continuum exists. The first embedding continuum is formed by the superposition of the potentials of distant particles. This type of binding produces fermions. The second embedding continuum is formed by the superposition of the potentials of local particles. This type of binding produces bosons.

It appears as if the correlation mechanism creates two rather than one distribution of step stones in which the descriptor of the first one plays the role of the quantum state function, while the descriptor of the second one plays the role of a mirror that has the sign flavor of the embedding continuum.

If the first Qpattern oscillates, then the second Qpattern oscillates asynchronous or partly in synchrony. This situation may differ per dimension. This results in 64 elementary particle types and 64 anti-particle types. Besides of that exist 8 oscillating potential types.

The coupling has a small set of observable properties:

- coupling strength,
- electric charge,
- color charge and
- spin.

Due to the fact that the enumerator creation occurs in configuration space, the coupling affects the local curvature of the involved Palestras.

Qpattern QPAD's that belong to the same generation have the same shape. This is explained in the paragraph on the enumeration process. The difference between the coupling partners resides in the discrete symmetry sets. ***Thus, the properties of the coupled pair are completely determined by the sign flavors of the partners.***

HYPOTHESIS 4: *If the quaternionic quantum state function of an elementary particle couples to an embedding continuum that is formed by distant particles, then the particle is a* ***fermion****, otherwise*

*it is a **boson**. The quantum state functions of anti-particles are coupled to canonical conjugates of the corresponding embedding continuums.*

The fact that for fermions both the reference continuum and the reference enumerator set play a crucial role may indicate ***that the Pauli principle is based on this fact.***

This paper does not give an explanation for the influence on the spin by the fact that the quantum state function is connected to an isotropic or an anisotropic Qpattern.

Photons and gluons are not coupled. They modulate the ultra-high frequency fields that constitute particle potentials.

In the standard model the eight gluons are constructed from superpositions of the six base gluons.

14.1 Reference frames

Each Qpattern possesses a reference frame that represents its current location, its orientation and its discrete symmetry. The reference frame corresponds with a Cartesian coordinate system that has a well-defined origin. Reference frames of different Qpatterns have a relative position. A Qpattern does not move with respect to its own reference frame. However, reference frames of different Qpatterns may move relative to each other. The reference frames reside in an affine space. Interaction can take place between reference frames that reside in different HBM pages and that are within the range of the interaction speed. Within the same HBM page no interaction is possible. Interaction runs from a reference frame to a frame that lays in the future of the sender.

Coupling into elementary particles puts the origins of the reference frames of the coupled Qpatterns at the same location. At the same location reference frames are parallel. That does not mean that the axes have the same sign.

14.2 Coupling Qpatterns

This section uses the fact that coupling is caused by interfering with the embedding continuum. Fermions couple to the embedding continuum that is formed by the superposition of the potentials of distant particles. Bosons couple to the embedding continuum that is formed by the superposition of the potentials of local particles.

The coupling is represented by pairs $\{\psi^x, \psi^y\}$ of two sign flavors of the same basic QPAD $\psi^{⓪}$. Thus the corresponding coupling equation runs:

$$\nabla\psi^x = \psi^y \tag{1}$$

The corresponding anti-particles obey

$$(\nabla\psi^x)^* = m\,(\psi^y)^* \tag{2}$$

The partial anti-phase couplings must use different sign flavors.

The coupling and its effect on local curvature is treated in the section on the enumeration process.

In the figure below $\psi^{⓪}$ and color *N* act as the reference sign flavor.

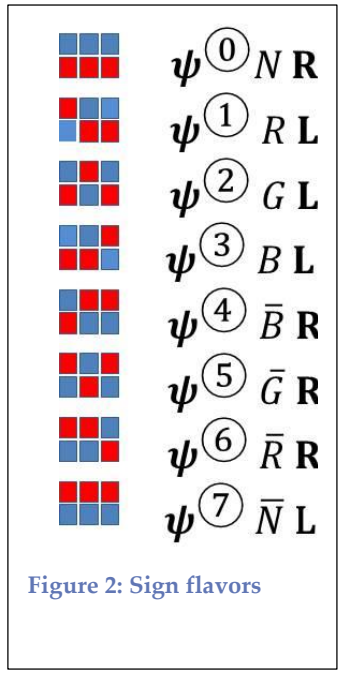


Figure 2: Sign flavors

Eight sign flavors
(discrete symmetries)
Colors N, R, G, B, $\bar{R}$, $\bar{G}$, $\bar{B}$, W
Right or Left handedness **R,L**

14.3 Elementary particle properties

Elementary particles retain their discrete properties when they are contained in composite particles.

14.3.1 Spin

HYPOTHESIS 5: *The size of the spin relates to the fact whether the coupled Qpattern is the reference Qpattern. The reference Qpattern QPAD has the reference sign flavor* $\psi^{⓪}$.

Each generation has its own reference Qpattern. Fermions couple to the reference Qpattern. Fermions have half integer spin. Bosons have integer spin.

The spin of a composite equals the sum of the spins of its components.

14.3.2 Electric charge

HYPOTHESIS 6: *Electric charge depends on the difference and direction of the imaginary base vectors for the Qpattern pair. Each sign difference stands for one third of a full electric charge. Further it depends on the fact whether the handedness differs. If the handedness differs then the sign of the count is changed as well.*

The electric charge of a composite is the sum of the electric charge of its components.

14.3.3 Color charge

HYPOTHESIS 7: *Color charge is related to the direction of the anisotropy of the considered Qpattern with respect to the reference Qpattern. The anisotropy lays in the discrete symmetry of the imaginary part. The color charge of the reference Qpattern is white. The corresponding anti-color is black. The color charge of the coupled pair is determined by the colors of its members.*

All composite particles are black or white. The neutral colors black and white correspond to Qpatterns that are isotropic with respect to the reference sign flavor.

Currently, color charge cannot be measured. In the Standard Model the existence of color charge is derived via the Pauli principle.

14.3.4 Mass

Mass is related to the internal energy of the Qpattern. More precisely stated, mass is related to the square root of the volume integral of the square of the local field energy $E^2 = |\nabla\psi|^2$. Any internal kinetic energy is included in E.

$$m^2 = \langle \nabla\psi | \nabla\psi \rangle = \int_V |\nabla\psi|^2 \, dV \tag{1}$$

The same mass rule holds for composite particles. The fields of the composite particles are dynamic superpositions of the fields of their components.

14.4 Elementary object samples

With these ingredients we can look for agreements with the standard model. It appears that the coverage is (over)complete. The larger diversity of this HBM table appears to be not (yet) measurable.

For the same generation, the real parts of the QPAD's (that contain the scalar density distribution) are all born the same way! In this way the Qpatterns become micro states.

Elementary particles are represented by couplings of two QPAD's that may differ in their discrete symmetries. The differences between the discrete symmetries determine the discrete properties of the particle.

14.4.1 Photons and gluons

Photons and gluons modulate the ultra-high frequency fields that constitute particle potentials. Once emitted, they flow freely. When the potential emitting particle oscillates, the photons or particle. When the potential emitting potentials annihilate, then the potentials keep spreading and flee from their original source. In that way special kinds of photons and gluons are created.

In the standard model the eight gluons are constructed from superpositions of the six HBM base gluons.

type	s-type	e-charge	c-charge	Handedness	SM Name
$\{\psi^{⑦}\}$	boson	0	N	**R**	photon
$\{\psi^{⓪}\}$	boson	0	W	**L**	photon
$\{\psi^{⑥}\}$	boson	0	$\overline{\mathrm{R}}$	**R**	gluon
$\{\psi^{①}\}$	boson	0	R	**L**	gluon
$\{\psi^{⑤}\}$	boson	0	$\overline{\mathrm{G}}$	**R**	gluon
$\{\psi^{②}\}$	boson	0	G	**L**	gluon

$\{\psi^{④}\}$	boson	0	$\overline{\mathrm{B}}$	**R**	gluon
$\{\psi^{③}\}$	boson	0	B	**L**	gluon

Only at the instant of their generation or annihilation photons and gluons couple to the emitter or absorber.

Two types of photons exist. One fades away from its point of generation. The other concentrates until it reaches the absorber.

The act of interaction can be interpreted as a Fourier transform. The Fourier transforms converts a distribution in configuration space into a distribution in its canonical conjugated space or vice versa.

For gluons similar things occur.

14.4.2 Leptons and quarks

According to the Standard Model both leptons and quarks comprise three generations. They form 22 particles. Neutrinos will be treated separately.

14.4.2.1 Leptons

Pair	s-type	e-charge	c-charge	Handedness	SM Name
$\{\psi^{⑦}, \psi^{⓪}\}$	fermion	-1	N	**LR**	electron
$\{\psi^{⓪}, \psi^{⑦}\}$	Anti-fermion	+1	W	**RL**	positron

The generations contain the muon and tau generations of the electrons. The Qpatterns quasi-oscillate asynchronous in three dimensions.

14.4.2.2 Quarks

Pair	s-type	e-charge	c-charge	Handedness	SM Name
$\{\psi^{①}, \psi^{⓪}\}$	fermion	-1/3	R	**LR**	down-quark
$\{\psi^{⑥}, \psi^{⑦}\}$	Anti-fermion	+1/3	$\overline{\mathrm{R}}$	**RL**	Anti-down-quark
$\{\psi^{②}, \psi^{⓪}\}$	fermion	-1/3	G	**LR**	down-quark
$\{\psi^{⑤}, \psi^{⑦}\}$	Anti-fermion	+1/3	$\overline{\mathrm{G}}$	**RL**	Anti-down-quark

$\{\psi^{③}, \psi^{⓪}\}$	fer-mion	-1/3	B	**LR**	down-quark
$\{\psi^{④}, \psi^{⑦}\}$	Anti-fer-mion	+1/3	$\overline{B}$	**RL**	Anti-down-quark
$\{\psi^{④}, \psi^{⓪}\}$	fer-mion	+2/3	$\overline{B}$	**RR**	up-quark
$\{\psi^{③}, \psi^{⑦}\}$	Anti-fer-mion	-2/3	B	**LL**	Anti-up-quark
$\{\psi^{⑤}, \psi^{⓪}\}$	fer-mion	+2/3	$\overline{G}$	**RR**	up-quark
$\{\psi^{②}, \psi^{⑦}\}$	Anti-fer-mion	-2/3	G	**LL**	Anti-up-quark
$\{\psi^{⑥}, \psi^{⓪}\}$	fer-mion	+2/3	$\overline{R}$	**RR**	up-quark
$\{\psi^{①}, \psi^{⑦}\}$	Anti-fer-mion	-2/3	R	**LL**	Anti-up-quark

The generations contain the charm and top versions of the up-quark and the strange and bottom versions of the down-quark. The Qpatterns quasi-oscillate asynchronous in one or two dimensions.

14.4.2.3 Reverse quarks

Pair	s-type	e-charge	c-charge	Handedness	SM Name
$\{\psi^{⓪}, \psi^{①}\}$	fer-mion	+1/3	R	**RL**	down-r-quark

$\{\psi^{⑦},\psi^{⑥}\}$	Anti-fer-mion	-1/3	$\overline{\mathrm{R}}$	**LR**	Anti-down-r-quark
$\{\psi^{⓪},\psi^{②}\}$	fer-mion	+1/3	G	**RL**	down-r-quark
$\{\psi^{⑦},\psi^{⑤}\}$	Anti-fer-mion	-1/3	$\overline{\mathrm{G}}$	**LR**	Anti-down-r-quark
$\{\psi^{⓪},\psi^{③}\}$	fer-mion	+1/3	B	**RL**	down-r-quark
$\{\psi^{⑦},\psi^{④}\}$	Anti-fer-mion	-1/3	$\overline{\mathrm{B}}$	**LR**	Anti-down-r_quark
$\{\psi^{⓪},\psi^{④}\}$	fer-mion	-2/3	$\overline{\mathrm{B}}$	**RR**	up-r-quark
$\{\psi^{⑦},\psi^{③}\}$	Anti-fer-mion	+2/3	B	**LL**	Anti-up-r-quark
$\{\psi^{⓪},\psi^{⑤}\}$	fer-mion	-2/3	$\overline{\mathrm{G}}$	**RR**	up-r-quark
$\{\psi^{⑦},\psi^{②}\}$	Anti-fer-mion	+2/3	G	**LL**	Anti-up-r-quark
$\{\psi^{⓪},\psi^{⑥}\}$	fer-mion	-2/3	$\overline{\mathrm{R}}$	**RR**	up-r-quark
$\{\psi^{⑦},\psi^{①}\}$	Anti-fer-mion	+2/3	R	**LL**	Anti-up-r-quark

The generations contain the charm and top versions of the up-r-quark and the strange and bottom versions of the down-r-quark. The Qpatterns oscillate asynchronous in one or two dimensions.

14.4.2.4 Neutrinos

Neutrinos are fermions and have zero electric charge. They are leptons, but they seem to belong to a separate low-weight family of (three) generations. Their quantum state function couples to a QPAD that has the same sign-flavor. The lowest generation has a very small rest mass.

type	s-type	e-charge	c-charge	Handedness	SM Name
$\{\psi^{⑦}, \psi^{⑦}\}$	fermion	0	NN	**RR**	neutrino
$\{\psi^{⓪}, \psi^{⓪}\}$	Anti-fermion	0	WW	**LL**	neutrino
$\{\psi^{⑥}, \psi^{⑥}\}$	boson?	0	$\overline{\mathrm{RR}}$	**RR**	neutrino
$\{\psi^{①}, \psi^{①}\}$	Anti-boson?	0	RR	**LL**	neutrino
$\{\psi^{⑤}, \psi^{⑤}\}$	boson?	0	$\overline{\mathrm{GG}}$	**RR**	neutrino
$\{\psi^{②}, \psi^{②}\}$	Anti-boson?	0	GG	**LL**	neutrino
$\{\psi^{④}, \psi^{④}\}$	boson?	0	$\overline{\mathrm{BB}}$	**RR**	neutrino
$\{\psi^{③}, \psi^{③}\}$	Anti-boson?	0	BB	**LL**	neutrino

14.4.3 W-particles

The 18 W-particles have indiscernible color mix. W_+ and W_- are each other's anti-particle.

Pair	s-type	e-charge	c-charge	Hand-edness	SM Name
$\{\psi^{⑥},\psi^{①}\}$	boson	-1	$\overline{\mathrm{R}}\mathrm{R}$	**RL**	W_-
$\{\psi^{①},\psi^{⑥}\}$	Anti-boson	+1	$\mathrm{R}\overline{\mathrm{R}}$	**LR**	W_+
$\{\psi^{⑥},\psi^{②}\}$	boson	-1	$\overline{\mathrm{R}}\mathrm{G}$	**RL**	W_-
$\{\psi^{②},\psi^{⑥}\}$	Anti-boson	+1	$\mathrm{G}\overline{\mathrm{R}}$	**LR**	W_+
$\{\psi^{⑥},\psi^{③}\}$	boson	-1	$\overline{\mathrm{R}}\mathrm{B}$	**RL**	W_-
$\{\psi^{③},\psi^{⑥}\}$	Anti-boson	+1	$\mathrm{B}\overline{\mathrm{R}}$	**LR**	W_+
$\{\psi^{⑤},\psi^{①}\}$	boson	-1	$\overline{\mathrm{G}}\mathrm{G}$	**RL**	W_-
$\{\psi^{①},\psi^{⑤}\}$	Anti-boson	+1	$\mathrm{G}\overline{\mathrm{G}}$	**LR**	W_+
$\{\psi^{⑤},\psi^{②}\}$	boson	-1	$\overline{\mathrm{G}}\mathrm{G}$	**RL**	W_-
$\{\psi^{②},\psi^{⑤}\}$	Anti-boson	+1	$\mathrm{G}\overline{\mathrm{G}}$	**LR**	W_+
$\{\psi^{⑤},\psi^{③}\}$	boson	-1	$\overline{\mathrm{G}}\mathrm{B}$	**RL**	W_-
$\{\psi^{③},\psi^{⑤}\}$	Anti-boson	+1	$\mathrm{B}\overline{\mathrm{G}}$	**LR**	W_+
$\{\psi^{④},\psi^{①}\}$	boson	-1	$\overline{\mathrm{B}}\mathrm{R}$	**RL**	W_-
$\{\psi^{①},\psi^{④}\}$	Anti-boson	+1	$\mathrm{R}\overline{\mathrm{B}}$	**LR**	W_+
$\{\psi^{④},\psi^{②}\}$	boson	-1	$\overline{\mathrm{B}}\mathrm{G}$	**RL**	W_-
$\{\psi^{②},\psi^{④}\}$	Anti-boson	+1	$\mathrm{G}\overline{\mathrm{B}}$	**LR**	W_+

$\{\psi^{④}, \psi^{③}\}$	boson	-1	$\overline{\mathrm{B}}\mathrm{B}$	**RL**	W_{-}
$\{\psi^{③}, \psi^{④}\}$	Anti-boson	+1	$\mathrm{B}\overline{\mathrm{B}}$	**LR**	W_{+}

The Qpatterns oscillate differently in multiple dimensions.

14.4.4 Z-candidates

The 12 Z-particles have indiscernible color mix.

Pair	s-type	e-charge	c-charge	Hand-edness	SM Name
$\{\psi^{②}, \psi^{①}\}$	boson	0	GR	**LL**	Z
$\{\psi^{⑤}, \psi^{⑥}\}$	Anti-boson	0	$\overline{\mathrm{G}}\overline{\mathrm{R}}$	**RR**	Z
$\{\psi^{③}, \psi^{①}\}$	boson	0	BR	**LL**	Z
$\{\psi^{④}, \psi^{⑥}\}$	Anti-boson	0	$\overline{\mathrm{R}}\overline{\mathrm{B}}$	**RR**	Z
$\{\psi^{③}, \psi^{②}\}$	boson	0	BR	**LL**	Z
$\{\psi^{④}, \psi^{⑤}\}$	Anti-boson	0	$\overline{\mathrm{R}}\overline{\mathrm{B}}$	**RR**	Z
$\{\psi^{①}, \psi^{②}\}$	boson	0	RG	**LL**	Z
$\{\psi^{⑥}, \psi^{⑤}\}$	Anti-boson	0	$\overline{\mathrm{R}}\overline{\mathrm{G}}$	**RR**	Z
$\{\psi^{①}, \psi^{③}\}$	boson	0	RB	**LL**	Z
$\{\psi^{⑥}, \psi^{④}\}$	Anti-boson	0	$\overline{\mathrm{R}}\overline{\mathrm{B}}$	**RR**	Z
$\{\psi^{②}, \psi^{③}\}$	boson	0	RB	**LL**	Z
$\{\psi^{⑤}, \psi^{④}\}$	Anti-boson	0	$\overline{\mathrm{R}}\overline{\mathrm{B}}$	**RR**	Z

The Qpatterns oscillate differently in multiple dimensions.

15 Fields

15.1 Physical fields

Elementary particles conserve their properties in higher level bindings. These properties are sources to fields that are exposed as dedicated potentials. Examples are the gravitational potential field and the electrostatic potential field. As soon as they leave the particle, the corresponding waves start their own life and keep flowing away from their source. These waves feature a fixed ultra-high frequency. If the particle oscillates or annihilates, then their amplitude can be modulated. We know these amplitude modulations as photons and gluons.

If the source stays at rest, then the waves superpose as a static potential. If the source oscillates, then the emitted stream oscillates as well. The corresponding amplitude modulation has a lower frequency.

If in a certain region a coherent distribution of property carriers exist, then that distribution can again be described by a QPAD. These fields are secondary fields. These new fields can be described by quaternionic distributions and when they cover large numbers of particles they can be described with quaternionic distributions that contain a scalar potential and a vector potential like the QPAD's that describe elementary particles.

Besides the photons and the gluons these secondary fields are the physical fields that we know.

15.2 Gravitation field

One of the physical fields, the gravitation field describes the local curvature of the reference Palestra. It equals the scalar potential field that corresponds to the real part of the quantum state function.

Now let ϕ represent the quaternionic potential of a set of massive particles. It is a superposition of single charge potentials.

$$\phi = \phi_0 + \boldsymbol{\phi} = \sum_i \phi_i = \sum_i m_i \, \varphi_i \tag{1}$$

The particles may represent composites. In that case the mass m_i includes the internal kinetic energy of the corresponding particle. All massive particles attract each other. In superpositions, gravitational fields tend to enforce each other.

15.3 Electromagnetic fields

The electric charge e_i is represented similarly as m_i, but where m_i is always positive, the electric charge e_i can be either positive or negative. Equal signs repel, opposite signs attract each other. Superposition of the fields must include the sign. In superpositions, arbitrary electronic fields tend to neutralize each other. Moving electric charges correspond to a vector potential and the curl of this vector potential corresponds to a magnetic field.

$$\phi = \phi_0 + \boldsymbol{\phi} = \sum_i e_i \, \varphi_i \tag{1}$$

Here ϕ is the quaternionic electro potential. It is a superposition of single charge potentials ϕ_i. ϕ_0 is the scalar potential. $\boldsymbol{\phi}$ is the vector potential. The values of the electric charge sources e_i are included in ϕ.

$$\boldsymbol{E} = \nabla_0 \boldsymbol{\phi} + \boldsymbol{\nabla} \phi_0 \tag{2}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{\phi} \tag{3}$$

15.4 Photons and gluons

Photons and gluons can be described by quaternionic functions.

In configuration space they obey

$$\nabla \psi = 0 \tag{1}$$

$$\nabla^2 \psi = 0 \tag{2}$$

Ensembles of photons and/or gluons are better considered as QPAD's in the canonical conjugated space of the configuration space.

15.5 Anisotropic potentials

15.5.1 Huygens principle for odd and even number of spatial dimension

The following is taken from http://www.math-pages.com/home/kmath242/kmath242.htm

The spherically symmetrical wave equation in n spatial dimensions can be written as

$$\frac{\partial^2\psi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\psi}{\partial r}=\frac{\partial^2\psi}{\partial t^2} \tag{1}$$

Now suppose we define a new scalar field ϕ by the relation

$$\phi(r,t)=r^{(n-1)/2}\psi(r,t) \tag{2}$$

This leads to

$$\frac{\partial^2\phi}{\partial r^2}+\frac{(n-1)(n-3)}{4r^2}\phi=\frac{\partial^2\phi}{\partial t^2} \tag{3}$$

If n equals 1, meaning that we have just a single space dimension, then the second term on the left hand side vanishes, leaving us with a one-dimensional wave equation, with has the well-known general solution

$$\psi(r,t)=f(r-t)+g(r+t) \tag{4}$$

for arbitrary functions f and g.

if n equals 3, i.e., in the case of three spatial dimensions, the spherically symmetrical wave equation reduces again to a one-parametric wave equation, in the modified wave function $\phi = r\psi$. Hence the general solution in three space dimensions is

$$\psi(r,t) = \frac{f(r-t)}{r} + \frac{g(r+t)}{r} \tag{5}$$

The fact that this solution is divided by r signifies that the magnitude of the wave tends to drop as r increases (unlike the one-dimensional case, in which a wave would theoretical propagate forever with undiminished strength). Focusing on just the "retarded" component of the wave, $f(r-t)/r$, the fact that the time parameter t appears only in the difference $r-t$ implies that the (attenuated) wave propagates in time with a phase velocity of precisely 1, because for any fixed phase β we have $r - t = \beta$ and so dr/dt for this phase point is 1. Consequently if f is a single pulse, it will propagate outward in a spherical shell at precisely the speed 1, i.e., on the light cone. Conversely, it can be shown that the wave function at any point in space and time is fully determined by the values and derivatives of that function on the past light cone of the point.

Any wave equation for which this is true (i.e., for which disturbances propagate at a single precise speed) is said to satisfy Huygens' Principle. The connection with Huygens' original statement about secondary wavelets is that each wavelet - with the same speed as the original wave - represents a tiny light cone at that point, and Huygens' principle asserts that light is confined to those light cones.

For n equals 2 the extra term in equation (3) does not vanish. We can still solve the wave equation, but the solution is not just a simple spherical wave propagating with unit velocity. Instead, we find that

there are effectively infinitely many velocities, in the sense that a single pulse disturbance at the origin will propagate outward on infinitely many "light cones" (and sub-cones) with speeds ranging from the maximum down to zero. Hence if we lived in a universe with two spatial dimensions (instead of three), an observer at a fixed location from the origin of a single pulse would "see" an initial flash but then the disturbance "afterglow" would persist, becoming less and less intense, but continuing forever, as slower and slower subsidiary branches arrive.

15.5.2 The case of even spatial dimensions

Now again start from equation (1) and try a solution in the form:

$$\psi(r,t) = f(r)g(t) \tag{1}$$

Inserting this into the wave equation and expanding the derivatives by the product rule gives

$$g\frac{\partial^2 f}{\partial r^2} + \frac{n-1}{r}g\frac{\partial f}{\partial r} = f\frac{\partial^2 g}{\partial t^2} \tag{2}$$

Dividing through by fg gives

$$\frac{1}{f}\frac{\partial^2 f}{\partial r^2} + \frac{n-1}{f\,r}\frac{\partial f}{\partial r} = \frac{1}{g}\frac{\partial^2 g}{\partial t^2} \tag{3}$$

This decouples into two equations

$$\frac{\partial^2 f}{\partial r^2} + \frac{n-1}{r}\frac{\partial f}{\partial r} = k\,f \tag{3}$$

And

$$\frac{\partial^2 g}{\partial t^2} = k\, g \tag{4}$$

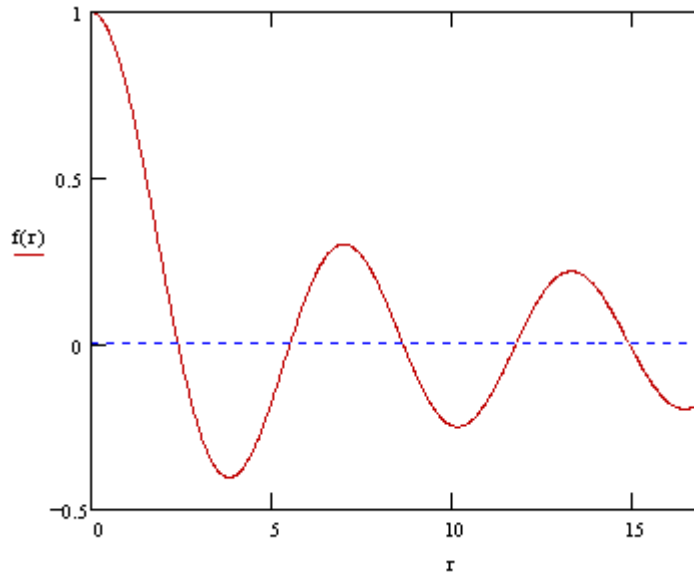
If k is positive or zero the right hand equation gives "run-away" solutions for $g(t)$, whereas if k is negative we can choose scaling so that $k = -1$ and then $g(t)$ satisfies the simple harmonic equation, whose solutions include functions of the form $sin(ct)$ and $cos(ct)$. In that case equation (9) can be re-written in the form

$$r\frac{\partial^2 f}{\partial r^2} + (n-1)\frac{\partial f}{\partial r} + r\, f = 0 \tag{5}$$

This is the form of a Bessel's equation. In fact for n=2 the solution is the zero order Bessel function $J_0(r)$.

$$J_0(r) = \frac{2}{\pi}\int_0^{\infty} \sin(\cosh(\theta)\, r)\, d\theta \tag{6}$$

A plot of $J_0(r)$ is shown below.



Inserting $g(t) = sin(ct)$ gives

$$\psi(r,t) = \frac{1}{\pi}\int_0^\infty [\cos(\cosh(\theta)\, r - ct) - \cos(\cosh(\theta)\, r + ct)]d\theta \tag{7}$$

Hence, instead of the solution being purely a function of $r \pm ct$ as in the case of odd dimensions, we find that it is an integral of functions of $cosh(\theta)r \pm ct$. Each value of θ corresponds to a propagation speed of $c/cosh(\theta)$, so the speeds vary from c down to zero. This signifies that the wave function at any event is correlated not

just with the wave function on its "light cone", but with the wave function at every event inside its light cone.

In two dimensions the Huygens principle corresponds to a centripetal force[73] with potential

$$V = -\frac{\hbar}{8Mr^2}. \tag{8}$$

15.5.3 Huygens principle applied

HYPOTHESIS 8: *Particles transmit waves in dimensions where the discrete symmetry of the quantum state function differs from the discrete symmetry of the embedding background.*

The correlation mechanism uses the Huygens principle in order to restore the potentials at each progression step. The Huygens principle works differently depending on the number of dimensions in which the waves are transmitted.

The characteristics of the potentials that are emitted or absorbed by elementary particles are determined by the differences between the symmetry set of the quantum state function of the particle and the symmetry set of the coupled QPAD that represents the embedding continuum. This difference determines whether the potentials act in 1, 2 or 3 dimensions. In odd dimensions the persistance of the potentials can be explained by the common interpretation of the Huygens principle. This common interpretation is that at every point of each wave front new waves are generated. This does not work for particles that send their waves in two dimensions. This includes quarks, W-particles and Z-particles. The corresponding messengers

[73] http://cds.cern.ch/record/514621/files/0108083.pdf

are gluons. For these objects the potentials also act in two dimensions. In even dimensions the Huygens principle does not act in its normal way.

The same conditions that determine whether waves are emitted in 1, 2, or 3 dimensions also determine whether the particle has 1/3, 2/3 or 3/3 integer electric charge.

The re-emitted waves consist out of a retarded component and an advanced component. These components correspond to outbound interactions and inbound interactions.

15.6 Discussion

This particular behavior of the Huygens principle for potential contributions that cover even dimensions might explain the exceptional strength of the corresponding strong force mechanism.

It appears that leptons with electric charges of $\pm n/3$ e produce n dimensional waves that contribute to their electrostatic potential.

For n=3 the Green's function is of form 1/r.

For n=2 the Green's function is a zero order Bessel function.

For n=1 the Green's function is a constant.

The gravitation potential is not influenced by the discrete symmetries. The corresponding potential contributions are always transmitted isotropic in three dimensions.

The electric potential is controlled by the discrete symmetry sets. Depending on the resulting electric charge of the particle the electric potential contributions are transmitted in 1, 2 or 3 dimensions.

The correlation mechanism applies the Huygens principle for the recreation in each progression step of the corresponding potentials.

16 Inertia

We use the ideas of Denis Sciama[74][75][76].

16.1 Inertia from coupling equation

In order to discuss inertia we must reformulate the coupling equation.

$$\nabla\psi = m\,\varphi \tag{1}$$

$$\nabla_0\psi_0 - \langle\nabla,\psi\rangle = m\,\varphi_0 \tag{2}$$

$$\nabla_0\psi + \nabla\psi_0 + \nabla\times\psi = \mathfrak{E} + \mathfrak{B} = m\,\varphi \tag{3}$$

We will write ψ as a superposition

$$\psi = \chi + \chi_0\,\boldsymbol{v} \tag{4}$$

$$\psi_0 = \chi_0 \tag{5}$$

$$\boldsymbol{\psi} = \boldsymbol{\chi} + \chi_0\,\boldsymbol{v}$$

χ represents the rest state of the object. With respect to progression, it is a constant.

$$\nabla_0\chi = 0 \tag{6}$$

For the elementary particles the coupled distributions $\{\,\psi\,,\varphi\,\}$ have the same real part.

$$\psi_0 = \varphi_0 \tag{7}$$

$$\nabla_0\boldsymbol{\psi} = \chi_0\,\dot{\boldsymbol{v}} \tag{8}$$

Remember

$$\mathfrak{E} = \nabla_0\boldsymbol{\psi} + \boldsymbol{\nabla}\psi_0 \tag{9}$$

$$\chi_0\,\dot{\boldsymbol{v}} = \mathfrak{E} - \boldsymbol{\nabla}\psi_0 \tag{10}$$

In static conditions $\boldsymbol{v}$ represents a uniform speed of linear movement. However, if the uniform speed turns into acceleration $\dot{\boldsymbol{v}} \neq \boldsymbol{0}$, then an extra field of size $\chi_0\dot{\boldsymbol{v}}$ is generated that counteracts the acceleration. The Qpattern does not change, thus $\boldsymbol{\nabla}\psi_0$ does not change.

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[74] http://arxiv.org/abs/physics/0609026v4.pdf

[75] http://www.adsabs.harvard.edu/abs/1953MNRAS.113...34S

[76] http://rmp.aps.org/abstract/RMP/v36/i1/p463_1

Also $\mathfrak{B}$ does not change. This means that the acceleration of the particle corresponds to an extra $\mathfrak{E}$ field that counteracts the acceleration. On its turn it corresponds with a change of the coupling partner φ. That change involves the coupling strength m. The counteraction is felt as inertia.

16.2 Information horizon

The terms in the integral continuity equation

$$\Phi = \int_V \nabla\psi \, dV = \int_V \phi \, dV \tag{1}$$

can be interpreted as representing the influence of a local object onto the rest of the universe or as the influence of the rest of the universe onto a local object. In the second case the influence diminishes with distance and the number of influencers increases such that the most distant contributors together poses the largest influence. These influencers sit at the information horizon. In the history of the model they are part of the birth state of the current episode of the universe. This was a state of densest packaging.

The local Qpattern that is described by ψ couples to the historic Qpattern φ for which the RQE acts as a Qpatch and as a Qtarget. This historic Qpattern resided in the reference page of the HBM.

17 Gravitation as a descriptor

The gravitation field describes the local curvature. The sharp allocation function can act as the base of a quaternionic gravitation theory. The sharp allocation function has sixteen partial derivatives that combine in a differential.

17.1 Palestra

All quantum state functions share their parameter space as affine spaces. Due to the fact that the coupling of Qpatterns affects this parameter space, the Palestra is curved. The curvature is not static. With other words the Qpatches in the parameter space move and densities in the distribution of these patches change. For potential observers, the Palestra is the place where everything classically happens. The Palestra comprises the whole universe.

17.1.1 Spacetime metric

The Palestra is defined with respect to a flat parameter space, which is spanned by the ***rational*** quaternions[77]. We already introduced the existence of a smallest rational number, which is used to arrange interspace freedom. The specification of the set of Qpatches is performed by a continuous quaternionic distribution $\wp(x)$ that acts as a (partial) allocation function. This allocation function defines a quaternionic infinitesimal interval ds. On its turn this definition defines a metric[78].

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[77] http://en.wikipedia.org/wiki/Quaternion_algebra#Quaternion_algebras_over_the_rational_numbers

[78] The intervals that are constituted by the smallest rational numbers represent the infinitesimal steps. Probably the hair of mathematicians are raised when we treat the interspacing as an infinitesimal steps. I apologize for that.

$$ds(x) = ds^{\nu}(x)e_{\nu} = d\wp = \sum_{\mu=0\ldots3} \frac{\partial\wp}{\partial x_{\mu}} dx_{\mu} = q^{\mu}(x)dx_{\mu} \tag{1}$$

$$= \sum_{\mu=0\ldots3} \sum_{\nu=0,\ldots3} e_{\nu} \frac{\partial\wp_{\nu}}{\partial x_{\mu}} dx_{\mu}$$

$$= \sum_{\mu=0\ldots3} \sum_{\nu=0,\ldots3} e_{\nu} q_{\nu}^{\mu} dx_{\mu}$$

The base e_{ν} and the coordinates x_{μ} are taken from the **flat** parameter space of $\wp(x)$. That parameter space is spanned by the quaternions. The definition of the quaternionic metric uses a full derivative $d\wp$ of the (partial) allocation function $\wp(x)$. This full derivative differs from the quaternionic nabla ∇, which ignores the curvature of the parameter space. On its turn $d\wp$ ignores the blur of $\mathcal{P}$.

The allocation function $\wp(x)$ may include an isotropic scaling function $a(\tau)$ that only depends on progression τ. It defines the expansion/compression of the Palestra.

ds is the infinitesimal quaternionic step that results from the combined real valued infinitesimal dx_{μ} steps that are taken along the e_{μ} base axes in the (flat) parameter space of $\wp(x)$.

$dx_0 = c\, d\tau$ plays the role of the infinitesimal space time interval ds_{st}[79]. It is a physical invariant. $d\tau$ plays the role of the proper time

[79] Notice the difference between the quaternionic interval ds and the spacetime interval ds_{st}

interval and it equals the infinitesimal progression interval. The progression step is an HBM invariant. Without curvature, dt in $\|ds\| = c\,dt$ plays the role of the infinitesimal coordinate time interval.

$$c^2\,dt^2 = ds\,ds^* = dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \tag{2}$$

$$dx_0^2 = ds_{st}{}^2 = c^2\,dt^2 - dx_1^2 - dx_2^2 - dx_3^2 \tag{3}$$

dx_0^2 is used to define the local spacetime metric tensor. With that metric the Palestra is a pseudo-Riemannian manifold that has a Minkowski signature. When the metric is based on ds^2, then the Palestra is a Riemannian manifold with a Euclidean signature. The Palestra comprises the whole universe. It is the arena where everything happens.

For the (partial) allocation function holds

$$\frac{\partial^2 \wp}{\partial x_\mu \partial x_\nu} = \frac{\partial^2 \wp}{\partial x_\nu \partial x_\mu} \tag{4}$$

And similarly for higher-order derivatives. Due to the spatial continuity of the allocation function $\wp(x)$, the quaternionic metric as it is defined above is far more restrictive than the metric tensor that that is used in General Relativity:

$$ds^2 = g_{ik}\,dx^i\,dx^k \tag{5}$$

Still

$$g_{ik} = g_{ki} \tag{6}$$

17.1.2 The Palestra step

When nature steps with universe (Palestra) wide steps during a progression step Δx_0, then in the Palestra a quaternionic step $\Delta s_\wp$ will be taken that differs from the corresponding flat step Δs_f

$$\Delta s_f = \Delta x_0 + \boldsymbol{i}\,\Delta x_1 + \boldsymbol{j}\,\Delta x_2 + \boldsymbol{k}\,\Delta x_3 \tag{1}$$

$$\Delta s_\wp = q^0 \Delta x_0 + q^1\,\Delta x_1 + q^2\,\Delta x_2 + q^3\,\Delta x_3 \tag{2}$$

The coefficients q^{μ} are quaternions. The Δx_{μ} are steps taken in the (flat) parameter space of the (partial) allocation function $\wp(x)$.

17.1.3 Pacific space and black regions.

If we treat the Palestra as a continuum, then the parameter space of the allocation function is a flat space that it is spanned by the number system of the quaternions. This parameter space gets the name "**Pacific space**". This is the space where the RQE's live. If in a certain region of the Palestra no matter is present, then in that region the Palestra is hardly curved. It means that in this region the Palestra is nearly equal to the parameter space of the allocation function.

The Pacific space has the advantage that when distributions are converted to this parameter space the Fourier transform of the converted distributions is not affected by curvature.

In a region where the curvature is high, the Palestra step comes close to zero. At the end where the Palestra step reaches the smallest rational value, an information horizon is established. For a distant observer, nothing can pass that horizon. The information horizon encloses a **black region**. Inside that region the quantum state functions are so densely packed that they lose their identity. However, they do not lose their sign flavor. The result is the formation of a single quantum state function that consists of the superposition of all contributing quantum state functions. The resulting black body has mass, electric charge and angular momentum. The quantum state function of a black region is quantized. Due to the fact that no information can escape through the information horizon, the inside of the horizon is obscure. No experiment can reveal its content. It does not contain a singularity at its center. All characteristics of the black region are contained in its quantum state function[80].

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[80] See Cosmological hstory

The (partial) allocation function $\wp(x)$ is a continuous quaternionic distribution. Like all continuous quaternionic distributions it contains two fields. It is NOT a QPAD. It does not contain density distributions.

17.1.4 Start of the universe.

At the start of the universe the package density was so high that also in that condition only one mixed QPAD can exist. That QPAD was a superposition of QPAD's that have different sign flavors. Only when the universe expands enough, multiple individual Qpatterns may have been generated. In the beginning, these QPAD's where uncoupled.

18 Modularization

A very powerful influencer is modularization. Together with the corresponding ***encapsulation*** it has a very healthy influence on the ***relational complexity*** of the ensemble of objects on which modularization works. The encapsulation takes care of the fact that most relations are kept internal to the module. When relations between modules are reduced to a few types, then the module becomes ***reusable***. The most Influential kind of modularization is achieved when modules can be ***configured from lower order modules***.

Elementary particles can be considered as the lowest level of modules. All composites are higher level modules.

When ***sufficient resources*** in the form of reusable modules are present, then modularization can reach enormous heights. On earth it was capable to generate ***intelligent species***.

18.1 Complexity

Potential complexity of a set of objects is a measure that is defined by the number of potential relations that exist between the members of that set.

If there are n elements in the set, then there exist n*(n-1) potential relations.

Actual complexity of a set of objects is a measure that is defined by the number of relevant relations that exist between the members of the set.

In human affairs and with intelligent design it takes time and other resources to determine whether a relation is relevant or not. Only an expert has the knowledge that a given rela-

tion is relevant. Thus it is advantageous to have as little irrelevant potential relations as is possible, such that mainly relevant and preferably usable relations result.

Physics is based on relations. Quantum logic is a set of axioms that restrict the relations that exist between quantum logical propositions. Via its isomorphism with Hilbert spaces quantum logic forms a fundament for quantum physics. Classical logic is a similar set of restrictions that define how we can communicate logically. Like classical logic, quantum logic only describes static relations. Traditional quantum logic does not treat physical fields and it does not touch dynamics. However, the model that is based on traditional quantum logic can be extended such that physical fields are included as well and by assuming that dynamics is the travel along subsequent versions of extended quantum logics, also dynamics will be treated. The set of propositions of traditional quantum logic is isomorphic with the set of closed subspaces of a Hilbert space. This is a mathematical construct in which quantum physicists do their investigations and calculations. In this way fundamental physics can be constructed. Here holds very strongly that only relevant relations have significance.

18.2 Relationalcomplexity

We define ***relational complexity*** as the ratio of the number of actual relations divided by the number of potential relations.

18.3 Interfaces

Modules connect via interfaces. Interfaces are used by interactions. Interactions run via (relevant) relations. Relations that act within modules are lost to the outside world of the module. Thus interfaces are collections of relations that are used by interactions. Inbound interactions come from the past. Outbound interactions go to the future. Two-sided interactions are cyclic. They are either oscillations or rotations of the inter-actor.

Interactions are implemented by potentials. The solutions in the Huygens principle cover both outgoing as well as incoming waves. The outbound waves implement outbound interfaces of elementary particles. The inbound waves implement inbound interfaces of elementary particles.

18.4 Interface types

Apart from the fact that they are inbound, outbound or cyclic the interfaces can be categorized with respect to the type of relations that they represent. Each category corresponds to an interface type. An interface that possesses a type and that installs the possibility to couple the corresponding module to other modules is called a standard interface.

18.5 Modular subsystems

Modular subsystems consist of connected modules. They need not be modules. They become modules when they are encapsulated and offer standard interfaces that makes the encapsulated system a reusable object.

The cyclic interactions bind the corresponding modules together. Like the coupling factor of elementary particles characterizes the binding of the pair of Qpatterns will a similar characteristic characterize the binding of modules.

This binding characteristic directly relates to the total energy of the constituted sub-system. Let ψ represent the renormalized superposition of the involved distributions.

$$\nabla\psi = \phi = m\,\varphi \tag{1}$$

$$\int_V |\psi|^2\, dV = \int_V |\varphi|^2\, dV = 1 \tag{2}$$

$$\int_V |\phi|^2\, dV = m^2 \tag{3}$$

Here again m represents total energy.

The binding factor is the total energy of the sub-system minus the sum of the total energies of the separate constituents.

18.6 Relational complexity indicators

The inner product of two Hilbert vectors is a measure of the relational complexity of the combination.

A Hilbert vector represents a linear combination of atoms. When all coefficients are equal, then the vector represents an assembly of atoms. When the coefficients are not equal, then the vector represents a ***weighted assembly*** of atoms.

For two normalized vectors $|a\rangle$ and $|b\rangle$:

$\langle a|a\rangle = 1$ (1)

$\langle b|b\rangle = 1$ (2)

$\langle a|b\rangle = 0$ means $|a\rangle$ and $|b\rangle$ are not related. (3)

$\langle a|b\rangle \neq 0$ means $|a\rangle$ and $|b\rangle$ are related. (4)

$|\langle a|b\rangle| = 1$ means $|a\rangle$ and $|b\rangle$ are optimally related. (5)

18.7 Modular actions

Subsystems that have the ability to choose their activity can choose to organize their actions in a modular way. As with static relational modularization the modular actions reduce complexity and for the decision maker it eases control.

18.8 Random design versus intelligent design

At lower levels of modularization nature design modular structures in a stochastic way. This renders the modularization process rather slow. It takes a huge amount of progression steps in order to achieve a relatively complicated structure. Still the complexity of that structure can be orders of magnitude less than the complexity of an equivalent monolith.

As soon as more intelligent sub-systems arrive, then these systems can design and construct modular systems in a more intelligent way. They use resources efficiently. This speeds the modularization process in an enormous way.

19 Functions that are invariant under Fourier transformation.

A subset of the (quaternionic) distributions have the same shape in configuration space and in the linear canonical conjugated space.

We call them dual space distributions. It are functions that are invariant under Fourier transformation[81]. These functions are not eigenfunctions.

The Qpatterns and the harmonic and spherical oscillations belong to this class.

Fourier-invariant functions show iso-resolution, that is, $\Delta_p = \Delta_q$ in the Heisenberg's uncertainty relation.

19.1 Natures preference

Nature seems to have a preference for quaternionic distributions that are invariant under Fourier transformation.

A possible explanation is the two-step generation process, where the first step is realized in configuration space and the second step is realized in canonical conjugated space. The whole pattern is generated two-step by two-step.

The only way to keep coherence between a distribution and its Fourier transform that are both generated step by step is to generate them in pairs.

[81] Q-Formulæ contains a section about functions that are invariant under Fourier transformation.

20 Events

20.1 Generations and annihilations

At the instant of generation or annihilation, the enumerator generator will change its mode and the Qpattern that will be generated changes its mode as well.

If the number of enumerator creations per step that contributes to a Qpattern is left open and if this number is larger than one, then it is difficult to understand that at a given instant the whole Qpattern changes its mode. The Qpattern has no knowledge of the mode that its members are in. The individual members might have that knowledge. In that case it is part of their charge.

So, from now on we suppose that the Qpatterns will be generated such that one member, the Qtarget, is generated per progression step. An event then indicates that the enumeration generator changes its generation mode.

For example, when a particle is annihilated the generator switches from generating a Qpattern in configuration space to generating an equivalent pattern in the canonical conjugated space. The result is that the pattern is no longer coupled and becomes a photon or a gluon. Of course the reverse procedure occurs at the generation of a particle.

In the original space, the object that corresponds to the Qpattern is annihilated while in the new space the transformed object is generated. Since the Qpattern is generated with a Qtarget at each progression step the event has immediate consequences.

Conservation laws govern the annihilation and creation processes.

20.2 Emissions and absorptions

When only a part of a composite annihilates, then a similar process can take place. A sub-module is annihilated and either the whole energy is emitted in the form of radiation or only part of the energy

is emitted and the rest is used to constitute a new particle at a lower energy level.

It is also possible that a complete sub-module is emitted. This can be done in a two-step mode, where first the sub-module or part of it is converted into radiation and subsequently the sub-module is re-generated.

Absorption is described as the reverse process.

20.3 Oscillating interactions

Oscillating interactions are implemented by cyclic interfaces. They consist of a sequence of annihilations and generations, where the locations alternate.

20.4 Movements

The fact that a particle moves, and the fact that a Qpattern is generated with only one Qtarget per progression step means that during a movement the Qpattern is spread along the path of movement.

20.5 Curvature

When the generator operates in one space and produces there a compact distribution then it affects the curvature of that space. It also has consequences in the canonical conjugated space. However, there the corresponding distribution will be spread out. Its effect on space curvature will also be spread. As a result the effect on space curvature in this canonical conjugated space will be negligible.

21 Entanglement

In the Hilbert Book Model, entanglement enters the model only after a huge number of extension steps. It is due to the fact that nature's building blocks have a set of discrete properties that can be observed via indirect means, while the building block may extend over rather large distances. So measuring the same property at nearly the same instant at quite different locations will give the same result. When the property is changed shortly before these measurements were performed, then it might give the impression that an instant action at a distance occurred, because light could not bridge these locations in the period between the two measurements. The explanation is that the building block at each progression instant moves to a different step stone and that these step stones may lay far apart. Apart from the property measurements, in this process no information transfer needs to take place. The measurements must be done without affecting the building block. At each arrival at a step stone the building block transmits contributions to its potentials. If the measurement uses these potentials, then the building block is not affected. According to this explanation, at least one progression step must separate the two measurements.

22 Cosmology

22.1 Cosmological view

Even when space was fully densely packed with matter (or another substance) then nothing dynamic would happen. Only when sufficient interspacing comes available dynamics becomes possible.

The Hilbert Book Model exploits this possibility. It sees black regions as local returns to the original condition.

22.2 The cosmological equations

The integral equations that describe cosmology are:

$$\frac{d}{d\tau}\int_V \rho\, dV + \oint_S \hat{\boldsymbol{n}}\rho\, dS = \int_V s\, dV \tag{1}$$

$$\int_V \nabla\rho\, dV = \int_V s\, dV \tag{2}$$

Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface S, $\boldsymbol{v}(\tau, \boldsymbol{q})$ is the velocity at which the charge density $\rho_0(\tau, \boldsymbol{q})$ enters volume V and s_0 is the source density inside V. In the above formula ρ stands for

$$\rho = \rho_0 + \boldsymbol{\rho} = \rho_0 + \frac{\rho_0 \boldsymbol{v}}{c} \tag{3}$$

It is the flux (flow per unit of area and per unit of progression) of ρ_0 . t stands for progression (not coordinate time).

22.3 Inversion surfaces

An inversion surface S is characterized by:

$$\oint_S \hat{\boldsymbol{n}}\rho\, dS = 0 \tag{1}$$

22.4 Cosmological history

The inversion surfaces divide universe into compartments. Think that these universe pockets contain matter that is on its way back to its natal state. If there is enough matter in the pocket this state forms a black region. The rest of the pocket is cleared from its mass content. Still the size of the pocket may increase. This represents the expansion of the universe. Inside the pocket the holographic principle governs. The black region represents the densest packaging mode of entropy.

The pockets may merge. Thus at last a very large part of the universe may return to its birth state, which is a state of densest packaging of entropy.

Then the resulting mass which is positioned at a huge distance will enforce a uniform attraction. This uniform attraction will install an isotropic extension of the central package. This will disturb the densest packaging quality of that package. The motor behind this is formed by the combination of the attraction through distant massive particles, which installs an isotropic expansion and the influence of the small scale random localization which is present even in the state of densest packaging.

This describes an eternal process that takes place in and between the pockets of an affine space.

22.5 Entropy

As a whole, universe expands. Locally regions exist where contraction overwhelms the global expansion. These regions are separated by inversion surfaces. The regions are characterized by their inversion surface. Within these regions the holographic principle resides. The fact that the universe as a whole expands means that the average size of the encapsulated regions increases.

The ***holographic principle*** says that the total entropy of the region equals the entropy of a black region that would contain all matter in the region. Black regions represent regions where entropy is optimally packed.

Thus entropy is directly related to the interspacing between enumerators. With other words, local entropy is related to local curvature.

23 Recapitulation

The model starts by taking quantum logic as its foundation. Next quantum logic is refined to Hilbert logic. It could as well have started by taking an infinite dimensional separable Hilbert space as its foundation. However, in that case the special role of base vectors would not so easily have been brought to the front. It appears that the atoms of the logic system and the base vectors of the Hilbert space play a very crucial role in the model. They represent the lowest level of objects in nature that play the theater of our observation.

The atoms are only principally unordered at very small "distances". They have content. The Hilbert space offers built-in enumerator machinery that defines the distances and that specifies the content of the represented atoms. The same can be achieved in a refined version of quantum logic that we call Hilbert logic.

In fact we focus on a compartment of universe, where universe is an affine space. The isotropic scaling factor that was assumed in the early phases of the model appears to relate to mass carrying particles that exist at huge distances. In the considered compartment an enumeration process establishes a kind of coordinate system. The master of the enumeration process is the allocation function $\mathcal{P}$. This function has a flat parameter space.

$$\mathcal{P} = \wp \circ \mathcal{S} \tag{1}$$

At small scales this function becomes a stochastic spatial spread function $\mathcal{S}$ that governs the quantum physics of the model. The whole function $\mathcal{P}$ is a convolution of a sharp part $\wp$ and the spread function $\mathcal{S}$. The differential of $\wp$ delivers a local metric. The spread function appears to be generated by a Poisson generator which produces Qpatterns.

After a myriad of progression steps the original ordering of the natal state of the model is disturbed so much that the natal large and medium scale ordering is largely lost. However, this natal ordering is returning in the black regions that constitute pockets that surround

them in universe. When the pockets merge into a huge black region, the history might restart enforced by the still existing low scale randomization and by the isotropic expansion factor, which is the consequence of the existence of massive particles at huge distances in the affine space.

The model uses a first part where elementary particles are formed by the representatives of the atomic propositions of the logic.

In a second part the formation of composites is described by a process called modularization. In that stage, in places where sufficient resources are present, the modularization process is capable of forming intelligent species.

This is the start of a new phase of evolution in which the intelligent species get involved in the modularization process and shift the mode from random design to intelligent design. Intelligent design runs much faster and uses its resources in a more efficient and conscientious way.

24 Conclusion

With respect to conventional physics, this simple model contains extra layers of individual objects. The most interesting addition is formed by the RQE's, the Qpatches, the Qtargets and the Qpatterns. They represent the atoms of the quantum logic sub-model.

The model gives an acceptable explanation for the existence of an (average) maximum velocity of information transfer. The two prepositions:

- Atomic quantum logic fundament
- Correlation vehicle

Lead to the existence of fuzzy interspacing of enumerators of the Hilbert space base vectors and to dynamically varying space curvature when compared to a flat reference continuum.

Without the freedom that is introduced by the interspacing fuzziness and which is used by the dynamic curvature, no dynamic behavior would be observable in the Palestra.

In the generation of the model the enumeration process plays a crucial role, but we must keep in mind that the choice of the enumerators and therefore the choice of the type of correlation vehicle is to a large degree arbitrary. It means that the Palestra has no natural origin. It is an affine space. The choice for quaternions as enumerators seems to be justified by the fact that the sign flavors of the quaternions explain the diversity of elementary particles.

Physicist that base their model of physics on an equivalent of the Gelfand triple which lacks a mechanism that creates the freedom that flexible interspaces provide, are using a model in which no natural curvature and fuzziness can occur. Such a model cannot feature dynamics.

Attaching a progression parameter to that model can only create the illusion of dynamics. However, that model cannot give a proper explanation of the existence of space curvature, space expansion,

quantum physics or even the existence of a maximum speed of information transfer.

Physics made its greatest misstep after the nineteen thirties when it turned away from the fundamental work of Garret Birkhoff and John von Neumann. This deviation did not prohibit pragmatic use of the new methodology. However, it did prevent deep understanding of that technology because the methodology is ill founded.

Doing quantum physics in continuous function spaces is possible, but it makes it impossible to find the origins of dynamics, curvature and inertia. Most importantly it makes it impossible to find the reason of existence of quantum physics.

Only the acceptance of the fact that ***physics is fundamentally countable*** can solve this dilemma.

Please attack the

Q-FORMULÆ

1 Introduction

This compilation starts with sections on quantum logic and Hilbert Logic.

Next lists of formulas for quaternionic algebra and quaternionic differentials will be given. These formulae are for a significant part derived from Bo Thidé's book "Electromagnetic Field Theory"; http://www.plasma.uu.se/CED/Book. I have merely converted the vector formula into quaternionic format.

Two types of quaternionic differentiation exist.

- Flat differentiation uses the quaternionic nabla and ignores the curvature of the parameter space.
- Full differentiation uses the allocation function $\wp(x)$ that defines the curvature of the parameter space.

The text focuses at applications in quantum mechanics, in electrodynamics and in fluid dynamics.

2 Quantum logic

Elementary particles behave non-classical. They can present themselves either as a particle or as a wave. A measurement of the particle properties of the object destroys the information that was obtained from an earlier measurement of the wave properties of that object.

With elementary particles it becomes clear that that nature obeys a different logic than our old trusted classical logic. The difference resides in the modularity axiom. That axiom is weakened. The classical logic is congruent to an orthocomplemented modular lattice. The quantum logic is congruent to an orthocomplemented weakly modular lattice. Another name for that lattice is orthomodular lattice.

2.1 Lattices

A subset of the axioms of the logic characterizes it as a half ordered set. A larger subset defines it as a lattice.

A lattice is a set of elements $a, b, c, ...$that is closed for the connections $\cap$ and $\cup$. These connections obey:

- The set is partially ordered. With each pair of elements a, b belongs an element c, such that $a \subset c$ and $b \subset c$.
- The set is a $\cap$ half lattice if with each pair of elements a, b an element c exists, such that $c = a \cap b$.
- The set is a $\cup$ half lattice if with each pair of elements a, b an element c exists, such that $c = a \cup b$.
- The set is a lattice if it is both a $\cap$ half lattice and a $\cup$ half lattice.

The following relations hold in a lattice:

$$a \cap b = b \cap a \tag{1}$$

$$(a \cap b) \cap c = a \cap (b \cap c) \tag{2}$$

$$a \cap (a \cup b) = a \tag{3}$$

$$a \cup b = b \cup a \tag{4}$$

$$(a \cup b) \cup c = a \cup (b \cup c) \tag{5}$$

$$a \cup (a \cap b) = a \tag{6}$$

The lattice has a partial order inclusion $\subset$:

$$a \subset b \Leftrightarrow a \subset b = a \tag{7}$$

A complementary lattice contains two elements n and e with each element a an complementary element a' such that:

$$a \cap a' = n \tag{8}$$

$$a \cap n = n \tag{9}$$

$$a \cap e = a \tag{1}$$

$$a \cup a' = e \tag{11}$$

$$a \cup e = e \tag{12}$$

$$a \cup n = a \tag{13}$$

An orthocomplemented lattice contains two elements n and e and with each element a an element a'' such that:

$$a \cup a'' = e \tag{14}$$

$$a \cap a'' = n \tag{15}$$

$$(a'')'' = a \tag{16}$$

$$a \subset b \Leftrightarrow b'' \subset a'' \tag{17}$$

e is the unity element; n is the null element of the lattice

A distributive lattice supports the distributive laws:

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \tag{18}$$

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c) \tag{19}$$

A modular lattice supports:

$$(a \cap b) \cup (a \cap c) = a \cap (b \cup (a \cap c)) \tag{20}$$

A weak modular lattice supports instead:

There exists an element d such that

$$a \subset c \Leftrightarrow (a \cup b) \cap c = a \cup (b \cap c) \cup (d \cap c) \tag{21}$$

where d obeys:

$$(a \cup b) \cap d = d \tag{22}$$

$$a \cap d = n \tag{23}$$

$$b \cap d = n \tag{24}$$

$$[(a \subset g) \, and \, (b \subset g) \Leftrightarrow d \subset g \tag{25}$$

In an atomic lattice holds

$$\exists_{p \in L} \forall_{x \in L} \{x \subset p \Rightarrow x = n\} \tag{26}$$

$$\forall_{a \in L} \forall_{x \in L} \{(a < x < a \cap p) \Rightarrow (x = a \, or \, x = a \cap p)\} \tag{27}$$

p is an atom

Both the set of propositions of quantum logic and the set of subspaces of a separable Hilbert space **H** have the structure of an orthomodular lattice. In this respect these sets are congruent.

In Hilbert space, an atom is a pure state (a ray spanned by a single vector).

Classical logic has the structure of an orthocomplemented distributive modular and atomic lattice.

Quantum logic has the structure of an orthomodular lattice. That is an orthocomplented weakly modular and atomic lattice. The set of closed subspaces of a Hilbert space also has that structure.

2.2 Proposition

In Aristotelian logic a proposition is a particular kind of sentence, one which affirms or denies a predicate of a subject. Propositions have binary values. They are either true or they are false.

Propositions take forms like "*This is a particle or a wave*". In quantum logic "*This is a particle.*" is not a proposition.

In mathematical logic, propositions, also called "propositional formulas" or "statement forms", are statements that do not contain quantifiers. They are composed of well-formed formulas consisting entirely of atomic formulas, the five logical connectives[82], and symbols of grouping (parentheses etc.).

[82] http://en.wikipedia.org/wiki/Logical_connective

Propositional logic is one of the few areas of mathematics that is totally solved, in the sense that it has been proven internally consistent, every theorem is true, and every true statement can be proved. Predicate logic is an extension of propositional logic, which adds variables and quantifiers.

In Hilbert space a vector is either inside or not inside a closed subspace. A proper quantum logical proposition is "*Vector |f> is inside state s*".

In Hilbert space, an atomic predicate corresponds with a subspace that is spanned be a single vector.

Predicates may accept attributes and quantifiers. The predicate logic is also called first order logic. A dynamic logic can handle the fact that predicates may influence each other when atomic predicates are exchanged.

2.3 Observation

In physics, particularly in quantum physics, a system **observable** is a property of the system state that can be determined by some sequence of physical operations. An observable can exist without being observed. This paper distinguishes between measurement data and observables.

- The state is considered as a linear combination of eigenvectors of an observable. The value of an observable returns the statistical expectation value of the eigenvalue of the observable.
- A measurement transforms the observed state to one of the eigenvectors of the observable. What happens depends on the characteristics of the measuring equipment. The measurement results in one or more measurement data.

A particle can reveal its existence in the form of potentials. Measuring a potential does not affect the state of the particle. In general, measuring an eigenvalue will alter the state of the particle. This can go as far as the annihilation of the particle.

3 Hilbert logic

The set of propositions of traditional quantum logic is lattice isomorphic with the set of closed subspaces of a separable Hilbert space. However there exist still significant differences between this logic system and the Hilbert space. This gap can be closed by a small expansion of the quantum logic system.

Step 1: Require that linear combinations of atomic propositions also belong to the logic system. Call such propositions ***linear propositions***.

Step 2: introduce the notion of ***relational coupling*** between two linear propositions. This measure has properties that are similar to the inner product of Hilbert space vectors.

Step 3: Close the subsets of the new logic system with respect to this relational coupling measure.

The relational coupling measure can have values that are taken from a suitable division ring. The resulting logic system will be called Hilbert logic.

The Hilbert logic is lattice isomorphic as well topological isomorphic with the corresponding Hilbert space.

Due to this similarity the Hilbert logic will also feature operators[83].

In a Hilbert logic linear operators can be defined that have linear atoms as their eigen-propositions. The eigenspace of these operators is countable.

Linear propositions are the equivalents of Hilbert vectors. General quantum logic propositions are the equivalents of (closed) subspaces of a Hilbert space.

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[83] The Hilbert logic does not feature dynamic operators.

The measure of the relational coupling between two linear propositions is the equivalent of the inner product between two Hilbert vectors.

In a Hilbert logic system the ***superposition principle*** holds. A linear combination of linear proposition is again a linear proposition.

4 Hilbert space isomorphism

The set of propositions in a quantum logic system is lattice isomorphic with the set of closed subspaces of an infinite dimensional separable Hilbert space.

The set of Hilbert propositions in a Hilbert logic system is isomorphic with the set of Hilbert vectors of an infinite dimensional separable Hilbert space.

The set of eigenvectors of a normal operator in Hilbert space is isomorphic to the set of eigen-atoms of a corresponding operator in the Hilbert logic system.

A coherent distribution of objects that is represented by a QPAD corresponds to a set of Hilbert logic propositions that are eigen-atoms of a normal operator that resides in the Hilbert logic. In this way it also corresponds to a set of Hilbert space base vectors that are eigenvectors of a normal operator that resides in the Hilbert space. The coherent distribution corresponds to a closed subspace of the Hilbert space. It also corresponds to a quantum logic proposition. That quantum logic proposition concerns a building block.

The atomic Hilbert propositions that span this quantum logic proposition form the constituents of the building block. In this way it also corresponds to a set of Hilbert space base vectors that are eigenvectors of a normal operator that resides in the Hilbert space. The coherent distribution corresponds to a closed subspace of the Hilbert space. It also corresponds to a quantum logic proposition. That quantum logic proposition concerns a building block.

5 About quaternions

5.1 Notation

Let **x** be the position vector (radius vector, coordinate vector) from the origin of the Euclidean space $\mathbb{R}^3$ coordinate system to the coordinate point $(x_1;\ x_2;\ x_3)$ in the same system and let $|x|$ denote the magnitude ('length') of x. Let further $\alpha(\boldsymbol{x}), \beta(\boldsymbol{x}), \gamma(\boldsymbol{x}), \ldots$, be arbitrary scalar fields, $\boldsymbol{a}(\boldsymbol{x}), \boldsymbol{b}(\boldsymbol{x}), \boldsymbol{c}(\boldsymbol{x}), \ldots$, arbitrary vector fields, and $\boldsymbol{A}(\boldsymbol{x}), \boldsymbol{B}(\boldsymbol{x}), \boldsymbol{C}(\boldsymbol{x}), \ldots$,arbitrary rank two tensor fields in this space.

Let q be the position relative to the origin of the space $\mathbb{H}$ that is spanned by the quaternions and that is given by the coordinate point $(q_0;\ q_1;\ q_2;\ q_3)$) and let $|q|$ denote the norm of q.

Let * denote complex or quaternionic conjugate and † denote Hermitian conjugate (transposition and, where applicable, complex or quaternionic conjugation).

5.2 Cayley-Dickson construction

The Cayley-Dickson construction formula enable the generation of a quaternion from two complex numbers:

$$p = a_0 + a_1\mathbf{k} + \mathbf{i}(b_0 + b_1\mathbf{k}) \tag{1}$$

$$q = c_0 + c_1\mathbf{k} + \mathbf{i}(d_0 + d_1\mathbf{k}) \tag{2}$$

$$(a, b)\,(c, d) = (ac - db^*;\ a^*d + cb) \tag{3}$$

$$r = pq \tag{4}$$

$$r_0 = a_0c_0 - a_1c_1 - b_0d_0 - b_1d_1 \tag{5}$$

$$r_{\mathbf{k}} = a_0c_1 - a_1c_0 - b_0d_1 + b_1d_0 \tag{6}$$

$$r_{\mathbf{i}} = a_0d_0 + a_1d_1 + b_0c_0 - b_1c_1 \tag{7}$$

$$r_{\mathbf{j}} = -a_1d_0 + a_0d_1 + b_0c_1 + b_1c_0 \tag{8}$$

5.3 Warren Smith's numbers

All hyper-complex numbers are based on real numbers. Two main construction formulas for hyper-complex numbers exist. The Cayley-Dickson construction is the most widely known. The Warren-Smith construction gives best algorithmic properties at higher dimensions. Until the octonions both construction formulas deliver the same results.

The quaternions are the highest dimensional hyper-complex numbers that deliver a division ring.

5.3.1 2^n-on construction

The 2^n-ons use the following doubling formula

$$\begin{aligned}(a,b)(c,d) \;=\; & (a\,c - (b\,d^*)^*, (b^*c^*)^* \\ & + (b^*(a^*((b^{-1})^*d^*)^*)^*)^*)\end{aligned} \tag{1}$$

Up until the 16-ons the formula can be simplified to

$$(a,b)(c,d) = (a\,c - b\,d^*, c\,b + (a^*\,b^{-1})\,(b\,d)) \quad (2)$$

Up to the octonions the Cayley Dickson construction delivers the same as the 2^n-on construction. From n>3 the 2^n-ons are 'nicer'.

5.3.1.1 2^n-ons

Table of properties of the 2^n-ons.

See http://www.scorevoting.net/WarrenSmithPages/homepage/nce2.pdf.

Type	***name***	**Lose**
1-ons	*Reals*.	
2-ons	*Complex numbers*	$z^* = z$ (the * denotes conjugating); the ordering properties that both {z > 0, -z > 0, or z = 0} and {w > 0, z > 0 implies w + z > 0, wz > 0}.
4-ons	*Quaternions*	commutativity ab = ba; the algebraic closedness property that every univariate polynomial equation has a root.
8-ons	*Octonions*	associativity ab · c = a · bc.
16-ons	(*not Sedenions!*)	right-alternativity x · yy = xy · y; right-cancellation $x = xy \cdot y^{-1}$; flexibility x · yx = xy · x; left-linearity (b + c)a = ba + ca; anti-automorphism ab = ba, $(ab)^{-1} = b^{-1} a^{-1}$; left-linearity (b + c)a = ba + ca; continuity of the map x → xy; Moufang and Bol identities; diassociativity

32-ons		generalized-smoothness of the map $x \to xy$; right-division properties that $xa = b$ has (generically) a solution x, and the uniqueness of such an x; the "fundamental theorem of algebra" that every polynomial having a unique "asymptotically dominant monomial" must have a root; Trotter's formula: $\lim_{n\to\infty}\left[e^{x/n}e^{y/n}\right]^n = \lim_{n\to\infty}\left(1+\frac{x+y}{n}\right)^n = e^{x+y}$

Type	**Retain**
2^n-ons	Unique 2-sided multiplicative & additive identity elements 1 & 0; Norm-multiplicativity $\|xy\|^2 = \|x\|^2 \cdot \|y\|^2$; Norm-subadditivity $\|a + b\| \leq \|a\| + \|b\|$; 2-sided inverse $a^{-1} = a^*/\|a\|^2$ ($a \# 0$); $a^{**} = a$; $(x \pm y)^* = x^* \pm y^*$; $(a^{-1})^{-1} = a$; $(a^*)^{-1} = (a^{-1})^*$; $\|a\|^2 = \|a\|^2 = a^*a$; Left-alternativity $yy \cdot x = y \cdot yx$; Left-cancellation $x = y^{-1} \cdot yx$; Right-linearity $a(b + c) = ab + ac$; r^{th} power-associativity $a^n a^m = a^{n+m}$; Scaling $s \cdot ab = sa \cdot b = as \cdot b = a \cdot sb = a \cdot bs = ab \cdot s$ (s real); Power-distributivity $(ra^n + sa^m)b = ra^n b + sa^m b$ (r, s real); Vector product properties of the imaginary part: **ab** - re(**ab**) of the product for pure-imaginary 2^n-ons **a**,**b** regarded as $(2^n - 1)$-vectors;

	$\langle xa,b\rangle = \langle a,x^*b\rangle$, $\langle xa,xb\rangle = \|x\|^2\cdot\langle a,b\rangle$ and $\langle x,y\rangle = \langle x^*,y^*\rangle$ Numerous weakened associativity, commutativity, distributivity, antiautomorphism, and Moufang and Bol properties including 9-coordinate ``niner'' versions of most of those properties; contains 2^{n-1}-ons as subalgebra.

5.3.1.1.1 The most important properties of 2^n-ons

If a,b,x,y are 2^n-ons, $n \geq 0$, and s and t are scalars (i.e. all coordinates are 0 except the real coordinate) then

unit: A unique 2^n-on 1 exists, with $1\cdot x = x\cdot 1 = x$.

zero: A unique 2^n-on 0 exists, with $0 + x = x + 0 = x$ and $0\cdot x = x\cdot 0 = 0$.

additive properties: $x+y = y+x$, $(x+y)+z = x+(y+z)$;

$-x$ exists with $x + (-x) = x - x = 0$.

norm: $|x|^2 = xx^* = x^*x$.

norm-multiplicativity: $|x|^2\cdot|y|^2 = |x\cdot y|^2$.

scaling: $s\cdot x\cdot y = s\cdot x\cdot y = x\cdot s\cdot y = x\cdot s\cdot y = x\cdot y\cdot s$.

weak-linearity: $(x + s)\cdot y = x\cdot y + s\cdot y$ and $x\cdot(y + s) = x\cdot y + x\cdot s$.

right-linearity: $x\cdot(y + z) = x\cdot y + x\cdot z$.

inversion: If $x \neq 0$ then a unique x^{-1} exists, obeying $x^{-1}\cdot x = x\cdot x^{-1} = 1$. It is $x^{-1} = x\cdot|x|^{-2}$.

left-alternativity: $x\cdot xy = x^2\cdot y$.

left-cancellation: $x\cdot x^{-1}\cdot y = y$.

effect on inner products: $\langle x\cdot a,b\rangle = \langle a, x^*\cdot b\rangle$, $\langle x,y\rangle = \langle x^*, y^*\rangle$, $\langle x^*\cdot a, x^{-1}\cdot b\rangle = \langle a,b\rangle$,

and $\langle x\cdot a,x\cdot b\rangle = |x|^2\cdot\langle a,b\rangle$.

Conjugate of inverse: $(x^{-1})^* = (x^*)^{-1}$.

Near-anticommutativity of unequal basis elements: $e_k^2 = -1$ and $e_k\cdot e_l^* = -e_l\cdot e_k^*$ if $k \neq l$.

(Note: the case k; $l > 0$ shows that unequal pure-imaginary basis elements anticommute.)

Alternative basis elements: $e_k \cdot e_l \cdot e_k = e_k \cdot e_l \cdot e_k$, $e_l \cdot e_k \cdot e_k = e_l \cdot e_k \cdot e_k$, and $e_k \cdot e_k \cdot e_l = e_k \cdot e_k \cdot e_l$. (However, when $n \geq 4$ the 2^n-ons are not flexible i.e. it is not generally true that $x \cdot y \cdot x = x \cdot y \cdot x$ if x and y are 16-ons that are not basis elements. They also are not right-alternative.)

Quadratic identity: If x is a 2^n-on (over any field F with charF $\neq$ 2), then $x^2 + |x|^2 = 2 \cdot x \text{ re } x$

Squares of imaginaries: If x is a 2^n-on with re x = 0 ("pure imaginary") then $x^2 = -|x|^2$ is nonpositive pure-real.

Powering preserves im*x* direction

5.3.1.1.2 Niners

Niners are 2n-ons whose coordinates with index > 8 are zero. The index starts with 0.

9-**flexibility** $xp \cdot x = x \cdot px$, $px \cdot p = p \cdot xp$.

9-**similitude unambiguity** $xp \cdot x^{-1} = x \cdot px^{-1}$, $px \cdot p^{-1} = p \cdot xp^{-1}$.

9-**right-alternativity** $xp \cdot p = x \cdot p^2$, $px \cdot x = p \cdot x^2$.

9-**right-cancellation** $xp^{-1} \cdot p = x$, $px^{-1} \cdot x = p$.

9-**effect on inner products** $\langle x, yp\rangle = \langle xp, y\rangle$, $\langle xp, yp\rangle = |p|^2\langle x, y\rangle$.

9-**left-linearity** $(x + y)p = xp + yp$, $(p + q)x = px + qx$.

9-**Jordan-identity** $xp \cdot xx = x(p \cdot xx)$, $py \cdot pp = p(y \cdot pp)$.

9-**coordinate-distributivity** $([x + y]z)_{0;\ldots;8} = (xz + yz)_{0;\ldots;8}$.

9-**coordinate-Jordan-identity** $[xy \cdot xx]_{0;\ldots;8} = [x(y \cdot xx)]_{0;\ldots;8}$.

9-**anticommutativity for orthogonal imaginary 2^n-ons**

If $\langle p, x\rangle = \text{re } p = \text{re } x = 0$ then $px = -xp$.

9-**reflection** If $|a| = 1$ and the geometric reflection operator is defined below then $-(\text{refl}[a](y))_{0;\ldots;8} = (a \cdot y^* a)_{0;\ldots;8}$, and $-\{\text{refl}[a](y)\}^*_{0;\ldots;8} = (a^* y \cdot a^*)_{0;\ldots;8}$, and

if either a or y is a niner then −refl[a](y) = a · y*a and −refl[a](y) = a*y · a*.

$$\text{refl}[\vec{x}](\vec{t}) \stackrel{\text{def}}{=} \vec{t} - \frac{2\langle \vec{x}, \vec{t} \rangle}{|\vec{x}|^2} \vec{x} \tag{1}$$

What holds for the niners, also holds for the octonions.

5.4 Waltz details

The 16-ons lose the continuity of the map $x \Rightarrow xy$. Also, in general holds $(x\ y)x \neq x\ (y\ x)$ for 16-ons. However, for all 2^n-ons the base numbers fulfill $(e_i\ e_j)\ e_i = e_i\ (e_j\ e_i)$. All 2^n-ons feature a conjugate and an inverse. The inverse only exists for non-zero numbers. The 2^n-ons support the **number waltz**

$$c = a\ b/a. \tag{}$$

Often the number waltz appears as a unitary number waltz

$$c = u^* b\ u \tag{}$$

where u is a unit size number and u^* is its conjugate $u\ u^* = 1$.

In quaternion space the **quaternion waltz** $a\ b/a$ can be written as

$$\begin{aligned} a\ b\ /\ a &= exp(2\ \pi\ \tilde{\imath}\ \varphi)\ b\ exp(-2\ \pi\ \tilde{\imath}\ \varphi) \\ &= b - \boldsymbol{b}_\perp + exp(2\ \pi\ \tilde{\imath}\ \varphi)\ \boldsymbol{b}_\perp\ exp(-2\ \pi\ \tilde{\imath}\ \varphi) \\ &= b - \boldsymbol{b}_\perp + exp(4\ \pi\ \tilde{\imath}\ \varphi)\boldsymbol{b}_\perp \end{aligned} \tag{}$$

$$\Delta b = (exp(4\ \pi\ \tilde{\imath}\ \varphi) - 1)\boldsymbol{b}_\perp \tag{}$$

$$= (cos(4\,\pi\,\varphi) + \tilde{\imath}\,sin(4\,\pi\,\varphi) - 1)\,\boldsymbol{b}_{\perp}$$

$$= exp(2\,\pi\,\tilde{\imath}\,\varphi)\;2\;\tilde{\imath}\,sin\,(2\,\pi\,\varphi)\,\boldsymbol{b}_{\perp}$$

$$\|\Delta b\| \;=\; \|2\,sin(2\,\pi\,\varphi)\,\boldsymbol{b}_{\perp}\| \qquad ($$

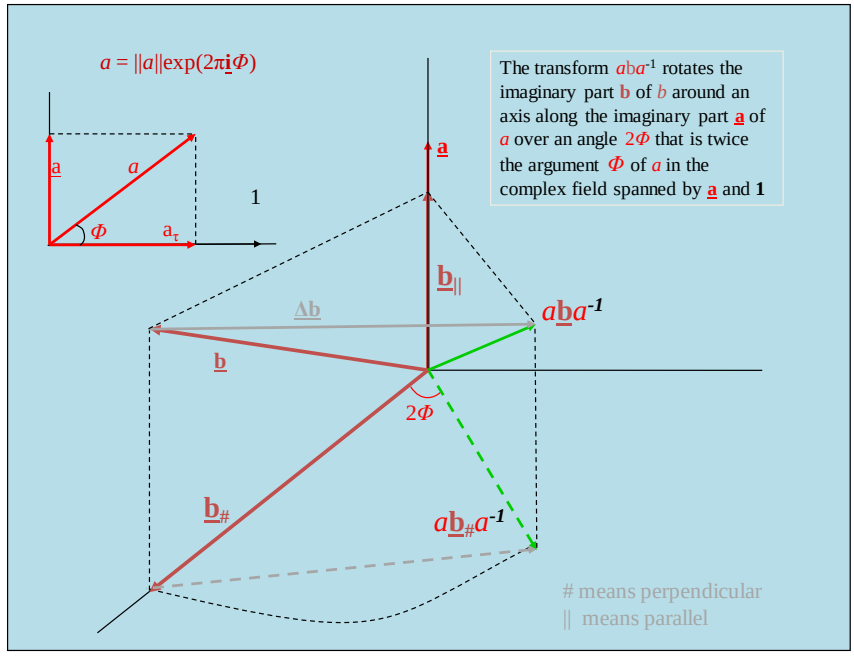

Figure 1. The rotation of a quaternion by a second quaternion.

Another way of specifying the difference is:

$$\Delta b \;=\; (a \cdot b - b \cdot a)/a \;=\; 2 \cdot (\boldsymbol{a} \times \boldsymbol{b})/a \qquad ($$

$$\|\Delta b\| \;= 2\, \|\boldsymbol{a} \times \boldsymbol{b}\| / \, \|a\| \qquad ($$

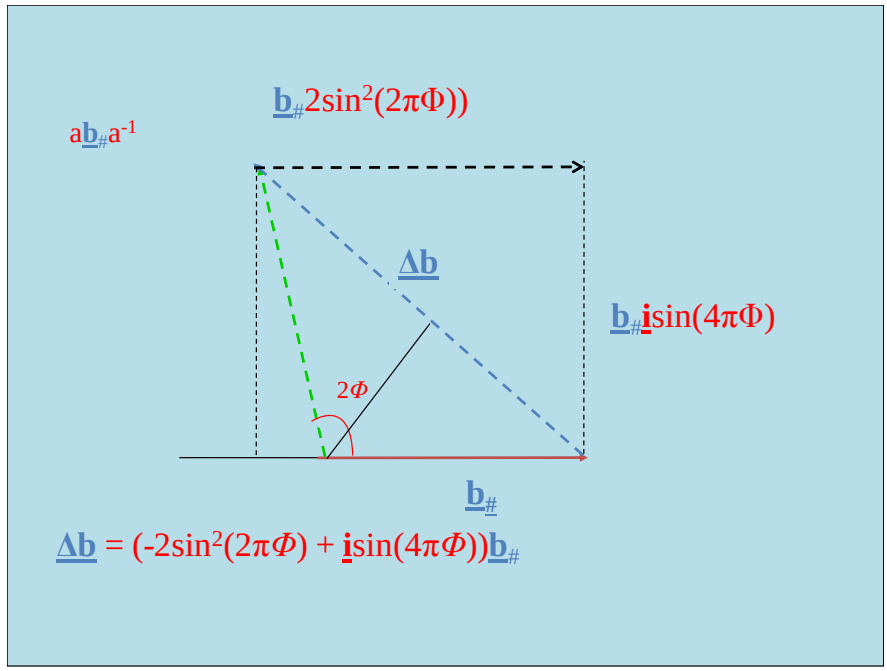

Figure 2: The difference after rotation

5.5 Spinors and matrices

In contemporary physics complex probability amplitude distributions (CPAD's) are used rather than quaternionic probability amplitude distributions (QPAD's). Spinors and matrices are used to simulate QPAD behavior for CPAD's.

5.5.1 Symmetries

The quaternionic number system exists in sixteen discrete symmetry sets (sign flavors). When the real part is ignored, then eight different symmetry sets result. The values of a continuous function all belong to the same symmetry set. The parameter space of the function may belong to a different symmetry set.

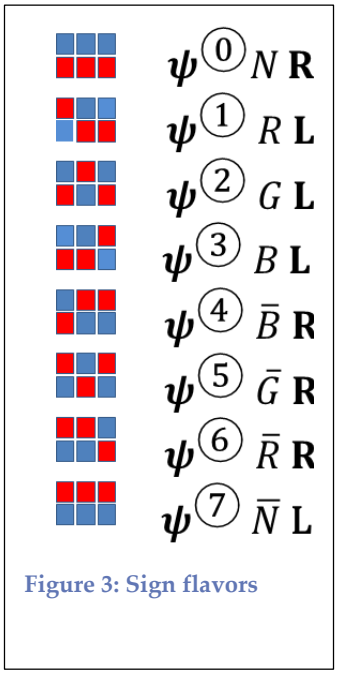

Figure 3: Sign flavors

Eight sign flavors
(discrete symmetries)
Colors N, R, G, B, $\overline{R}$, $\overline{G}$, $\overline{B}$, W
Right or Left handedness R,L

[84]The red blocks indicates sign up or down with respect to the base sign flavor. For quaternionic distributions the (quaternionic) parameter space acts as base sign flavor.

Quaternionic functions can be interpreted as the combination of a scalar function and a 3D vector function. The scalar part can be interpreted as the representation of an object density distribution. In that case the vector function can be thought to correspond to an associated current density distribution. The discrete symmetry values control the direction of the currents. This must be determined relative to a reference.

If we ignore the real part, then only eight discrete symmetries result. The next table lists these symmetries in text format;

[84] This picture has been changed!

||ddd||n u=up;d=down;
||RH|| n=neutral;r=red;g=green;b=blue;
||udd||r||L B=anti.blue;G=anti.green;R=anti.r
H|| ed;N=anti-neutral
||dud||g||L RH=right handed; LH= left handed.
H||
||ddu||b||L
H||
||duu||B||R
H||
||udu||G||R
H||
||uud||R||R
H||
||uuu||N||L
H||

The 3D Kronecker delta tensor

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \quad (1)$$

The fully antisymmetric Levi-Civita tensor

$$\epsilon_{ijk} = \begin{cases} 1 \text{ if } i,j,k \text{ is an even permutation of 1,2,3} \\ 0 \text{ if at least two of } i,j,k \text{ are equal} \\ -1 \text{ if } i,j,k \text{ is an odd permutation of 1,2,3} \end{cases} \quad (2)$$

5.5.2 Spinor

We use square brackets for indicating spinors. Spinors use real component functions ψ_i. . Complex component functions $\psi_i$would

result in spinor representations of bi-quaternions. Bi-quaternions do not form a division ring[85].

A 2×2 spinor is defined by the row:

$$[\psi] \equiv \left[[\psi_0][\boldsymbol{\Psi}]\right] \tag{1}$$

$$[\psi]^{\ddagger} \equiv \left[[\boldsymbol{\Psi}][\psi_0]\right] \tag{2}$$

Where

$$[\psi_0] \equiv \begin{bmatrix} \Psi_0 & 0 \\ 0 & \Psi_0 \end{bmatrix} \tag{3}$$

$$[\boldsymbol{\Psi}] \equiv \begin{bmatrix} \Psi_3 & \Psi_1 - i\Psi_2 \\ \Psi_1 + i\Psi_2 & -\Psi_3 \end{bmatrix} \tag{4}$$

Spinors obey[86]

$$[\boldsymbol{\Psi}] + [\boldsymbol{\phi}] = 2[\langle \boldsymbol{\Psi}, \boldsymbol{\phi} \rangle] \tag{5}$$

$$[\boldsymbol{\Psi}] - [\boldsymbol{\phi}] = 2\, i[\boldsymbol{\Psi} \times \boldsymbol{\phi}] \tag{6}$$

5.5.2.1 Sign flavors

The relation with the sign flavors is

$$[\boldsymbol{\Psi}] = [\boldsymbol{\Psi}]^{⓪} = \left[\boldsymbol{\psi}^{⓪}\right] \tag{1}$$

$$[\boldsymbol{\Psi}]^{①} = \left[\boldsymbol{\psi}^{①}\right] \tag{2}$$

$$[\boldsymbol{\Psi}]^{②} = \left[\boldsymbol{\psi}^{②}\right] \tag{3}$$

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[85] The author uses its own notation for spinors and sign flavors

[86] http://en.wikipedia.org/wiki/Spinors_in_three_dimensions

$$[\boldsymbol{\Psi}]^{③} = \left[\boldsymbol{\psi}^{③}\right] \tag{4}$$
$$[\boldsymbol{\Psi}^{*}] = [\boldsymbol{\Psi}^{*}]^{⓪} = \left[\boldsymbol{\psi}^{⑦}\right] \tag{5}$$
$$[\boldsymbol{\Psi}^{*}]^{①} = \left[\boldsymbol{\psi}^{⑥}\right] \tag{6}$$
$$[\boldsymbol{\Psi}^{*}]^{②} = \left[\boldsymbol{\psi}^{⑤}\right] \tag{7}$$
$$[\boldsymbol{\Psi}^{*}]^{③} = \left[\boldsymbol{\psi}^{④}\right] \tag{8}$$

5.5.3 Dirac spinors

The 4×4 spinors target the application in the Dirac equation.
A general 4×4 spinor is defined by the column:

$$\begin{bmatrix} [\psi] \\ [\phi^{*}]^{\ddagger} \end{bmatrix} \equiv \begin{bmatrix} [\Psi_0] & [\boldsymbol{\Psi}] \\ [-\boldsymbol{\phi}] & [\phi_0] \end{bmatrix} \tag{1}$$

A compacted spinor $]\Psi[$ is a 1×4 matrix consisting of real functions that represent all sixteen sign flavors of a QPAD.

$$]\Psi[\equiv \begin{bmatrix} [\psi] \\ [\psi^{*}]^{\ddagger} \end{bmatrix} = \begin{bmatrix} [\Psi_0] & [\boldsymbol{\Psi}] \\ [-\boldsymbol{\Psi}] & [\Psi_0] \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} \Psi_0 & 0 & \Psi_3 & \Psi_1 - i\Psi_2 \\ 0 & \Psi_0 & \Psi_1 + i\Psi_2 & -\Psi_3 \\ -\Psi_3 & -\Psi_1 + i\Psi_2 & \Psi_0 & 0 \\ -\Psi_1 - i\Psi_2 & +\Psi_3 & 0 & \Psi_0 \end{bmatrix}$$

5.5.4 Spinor base

The $\boldsymbol{\alpha}$ and β matrices form the base of spinor $]\Psi[$ and its elements

$$\alpha_1 \equiv \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{1}$$

$$\alpha_2 \equiv \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{2}$$

$$\alpha_3 \equiv \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{3}$$

$$\beta \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4}$$

$\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ represent imaginary base vectors of the simulated quaternion. β represents the conjugation action for the spinor.

A relation exist between $\alpha_1, \alpha_2, \alpha_3$ and the Pauli[87] matrices $\sigma_1, \sigma_2, \sigma_3$:

$$\sigma_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{5}$$

$$1 \mapsto I, \quad \boldsymbol{i} \mapsto \sigma_1, \quad \boldsymbol{j} \mapsto \sigma_2, \quad \boldsymbol{k} \mapsto \sigma_3 \tag{6}$$

5.5.5 Gamma matrices

This combination is usually represented in the form of gamma matrices.

In Dirac representation, the four contravariant gamma matrices are

$$\gamma^0 \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \gamma^1 \equiv \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \tag{1}$$

[87] http://en.wikipedia.org/wiki/Pauli_matrices

$$\gamma^2 \equiv \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \qquad \gamma^3 \equiv \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

It is useful to define the product of the four gamma matrices as follows:

$$\gamma^5 \equiv i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{2}$$

The gamma matrices as specified here are appropriate for acting on Dirac spinors written in the Dirac basis; in fact, the Dirac basis is defined by these matrices. In the Dirac basis[88]:

$$\gamma^0 \equiv \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \qquad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}, \qquad \gamma^5 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{3}$$

This corresponds with $\alpha_k = \gamma^k, \beta = \gamma^5$.
Apart from the Dirac basis, a Weyl basis exists

$$\gamma^0 = \gamma^\beta = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \qquad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}, \qquad \gamma^5 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \tag{4}$$

[88] http://en.wikipedia.org/wiki/Gamma_matrices#Dirac_basis

The Weyl basis has the advantage that its chiral projections[89] take a simple form:

$$\psi_L = ½\,(1 - \gamma^5)[\psi] = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}[\psi] \tag{5}$$

$$\psi_R = ½\,(1 + \gamma^5)[\psi] = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}[\psi] \tag{6}$$

$$[\psi^*] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}[\psi] \tag{7}$$

[89] http://en.wikipedia.org/wiki/Chirality_(physics)

6 Quaternionic differentiation

6.1 Differentiation in flat space

We treat quaternionic distributions as if they possess a continuous parameter space. The differential vector operator $\boldsymbol{\nabla}$ is in Cartesian coordinates given by

$$\boldsymbol{\nabla} = \sum_{i=1}^{3} \boldsymbol{e}_i \frac{\partial}{\partial x_i} \tag{1}$$

The flat quaternionic differential operator ∇ is in Cartesian coordinates given by

$$\nabla = \sum_{i=0}^{3} e_i \nabla_i = \sum_{i=0}^{3} e_i \frac{\partial}{\partial x_i}; \quad e = (1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}) \tag{2}$$

$$\nabla f = \sum_{i=0}^{3} \sum_{j=0}^{3} e_i e_j \frac{\partial f_j}{\partial x_i} \tag{3}$$

6.2 Differentiation in curved space

The allocation function $\wp(x)$ has a flat parameter space that is spanned by the rational or the real quaternions[90]. However, in this section we treat the E-type $\wp(x)$ as if it has a continuous parameter space[91]. That makes it possible to use regular differential calculus. The full quaternionic difference operator $\mathrm{d}\wp$ is given by

$$\mathrm{d}\wp = \sum_{\mu=0}^{3} q^{\mu}\, dx_{\mu} = \sum_{\mu=0}^{3} \frac{\partial \wp}{\partial x_{\mu}} dx_{\mu} \qquad (1)$$

$$= \sum_{\nu=0}^{3} e_{\mu} \sum_{\mu=0}^{3} \frac{\partial \wp_{\nu}}{\partial x_{\mu}} dx_{\mu}$$

Here the coefficients q^{μ} are quaternionic coefficients, which are determined by the quaternionic allocation function $\wp(x)$.

$\wp(x)$ defines a curved target space. This curved space can act as parameter space to other quaternionic distributions. (2)

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[90] http://en.wikipedia.org/wiki/Quaternion_algebra#Quaternion_algebras_over_the_rational_numbers

[91] See section on quaternionic distributions.

$$q^{\mu} = \frac{\partial \wp}{\partial x_{\mu}}; \quad \wp = \sum_{\nu=0}^{3} e_{\nu} \wp_{\nu}$$

The allocation function $\wp(x)$ may include an isotropic scaling function $a(\tau)$ that only depends on progression τ. It defines the expansion/compression of the curved space.

The quaternionic infinitesimal interval $d\wp$ defines the quaternionic metric of the curved space that is defined by $\wp(x)$.

~~In this way, the quaternionic function g(ζ), which has a curved parameter space defined by $\zeta = \wp(x)$ corresponds to a new function h(x)= g(℘(x)), which has a flat parameter space. The flattened nabla $\breve{\nabla}$ is defined as:~~

$$\breve{\nabla} g = \sum_{\nu=0}^{3} e_{\nu} \frac{\partial g(\zeta)}{\partial x_{\nu}} = \sum_{\nu=0}^{3} e_{\nu} \sum_{\lambda=0}^{3} e_{\lambda} \frac{\partial g_{\lambda}}{\partial x_{\nu}} \tag{3}$$

$$= \sum_{\nu=0}^{3} e_{\nu} \sum_{\lambda=0}^{3} e_{\lambda} \sum_{\mu=0}^{3} \frac{\partial g_{\lambda}}{\partial \zeta_{\mu}} e_{\mu} \frac{\partial \zeta_{\mu}}{\partial x_{\nu}}$$

$$= \sum_{\nu=0}^{3} \sum_{\lambda=0}^{3} \sum_{\mu=0}^{3} e_{\nu} e_{\lambda} e_{\mu} \frac{\partial g_{\lambda}}{\partial \zeta_{\mu}} \frac{\partial \wp_{\mu}}{\partial x_{\nu}}$$

7 Coordinate systems

7.1 Cylindrical circular coordinates

7.1.1 Base vectors

7.1.2 Cartesian to cylindrical circular

$$\rho = x_1\, cos(\,\theta) + x_2\, sin(\,\theta) \tag{1}$$

$$\varphi = -x_1\, sin(\,\theta) + x_2\, cos(\,\theta) \tag{2}$$

$$z = x_3 \tag{3}$$

7.1.3 Cylindrical circular to Cartesian

$$x_1 = \boldsymbol{\rho}\, cos(\,\theta) - \boldsymbol{\varphi}\, sin(\,\theta) \tag{1}$$

$$x_2 = \rho\, sin(\,\theta) + \boldsymbol{\varphi}\, cos(\,\theta) \tag{2}$$

$$x_3 = z \tag{3}$$

7.1.4 Directed line element

$$dl \;=\; dx\,\frac{\boldsymbol{x}}{|\boldsymbol{x}|} = \boldsymbol{e_\rho} d\rho \;+ \boldsymbol{e_\varphi} \rho d\varphi \;+ \boldsymbol{e_z} dz \tag{1}$$

7.1.5 Solid angle element

$$d\Omega = \sin(\theta)\, d\theta\, d\varphi \tag{1}$$

7.1.6 Directed area element

$$d\boldsymbol{S} = \boldsymbol{e_r}\, r^2\, d\Omega + \boldsymbol{e_\theta}\, r\, sin(\theta)\, dr\, d\varphi + \boldsymbol{e_\varphi}\; r\, dr\, d\theta \tag{1}$$

7.1.7 Volume element

$$dV = dx^3 = dr\, r^2\, d\Omega \tag{1}$$

7.1.8 Spatial differential operators

$$\alpha = \alpha(r, \theta, \varphi) \tag{1}$$
$$\boldsymbol{a} = \boldsymbol{a}(r, \theta, \varphi) \tag{2}$$

Gradient

$$\nabla\alpha = \boldsymbol{e_r}\, \frac{\partial\alpha}{\partial r} + \boldsymbol{e_\theta}\, \frac{1}{r}\frac{\partial\alpha}{\partial\theta} + \mathbf{e_\varphi}\frac{1}{r\, sin(\theta)}\frac{\partial\alpha}{\partial\varphi} \tag{3}$$

Divergence

$$\langle\boldsymbol{\nabla}, \boldsymbol{a}\rangle = \frac{1}{r^2}\frac{\partial(r^2\alpha_r)}{\partial r} + \frac{1}{r\, sin(\theta)}\frac{\partial(a_\theta\, sin(\theta))}{\partial\theta} + \frac{1}{r\, sin(\theta)}\frac{\partial a_\varphi}{\partial\varphi} \tag{4}$$

Curl

$$\boldsymbol{\nabla}\times\boldsymbol{a} = \boldsymbol{e_r}\, \frac{1}{r\, sin(\theta)}\left(\frac{\partial\left(a_\varphi\, sin(\theta)\right)}{\partial\theta} - \frac{\partial a_\varphi}{\partial\varphi}\right) + \boldsymbol{e_\theta}\, \frac{1}{r}\left(\frac{1}{sin(\theta)}\frac{\partial\alpha_r}{\partial\varphi} - \frac{\partial a_\varphi}{\partial r}\right) + \boldsymbol{e_\varphi}\frac{1}{r}\left(\frac{\partial r\, a_\varphi}{\partial r} - \frac{\partial a_r}{\partial\theta}\right) \tag{5}$$

The Laplacian

(6)

$$\nabla^2\alpha = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\alpha}{\partial r}\right) + \frac{1}{r^2\,sin(\theta)}\frac{\partial}{\partial\theta}\left(sin(\theta)\frac{\partial\alpha}{\partial\theta}\right) + \frac{1}{r^2\,sin^2(\theta)}\frac{\partial^2\alpha}{\partial\varphi^2}$$

7.2 Polar coordinates

The equivalent to rectangular coordinates in quaternion space is (a_τ, a_x, a_y, a_z)

$$a = a_\tau + \boldsymbol{i}\,a_x + \boldsymbol{j}\,a_y \pm \boldsymbol{i}\boldsymbol{j}\,a_z \tag{1}$$

The equivalent to polar coordinates in quaternion space is

$$\mathrm{a}_\tau = \|\mathrm{a}\|\cos(\psi) \tag{2}$$

$$\mathrm{a}_\mathrm{x} = \|\mathrm{a}\|\sin(\psi)\sin(\theta)\cos(\varphi) \tag{3}$$

$$\mathrm{a}_\mathrm{y} = \|\mathrm{a}\|\sin(\psi)\sin(\theta)\sin(\varphi) \tag{4}$$

$$a_z = \|a\|\,sin(\psi)\,cos(\theta) \tag{5}$$

$sin(\psi)$, where $\psi = (0,\pi)$, is known as the (imaginary) amplitude of the quaternion.

Angle $\theta = (0,\pi)$ is the (co-)latitude and angle $\varphi = (0,2\pi)$ is the longitude.

For any fixed value of ψ, θ and φ parameterize a 2-sphere of radius $sin(\psi)$, except for the degenerate cases, when ψ equals 0 or π, in which case they describe a point.

This suggests the following structure of the argument $\boldsymbol{\Lambda} = \tilde{\mathrm{i}} \cdot \psi$

$$a = \|a\| \, exp(\tilde{\imath} \cdot \psi) \tag{6}$$

$$= \|a\| \, (cos(\psi) + \tilde{\imath} \, sin(\psi)) \tag{7}$$

$$= a_\tau + \|a\| \, \tilde{\imath} \, sin(\psi) = a_\tau + \boldsymbol{a} \tag{8}$$

The imaginary number ĩ may take any direction. This shows that for quaternions exponential functions only work for (local) abstractions to complex number sub-systems. It also means that the notions of Lie groups works in complex number systems, but not in general in quaternionic number systems.

7.3 3 sphere

A 3-sphere is a compact, connected, 3-dimensional manifold without boundary. It is also simply-connected. What this means, loosely speaking, is that any loop, or circular path, on the 3-sphere can be continuously shrunk to a point without leaving the 3-sphere. The Poincaré conjecture[92] proposes that the 3-sphere is the only three dimensional manifold with these properties (up to homeomorphism)[93].

The round metric on the 3-sphere in these coordinates is given by

$$ds^2 = d\psi^2 + sin^2(\psi) \, (d\theta^2 + sin^2(\theta) d\varphi^2) \tag{1}$$

The volume form is given by

$$dV = sin^2(\psi) \, sin(\theta) \, d\psi \wedge d\theta \wedge d\varphi \tag{2}$$

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[92] http://en.wikipedia.org/wiki/Poincar%C3%A9_conjecture

[93] http://en.wikipedia.org/wiki/3-sphere

The 3-dimensional volume (or **hyperarea**) of a 3-sphere of radius r is

$$2\,\pi^2\,r^3 \tag{3}$$

The 4-dimensional **hypervolume** (the volume of the 4-dimensional region bounded by the 3-sphere) is

$$½\,\pi^2\,r^4 \tag{4}$$

The 3-sphere has constant positive sectional curvature equal to $1/r^2$.

The 3-sphere has a natural Lie group structure SU(2) given by quaternion multiplication.

The 3-sphere admits non-vanishing vector fields (sections of its tangent bundle). One can even find three linearly-independent and non-vanishing vector fields. These may be taken to be any left-invariant vector fields forming a basis for the Lie algebra of the 3-sphere. This implies that the 3-sphere is parallelizable. It follows that the tangent bundle of the 3-sphere is trivial.

There is an interesting action of the circle group $\mathbb{T}$ on $\mathbb{S}^3$ giving the 3-sphere the structure of a principal circle bundle known as the Hopf bundle. If one thinks of $\mathbb{S}^3$ as a subset of $\boldsymbol{C}^2$, the action is given by

$$(z_1, z_2)\,\lambda\; =\; (z_1\,\lambda, z_2\,\lambda)\;\forall_{\lambda\,\in\,\mathbb{T}}. \tag{5}$$

The orbit space of this action is homeomorphic to the two-sphere $\mathbb{S}^2$. Since $\mathbb{S}^3$ is not homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, the Hopf bundle is nontrivial.

7.4 Hopf coordinates

Another choice of hyperspherical coordinates, (η, ξ_1, ξ_2), makes use of the embedding of $\mathbb{S}^3$ in $\boldsymbol{C}^2$. In complex coordinates $(z_1, z_2) \in \boldsymbol{C}^2$ we write

$$z_1 = exp(\tilde{\imath}\, \xi_1)\, sin(\eta) \tag{1}$$

$$z_2 = exp(\tilde{\imath}\, \xi_2)\, cos(\eta) \tag{2}$$

Here η runs over the range 0 to $\pi/2$, and ξ_1 and ξ_2 can take any values between 0 and 2π. These coordinates are useful in the description of the 3-sphere as the Hopf bundle

$$\mathbb{S}^1 \rightarrow \mathbb{S}^3 \rightarrow \mathbb{S}^2 \tag{3}$$

For any fixed value of η between 0 and $\pi/2$, the coordinates (ξ_1, ξ_2) parameterize a 2-dimensional torus. In the degenerate cases, when η equals 0 or $\pi/2$, these coordinates describe a circle.

The round metric on the 3-sphere in these coordinates is given by

$$ds^2 = d\eta^2 + sin^2(\eta)\, (d\zeta_1^2 + cos^2(\eta)\, d\, \zeta_2^2) \tag{4}$$

and the volume form by

$$dV = sin(\eta)\, cos(\eta)\, d\eta \hat{} d\zeta_1 \hat{} d\zeta_2 \tag{5}$$

7.5 Group structure

Because the set of unit quaternions is closed under multiplication, $\mathbb{S}^3$ takes on the structure of a group. Moreover, since quaternionic multiplication is smooth, $\mathbb{S}^3$ can be regarded as a real Lie group. It is a non-abelian, compact Lie group of dimension 3. When thought of as a Lie group $\mathbb{S}^3$ is often denoted $Sp(1)$ or $\mathrm{U}(1, \mathbb{H})$.

It turns out that the only spheres which admit a Lie group structure are $\mathbb{S}^1$, thought of as the set of unit complex numbers, and $\mathbb{S}^3$, the set of unit quaternions. One might think that $\mathbb{S}^7$, the set of unit octonions, would form a Lie group, but this fails since octonion multiplication is non-associative. The octonionic structure does give $\mathbb{S}^7$ one important property: *parallelizability*[94]. It turns out that the only spheres which are parallelizable are $\mathbb{S}^1$, $\mathbb{S}^3$, and $\mathbb{S}^7$.

By using a matrix representation of the quaternions, $\mathbb{H}$, one obtains a matrix representation of $\mathbb{S}^3$. One convenient choice is given by the Pauli matrices:

$$\left(a_\tau + a_x \cdot \mathbf{i} + a_y \cdot \mathbf{j} + a_z \cdot \mathbf{k}\right) = \begin{bmatrix} a_\tau + \tilde{\imath} \cdot a_x & a_y + \tilde{\imath} \cdot a_z \\ -a_y + \tilde{\imath} \cdot a_z & a_\tau - \tilde{\imath} \cdot a_x \end{bmatrix} \tag{1}$$

This map gives an injective algebra homomorphism from $\mathbf{H}$ to the set of 2×2 complex matrices. It has the property that the absolute value of a quaternion q is equal to the square root of the determinant of the matrix image of q.

The set of unit quaternions is then given by matrices of the above form with unit determinant. This matrix subgroup is precisely the special unitary group SU(2). Thus, $\mathbb{S}^3$ as a Lie group is isomorphic to SU(2).

Using our hyperspherical coordinates (η, ξ_1, ξ_2) we can then write any element of SU(2) in the form

$$\begin{bmatrix} \exp(\tilde{\imath} \cdot \xi_1) \cdot \sin(\eta) & \exp(\tilde{\imath} \cdot \xi_2) \cdot \cos(\eta) \\ -\exp(\tilde{\imath} \cdot \xi_2) \cdot \cos(\eta) & \exp(-\tilde{\imath} \cdot \xi_1) \cdot \sin(\eta) \end{bmatrix} \tag{2}$$

[94] http://en.wikipedia.org/wiki/Parallelizability

Another way to state this result is if we express the matrix representation of an element of SU(2) as a linear combination of the Pauli matrices. It is seen that an arbitrary element U ∈ SU(2) can be written as

$$U = \alpha_\tau \cdot 1 + \sum_{n=x,y,z} \alpha_n \boldsymbol{I_n} \tag{3}$$

The condition that the determinant of U is +1 implies that the coefficients α_n are constrained to lie on a 3-sphere.

7.6 Versor

Any **unit quaternion** u can be written as a **versor**:

$$u = exp(\tilde{\imath}\ \psi) = cos(\psi) + \tilde{\imath}\ sin(\psi) \tag{1}$$

This is the quaternionic analogue of Euler's formula. Now the unit imaginary quaternions all lie on the unit 2-sphere in Im $\mathbb{H}$ so any such $\tilde{\mathbf{i}}$ can be written:

$$\tilde{\imath} = \boldsymbol{i}\, cos(\varphi)\, sin(\theta) + \boldsymbol{j}\, sin(\varphi)\, sin(\theta) + \boldsymbol{k}\, cos(\theta) \tag{2}$$

7.7 Symplectic decomposition

Quaternions can be written as the combination of two complex numbers and an imaginary number k with unit length.

$q = a + b\boldsymbol{j}$; where $a = w + x\boldsymbol{i}$; and $b = y + z\boldsymbol{i}$

$$q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

7.8 Quaternionic algebra

$$a = (a_0, a_1, a_2, a_3) = \sum_{\mu=0}^{3} e_\mu \, \mathrm{a}_\mu$$

$$= a_0 + \boldsymbol{i}\, a_1 + \boldsymbol{j}\, a_2 + \boldsymbol{k}\, a_3 = \; a_0 + \boldsymbol{a}$$

$$a^* = a_0 - \boldsymbol{a} \tag{2}$$

$$a^* a = a\, a^* = |a|^2 \tag{3}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_{\mu=1}^{3} a_\mu \, b_\mu = \delta_{\mu\nu} a_\mu b_\nu = |\boldsymbol{a}||\boldsymbol{b}| \cos(\theta) \tag{4}$$

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} = \pm\left(\in_{\boldsymbol{ijk}} \; \boldsymbol{e_i} a_j b_k\right) \tag{5}$$

$$a\, b = \; a_0 \boldsymbol{b} + \; b_0 \, \boldsymbol{a} - \langle \boldsymbol{a}, \boldsymbol{b} \rangle \pm \boldsymbol{a} \times \boldsymbol{b} \tag{6}$$

The colored $\pm$ indicates the handedness of the vector cross product.

$$\boldsymbol{a}\, \boldsymbol{b} = \; -\langle \boldsymbol{a}, \boldsymbol{b} \rangle \pm \boldsymbol{a} \times \boldsymbol{b} \tag{7}$$

$$a\,(b + c) = a\, b + a\, c \tag{8}$$

$$(a \; + \; b)\, c \; = \; a\, c \; + \; b\, c \tag{9}$$

$$(a\, b) c \; = \; a(b\, c) \tag{1}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \times \boldsymbol{c} \rangle = \; \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{c} \rangle \tag{1}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \langle \boldsymbol{a}, \boldsymbol{c} \rangle - \boldsymbol{c} \langle \boldsymbol{a}, \boldsymbol{b} \rangle \tag{1}$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{b} \langle \boldsymbol{a}, \boldsymbol{c} \rangle - \boldsymbol{a} \langle \boldsymbol{b}, \boldsymbol{c} \rangle \tag{1}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = 0 \tag{1}$$

$$\langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{c} \times \boldsymbol{d} \rangle = \langle \boldsymbol{a}, \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d}) \rangle \tag{1}$$

$$= \langle \boldsymbol{a}, \boldsymbol{c} \rangle \langle \boldsymbol{b}. \boldsymbol{d} \rangle - \langle \boldsymbol{a}, \boldsymbol{d} \rangle \langle \boldsymbol{b}. \boldsymbol{c} \rangle$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{d} \rangle \boldsymbol{c} - \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{c} \rangle \boldsymbol{d} \tag{16}$$

8 Quaternionic distributions

We consider four kinds of quaternionic distributions

A. Distributions of rational quaternions with a discrete parameter space. That parameter space must be flat and it is spanned by the rational quaternions[95]. The A-type quaternionic distribution has a countable set of values.
B. Distributions of rational quaternions with a continuous parameter space. That parameter space may be curved. The curvature is defined by a continuous quaternionic function. The B-type quaternionic distribution has a countable set of values. It inherits the sign flavor of the quaternionic function that defines the curvature of its parameter space.
C. Continuous quaternionic distributions with a continuous parameter space. That parameter space may be curved. The curvature is defined by a continuous quaternionic function. The C-type quaternionic distributions inherit the sign flavor of the quaternionic distribution that defines the curvature of their parameter space. The C-type quaternionic function can be split in a real scalar function and a real 3D vector

[95] http://en.wikipedia.org/wiki/Quaternion_algebra#Quaternion_algebras_over_the_rational_numbers

function. The real scalar function can be interpreted as an object density distribution and the 3D vector function can be interpreted as the associated current density distribution.

D. A convolution of a continuous quaternionic function with a discrete flat parameter space consisting of rational quaternions and a 3D stochastic generator of rational quaternionic target values. The D-type quaternionic distribution has a countable set of values. It corresponds to a collection of coherent B-type distributions, where the continuous function generates the curvature of the parameter space for the B-type distributions, which are generated by the stochastic process.

8.1 Basic properties of continuous quaternionic distributions

For simplicity we confine to quaternionic distributions with flat parameter space. A continuous quaternionic distribution contains a scalar field in its real part and a vector field in its imaginary part.

$$f(x) = f_0(x) + \boldsymbol{f}(x) \tag{3}$$

$$a\,f(x) = a_0\boldsymbol{f}(x) + f_0(x)\,\boldsymbol{a} - \langle \boldsymbol{a}, \boldsymbol{f}(x)\rangle \pm \boldsymbol{a} \times \boldsymbol{f}(x) \tag{2}$$

$$f(x)\,b = f_0(x)\boldsymbol{b} + b_0\,\boldsymbol{f}(x) - \langle \boldsymbol{f}(x), \boldsymbol{b}\rangle \pm \boldsymbol{f}(x) \times \boldsymbol{b} \tag{3}$$

The distributions follow the rules for the quaternion algebra.

$$a\,(f(x) + g(x)) = a\,f(x) + a\,g(x) \tag{4}$$

$$(a + b)f(x) = a\,f(x) + b\,f(x) \tag{5}$$

(6)

$$f(x)\,g(x) = f_0(x)\boldsymbol{g}(x) + g_0(x)\boldsymbol{f}(x) - \langle \boldsymbol{f}(x), \boldsymbol{g}(x)\rangle \pm \boldsymbol{f}(x) \times \boldsymbol{g}(x)$$

$$\big(f(x)g(x)\big)h(x) = f(x)(g(x)\,h(x)) \tag{7}$$

8.1.1 Symmetries

Continuous quaternionic distributions keep the same discrete symmetries (sign flavor) throughout their domain. The sign flavor of the parameter space acts as reference sign flavor.

8.1.2 Differentials

The quaternionic nabla acts similarly as a normal quaternion

$$\nabla\,(f(x) + g(x)) = \nabla\, f(x) + \nabla\, g(x) \tag{1}$$

$$\nabla\, f(x) = \nabla_0 \boldsymbol{f}(x) + \boldsymbol{\nabla} f_0(x) - \langle \boldsymbol{\nabla}, \boldsymbol{f}(x)\rangle \pm \boldsymbol{\nabla} \times \boldsymbol{f}(x) \tag{2}$$

However

$$\nabla(b\,c) \neq (\nabla\, b)c \tag{3}$$

and

$$\nabla(b\,c) \neq (\nabla\, b)c + b\,\nabla\, c \tag{4}$$

Further

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle \alpha \equiv \boldsymbol{\nabla}^2 \alpha \tag{5}$$

$$\langle \boldsymbol{\nabla} \times \boldsymbol{\nabla}, \mathbf{a}\rangle = 0 \tag{6}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{a}\rangle = 0 \tag{7}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\alpha = \mathbf{0} \tag{8}$$

$$\boldsymbol{\nabla}\, \boldsymbol{b} = -\langle \boldsymbol{\nabla}, \boldsymbol{b}\rangle \pm \boldsymbol{\nabla} \times \boldsymbol{b} \tag{9}$$

$$\boldsymbol{\nabla}\,(\alpha\,\beta) = \alpha \boldsymbol{\nabla}\,\beta + \beta \boldsymbol{\nabla}\,\alpha \tag{1}$$

$$\boldsymbol{\nabla}\,(\alpha\,\boldsymbol{a}) = \alpha \boldsymbol{\nabla} \times \boldsymbol{a} - \alpha\langle \boldsymbol{\nabla}, \boldsymbol{a}\rangle + (\boldsymbol{\nabla}\,\alpha)\boldsymbol{a} \tag{1}$$

$$\langle \boldsymbol{\nabla}, \alpha\,\boldsymbol{a}\rangle = \boldsymbol{a}\boldsymbol{\nabla}\alpha + \alpha\langle \boldsymbol{\nabla}, \boldsymbol{a}\rangle \tag{1}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{a} \times \boldsymbol{b}\rangle = \langle \boldsymbol{b}, \boldsymbol{\nabla} \times \boldsymbol{a}\rangle - \langle \boldsymbol{a}, \boldsymbol{\nabla} \times \boldsymbol{b}\rangle \tag{1}$$

$$\langle \boldsymbol{\nabla}\,\alpha, \boldsymbol{\nabla}\,\beta\rangle = \langle \boldsymbol{\nabla}, \alpha \boldsymbol{\nabla}\,\beta\rangle - \boldsymbol{\alpha}\boldsymbol{\nabla}^2\beta \tag{1}$$

$$\langle \boldsymbol{\nabla}\,\alpha, \boldsymbol{\nabla} \times \boldsymbol{a}\rangle = -\boldsymbol{\nabla}\,, \boldsymbol{a} \times \boldsymbol{\nabla}\alpha \tag{1}$$

$$\langle \boldsymbol{\nabla} \times\, \boldsymbol{a}, \boldsymbol{\nabla} \times\, \boldsymbol{b}\rangle = \langle \boldsymbol{b}, \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a})\rangle - \langle \boldsymbol{a}, \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{b})\rangle \tag{1}$$

$$\boldsymbol{\nabla} \times (\alpha\mathbf{a}) = \alpha \boldsymbol{\nabla} \times \boldsymbol{a} - \boldsymbol{a} \times \boldsymbol{\nabla}\alpha \tag{1}$$

$$\boldsymbol{\nabla} \times (\alpha \boldsymbol{\nabla} \beta) = (\boldsymbol{\nabla} \alpha) \times \nabla \beta \qquad (1$$

9 The separable Hilbert space Ң

We will specify the characteristics of a generalized quaternionic infinite dimensional separable Hilbert space. The adjective "quaternionic" indicates that the inner products of vectors and the eigenvalues of operators are taken from the number system of the quaternions. Separable Hilbert spaces can be using real numbers, complex numbers or quaternions. These three number systems are division rings.

9.1 Notations and naming conventions

$\{f_x\}_x$ means ordered set of f_x . It is a way to define functions.

The use of bras and kets differs slightly from the way Dirac uses them.

|f> is a ket vector, f> is the same ket
<f| is a bra vector, <f is the same bra

A is an operator.
|A is the same operator
A† is the adjoint operator of operator A.
A| is the same operator as A†
| on its own, is a nil operator
|A| is a self-adjoint (Hermitian) operator

We will use capitals for operators and lower case for quaternions, eigenvalues, ket vectors, bra vectors and eigenvectors. Quaternions and eigenvalues will be indicated with *italic* characters. Imaginary and anti-Hermitian objects are often underlined and/or indicated in **bold** text.

$\sum_k$ means: sum over all items with index k.

$\int_x$ means: integral over all items with parameter x.

9.2 Quaternionic Hilbert space

The Hilbert space is a **linear space**. That means for the elements |f>, |g> and |h> and numbers *a* and *b*:

9.2.1 Ket vectors

For **ket** vectors hold

$$|f\rangle + |g\rangle = |g\rangle + |f\rangle = |g + f\rangle \tag{1}$$

$$(|f\rangle + |g\rangle) + |h\rangle = |f\rangle + (|g\rangle + |h\rangle) \tag{2}$$

$$|(a + b)\, f\rangle = |f\rangle \cdot a + |f\rangle \cdot b \tag{3}$$

$$(|f\rangle + |g\rangle) \cdot a = |f\rangle \cdot a + |g\rangle \cdot a \tag{4}$$

$$|f\rangle \cdot 0 = |0\rangle \tag{5}$$

$$|f\rangle \cdot 1 = |f\rangle \tag{6}$$

Depending on the number field that the Hilbert space supports, *a* and *b* can be real numbers, complex numbers or (real) quaternions.

9.2.2 Bra vectors

The **bra** vectors form the dual Hilbert space $\mathbf{H}^\dagger$ of $\mathbf{H}$.

$$\langle f| + \langle g| = \langle g| + \langle f| = |g + f\rangle \tag{1}$$

$$(\langle f| + \langle g|) + \langle h| = \langle f| + (\langle g| + \langle h|) \tag{2}$$

$$\langle f\,(a + b)\rangle = \langle f| \cdot a + \langle f| \cdot b = a^* \cdot \langle f| + b^* \cdot \langle f|$$

(3)

$$(\langle f| + \langle g|)\cdot a = \langle f|\cdot a + \langle g|\cdot a = a^*\cdot\langle f| + a^*\cdot\langle g| \quad (4)$$

$$0\cdot\langle f| = \langle 0| \quad (5)$$

$$1\cdot\langle f| = \langle f| \quad (6)$$

9.2.3 Scalar product

The Hilbert space contains a **scalar product,** also called **inner product,** $\langle f|g\rangle$ that combines $\mathbf{H}$ and $\mathbf{H}^\dagger$ in a direct product that we also indicate with $\mathbf{H}$.

For Hilbert spaces the values of inner products are restricted to elements of a division ring.

The scalar product $\langle f|g\rangle$ satisfies:

$$\langle f|g + h\rangle = \langle f|g\rangle + \langle f|h\rangle \quad (1)$$

$$\langle f|\{|g\rangle\cdot a\}_g = \{\langle f|g\rangle\}_g\cdot a \quad (2)$$

With each ket vector $|g\rangle$ in $\mathbf{H}$ belongs a bra vector $\langle g|$ in $\mathbf{H}^\dagger$ such that for all bra vectors $\langle f|$ in $\mathbf{H}^\dagger$

$$\langle f|g\rangle = \langle g|f\rangle^* \quad (3)$$

$$\langle f|f\rangle = 0 \text{ when } |f\rangle = |0\rangle \quad (4)$$

$$\langle f|a\ g\rangle = \langle f|g\rangle\cdot a = \langle g|f\rangle^*\cdot a = \langle g\ a|f\rangle^* = (a^*\cdot\langle g|f\rangle)^* = \langle f|g\rangle\cdot a \quad (5)$$

In general is <f|a g> ≠ <f a|g>. However for real numbers r holds <f|r g> = <f r|g>

Remember that when the number field consists of quaternions, then also <f|g> is a quaternion and a quaternion q and <f|g> do in general not commute.

The scalar product defines a **norm**:

$$||f|| = \sqrt{(<f|f>)} \tag{6}$$

And a **distance**:

$$D(f,g) = ||f - g|| \tag{7}$$

The Hilbert space **H** is closed under its norm. Each converging row of elements of converges to an element of this space.

9.2.4 Separable

In mathematics a topological space is called separable if it contains a countable dense subset; that is, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence.

Every continuous function on the separable space **H** is determined by its values on this countable dense subset.

9.2.5 *Base vectors*

The Hilbert space **H** is **separable**. That means that a countable row of elements $\{f_n>\}$ exists that **spans** the whole space.

If $\langle f_n|f_m\rangle = \delta(m,n) = [1 \text{ when } n = m;\ 0 \text{ otherwise}]$
then $\{|f_n\rangle\}$ forms an **orthonormal base** of the Hilbert space.

A ket base $\{|k\rangle\}$ of $\mathbf{H}$ is a minimal set of ket vectors $|k\rangle$ that together span the Hilbert space $\mathbf{H}$.

Any ket vector $|f\rangle$ in $\mathbf{H}$ can be written as a linear combination of elements of $\{|k\rangle\}$.

$$|f\rangle = \sum_k (|k\rangle \cdot \langle k|f\rangle) \tag{1}$$

A bra base $\{\langle b|\}$ of $\mathbf{H}^\dagger$ is a minimal set of bra vectors $\langle b|$ that together span the Hilbert space $\mathbf{H}^\dagger$.

Any bra vector $\langle f|$ in $\mathbf{H}^\dagger$ can be written as a linear combination of elements of $\{\langle b|\}$.

$$\langle f| = \sum_b (\langle f|b\rangle \cdot \langle b|) \tag{2}$$

Usually base vectors are taken such that their norm equals 1. Such a base is called an othonormal base.

9.2.6 *Operators*

Operators act on a subset of the elements of the Hilbert space.

9.2.6.1 Linear operators

An operator Q is linear when for all vectors $|f\rangle$ and $|g\rangle$ for which Q is defined and for all quaternionic numbers a and b:

$$|Q\cdot a\, f\rangle + |Q\cdot b\, g\rangle = |a\cdot Q\, f\rangle + |b\cdot Q\, g\rangle = |Q\, f\rangle\cdot a + |Q\, g\rangle\cdot b \tag{1}$$

=

$$Q\,(|f\rangle\cdot a + |g\rangle\cdot b) = Q\,(|a\ f\rangle + |b\ g\rangle) \tag{2}$$

B is **colinear** when for all vectors $|f\rangle$ for which B is defined and for all quaternionic numbers a there exists a quaternionic number c such that:

$$|B\cdot a\ f\rangle = |a\cdot B\ f\rangle = |B\ f\rangle\ c\cdot a\cdot c^{-1} \tag{3}$$

If $|f\rangle$ is an eigenvector of operator A with quaternionic eigenvalue a, then is $|b\ f\rangle$ an eigenvector of A with quaternionic eigenvalue $b\cdot a\cdot b^{-1}$.

$A| = A^{\dagger}$ is the **adjoint** of the **normal** operator A. $|A$ is the same as A.

$$\langle f\ A|\ g\rangle = \langle fA^{\dagger}|g\rangle^{*} \tag{4}$$

$$A^{\dagger\,\dagger} = A \tag{5}$$

$$(A\cdot B)^{\dagger} = B^{\dagger}\cdot A^{\dagger} \tag{6}$$

$|B|$ is a **self adjoint** operator.

$|$ is a nil operator.

The construct $|f\rangle\langle g|$ acts as a linear operator. $|g\rangle\langle f|$ is its adjoint operator.

$$\sum_n \{|f_n\rangle\cdot a_n\cdot\langle f_n|\}, \tag{7}$$

where a n is real and acts as a density function.

The set of eigenvectors of a normal operator form an orthonormal base of the Hilbert space.

A self adjoint operator has real numbers as eigenvalues.

$\{<q|f>\}_q$ is a function f(q) of parameter q.
$\{<g|q>\}_q$ is a function g(q) of parameter q.

When possible, we use the same letter for identifying eigenvalues, eigenvalues and the corresponding operator.

So, usually |q> is an eigenvector of a normal operator Q with eigenvalues q.

{q} is the set of eigenvalues of Q.
$\{q\}_q$ is the ordered field of eigenvalues of q.
$\{|q>\}_q$ is the ordered set of eigenvectors of Q.
$\{<q|f>\}_q$ is the **Q view** of |f>.

9.2.6.2 Normal operators

The most common definition of continuous operators is:

A **continuous** operator is an operator that creates images such that the inverse images of open sets are open.

Similarly, a **continuous** operator creates images such that the inverse images of closed sets are closed.

If |a> is an eigenvector of normal operator A with eigenvalue a then

$$< a|A|a > = < a|a|a > = < a|a > \ a$$

indicates that the eigenvalues are taken from the same number system as the inner products.

A normal operator is a continuous linear operator.

A normal operator in **Ң** creates an image of **Ң** onto **Ң.** It transfers closed subspaces of **Ң** into closed subspaces of **Ң**.

Normal operators represent continuous quantum logical observables.

The normal operators N have the following property.

$$N: \mathbf{Ң} \Rightarrow \mathbf{Ң} \tag{1}$$

N commutes with its **(Hermitian) adjoint** $N^{\dagger}$

$$N \cdot N^{\dagger} = N^{\dagger} \cdot N \tag{2}$$

Normal operators are important because the spectral theorem holds for them.

Examples of normal operators are

- **unitary** operators: $U^{\dagger} = U^{-1}$, unitary operators are bounded;
- **Hermitian** operators (i.e., self-adjoint operators): $N^{\dagger} = N$;
- **Anti-Hermitian** or anti-self-adjoint operators: $N^{\dagger} = -N$;
- **Anti-unitary** operators: $I^{\dagger} = -I = I^{-1}$, anti-unitary operators are bounded;
- **positive operators**: $N = MM^{\dagger}$
- **orthogonal projection** operators: $N = N^{\dagger} = N^2$

9.2.6.3 Spectral theorem

For every compact self-adjoint operator *T* on a real, complex or quaternionic Hilbert space **Ң**, there exists an orthonormal basis of **Ң** consisting of eigenvectors of T. More specifically, the orthogonal

complement of the kernel (null space) of T admits, either a finite orthonormal basis of eigenvectors of T, or a countable infinite orthonormal basis {en} of eigenvectors of T, with corresponding eigenvalues $\{\lambda_n\} \subset R$, such that $\lambda_n \to 0$. Due to the fact that $\mathbf{H}$ is separable the set of eigenvectors of T can be extended with a base of the kernel in order to form a complete orthonormal base of $\mathbf{H}$.

If T is compact on an infinite dimensional Hilbert space $\mathbf{H}$, then T is not invertible, hence $\sigma(T)$, the spectrum of T, always contains 0. The spectral theorem shows that $\sigma(T)$ consists of the eigenvalues $\{\lambda_n\}$ of T, and of 0 (if 0 is not already an eigenvalue). The set $\sigma(T)$ is a compact subset of the real line, and the eigenvalues are dense in $\sigma(T)$.

A normal operator has a set of eigenvectors that spans the whole Hilbert space $\mathbf{H}$.

In quaternionic Hilbert space a normal operator has quaternions as eigenvalues.

The set of eigenvalues of a normal operator is NOT compact. This is due to the fact that $\mathbf{H}$ is separable. Therefore the set of eigenvectors is countable. As a consequence the set of eigenvalues is countable. Further, in general the eigenspace of normal operators has no finite diameter.

A continuous bounded linear operator on $\mathbf{H}$ has a compact eigenspace. The set of eigenvalues has a closure and it has a finite diameter.

9.2.6.4 Eigenspace

The set of eigenvalues $\{q\}$ of the operator Q form the eigenspace of Q

9.2.6.5 Eigenvectors and eigenvalues

For the eigenvector |q> of normal operator Q holds

$$|Q\ q\rangle = |q\ q\rangle = |q\rangle \cdot q \tag{1}$$

$$\langle q\ Q^{\dagger}| = \langle q\ q^{*}| = q^{*} \cdot \langle q| \tag{2}$$

$$\forall_{|f\rangle \in \mathrm{H}} [\{\langle f|Q\ q\rangle\}_q = \{\langle f|q > q\}_q = \{\langle q\ Q^{\dagger}|f >^{*}\}_q = \{q^{*} \langle q|f >^{*}\}_q] \tag{3}$$

The eigenvalues of 2^n-on normal operator are 2^n-ons. For Hilbert spaces the eigenvalues are restricted to elements of a division ring.

$$Q = \sum_{j=0}^{n-1} \mathrm{I}_j Q_i \tag{4}$$

The Q_j are self-adjoint operators.

9.2.6.6 Generalized Trotter formula

For bounded operators $\{A_j\}$ hold:

$$\lim_{n\to\infty}\left(\prod_{j=1}^{p} e^{A_j/n}\right)^n = \exp\left(\sum_{j=1}^{p} A_j\right) = \lim_{n\to\infty}\left(1+\frac{\sum_{j=1}^{p} A_j}{n}\right)^n \tag{1}$$

In general

$$\exp\left(\sum_{j=1}^{p} A_j\right) \neq \prod_{j=1}^{p} e^{A_j} \tag{2}$$

In the realm of quaternionic notion the Trotter formula is confusing.

9.2.6.7 Unitary operators

For unitary operators holds:

$$U^{\dagger} = U^{-1} \tag{1}$$

Thus

$$U\cdot U^{\dagger} = U^{\dagger}\cdot U = 1 \tag{2}$$

Suppose $U = I + C$ where U is unitary and C is compact. The equations $U\,U^* = U^*U = I$ and $C = U - I$ show that C is normal. The spectrum of C contains 0, and possibly, a finite set or a sequence

tending to 0. Since $U = I + C$, the spectrum of U is obtained by shifting the spectrum of C by 1.

The unitary transform can be expressed as:

$$U = \exp(\underline{\tilde{\mathbf{I}}} \cdot \Phi/\hbar) \tag{3}$$

$$\hbar = h/(2 \cdot \pi) \tag{4}$$

Φ is Hermitian. The constant h refers to the granularity of the eigenspace.

Unitary operators have eigenvalues that are located in the unity sphere of the 2^n-ons field.

The eigenvalues have the form:

$$u = \exp(\underline{\mathbf{i}} \cdot \varphi/\hbar) \tag{5}$$

φ is real. $\underline{\mathbf{i}}$ is a unit length imaginary number in 2^n-on space. It represents a direction.

u spans a sphere in 2^n-on space. For constant $\underline{\mathbf{i}}$, u spans a circle in a complex subspace.

9.2.6.7.1 Polar decomposition

Normal operators N can be split into a real operator A and a unitary operator U. U and A have the same set of eigenvectors as N.

$$N = ||N|| \cdot U = A \cdot U \tag{1}$$

$$N = A \cdot U = U \cdot A \tag{2}$$

$$= A \cdot \exp(\underline{\tilde{\mathbf{I}}} \cdot \Phi)/\hbar)$$

$$= \exp\left(\Phi_r + \underline{\tilde{\mathbf{I}}} \cdot \Phi\right)/\hbar)$$

Φ_r is a positive normal operator.

9.2.6.8 Ladder operator

9.2.6.8.1 General formulation

Suppose that two operators X and N have the commutation relation:

$$[N, X] = c \cdot X \tag{1}$$

for some scalar c. If $|n\rangle$ is an eigenstate of N with eigenvalue equation,

$$|N\, n\rangle = |n\rangle \cdot n \tag{2}$$

then the operator X acts on $|n\rangle$ in such a way as to shift the eigenvalue by c:

$$\begin{aligned} |N \cdot X\, n\rangle &= |(X \cdot N + [N, X])\, n\rangle \\ &= |(X \cdot N + c \cdot X)\, n\rangle \\ &= |X \cdot N\, n\rangle + |X\, n\rangle \cdot c \\ &= |X\, n\rangle \cdot n + |X\, n\rangle \cdot c \\ &= |X\, n\rangle \cdot (n+c) \end{aligned} \tag{3}$$

In other words, if $|n\rangle$ is an eigenstate of N with eigenvalue n then $|X\, n\rangle$ is an eigenstate of N with eigenvalue $n + c$.

The operator X is a *raising operator* for N if c is real and positive, and a *lowering operator* for N if c is real and negative.

If N is a Hermitian operator then c must be real and the Hermitian adjoint of X obeys the commutation relation:

$$[N, X^{\dagger}] = - c \cdot X^{\dagger} \tag{4}$$

In particular, if X is a lowering operator for N then $X^{\dagger}$ is a raising operator for N and vice-versa.

9.2.7 Unit sphere of Ң

The ket vectors in **Ң** that have their norm equal to one form together the **unit sphere** Θ of **Ң**.

Base vectors are all member of the unit sphere. The eigenvectors of a normal operator are all member of the unit sphere.

The end points of the eigenvectors of a normal operator form a **grid** on the unit sphere Θof **Ң**.

9.2.8 Bra-ket in four dimensional space

The Bra-ket formulation can also be used in transformations of the four dimensional curved spaces.

The bra $\langle f$ is then a covariant vector and the ket $g\rangle$ is a contravariant vector. The inner product acts as a metric.

$$s = \langle f|g\rangle \tag{1}$$

The effect of a linear transformation L is then given by

$$s_L = \langle f|Lg\rangle \tag{2}$$

The effect of a the transpose transformation $L^{\dagger}$ is then given by

$$\langle fL^{\dagger} \, | g \rangle = \langle f | Lg \rangle \tag{3}$$

For a unitary transformation U holds:

$$\langle Uf | Ug \rangle = \langle f | g \rangle \tag{4}$$

These definitions work for curved spaces with a Euclidian signature as well as for curved spaces with a Minkowski signature.

$$\langle \nabla f | \nabla g \rangle = \langle f | \nabla^2 \mathrm{g} \rangle = \langle f | \square \mathrm{g} \rangle \tag{5}$$

9.2.9 Closure

The closure of $\mathbf{H}$ means that converging rows of vectors converge to a vector of $\mathbf{H}$.

In general converging rows of eigenvalues of Q do not converge to an eigenvalue of Q.

Thus, the set of eigenvalues of Q is open.

At best the density of the coverage of the set of eigenvalues is comparable with the set of 2^n-ons that have rational numbers as coordinate values.

With other words, compared to the set of real numbers the eigenvalue spectrum of Q has holes.

The set of eigenvalues of operator Q includes 0. This means that Q does not have an inverse.

The rigged Hilbert space $\mathbf{H}$ can offer a solution, but then the direct relation with quantum logic is lost.

9.2.10 Canonical conjugate operator P

The existence of a canonical conjugate represents a stronger requirement on the continuity of the eigenvalues of canonical eigenvalues.

Q has eigenvectors $\{|q>\}_q$ and eigenvalues q.

P has eigenvectors $\{|p>\}_p$ and eigenvalues p.

For each eigenvector |q> of Q we define an eigenvector |p> and eigenvalues p of P such that:

$$< q|p > = < p|q >^* = exp\ (\hat{\imath} \cdot p \cdot q/\hbar) \tag{1}$$

$\hbar = h/(2\pi)$ is a scaling factor. $< q|p >$ is a quaternion. $\hat{\imath}$ is a unit length imaginary quaternion.

9.2.11 Displacement generators

Variance of the scalar product gives:

$$\boldsymbol{i}\,\hbar\,\delta < q|p > = -p < q|p > \delta q \tag{1}$$

$$\boldsymbol{i}\,\hbar\,\delta < p|q > = -q < p|q > \delta p \tag{2}$$

In the rigged Hilbert space $\mathbf{H}$ the variance can be replaced by differentiation.

Partial differentiation of the function <q|p> gives:

$$\boldsymbol{i}\,\hbar\,\partial/\partial q_s < q|p > = -p_s < q|p > \tag{3}$$

$$\boldsymbol{i}\,\hbar\frac{\partial}{\partial p_s} < p|q > = -q_s < p|q > \tag{4}$$

9.3 Quaternionic L² space

The space of quaternionic measurable functions is a separable quaternionic Hilbert space. For example quaternionic probability amplitude distributions are measurable.[96]

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[96] http://en.wikipedia.org/wiki/Lp_space#Lp_spaces

This space is spanned by an orthonormal basis of quaternionic measurable functions. The shared affine versions of the parameter space of these functions is called **Palestra**[97]. When the Palestra is non-curved, then this base has a canonical conjugate, which is the quaternionic Fourier transform of the original base.

As soon as curvature of the Palestra arises, this relation is disturbed.

With other words: “In advance the Palestra has a virgin state.”

[97] The name Palestra is suggested by Henning Dekant’s wive Sarah. It is a name from Greek antiquity. It is a public place for training or exercise in wrestling or athletics

10 Gelfand triple

The separable Hilbert space only supports countable orthonormal bases and countable eigenspaces. The rigged Hilbert space $\mathbf{H}$ that belongs to a separable Hilbert space $\mathbf{H}$ is a Gelfand triple. It supports non-countable orthonormal bases and continuum eigenspaces.

A rigged Hilbert space is a pair (H, Φ) with $\mathbf{H}$ a Hilbert space, Φ a dense subspace, such that Φ is given a topological vector space structure for which the inclusion map i is continuous. Its name is not correct, because it is not a Hilbert space.

Identifying $\mathbf{H}$ with its dual space $\mathbf{H}^*$, the adjoint to i is the map

$$i^*: \mathrm{H} = \mathrm{H}^* \to \Phi^* \tag{1}$$

The duality pairing between Φ and Φ^* has to be compatible with the inner product on $\mathbf{H}$, in the sense that:

$$\langle u, v \rangle_{\Phi \times \Phi^*} = (u, v)_{\mathrm{H}} \tag{2}$$

whenever $u \in \Phi \subset \mathrm{H}$ and $v \in \mathrm{H} = \mathrm{H}^* \subset \Phi^*$.

The specific triple $(\Phi \subset \mathrm{H} \subset \Phi^*)$ is often named after the mathematician Israel Gelfand).

Note that even though Φ is isomorphic to Φ^* if Φ is a Hilbert space in its own right, this isomorphism is *not* the

same as the composition of the inclusion i with its adjoint i^*

$$i^* i : \Phi \subset \text{Ң} = \text{Ң}^* \rightarrow \Phi^* \tag{3}$$

10.1 Understanding the Gelfand triple

The Gelfand triple of a real separable Hilbert space can be understood via the enumeration model of the real separable Hilbert space. This enumeration is obtained by taking the set of eigenvectors of a normal operator that has rational numbers as its eigenvalues. Let the smallest enumeration value of the rational enumerators approach zero. Even when zero is reached, then still the set of enumerators is countable. Now add all limits of converging rows of rational enumerators to the enumeration set. After this operation the enumeration set has become a continuum and has the same cardinality as the set of the real numbers. This operation converts the Hilbert space into its Gelfand triple and it converts the normal operator in a new operator that has the real numbers as its eigenspace. It means that the orthonormal base of the Gelfand triple that is formed by the eigenvectors of the new normal operator has the cardinality of the real numbers. It also means that linear operators in this Gelfand triple have eigenspaces that are continuums and have the cardinality of the real numbers[98]. The same reasoning holds for complex number based Hilbert spaces and quaternionic Hilbert spaces and their respective Gelfand triples.

[98] This story also applies to the complex and the quaternionic Hilbert spaces and their Gelfand triples.

11 Fourier transform

The Fourier transformation is a linear operator. This transform transfers functions to another parameter space. As a consequence the Fourier transform has no eigenvalues, but the Fourier transform knows functions that are invariant under Fourier transformation.

The Fourier transform cannot cope with functions that have curved parameter spaces. However, it is possible to reduce the parameter space to a domain in which the Fourier transform keeps acceptable accuracy. Another possibility is that the target function is flattened, such that its parameter space becomes flat.

The Fourier transform transfer a orthonormal set of base functions into a new a orthonormal set such that each member of the new set can be written as a linear combination of members of the old set such that none of the coefficients is zero. In fact all coefficients have the same norm.

The Fourier transform converts the nabla operator into an operator that does not differentiate but multiplies the converted function with a factor. That operator will be called a momentum operator.

The Fourier transform has an inverse. It turns the momentum operator into the nabla operator.

The Fourier transform converts convolution of two functions into the multiplication of the two functions and vice versa.

In order to simplify the discussion we restrict it to the case that the parameter spaces of the functions are not curved.

11.1 Fourier transform properties

11.1.1 Linearity

The Fourier transform is a linear operator

$$\mathcal{F}\big(g(q)\big) = \tilde{g}(p) \tag{1}$$

$$\mathcal{F}(a\ g(q) + b\ h(q)) = \ a\ \tilde{g}(p) + b\ \tilde{h}(p) \tag{2}$$

11.1.2 Differentiation

Fourier transformation converts differentiation into multiplication with the canonical conjugated coordinate.

$$\mathrm{g}(q) = \nabla f(q) \tag{1}$$

$$\tilde{\mathrm{g}}(p) = \mathrm{p}\tilde{f}(p) \tag{2}$$

$$\begin{aligned}\mathrm{g}(q) = \nabla f(q) = \ & \nabla_0 f_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \pm \nabla_0 \boldsymbol{f}(q) \\ & + \ \boldsymbol{\nabla} f_0(q) \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q) \right)\end{aligned} \tag{3}$$

$$\begin{aligned}\tilde{\mathrm{g}}(k) = \mathrm{k}\tilde{f}(k) = \ & \mathrm{k}_0 \tilde{f}_0(k) \mp \langle \mathbf{k}, \tilde{\boldsymbol{f}}(k) \rangle \pm \mathrm{k}_0 \tilde{\boldsymbol{f}}(k) \\ & + \ \mathbf{k} \tilde{f}_0(k) \pm \left(\pm \mathbf{k} \times \tilde{\boldsymbol{f}}(k) \right)\end{aligned} \tag{4}$$

For the imaginary parts holds:

$$\mathbf{g}(q) = \ \pm \nabla_0 \boldsymbol{f}(q) + \ \boldsymbol{\nabla} f_0(q) \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q) \right) \tag{5}$$

$$\tilde{\mathbf{g}}(k) = \pm \mathrm{k}_0 \tilde{\boldsymbol{f}}(k) + \ \mathbf{k} \tilde{f}_0(k) \pm \left(\pm \mathbf{k} \times \tilde{\boldsymbol{f}}(k) \right) \tag{6}$$

By using

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} f_0(q) = \mathbf{0} \tag{7}$$

and

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{f}(q) \rangle = 0 \tag{8}$$

It can be seen that for the static part ($\nabla_0 f(q) = 0$) holds:

$$\mathbf{g}(q) = \boldsymbol{\nabla} f_0(q) \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q) \right) \tag{9}$$

$$\tilde{\mathbf{g}}(k) = \mathbf{k} \tilde{f}_0(k) \pm \left(\pm \mathbf{k} \times \tilde{\boldsymbol{f}}(k) \right) \tag{1}$$

11.1.3 Parseval's theorem

Parseval's theorem runs:

$$\int f^*(q) \cdot g(q) \cdot dV_q = \int \tilde{f}^*(p) \cdot \tilde{g}(p) \cdot dV_p \tag{1}$$

This leads to

$$\int |f(q)|^2 \cdot dV_q = \int \left| \tilde{f}(p) \right|^2 \cdot dV_p \tag{2}$$

11.1.4 Convolution

Through Fourier transformation a convolution changes into a simple product and vice versa.

$$\mathcal{F}(f(q) \circ g(q)) = \tilde{f}(p) \cdot \tilde{g}(p) \tag{1}$$

11.2 Helmholtz decomposition

The Helmholtz decomposition splits the **static** vector field $\boldsymbol{F}$ in a (transversal) divergence free part $\boldsymbol{F_t}$ and a (one dimensional longitudinal) rotation free part $\boldsymbol{F_l}$.

$$\boldsymbol{F} = \boldsymbol{F_t} + \boldsymbol{F_l} = \boldsymbol{\nabla} \times \boldsymbol{f} - \boldsymbol{\nabla} f_0 \tag{1}$$

Here f_0 is a scalar field and $\boldsymbol{f}$ is a vector field. In quaternionic terms f_0 and $\boldsymbol{f}$ are the real and the imaginary part of a quaternionic field f. $\boldsymbol{F}$ is an imaginary quaternionic distribution.

The significance of the terms "longitudinal" and "transversal" can be understood by computing the local three-dimensional Fourier transform of the vector field $\boldsymbol{F}$, which we call $\widetilde{\boldsymbol{F}}$. Next decompose this field, at each point $\boldsymbol{k}$, into two components, one of which points longitudinally, i.e. parallel to $\boldsymbol{k}$, the other of which points in the transverse direction, i.e. perpendicular to $\boldsymbol{k}$.

$$\widetilde{\boldsymbol{F}}(\boldsymbol{k}) = \widetilde{\boldsymbol{F}}_{\boldsymbol{l}}(\boldsymbol{k}) + \widetilde{\boldsymbol{F}}_{\boldsymbol{t}}(\boldsymbol{k}) \tag{2}$$

$$\langle \boldsymbol{k}, \widetilde{\boldsymbol{F}}_{\boldsymbol{t}}(\boldsymbol{k}) \rangle = 0 \tag{3}$$

$$\boldsymbol{k} \times \widetilde{\boldsymbol{F}}_{\boldsymbol{l}}(\boldsymbol{k}) = \boldsymbol{0} \tag{4}$$

The Fourier transform converts gradient into multiplication and vice versa. Due to these properties the inverse Fourier transform gives:

$$\boldsymbol{F} = \boldsymbol{F}_{\boldsymbol{l}} + \boldsymbol{F}_{\boldsymbol{t}} \tag{5}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{F}_{\boldsymbol{t}} \rangle = 0 \tag{6}$$

$$\boldsymbol{\nabla} \times \boldsymbol{F}_{\boldsymbol{l}} = \boldsymbol{0} \tag{7}$$

So, this split indeed conforms to the Helmholtz decomposition.

This interpretation relies on idealized circumstance in which the decomposition runs along straight lines. This idealized condition is

not provided in a curved parameter space. In curved parameter space the decomposition and the interpretation via Fourier transformation only work locally and with reduced accuracy.

11.2.1 Quaternionic Fourier transform split

The longitudinal Fourier transform represents only part of the full quaternionic Fourier transform. It depends on the selection of a radial line $\boldsymbol{k}(q)$ in p space that under ideal conditions runs along a straight line.

$$\mathcal{F}_{\mathbf{k}}\big(g(q)\big) = \mathcal{F}(g(q), \boldsymbol{k}(q)) \tag{1}$$

Or

$$\mathcal{F}_{\parallel}\big(g(q)\big) \stackrel{\text{def}}{=} \mathcal{F}\left(g_{\parallel}(q)\right) \tag{2}$$

It relates to the full quaternionic Fourier transform Ƒ

$$\mathcal{F}\big(g(q)\big) = \tilde{g}(p) \tag{3}$$

The inverse Fourier transform runs:

$$\mathcal{F}^{-1}(\tilde{g}(p)) = g(q) \tag{4}$$

The split in longitudinal and transverse Fourier transforms corresponds to a corresponding split in the multi-dimensional Dirac delta function.

11.3 Fourier integral

For the bra-ket inner product holds:

$$< q\left|\check{P}\, f >= \hbar \cdot \nabla_q < q\right|f >= \hbar \cdot \nabla_q f^*(q) = \mathrm{g}(q) \tag{1}$$

$$= \int_{\boldsymbol{p}} < q|p >\cdot< p|g >$$

The static imaginary part is

$$< q\left|\check{\boldsymbol{P}}\, f >= \hbar \cdot \boldsymbol{\nabla}_q < q\right|f >= \hbar \cdot \boldsymbol{\nabla}_q \boldsymbol{f}^*(q) = \mathbf{g}(q) \tag{2}$$

$$= Im\left(\int_{\boldsymbol{p}} < q|p >\cdot< p|\boldsymbol{g} >\right)$$

$$= \int_{\boldsymbol{p}} Im(< q|p >\cdot< p|\boldsymbol{g} >)$$

$$= \int_{\boldsymbol{p}} Im(< q|p >\cdot< p|\boldsymbol{g_l} >) + \int_{\boldsymbol{p}} Im(< q|p >\cdot< p|\boldsymbol{g_t} >)$$

$$= \int_{\boldsymbol{p}} Im(< q|p >\cdot \tilde{\boldsymbol{g}}_{\boldsymbol{l}}(p)) + \int_{\boldsymbol{p}} Im(< q|p >\cdot \tilde{\boldsymbol{g}}_{\boldsymbol{t}}(p))$$

The left part is the longitudinal inverse Fourier transform of field $\tilde{\boldsymbol{g}}(p)$.

The right part is the transverse inverse Fourier transform of field $\tilde{\boldsymbol{g}}(p)$.

For the Fourier transform of $\mathbf{g}(q)$ holds the split:

$$\tilde{\boldsymbol{g}}(p) = \int_{\boldsymbol{q}} Im(< p|q > \cdot \boldsymbol{g_l}(q)) + \int_{\boldsymbol{p}} Im(< p|q > \cdot \boldsymbol{g_t}(q)) = \int_{\boldsymbol{q}} Im(< p|q > \cdot \boldsymbol{g}(q)) \tag{3}$$

The longitudinal direction is a one dimensional (radial) space. The corresponding transverse direction is tangent to a sphere in 3D. Its direction depends on the field $\mathbf{g}(q)$ or alternatively on the combination of field f and the selected (ideal) coordinate system $\check{Q}$.

For a weakly curved coordinate system $\breve{\mathrm{Q}}$ the formulas hold with a restricted accuracy and within a restricted region.

11.3.1 Alternative formulation

The reference S. Thangavelu[99] provides an alternative specification of the multidimensional Fourier transform .

11.4 Functions invariant under Fourier transform

In this section we confine to a complex part of the Hilbert space.

See http://en.wikipedia.org/wiki/Hermite_polynomials.

[99] http://www.math.iitb.ac.in/atm/faha1/veluma.pdf

There exist two types of Hermite polynomials: (1,

1. The probalist's Hermite polynomials:

$$H_n^{prob}(z) = (-1)^n \exp(½z^2)\, \frac{d^n}{dz^n} \exp(-½z^2).$$

2. The physicist's Hermite polynomials

$$H_n^{phys}(z) = (-1)^n \exp(z^2) \frac{d^n}{dx^n} \exp(-z^2)$$
$$= \exp(½z^2) \left(z - \frac{d}{dz}\right) \exp(-½z^2)$$

These two definitions are *not* exactly equivalent; either is a rescaling of the other:

$$H_n^{phys}(z) = 2^{n/2}\, H_n^{prob}\left(z\sqrt{2}\right) \tag{3}$$

In the following we focus on the physicist's Hermite polynomials.

The Gaussian function φ(z) defined by

$$\varphi(x) \;=\; exp(-\pi\, z^2) \tag{4}$$

is an eigenfunction of F. It means that its Fourier transform has the same form.

As $\mathcal{F}^4 = \mathrm{I}$ any λ in its spectrum $\sigma\,(\mathcal{F})$ satisfies $\lambda^4 = 1$: Hence,

$$\sigma\,(\mathcal{F}) \;=\; \{1; -1;\; i; -i\}. \tag{5}$$

We take the Fourier transform of the expansion:

$$exp(-½\, z^2 + 2\, z\, c - c^2) = \sum_{n=0}^{\infty} exp(-½\, z^2)\, H_n(z)\, c^n /n! \tag{6}$$

First we take the Fourier transform of the left hand side:

$$\frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\infty} exp(-\boldsymbol{k}\, z\, p_z)\, exp(-½\, z^2 + 2\, z\, c - c^2)\, dz \tag{7}$$

$$= exp(-½\, p_z^2 - 2\, \boldsymbol{k}\, p_z\, c + c^2)$$

$$= \sum_{n=0}^{\infty} exp(-½\, p_z^2)\, H_n(p_z)\, (-\boldsymbol{k}\, c)^n /n!$$

The Fourier transform of the right hand side is given by

$$\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \int_{z=-\infty}^{\infty} exp(-\boldsymbol{k}\, z\, p_z) \cdot exp(-½\, z^2)\, H_n(z)\, c^n /n!\, dz \tag{8}$$

Equating like powers of c in the transformed versions of the left- and right-hand sides gives

$$\frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\infty} \exp(-\boldsymbol{k}\, z\, p_z) \cdot exp(-½\, z^2)\, H_n(z)\, c^n/n!\; dz = (-\boldsymbol{k})^n \cdot \exp(-½\, p_z^2)\; H_n(p_z)\, \frac{c^n}{n!} \tag{9}$$

Let us define the Hermite functions $\psi_n(z)$

$$\psi_n(z) \stackrel{\text{def}}{\equiv} < z|\psi_n > = \mathrm{c}_n \exp(-½\, z^2)\; H_n(z) \tag{10}$$

$$|\mathcal{F}\, \psi_n > = |\psi_n > (-\boldsymbol{k})^n \tag{11}$$

with suitably chosen c_n so as to make

$$\|\psi_n\|^2 = 1 \tag{12}$$

$$\mathrm{c}_n = \frac{1}{\sqrt{2^n n!\, \sqrt{\pi}}} \tag{13}$$

The importance of the Hermite functions lie in the following theorem.

"The Hermite functions ψ_n; $n \in \mathsf{N}$ form an orthonormal basis for $L^2(\mathsf{R})$"

Consider the operator

$$H = -\tfrac{1}{2}\frac{d^2}{dz^2} + \tfrac{1}{2}z^2 \tag{14}$$

Apply this to $\psi_n(z)$:

$$H \cdot \psi_n(z) = (\tfrac{1}{2} + n)\,\psi_n(z) \tag{15}$$

Thus, ψ_n is an eigenfunction of H.

Let $f = \psi_{4k+j}$ be any of the Hermite functions. Then we have

$$\sum_{n=-\infty}^{\infty} f(y + n) \cdot \exp\left(-2\pi \boldsymbol{k}\, x\, (y+n)\right) = (-\boldsymbol{k})^j \sum_{n=-\infty}^{\infty} f(x + n)\, exp(2\pi \boldsymbol{k}\, n\, y) \tag{16}$$

The vectors $|\psi_n\rangle$ are eigenvectors of the Fourier transform operator with eigenvalues $(-\boldsymbol{k})^n$. The eigenfunctions $\psi_n(x)$ represent eigenvectors $|\psi_n\rangle$ that span the complex Hilbert space $\mathbf{H}_{\boldsymbol{k}}$.

For higher n the central parts of $\psi_n(x)$ and $|\psi_n(x)|^2$ become a sinusoidal form.

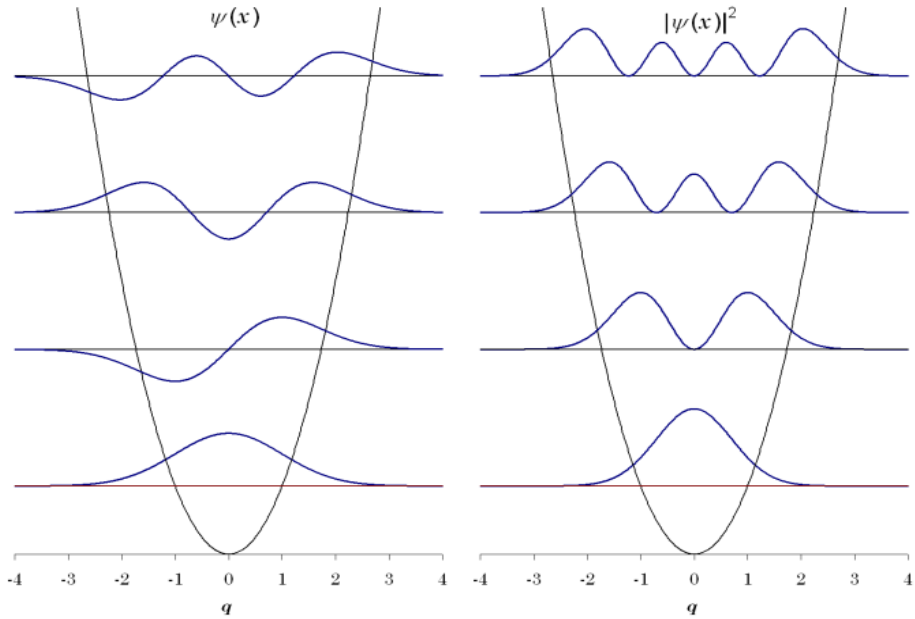

Figure 4

A **coherent state**[100] is a specific kind of state[101] of the quantum harmonic oscillator whose dynamics most closely resemble the oscillating behavior of a classical harmonic oscillator system. The ground state is a squeezed coherent state[102].

[100] http://en.wikipedia.org/wiki/Coherent_state

[101] States

[102] Canonical conjugate: Heisenberg's uncertainty

11.5 Special Fourier transform pairs

Functions that keep the same form through Fourier transformation are:

$$f(q) = \exp(-|q|^2) \tag{1}$$

$$f(q) = \frac{1}{|q|} \tag{2}$$

$$f(q) = comb(q) \tag{3}$$

The comb function consists of a set of equidistant Dirac delta functions.

Other examples of functions that are invariant under Fourier transformation are the linear and spherical harmonic oscillators and the solutions of the Laplace equation.

11.6 Complex Fourier transform invariance properties

Each even function $f(q) \iff \tilde{f}(p)$ induces a Fourier invariant:

$$h(q) = \sqrt{2\pi}\, f(q) + \tilde{f}(q). \tag{1}$$

$$\tilde{h}(q) = \sqrt{2\pi}\, h(q) \tag{2}$$

Each odd function $f(q) \iff \tilde{f}(p)$ induces a Fourier invariant:

$$h(q) = \sqrt{2\pi}\, f(q) - \tilde{f}(q). \tag{3}$$

A function $f(q)$ is invariant under Fourier transformation *if and only if* the function f satisfies the differential equation

$$\frac{\partial^2 f(q)}{\partial q^2} - t^2 f(q) = \alpha\, f(q), \text{ for some scalar } \alpha \in C. \tag{4}$$

The Fourier transform invariant functions are fixed apart from a scale factor. That scale factor can be 1, $\boldsymbol{k}$, -1 or $-\boldsymbol{k}$. $\boldsymbol{k}$ is an imaginary base number in the longitudinal direction.

Fourier-invariant functions show iso-resolution, that is, $\Delta_p = \Delta_q$ in the Heisenberg's uncertainty relation.

For proves see: http://www2.ee.ufpe.br/codec/isoresolution_vf.pdf.

12 Quaternionic probability amplitude distributions

Continuous quaternionic distributions contain a scalar field in their real part and an associated vector field in their imaginary part. In a quaternionic probability amplitude distribution (QPAD), the scalar field can be interpreted as a distribution of the density of property carriers. The associated vector field can be interpreted as a distribution of the current density of these carriers. The squared modulus of the value of the QPAD can be interpreted as the probability density of the presence of the carrier of the charge at the location that is specified by the parameter. The charge can be any property of the carrier or it stands for the ensemble of the properties of the carrier. The QPAD inherits the sign flavor of the quaternionic distribution that defines the curvature of its parameter space.

If a QPAD is an E-type quaternionic distribution, then a continuous quaternionic function defines the curvature of the parameter space of the QPAD. The carriers can be interpreted as the function values of this allocation function. In this case the carriers are tiny patches of the parameter space of the QPAD. Their charge is formed by the discrete symmetry set (sign flavor) of the QPAD. This type of QPAD is suitable for application in quantum fluid dynamics.

If a QPAD is a D-type quaternionic distribution, then a continuous quaternionic function defines the curvature of the parameter space of the QPAD. The carriers can be interpreted as elements of a medium like a gas or a fluid. This type of QPAD is suitable for application in conventional fluid dynamics.

12.1 Potential functions

Each charge carrier corresponds to a potential function. In combination the charge carriers correspond to an integral potential. If

the charge carrier distribution is sufficiently localized, then the integral potential function approaches the form of the single carrier potential function.

12.2 Dynamic potential

If the charge carrier distribution is generated in a rate of one temporary element per progression step, then the potential of the single carriers is transmitted at that same rate. This transmission is performed by spherical waves that extend in the embedding continuum. The waves slightly fold the continuum. An integration of these effects over a series of progression steps will then show the static integral potential function.

12.3 Differential equation

For QPAD's the equation for the differential can be interpreted as a differential continuity equation. Another name for continuity equation is balance equation. The differential continuity equation is paired by an integral continuity equation. The differential equation runs:

$$g(q) = g_0(q) + \boldsymbol{g}(q) = \nabla f(q)$$

$$= \nabla_0 f_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle$$

$$\pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) \pm (\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q))$$

12.4 Continuity equation

Let us approach the balance equation from the integral variety of the balance equation.

When $\rho_0(q)$ is interpreted as a charge density distribution, then the conservation of the corresponding charge[103] is given by the continuity equation:

Total change within V = flow into V + production inside V (1)

In formula this means:

$$\frac{d}{d\tau}\int_V \rho_0\, dV = \oint_S \hat{\boldsymbol{n}}\rho_0 \frac{\boldsymbol{v}}{c}\, dS + \int_V s_0\, dV$$

$$\int_V \nabla_0\rho_0\, dV = \int_V \langle \boldsymbol{\nabla}, \boldsymbol{\rho}\rangle\, dV + \int_V s_0\, dV \tag{3}$$

The conversion from formula (2) to formula (3) uses the Gauss theorem[104]. Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface S, $\boldsymbol{v}(\tau, \boldsymbol{q})$ is the velocity at which the charge density $\rho_0(\tau, \boldsymbol{q})$ enters volume V and s_0 is the source density inside V. In the above formula $\boldsymbol{\rho}$ stands for

$$\boldsymbol{\rho} = \rho_0 \boldsymbol{v}/c \tag{4}$$

It is the flux (flow per unit area and unit time) of ρ_0 .

The combination of $\rho_0(\tau, \boldsymbol{q})$ and $\boldsymbol{\rho}(\tau, \boldsymbol{q})$ is a quaternionic skew field $\rho(\tau, \boldsymbol{q})$ and can be seen as a probability amplitude distribution (QPAD).

$$\rho \stackrel{\text{def}}{=} \rho_0 + \boldsymbol{\rho} \tag{5}$$

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[103] Also see Noether's laws: http://en.wikipedia.org/wiki/Noether%27s_theorem

[104] http://en.wikipedia.org/wiki/Divergence_theorem

$\rho(\tau, \boldsymbol{q})\rho^*(\tau, \boldsymbol{q})$ can be seen as an overall probability density distribution of the presence of the carrier of the charge. $\rho_0(\tau, \boldsymbol{q})$ is a charge density distribution. $\boldsymbol{\rho}(\tau, \boldsymbol{q})$ is the current density distribution.

This results in the law of charge conservation:

$$\begin{aligned} s_0(\tau, \boldsymbol{q}) &= \nabla_0 \rho_0(\tau, \boldsymbol{q}) \\ &\quad \mp \langle \boldsymbol{\nabla}, \left(\rho_0(\tau, \boldsymbol{q})\boldsymbol{v}(\tau, \boldsymbol{q}) + \boldsymbol{\nabla} \times \boldsymbol{a}(\tau, \boldsymbol{q})\right)\rangle \\ &= \nabla_0 \rho_0(\tau, \boldsymbol{q}) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(\tau, \boldsymbol{q}) + \boldsymbol{A}(\tau, \boldsymbol{q})\rangle \\ &= \nabla_0 \rho_0(\tau, \boldsymbol{q}) \mp \langle \boldsymbol{v}(\tau, \boldsymbol{q}), \boldsymbol{\nabla}\rho_0(\tau, \boldsymbol{q})\rangle \\ &\quad \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(\tau, \boldsymbol{q})\rangle \, \rho_0(\tau, \boldsymbol{q}) \\ &\quad \mp \langle \boldsymbol{\nabla}, \boldsymbol{A}(\tau, \boldsymbol{q})\rangle \end{aligned} \tag{6}$$

The blue colored $\pm$ indicates quaternionic sign selection through conjugation of the field $\rho(\tau, \boldsymbol{q})$. The field $\boldsymbol{a}(\tau, \boldsymbol{q})$ is an arbitrary differentiable vector function.

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{a}(\tau, \boldsymbol{q})\rangle = 0 \tag{7}$$

$\boldsymbol{A}(\tau, \boldsymbol{q}) \stackrel{\text{def}}{=} \boldsymbol{\nabla} \times \boldsymbol{a}(\tau, \boldsymbol{q})$ is always divergence free. In the following we will neglect $\boldsymbol{A}(\tau, \boldsymbol{q})$.

Equation (6) represents a balance equation for charge density. What this charge actually is, will be left in the middle. It can be one of the properties of the carrier or it can represent the full ensemble of the properties of the carrier.

Up to this point the investigation only treats the real part of the full equation. The full continuity equation runs:

$$s(\tau,\boldsymbol{q}) = \nabla\rho(\tau,\boldsymbol{q}) = s_0(\tau,\boldsymbol{q}) + \boldsymbol{s}(\tau,\boldsymbol{q}) \tag{8}$$

$$\begin{aligned} &= \nabla_0\rho_0(\tau,\boldsymbol{q}) \mp \langle\boldsymbol{\nabla},\boldsymbol{\rho}(\tau,\boldsymbol{q})\rangle \pm \nabla_0\boldsymbol{\rho}(\tau,\boldsymbol{q}) \\ &\qquad + \boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q}) \\ &\qquad \pm \left(\pm\boldsymbol{\nabla}\times\boldsymbol{\rho}(\tau,\boldsymbol{q})\right) \end{aligned}$$

$$\begin{aligned} &= \nabla_0\rho_0(\tau,\boldsymbol{q}) \mp \langle\boldsymbol{v}(\tau,\boldsymbol{q}),\boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q})\rangle \\ &\qquad \mp \langle\boldsymbol{\nabla},\boldsymbol{v}(\tau,\boldsymbol{q})\rangle\,\rho_0(\tau,\boldsymbol{q}) \\ &\qquad \pm\nabla_0\boldsymbol{v}(\tau,\boldsymbol{q}) + \nabla_0\rho_0(\tau,\boldsymbol{q}) \\ &\qquad\qquad + \boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q}) \\ &\qquad \pm\big(\pm(\rho_0(\tau,\boldsymbol{q})\,\boldsymbol{\nabla}\times\boldsymbol{v}(\tau,\boldsymbol{q}) \\ &\qquad\qquad - \boldsymbol{v}(\tau,\boldsymbol{q}) \\ &\qquad\qquad \times\boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q})\big) \end{aligned}$$

$$\begin{aligned} s_0(\tau,\boldsymbol{q}) = 2\nabla_0\rho_0(\tau,\boldsymbol{q}) &\mp \langle\boldsymbol{v}(q),\boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q})\rangle \\ &\mp \langle\boldsymbol{\nabla},\boldsymbol{v}(\tau,\boldsymbol{q})\rangle\,\rho_0(\tau,\boldsymbol{q}) \end{aligned} \tag{9}$$

$$\boldsymbol{s}(\tau,\boldsymbol{q}) = \pm\nabla_0\boldsymbol{v}(\tau,\boldsymbol{q}) \pm \boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q}) \tag{1}$$

$$\pm\left(\pm\big(\rho_0(\tau,\boldsymbol{q})\,\boldsymbol{\nabla}\times\boldsymbol{v}(\tau,\boldsymbol{q}) - \boldsymbol{v}(\tau,\boldsymbol{q}) \times\boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q})\big)\right)$$

The red sign selection indicates a change of handedness by changing the sign of one of the imaginary base vectors. Conjugation also causes a switch of handedness. It changes the sign of all three imaginary base vectors.

In its simplest form the full continuity equation runs:

$$s(\boldsymbol{q},\tau) = \nabla\rho(\boldsymbol{q},\tau)$$

Thus the full continuity equation specifies a quaternionic distribution s as a flat differential $\nabla\rho$.

When we go back to the integral balance equation, then holds for the imaginary parts:

$$\frac{d}{d\tau}\int_V \boldsymbol{\rho}\, dV = -\oint_S \hat{\boldsymbol{n}}\rho_0\, dS - \oint_S \hat{\boldsymbol{n}} \times \boldsymbol{\rho}\, dS + \int_V \boldsymbol{s}\, dV \tag{4}$$

$$\int_V \nabla_0\, \boldsymbol{\rho}\, dV = -\int_V \boldsymbol{\nabla}\rho_0\, dV - \int_V \boldsymbol{\nabla} \times \boldsymbol{\rho}\, dV + \int_V \boldsymbol{s}\, dV \tag{5}$$

For the full integral equation holds:

$$\frac{d}{d\tau}\int_V \rho\, dV + \oint_S \hat{\boldsymbol{n}}\rho\, dS = \int_V s\, dV \tag{6}$$

$$\int_V \nabla\, \rho\, dV = \int_V s\, dV \tag{7}$$

Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface S, $\boldsymbol{v}(\tau,\boldsymbol{q})$ is the velocity at which the charge density $\rho_0(\tau,\boldsymbol{q})$ enters volume V and s_0 is the source density inside V. In the above formula ρ stands for

$$\rho = \rho_0 + \boldsymbol{\rho} = \rho_0 + \frac{\rho_0 \boldsymbol{v}}{c} \tag{8}$$

It is the flux (flow per unit of area and per unit of progression) of ρ_0 . t stands for progression (not coordinate time).

12.5 Fluid dynamics

The quaternionic continuity equation is the foundation of quaternionic fluid dynamics. Depending on the nature of the streaming medium, this branch of physics exists in two forms.

- In conventional fluid dynamics the streaming charge carriers are elements of a gas or a liquid.
- In quantum fluid dynamics the streaming charge carriers are tiny patches of the parameter space of the QPAD. They correspond to the target values of an E-type quaternionic allocation function $\wp(x)$. This function has a flat parameter space that is spanned by the rational quaternions.

It means that in quantum fluid dynamics the coupling of QPAD's can affect the local curvature.

12.5.1 Coupling equation

In its simplest form the continuity equation runs:

$$\nabla\psi = \varphi$$

The continuity equation couples the local distribution ψ to a source φ.

The coupling strength can be made explicit. This results in the coupling equation.

$$\nabla\psi = m\,\phi$$

Here m is the coupling factor and ϕ is the adapted source.

13 Conservation laws

The following holds for all QPAD's!!!

Only the interpretation tells whether the QPAD concerns a quantum state function, a photon, a gluon or the field of a single charge, a field of a set of charges or a field corresponding to the density distribution of eventually moving charge carriers.

13.1 Differential potential equations

Let $\phi(q)$ define a quaternionic potential. The potential corresponds to a charge density distribution $\phi_0(q)$ and a current density distribution $\boldsymbol{\phi}(q)$.

Note: This means that the following holds for any QPAD!

$$\phi(q) = \rho_0(q) + \boldsymbol{\rho}(q) = \rho_0(q) + \rho_0(q)\boldsymbol{\nu}(q) \tag{1}$$

The gradient and curl of $\phi(q)$ are related. In configuration space holds:

$$\begin{aligned}\mathfrak{F}(q) \stackrel{\text{def}}{=} \nabla\phi(q) = {} & \nabla_0\phi_0(q) \mp \langle\boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle \pm \nabla_0\boldsymbol{\phi}(q) \\ & \pm \boldsymbol{\nabla}\phi_0(q) \pm \left(\pm\boldsymbol{\nabla} \times \boldsymbol{\phi}(q)\right)\end{aligned} \tag{2}$$

$$\mathfrak{E}(q) \stackrel{\text{def}}{=} -\boldsymbol{\nabla}\phi_0(q) \tag{3}$$

$$\mathfrak{B}(q) \stackrel{\text{def}}{=} \boldsymbol{\nabla} \times \phi(q) \tag{4}$$

$$\mathfrak{F}(q) \stackrel{\text{def}}{=} \nabla\phi(q) = \mathfrak{F}_0(q) + \mathfrak{F}(q) \tag{5}$$

$$\mathfrak{F}_0(q) = \nabla_0\phi_0(q) \mp \langle\boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle \tag{6}$$

$$\mathfrak{F}(q) = \mp\mathfrak{E}(q) \pm \mathfrak{B}(q) \pm \nabla_0\boldsymbol{\phi}(q) \tag{7}$$

Note: When the velocity $\boldsymbol{v}$ in $\boldsymbol{\phi}$ changes, then an extra term $\nabla_0 \boldsymbol{\phi}(q)$ is added to equation (7).

13.1.1 Maxwell

In Maxwell equations, the electric field $\boldsymbol{E}(\boldsymbol{r},t)$ is defined as:

$$\boldsymbol{E}(\boldsymbol{r},t) \equiv -\boldsymbol{\nabla}\phi_0(\boldsymbol{r},t) - \frac{\partial \boldsymbol{\phi}(\boldsymbol{r},t)}{\partial t} = \mathfrak{E}(\boldsymbol{r},t) - \frac{\partial \boldsymbol{\phi}(\boldsymbol{r},t)}{\partial t} \tag{1}$$

This is a remarkable decision, because $\dot{\boldsymbol{\phi}}$ can have components along $\mathfrak{E}$ and components along $\mathfrak{B}$, while $\mathfrak{E}$ and $\mathfrak{B}$ are mutually perpendicular.

Further:

$$\begin{aligned} \langle \boldsymbol{\nabla}, \boldsymbol{E}(\boldsymbol{r},t)\rangle &= -\boldsymbol{\nabla}^2\phi_0(\boldsymbol{r},t) - \frac{\partial\langle \boldsymbol{\nabla}, \boldsymbol{\phi}(\boldsymbol{r},t)\rangle}{\partial t} \\ &= \frac{\rho_0(\boldsymbol{r},t)}{\varepsilon_0} - \frac{\partial\langle \boldsymbol{\nabla}, \boldsymbol{\phi}(\boldsymbol{r},t)\rangle}{\partial t} \end{aligned} \tag{2}$$

In Maxwell equations, $\boldsymbol{B}(\boldsymbol{r})$ is defined as:

$$\boldsymbol{B}(\boldsymbol{r},t) \equiv \boldsymbol{\nabla} \times \boldsymbol{\phi}(\boldsymbol{r},t) = \mathfrak{B}(\boldsymbol{r},t) \tag{3}$$

Further:

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t} \tag{4}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{B}(\boldsymbol{r},t)\rangle = 0 \tag{5}$$

$$\nabla \times \boldsymbol{B}(\boldsymbol{r},t) = \mu_0(\boldsymbol{\rho} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}) \tag{6}$$

13.2 Gravity and electrostatics

Gravity and electrostatics can be treated by the same equations.

Description	***Gravity***	***Electrostat-ics***
Field	$\boldsymbol{g} = -\nabla\,\varphi$	$\boldsymbol{E} = -\nabla\,\varphi$
Force	$\boldsymbol{F} = m\boldsymbol{g}$	$\boldsymbol{F} = Q\boldsymbol{E}$
Gauss law	$\langle \nabla, \mathrm{g} \rangle = -4\pi G\rho$	$\langle \nabla, \mathrm{E} \rangle = \frac{\rho}{\varepsilon}$
Poisson law $\Delta\varphi = \langle \nabla, \nabla\varphi \rangle$	$\Delta\varphi = 4\pi G\rho$	$\Delta\varphi = -\frac{\rho}{\varepsilon}$
Greens function	$\frac{-1}{\lvert\boldsymbol{r}\rvert}$	$\frac{1}{\lvert\boldsymbol{r}\rvert}$
Single charge potential	$\varphi = -\frac{4\pi G m}{\lvert\boldsymbol{r}\rvert}$	$\varphi = \frac{Q}{4\pi\varepsilon\lvert\boldsymbol{r}\rvert}$
Single charge field	$g = -\frac{4\pi G m}{\lvert\boldsymbol{r}\rvert^2}\boldsymbol{r}$	$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon\lvert\boldsymbol{r}\rvert^2}\boldsymbol{r}$
Two charge force	$\boldsymbol{F} = -\frac{4\pi G m_1 m_2}{\lvert\boldsymbol{r}\rvert^3}\boldsymbol{r}$	$\boldsymbol{F} = \frac{Q_1 Q_2}{4\pi\varepsilon\lvert\boldsymbol{r}\rvert^3}\boldsymbol{r}$
Mode	**attracting**	**repelling**

13.3 Flux vector

The longitudinal direction **k** of field $\mathfrak{E}(q)$ and the direction **i** of field $\mathfrak{B}(q)$ fix two mutual perpendicular directions. This generates curiosity to the significance of the direction $\mathbf{k} \times \mathbf{i}$**.** With other words what happens with $\mathfrak{E}(q) \times \mathfrak{B}(q)$.

The **flux vector** $\mathfrak{S}(q)$ is defined as:

$$\mathfrak{S}(q) \stackrel{\text{def}}{=} \mathfrak{E}(q) \times \mathfrak{B}(q) \tag{1}$$

13.4 Conservation of energy

$$\begin{aligned}\langle \boldsymbol{\nabla}, \mathfrak{S}(q)\rangle &= \langle \mathfrak{B}(q), \boldsymbol{\nabla} \times \mathfrak{E}(q)\rangle - \langle \mathfrak{E}(q), \boldsymbol{\nabla} \times \mathfrak{B}(q)\rangle \\ &= -\langle \mathfrak{B}(q), \nabla_0 \mathfrak{B}(q)\rangle - \langle \mathfrak{E}(q), \boldsymbol{\phi}(q)\rangle \\ &\qquad - \langle \mathfrak{E}(q), \nabla_0 \mathfrak{B}(q)\rangle \\ &= -\tfrac{1}{2}\nabla_0(\langle \mathfrak{B}(q), \mathfrak{B}(q)\rangle + \langle \mathfrak{E}(q), \mathfrak{E}(q)\rangle) \\ &\qquad - \langle \mathfrak{E}(q), \boldsymbol{\phi}(q)\rangle\end{aligned} \tag{1}$$

The **field energy density** is defined as:

$$\begin{aligned}u_{field}(q) &= \tfrac{1}{2}(\langle \mathfrak{B}(q), \mathfrak{B}(q)\rangle + \langle \mathfrak{E}(q), \mathfrak{E}(q)\rangle) \\ &= u_{\mathfrak{B}}(q) + u_{\mathfrak{E}}(q)\end{aligned} \tag{2}$$

$\mathfrak{S}(q)$ can be interpreted as the **field energy current density**.
The continuity equation for field energy density is given by:

$$\begin{aligned}\nabla_0 u_{field}(q) + \langle \boldsymbol{\nabla}, \mathfrak{S}(q)\rangle &= -\langle \mathfrak{E}(q), \boldsymbol{\phi}(q)\rangle \\ &= -\phi_0(q)\langle \mathfrak{E}(q), \boldsymbol{v}(q)\rangle\end{aligned} \tag{3}$$

This means that $\langle \mathfrak{E}(q), \boldsymbol{\phi}(q)\rangle$ can be interpreted as a source term.

13.4.1 Interpretation in physics

Despite the fact that the above equations hold for any QPAD, we give here the physical interpretations when $\mathfrak{E}$ is the electric field and $\mathfrak{B}$ is the magnetic field.

$\phi_0(q)\mathfrak{E}(q)$ represents **force** per unit volume.

$\phi_0(q)\langle \mathfrak{E}(q), \boldsymbol{v}(q)\rangle$ represents **work** per unit volume, or, in other words, the power density. It is known as the Lorentz power

density and is equivalent to the time rate of change of the mechanical energy density of the charged particles that form the current $\boldsymbol{\phi}(q)$.

$$\nabla_0 u_{field}(q) + \langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle = -\nabla_0 u_{mechanical}(q) \tag{4}$$

$$\nabla_0 u_{mechanical} = \langle \mathfrak{E}(q), \boldsymbol{\phi}(q) \rangle = \phi_0(q) \langle \mathfrak{E}(q), \boldsymbol{v}(q) \rangle \tag{5}$$

$$\nabla_0 \Big(u_{field}(q) + u_{mechanical}(q) \Big) = -\langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle \tag{6}$$

Total change within V = flow into V + production inside V (7)

$$\begin{aligned} u(q) &= u_{field}(q) + u_{mechanical}(q) \\ &= u_B(q) + u_E(q) + u_{mechanical}(q) \end{aligned} \tag{8}$$

$$\begin{aligned} U = U_{field} + U_{mechanical} &= U_B + U_E + U_{mechanical} \\ &= \int_V u \, dV \end{aligned} \tag{9}$$

$$\frac{d}{dt} \int_V u \, dV = \oint_S \langle \hat{\boldsymbol{n}}, \mathfrak{S} \rangle dS + \int_V s_0 \, dV \tag{1}$$

Here the source s_0 is zero.

13.4.2 How to interpret $U_{mechanical}$

$U_{mechanical}$ is the energy of the private field (state function) of the involved particle(s).

13.5 Conservation of linear momentum

$\mathfrak{S}(q)$ can also be interpreted as the **field linear momentum density**. The time rate change of the field linear momentum density is:

$$\nabla_0 \mathfrak{S}(q) = \boldsymbol{g}_{field}(q) = \nabla_0\, \mathfrak{E}(q) \times \mathfrak{B}(q) + \mathfrak{E}(q) \times \nabla_0 \mathfrak{B}(q) \tag{1}$$

$$= \big(\boldsymbol{\nabla} \times \mathfrak{B}(q) - \boldsymbol{\rho}(q)\big) \times\ \mathfrak{B}(q) - \mathfrak{E}(q) \times \boldsymbol{\nabla} \times \mathfrak{E}(q) \tag{2}$$

$$\boldsymbol{G}(\mathfrak{E}) = \mathfrak{E} \times (\boldsymbol{\nabla}\ \times\ \mathfrak{E}) = \langle \boldsymbol{\nabla}\mathfrak{E}\,, \mathfrak{E}\rangle - \langle \mathfrak{E}, \mathfrak{E}\rangle = \tfrac{1}{2}\boldsymbol{\nabla}\langle \mathfrak{E}\,, \mathfrak{E}\rangle - \langle \mathfrak{E}, \mathfrak{E}\rangle \tag{3}$$

$$= -\boldsymbol{\nabla}(\mathfrak{E}\mathfrak{E}) + \tfrac{1}{2}\boldsymbol{\nabla}\langle \mathfrak{E}\,, \mathfrak{E}\rangle + \langle \boldsymbol{\nabla}\,, \mathfrak{E}\rangle \mathfrak{E}$$

$$= -\boldsymbol{\nabla}(\mathfrak{E}\mathfrak{E} + \tfrac{1}{2}\mathbf{1}_3\langle \mathfrak{E}\,, \mathfrak{E}\rangle) + \langle \boldsymbol{\nabla}\,, \mathfrak{E}\rangle \mathfrak{E}$$

$$\boldsymbol{G}(\mathfrak{B}) = \mathfrak{B} \times (\boldsymbol{\nabla}\ \times\ \mathfrak{B}) = -\boldsymbol{\nabla}(\mathfrak{B}\mathfrak{B} + \tfrac{1}{2}\mathbf{1}_3\langle \mathfrak{B}\,, \mathfrak{B}\rangle) + \langle \boldsymbol{\nabla}\,, \mathfrak{B}\rangle \mathfrak{B} \tag{4}$$

$$\boldsymbol{H}(\mathfrak{B}) =\ -\boldsymbol{\nabla}(\mathfrak{B}\mathfrak{B} + \tfrac{1}{2}\mathbf{1}_3\langle \mathfrak{B}\,, \mathfrak{B}\rangle) \tag{5}$$

$$\nabla_0 \mathfrak{S}(q) = \boldsymbol{G}(\mathfrak{B}) + \boldsymbol{G}(\mathfrak{E}) - \boldsymbol{\rho}(q) \times\ \mathfrak{B}(q) \tag{6}$$

$$= \boldsymbol{H}(\mathfrak{E}) + \boldsymbol{H}(\mathfrak{B}) - \boldsymbol{\rho}(q) \times \mathfrak{B}(q) + \langle \boldsymbol{\nabla}\,, \mathfrak{B}\rangle \mathfrak{B} + \langle \boldsymbol{\nabla}\,, \mathfrak{E}\rangle \mathfrak{E}$$

$$= \boldsymbol{H}(\mathfrak{E}) + \boldsymbol{H}(\mathfrak{B}) - \boldsymbol{\rho}(q) \times\ \mathfrak{B}(q) - \rho_0(q)\, \mathfrak{E}(q)$$

$$= \boldsymbol{H}(\mathfrak{E}) + \boldsymbol{H}(\mathfrak{E}) - \boldsymbol{f}(q) = \boldsymbol{\mathcal{T}}(q) - \boldsymbol{f}(q)$$

$\mathcal{T}(\mathrm{q})$ is the linear momentum flux tensor.

The linear momentum of the field contained in volume V surrounded by surface S is:

$$\begin{aligned} \boldsymbol{P}_{field} &= \int_V \boldsymbol{g}_{field}\, dV \\ &= \int_V \rho_0 \boldsymbol{\phi}\, dV \\ &+ \int_V \langle \nabla \boldsymbol{\phi}, \mathfrak{E} \rangle\, dV + \oint_S \langle \hat{\boldsymbol{n}}, \mathfrak{E} \boldsymbol{A} \rangle dS \end{aligned} \tag{7}$$

$$\boldsymbol{f}(q) = \boldsymbol{\rho}(q) \times \mathfrak{B}(q) + \rho_0(q)\, \mathfrak{E}(q) \tag{8}$$

Physically, $\boldsymbol{f}(q)$ is the Lorentz force density. It equals the time rate change of the mechanical linear momentum density $\boldsymbol{g}_{mechanical}$.

$$\boldsymbol{g}_{mechanical}(q) = \rho_{0m}(q) \boldsymbol{v}(q) \tag{9}$$

The force acted upon a single particle that is contained in a volume V is:

$$\boldsymbol{F} = \int_V \boldsymbol{f}\, dV = \int_V (\boldsymbol{\rho} \times \mathfrak{B} + \rho_0\, \mathfrak{E})\, dV \tag{1}$$

Brought together this gives:

$$\nabla_0 \left(\boldsymbol{g}_{field}(q) + \boldsymbol{g}_{mechanical}(q) \right) = -\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{T}}(q) \rangle \tag{1}$$

This is the continuity equation for linear momentum.

The component $\mathcal{T}_{ij}$ is the linear momentum in the i-th direction that passes a surface element in the j-th direction per unit time, per unit area.

Total change within V = flow into V + production inside V (1

$$\boldsymbol{g}(q) = \boldsymbol{g}_{field}(q) + \boldsymbol{g}_{mechanical}(q) \quad (1$$

$$\boldsymbol{P} = \boldsymbol{P}_{field} + \boldsymbol{P}_{mechanical} = \int_V \boldsymbol{g}\, dV \quad (1$$

$$\frac{d}{dt}\int_V \boldsymbol{g}\, dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{\mathcal{T}} \rangle dS + \int_V \boldsymbol{s_g}\, dV \quad (1$$

Here the source $\boldsymbol{s_g} = 0$.

13.6 Conservation of angular momentum

13.6.1 *Field angular momentum*

The angular momentum relates to the linear momentum.

$$\boldsymbol{h}(\boldsymbol{q}_c) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{g}(q) \quad (1)$$

$$\boldsymbol{h}_{field}(\boldsymbol{q}_c) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{g}_{field}(q) \quad (2)$$

$$\boldsymbol{h}_{mechanical}(q) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{g}_{mechanical}(q) \quad (3)$$

$$\boldsymbol{\mathcal{K}}(\boldsymbol{q}_c) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{\mathcal{T}}(q) \quad (4)$$

This enables the balance equation for angular momentum:

$$\nabla_0 \left(\boldsymbol{h}_{field}(\boldsymbol{q}_c) + \boldsymbol{h}_{mechanical}(\boldsymbol{q}_c) \right) = -\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{K}}(\boldsymbol{q}_c) \rangle \tag{5}$$

Total change within V = flow into V + production inside V

$$\boldsymbol{J} = \boldsymbol{J}_{field} + \boldsymbol{J}_{mechanical} = \int_V \boldsymbol{h}\, dV \tag{6}$$

$$\frac{d}{dt} \int_V \boldsymbol{h}\, dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{\mathcal{K}} \rangle dS + \int_V \boldsymbol{s}_{\boldsymbol{h}}\, dV \tag{7}$$

Here the source $\boldsymbol{s}_{\boldsymbol{h}} = 0$.

For a localized charge density contained within a volume V holds for the mechanical torsion:

$$\begin{aligned} \tau(\boldsymbol{q}_c) &= \int_V (\boldsymbol{q}' - \boldsymbol{q}_c) \times \boldsymbol{f}(q') dV \\ &= \int_V (\boldsymbol{q}' - \boldsymbol{q}_c) \times \left(\rho_0(q') \mathfrak{E}(q') + \boldsymbol{j}(q') \times \mathfrak{B}(q') \right) dV \\ &= Q(\boldsymbol{q} - \boldsymbol{q}_c) \times (\mathfrak{E}(q) + \boldsymbol{v}(q) \times \mathfrak{B}(q)) \end{aligned} \tag{8}$$

$$\boldsymbol{J}_{field}(\boldsymbol{q}_c) = \boldsymbol{J}_{field}(\boldsymbol{0}) + \boldsymbol{q}_c \times \boldsymbol{P}(q) \tag{9}$$

Using

$$\langle \nabla \boldsymbol{a}, \boldsymbol{b} \rangle = \boldsymbol{n}_\nu \frac{\partial a_\mu}{\partial q_\nu} b_\mu \tag{1}$$

$$\langle \boldsymbol{b}, \nabla \boldsymbol{a} \rangle = \boldsymbol{n}_\mu \frac{\partial a_\mu}{\partial q_\nu} b_\mu \tag{1}$$

holds

$$\begin{aligned} \boldsymbol{J}_{field}(\mathbf{0}) &= \int_V \boldsymbol{q}' \times \mathfrak{S}(q')dV \\ &= \int_V \boldsymbol{q}' \times \mathfrak{E}(q') \times \nabla \times \boldsymbol{\phi}(q')\, dV \\ &= \int_V (\boldsymbol{q}' \times \langle (\nabla \boldsymbol{\phi}), \mathfrak{E} \rangle - \langle \boldsymbol{q}' \times \mathfrak{E}, (\nabla \boldsymbol{\phi}) \rangle)\, dV \\ &= \int_V \boldsymbol{q}' \times \langle (\nabla \boldsymbol{\phi}), \mathfrak{E} \rangle dV \\ &\quad + \int_V \mathfrak{E} \times \boldsymbol{\phi}\, dV - \int_V \langle \nabla, \mathfrak{E} \boldsymbol{q}' \\ &\quad \times \boldsymbol{\phi} \rangle dV \\ &\quad + \int_V (\boldsymbol{q}' \\ &\quad \times \boldsymbol{\phi}) \langle \nabla, \mathfrak{E} \rangle dV \end{aligned} \tag{1}$$

13.6.2 Spin

Define the non-local spin term, *which does not depend on* $\boldsymbol{q}'$ as:

$$\boldsymbol{\Sigma}_{field} = \int_V \mathfrak{E}(q) \times \boldsymbol{\phi}(q) dV \quad (1$$

Notice

$$\boldsymbol{\phi}(q) \times \boldsymbol{\nabla}\phi_0(q) = \phi_0 \boldsymbol{\nabla} \times \boldsymbol{\phi}(q) + \boldsymbol{\nabla} \times \left(\phi_0(q)\boldsymbol{\phi}(q)\right)$$

And

$$\boldsymbol{L}_{field}(\mathbf{0}) = \int_V \boldsymbol{q}' \times \langle(\boldsymbol{\nabla}\boldsymbol{\phi}), \mathfrak{E}\rangle dV + \int_V \boldsymbol{q}' \times \rho_0 \boldsymbol{\phi} dV \quad (1$$

Using Gauss:

$$\int_V \langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{a} \rangle dS \quad (1$$

And

$$\rho_0 = \langle \boldsymbol{\nabla}, \mathfrak{E} \rangle \quad (1$$

Leads to:

$$\boldsymbol{J}_{field}(\mathbf{0}) = \boldsymbol{\Sigma}_{field} + \boldsymbol{L}_{field}(\mathbf{0}) + \oint_S \langle \hat{\boldsymbol{n}}, \mathfrak{E}\boldsymbol{q}' \times \boldsymbol{\phi} \rangle dS \quad (1$$

13.6.3 Spin discussion

The spin term is defined by:

$$\boldsymbol{\Sigma}_{field} = \int_V \mathfrak{E}(q) \times \boldsymbol{\phi}(q) dV \quad (1)$$

In free space the charge density ρ_0 vanishes and the scalar potential $\boldsymbol{\phi}_0$ shows no variance. Only the vector potential $\boldsymbol{\phi}$ may vary with q_0. Thus:

$$\mathfrak{E} = \boldsymbol{\nabla}\phi_0 - \nabla_0\boldsymbol{\phi} \approx -\nabla_0\boldsymbol{\phi} \tag{2}$$

$$\boldsymbol{\Sigma}_{field} \approx \int_V (\nabla_0\boldsymbol{\phi}(q)) \times \boldsymbol{\phi}(q)dV \tag{3}$$

Depending on the selected field $\boldsymbol{\Sigma}_{field}$ has two versions that differ in their sign. These versions can be combined in a single operator:

$$\boldsymbol{\Sigma}_{field} = \begin{bmatrix} \boldsymbol{\Sigma}^{+}{}_{field} \\ \boldsymbol{\Sigma}^{-}{}_{field} \end{bmatrix} \tag{4}$$

If $\frac{\boldsymbol{\phi}(q)}{|\boldsymbol{\phi}(q)|}$ can be interpreted as tantrix (q_0)) and $\frac{\nabla_0\boldsymbol{\phi}(q)}{|\nabla_0\boldsymbol{\phi}(q)|}$ can be interpreted as the principle normal $\boldsymbol{N}(q_0)$, then $\frac{(\nabla_0\boldsymbol{\phi}(q))\times\boldsymbol{\phi}(q)}{|(\nabla_0\boldsymbol{\phi}(q))\times\boldsymbol{\phi}(q)|}$ can be interpreted as the binormal $\mathfrak{B}(q_0)$.

From these quantities the curvature and the torsion[105] can be derived.

$$\begin{bmatrix} \dot{\boldsymbol{T}}(t) \\ \dot{\boldsymbol{N}}(t) \\ \dot{\boldsymbol{B}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}(t) \\ \boldsymbol{N}(t) \\ \boldsymbol{B}(t) \end{bmatrix} \tag{5}$$

[105] Path characteristics