The Analysis of Harold White applied to the Natario Warp Drive Spacetime. From 10 times the mass of the Universe to arbitrary low levels of negative energy density using a continuous Natario shape function with power factors.Warp Drives with two warped regions

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Abstract

Warp Drives are solutions of the Einstein Field Equations that allows superluminal travel within the framework of General Relativity. There are at the present moment two known solutions: The Alcubierre warp drive discovered in 1994 and the Natario warp drive discovered in 2001. However as stated by both Alcubierre and Natario themselves the warp drive violates all the known energy conditions because the stress energy momentum tensor is negative implying in a negative energy density. While from a classical point of view the negative energy is forbidden the Quantum Field Theory allows the existence of very small amounts of it being the Casimir effect a good example as stated by Alcubierre himself.The major drawback concerning negative energies for the warp drive is the huge amount of negative energy able to sustain the warp bubble.Ford and Pfenning computed the amount of negative energy needed to maintain an Alcubierre warp drive and they arrived at the result of 10 times the mass of the entire Universe for a stable warp drive configuration rendering the warp drive impossible.However Harold White manipulating the parameter @ in the original shape function that defines the Alcubierre spacetime demonstrated that it is possible to low these energy density requirements.We repeat here the Harold White analysis for the Natario spacetime and we arrive at similar conclusions.From 10 times the mass of the Universe we also manipulated the parameter @ in the original shape function that defines the Natario spacetime and we arrived at arbitrary low results.We demonstrate in this work that both Alcubierre and Natario warp drives have two warped regions and not only one.We also discuss Horizons and Doppler Blueshifts.The main reason of this work is to demonstrate that Harold White point of view is entirely correct.

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1 Introduction

The Warp Drive as a solution of the Einstein Field Equations of General Relativity that allows superluminal travel appeared first in 1994 due to the work of Alcubierre.([1]) The warp drive as conceived by Alcubierre worked with an expansion of the spacetime behind an object and contraction of the spacetime in front.The departure point is being moved away from the object and the destination point is being moved closer to the object.The object do not moves at all[1].It remains at the rest inside the so called warp bubble but an external observer would see the object passing by him at superluminal speeds(pg 8 in [1])(pg 1 in [2]).

Later on in 2001 another warp drive appeared due to the work of Natario.([2]).This do not expands or contracts spacetime but deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [2]). Imagine the object being a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium whose walls do not expand or contract. An observer in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.

However the major drawback that affects the warp drive is the quest of large negative energy requirements enough to sustain the warp bubble.While from a classical point of view negative energy densities are forbidden the Quantum Field Theory allows the existence of very small quantities of such energies but unfortunately the warp drive requires immense amounts of it.Ford and Pfenning computed the negative energy density needed to maintain a warp bubble and they arrived at the conclusion that in order to sustain a stable configuration able to perform interstellar travel the amount of negative energy density is of about 10 times the mass of the Universe and they concluded that the warp drive is impossible.(see pg 10 in [3] and pg 78 in [5]).

Ford and Pfenning used in their calculations the piecewise shape function that is not exactly equal to the Alcubierre shape function.Both shape functions are dimensionless but there exists a crucial difference between the derivative of the Alcubierre shape function and the derivative of the Ford-Pfenning piecewise shape function:

- 1)-The derivative of the Alcubierre shape function remains also dimensionless but the derivative of the Ford-Pfenning shape function have dimensions and this affects the final calculation of the negative energy density.Also the Ford-Pfenning piecewise shape function is not analytical in all the trajectory points.

Another drawback that affects the warp drive is the quest of the interstellar navigation:Interstellar space is not empty and from a real point of view a ship at superluminal speeds would impact asteroids,comets,interstellar space dust and photons of Cosmic Background Radiation(COBE).

According to Clark,Hiscock and Larson a single collision between a ship and a COBE photon would release an amount of energy equal to the photosphere of a star like the Sun.(see pg 11 in [9]).And how many photons of COBE we have per cubic centimeter of space??

These highly energetic collisions would pose a very serious threat to the astronauts as pointed out by McMonigal,Lewis and O'Byrne (see pg 10 in [10]).

[1]do not violates Relativity

Another problem:these highly energetic collisions would raise the temperature of the warp bubble reaching the Hawking temperature as pointed out by Barcelo,Finazzi and Liberati.(see pg 6 in in [11]).At pg 9 they postulate that all future spaceships cannot bypass 99 percent of the light speed.

In section 5 we will see that these problems of interstellar navigation affects the Alcubierre warp drive but not the Natario one.

The last drawback raised against the warp drive is the fact that inside the warp bubble an astronaut cannot send signals with the speed of the light to control the front of the bubble because an Horizon(causally disconnected portion of spacetime)is established between the astronaut and the warp bubble.We discuss this in section 5 and in section 6 we discuss a possible way to overcome the Horizon problem using only General Relativity.

Recently Harold White discovered that by a manipulation of the parameter @ in the original shape function that defines the Alcubierre spacetime the amounts of negative energy density needed to maintain the warp drive can be lowered to more reasonable levels.(see pg 4 fig 2 in [8])(see pg 8 in [17]).

In this work we repeat the analysis of Harold White for the Natario warp drive spacetime and by a manipulation of the parameter @ in the original shape function that defines the Natario spacetime we arrive at similar conclusions.

We adopted the International System of Units where $G = 6,67 \times 10^{-11}$ $\frac{Newton \times meters^2}{kilograms^2}$ and $c = 3 \times 10^8$ $\frac{meters}{seconds}$ and not the Geometrized System of units in which $c = G = 1$.

We consider here a Natario warp drive with a radius $R = 100$ meters a thickness parameter @ $= 900$ and @ $= 5000$ moving with a speed 200 times faster than light implying in a $vs = 2 \times 10^2 \times 3 \times 10^8 = 6 \times 10^{10}$ and a $vs^2 = 3,6 \times 10^{21}$

We also adopt a warp factor as a dimensionless parameter in our Natario shape function $WF = 200$

This work is a companion work to our work [13] but this work is by far much more advanced.

This work is organized as follows:

- Section 2)-Outlines the differences between the Alcubierre original shape function and the Ford-Pfenning piecewise shape function.

- Section 3)-Outlines the problems of the immense magnitude in negative energy density when a ship travels with a speed of 200 times faster then light.

- Section 4)-The most important section in this work.Outlines the analysis of Harold White for the Natario warp drive spacetime.This section must be read together with the Appendices from B ro F.In these Appendices we will see that both Alcubierre and Natario warp drive spacetimes have two warped regions and not only one.

- Section 5)-Outlines the major advantages of the Natario warp drive spacetime when compared to its Alcubierre counterpart.The Natario warp drive can survive to the Horizons and Doppler Blueshift problem.It can also survive against the objections raised by Clark,Hiscock,Larson, McMonigal,Lewis,O'Byrne,Barcelo,Finazzi and Liberati

- Section 6)-Outlines the possibility of how to overcome the Horizon problem from an original point of view of General Relativity.

2 Differences between the Alcubierre shape function and the Ford-Pfenning piecewise shape function

The negative energy density distribution in the Alcubierre warp drive spacetime is given by the following expression(see eq 8 pg 6 in [3])[2][3]:

$$\langle T^{\mu\nu} u_{\mu} u_{\nu} \rangle = \langle T^{00} \rangle = \frac{1}{8\pi} G^{00} = -\frac{1}{8\pi} \frac{v_s^2(t)[y^2 + z^2]}{4r_s^2(t)} \left(\frac{df(r_s)}{dr_s} \right)^2, \tag{1}$$

In the expression above $f(rs)$ is the Alcubierre shape function defined as being 1 inside the warp bubble and 0 outside the warp bubble while being $1 > f(rs) > 0$ in the Alcubierre warped region according to eqs 6 and 7 pg 4 in [1] or top of pg 4 in [2].See also eq 3 pg 3 in [3].

The expressions for $f/rs)$ are given by:

$$rs = \sqrt{(x - xs)^2 + y^2 + z^2} \tag{2}$$

$$f(rs) = \frac{tanh[@(rs + R)] - tanh[@(rs - R)]}{2tanh(@R)} \tag{3}$$

In the expressions above rs defined using meters is the distance travelled by the Eulerian observer from the center of the bubble($rs = 0$)to the end of the warp bubble.R is the radius of the bubble also defined using meters and @ is a dimensionless parameter related to the thickness of the bubble.In the Appendix B a plot of $f(rs)$ can be seen for a warp bubble of 100 meters of radius $R = 100$ meters and a thickness parameter @ = 900 and not @ = 900 meters.Note that in pg 4 in [1] Alcubierre mentions the fact that both R and @ are arbitrary parameters although he do not mention the dimensionless nature of @.In the Appendix C a plot of $f(rs)$ can be seen for a warp bubble of 100 meters of radius $R = 100$ meters and a thickness parameter @ = 5000 and not @ = 5000 meters.

In these Appendices B and C the regions inside the bubble $f(rs) = 1$ outside the bubble $f(rs) = 0$ and in the Alcubierre warped region $1 > f(rs) > 0$ can be clearly seen.

Note that although Alcubierre defined the warped region as being $1 > f(rs) > 0$ the transition region between the interior and the exterior of the warp bubble(warp bubble walls) when rs approaches the neighborhoods of the bubble radius R but still inside the bubble($f(rs) = 1$) the derivative of the Alcubierre shape function is not zero.In the same way when rs leaves the bubble and moves further in the regions outside the bubble but also in the neighborhoods of the bubble radius ($f(rs) = 0$) the derivative of the Alcubierre shape function is again not zero.

[2] $f(rs)$ is the Alcubierre shape function.Equation written in the Geometrized System of Units $c = G = 1$

[3] Equation written in Cartezian Coordinates

This leads us to these important conclusions:

- 1)-The Alcubierre warp drive have two warped regions and not one:The first is the region where $1 > f(rs) > 0$ and the second is the region where the derivatives of the shape function are not zero meaning a non-flat spacetime and a negative energy distribution even inside and outside the bubble and not only in the region where $1 > f(rs) > 0$.
- 2)-When computing the total amount of negative energy needed to sustain a warp bubble these amounts of negative energy even inside and outside the bubble must be taken into account.

Plotting all the terms of the Alcubierre shape function individually we can see that the terms $tanh[@(rs + R)] = 1$ and $tanh(@R) = 1$ for a dimensionless thickness parameter $@ = 900$ or $@ = 5000$ or $@ = 50000$.So we can simplify the Alcubierre shape function.

The final expressions are given by:

$$f(rs) = \frac{1 - tanh[@(rs - R)]}{2} \tag{4}$$

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{5}$$

The derivatives of the Alcubierre shape function are given by:

$$f'(rs) = \frac{1}{2}[-\frac{@}{cosh^2[@(rs - R)]}] \tag{6}$$

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs - R)]}] \tag{7}$$

Note that all the terms in the expressions above are dimensionless and dividing dimensionless terms we will get a dimensionless result.Hence we can clearly see that the derivatives of the Alcubierre shape function are dimensionless.Another important thing:The Alcubierre shape function is analytical in all the points of its trajectory.[4]

[4] continuous and differentiable

The Ford-Pfenning piecewise shape function $f_{p.c.}(r_s)$ in order to resemble the Alcubierre shape function $f(rs)$ must be defined as being 1 inside the warp bubble and 0 outside the warp bubble while being $1 > f_{p.c.} > 0$ in the Alcubierre warped region according to eq 4 pg 3 in [3].The Ford-Pfenning piecewise shape function is almost equal to the Alcubierre shape function.

$$f_{p.c.}(r_s) = \begin{cases} 1 & r_s < R - \frac{\Delta}{2} \\ -\frac{1}{\Delta}(r_s - R - \frac{\Delta}{2}) & R - \frac{\Delta}{2} < r_s < R + \frac{\Delta}{2} \\ 0 & r_s > R + \frac{\Delta}{2} \end{cases} \tag{8}$$

In order to make $f_{p.c.}(r_s)$ dimensionless as $f(rs)$ and since R and rs have dimensions given in meters if we divide as shown above both R and rs by Δ then Δ must be also given in meters.

But the derivatives of $f_{p.c.}(r_s)$ with respect to r_s are not dimensionless:As a matter of fact we have:

$$f'_{p.c.}(r_s) = \begin{cases} 0 & r_s < R - \frac{\Delta}{2} \\ -\frac{1}{\Delta} & R - \frac{\Delta}{2} < r_s < R + \frac{\Delta}{2} \\ 0 & r_s > R + \frac{\Delta}{2} \end{cases} \tag{9}$$

$$[f'_{p.c.}(r_s)]^2 = \begin{cases} 0 & r_s < R - \frac{\Delta}{2} \\ \frac{1}{\Delta^2} & R - \frac{\Delta}{2} < r_s < R + \frac{\Delta}{2} \\ 0 & r_s > R + \frac{\Delta}{2} \end{cases} \tag{10}$$

These derivatives have dimensions in meters or in square meters due to the non-dimensionality of the term Δ.

Also this function is not analytical in the points where $rs = R - \frac{\Delta}{2}$ and $rs = R + \frac{\Delta}{2}$[5]

Looking again to the Alcubierre expression of the negative energy density and the derivatives of the Alcubierre shape function:

$$\langle T^{\mu\nu} u_\mu u_\nu \rangle = \langle T^{00} \rangle = \frac{1}{8\pi} G^{00} = -\frac{1}{8\pi} \frac{v_s^2(t)[y^2 + z^2]}{4r_s^2(t)} \left(\frac{df(r_s)}{dr_s} \right)^2, \tag{11}$$

$$f'(rs) = \frac{1}{2}[-\frac{@}{cosh^2[@(rs - R)]}] \tag{12}$$

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs - R)]}] \tag{13}$$

If we insert the derivatives of the Alcubierre shape function in the expression of the negative energy density we will get a result different than the one obtained if we insert the derivatives of the Ford-Pfenning shape function into the same expression due to differences in the dimensionality.

The total energy calculations must be made with the original Alcubierre shape function that is analytical in all the points of he trajectory.

[5] not continuous and not differentiable in these points

3 The Problem of the Negative Energy in the Natario Warp Drive Spacetime-The Unphysical Nature of Warp Drive

The negative energy density for the Natario warp drive is given by(see pg 5 in [2])

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{14}$$

Converting from the Geometrized System of Units to the International System we should expect for the following expression(see eqs 21 and 23 pg 6 in [4]):

$$\rho = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{rs}{2}n''(rs)\right)^2\sin^2\theta\right]. \tag{15}$$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for[6]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{16}$$

In the equatorial plane:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{17}$$

Note that in the above expressions the warp drive speed vs appears raised to a power of 2. Considering our Natario warp drive moving with $vs = 200$ which means to say 200 times light speed in order to make a round trip from Earth to a nearby star at 20 light-years away in a reasonable amount of time(in months not in years) we would get in the expression of the negative energy the factor $c^2 = (3 \times 10^8)^2 = 9 \times 10^{16}$ being divided by $6,67 \times 10^{-11}$ giving $1,35 \times 10^{27}$ and this is multiplied by $(6 \times 10^{10})^2 = 36 \times 10^{20}$ coming from the term $vs = 200$ giving $1,35 \times 10^{27} \times 36 \times 10^{20} = 1,35 \times 10^{27} \times 3,6 \times 10^{21} = 4,86 \times 10^{48}$!!!

A number with 48 zeros!!!

Our Earth have a mass[7] of about $6 \times 10^{24} kg$ and multiplying this by c^2 in order to get the total positive energy "stored" in the Earth according to the Einstein equation $E = mc^2$ we would find the value of $54 \times 10^{40} = 5,4 \times 10^{41} Joules$.

Earth have a positive energy of $10^{41} Joules$ and dividing this by the volume of the Earth(radius $R_{Earth} =$ 6300 km approximately) we would find the positive energy density of the Earth.Taking the cube of the Earth radius $(6300000m = 6,3 \times 10^6)^3 = 2,5 \times 10^{20}$ and dividing $5,4 \times 10^{41}$ by $(4/3)\pi R_{Earth}^3$ we would find the value of $4,77 \times 10^{20}\frac{Joules}{m^3}$. So Earth have a positive energy density of $4,77 \times 10^{20}\frac{Joules}{m^3}$ and we are talking about negative energy densities with a factor of 10^{48} for the warp drive while the quantum theory allows only microscopical amounts of negative energy density.

So we would need to generate in order to maintain a warp drive with 200 times light speed the negative energy density equivalent to the positive energy density of 10^{28} Earths!!!!

[6] see Appendix A

[7] see Wikipedia:The free Encyclopedia

A number with 28 zeros!!!.Unfortunately we must agree with the major part of the scientific community that says:"Warp Drive is impossible and unphysical!!"

However looking better to the expression of the negative energy density in the equatorial plane of the Natario warp drive:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{18}$$

We can see that a very low derivative and hence its square can perhaps obliterate the huge factor of 10^{48} ameliorating the negative energy requirements to sustain the warp drive.By manipulating the term @ in the original Alcubierre shape function Harold White lowered these requirements for the Alcubierre warp drive.

In the next section we will repeat the White analysis for the Natario warp drive manipulating also the term @ in the original Natario shape function ameliorating the negative energy requirements from 10 times the mass of the Universe that would render the warp drive as impossible and unphysical to arbitrary low values.

4 The Analysis of Harold White applied to the Natario Warp Drive Spacetime

According to Natario(pg 5 in [2]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive.

The Natario warp drive continuous shape function and its derivatives can be defined by:(see eq 38 pg 9 and eqs 39 and 40 pg 10 in [4]).

$$n(rs) = [\frac{1}{2}][1 - f(rs)]^{WF} \tag{19}$$

$$n'(rs) = -[\frac{1}{2}]WF[1 - f(rs)]^{WF-1}f'(rs) \tag{20}$$

$$n'(rs)^2 = [\frac{1}{4}]WF^2[1 - f(rs)]^{2(WF-1)}f'(rs)^2 \tag{21}$$

This shape function gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region according with our Microsoft Excel simulations .(see pg 5 in [2])(see also Appendices B,C and D)

Note that the Alcubierre shape function is being used to define its Natario counterpart.Below is presented the Alcubierre shape function and its derivatives.(see eqs 6 and 7 pg 4 in [1] or top of pg 4 in [2])[8].

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{22}$$

$$f'(rs) = \frac{1}{2}[-\frac{@}{cosh^2[@(rs - R)]}] \tag{23}$$

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs - R)]}] \tag{24}$$

.

$$rs = \sqrt{(x - xs)^2 + y^2 + z^2} \tag{25}$$

In the Alcubierre shape function xs is the center of the warp bubble where the ship resides. R is the radius of the warp bubble and @ is the Alcubierre dimensionless parameter related to the thickness.According to Alcubierre these can have arbitrary values.We outline here the fact that according to pg 4 in [1] the parameter @ can have arbitrary values.This is very important for the White analysis as we will see later.

[8] $tanh[@(rs * R)] = 1$,$tanh(@R) = 1$.See section 2

The shape function $f(rs)$ have a value of 1 inside the warp bubble and zero outside the warp bubble while being $0 < f(rs) < 1$ in the warp bubble walls.rs is the path of the so-called Eulerian observer that starts at the center of the bubble xs and ends up outside the warp bubble.In our case we consider the equatorial plane and we have for rs the following expression[9].

$$rs = \sqrt{(x - xs)^2} \tag{26}$$

$$rs = x - xs \tag{27}$$

The term WF in the Natario shape function is dimensionless too:it is the warp factor that will squeeze the region where the derivatives of the Natario shape function are different than 0.(see Appendices B,C and D)

It is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

Back again to the negative energy density in the Natario warp drive[10]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{28}$$

The total energy needed to sustain the Natario warp bubble is obtained by integrating the negative energy density ρ over the volume of the Natario warped region(The region where the derivatives of the Natario shape function are not null)(points a and b in the integral in the bottom of this page).
Since we are in the equatorial plane then only the term in rs accounts for and the total energy integral can be given by:

$$E = \int(\rho)drs = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\int(3(n'(rs))^2)drs = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\int((n'(rs))^2)drs \tag{29}$$

Above we placed the constant terms c,G and vs^2 outside the integral. But the integral now becomes:

$$\int((n'(rs))^2)drs = \int(\frac{1}{4}WF^2[1-f(rs)]^{2(WF-1)}f'(rs)^2)drs = \frac{1}{4}WF^2\int([1-f(rs)]^{2(WF-1)}f'(rs)^2)drs \tag{30}$$

Since WF ia also a constant.Then the total energy integral for the Natario warp drive is given by:

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\int((n'(rs))^2)drs = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{1}{4}WF^2\int([1-f(rs)]^{2(WF-1)}f'(rs)^2)drs \tag{31}$$

Unfortunately integrals of this form do not have known primitives and also the integration methods to compute the integral are not known.In order to compute the total energy needed to sustain the Natario warp bubble we must employ the numerical integration by the Trapezoidal Rule.(see pg 4 in [18])[11]

$$\int_a^b f(x)dx = (b-a)\frac{f(b)+f(a)}{2} \cong 0 \dashrightarrow b - a \cong 0 \dashrightarrow b \cong a \dashrightarrow f(b) \cong f(a) \tag{32}$$

[9]see Appendix A
[10]written in the International System of units
[11]see Wikipedia the Free Encyclopedia

The total energy integral needed to sustain the Natario warp bubble now becomes:

$$E = \int_a^b \rho drs = \int_a^b -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3n'(rs))^2\right] drs = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\int_a^b \left[n'(rs)^2\right] drs \tag{33}$$

$$\int_a^b \left[(n'(rs)^2\right] drs = (b-a)\frac{n'(b)^2 + n'(a)^2}{2} \tag{34}$$

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}(b-a)\frac{n'(b)^2 + n'(a)^2}{2} \simeq 0 \dashrightarrow b \simeq a \dashrightarrow b - a \simeq 0 \tag{35}$$

The dimensionless thickness parameter @ defines the width of the region where the derivatives of the Natario shape function are not null.Inside the bubble but in the bubble center or at the neighborhoods of the bubble center the derivatives are zero.If an Eulerian observer starts to move from the bubble center towards the bubble walls he will cross regions inside the bubble where the derivatives of the Natario shape function are zero.When he reaches the Natario warped region the derivatives of the shape function ceases to be zero and starts to grow to reach a maximum value and then decreases again.When the observer crosses the bubble radius leaving the bubble and entering in the regions outside the bubble the derivatives of the shape function in the regions outside the bubble but in the neighborhoods of the bubble radius are not zero but as far as the observer moves away from the bubble the derivatives decrease its values reaching again zero in distant regions outside the bubble.

The beginning of the region where the derivatives of the shape function ceases to be zero is the beginning of the Natario warped region for negative energy(point a).It coincides with the Natario warped region for the shape function($0 < n(rs) < \frac{1}{2}$)(see in the Appendices B,C and D the explanation for the two warped regions) .

The end of the region outside the bubble where the derivatives of the shape function still are not zero is the end of the Natario warped region for negative energy.It extends beyond the Natario warped region for the shape function.(point b)

Now a reader can see the power of the White idea.Compare the distance between the points a (beginning of the negative energy warped region) and b(end of the negative energy warped region) in the Appendices B and C.The points a and b are more close to each other in Appendix C than in the Appendix B.As higher the thickness parameter @ is as thicker or thinner the Natario negative energy warped region becomes approaching the point b to the point a.The thickness parameter passed from @ $= 900$ to @ $= 5000$.If we make @ $= 500.000$ or @ $= 5.000.000$ the points a and b will be so close that the difference between b and a will approach zero for real obliterating the value of 10^{48} or any other value.($b \simeq a$)($b - a \simeq 0$)

This was exactly what Harold White did:by manipulating @ to very large values he created Alcubierre warped regions so thinner that the point b is infinitely closed to a giving a very low (and acceptable) value for the negative energy in the Alcubierre warp drive able to obliterate factors of 10^{48}(see pg 4 fig 2 in [8])(see pg 8 in [17]).

However there exists an error margin in the integration by the Trapezoidal Rule.The correct procedure is to decompose the area to be integrated in slices and integrate numerically still by the Trapezoidal Rule separately each slice and in the end we sum the result of all the slices integrated.This method is known as the Composite Trapezoidal Rule.(see pg 15 in [18])(see Appendix D)

As higher the number of slices is as more precise the result of the integral by the Composite Trapezoidal Rule is.(see pgs 14 and 15 in [18]).Remember from the Appendix B that a warp bubble wether in the Alcubierre or Natario cases with a radius of 100 meters moving at 200 times light speed have the total amount of negative energy equal to the product of 10^{48} by the integral of the square derivatives of the shape function in the region between the point b(end of the warped region) and the point a(beginning of the warped region).If we want to integrate from the point a to point b using the Composite Trapezoidal Rule reducing the error margin we must divide the region between a and b in slices and integrate separately each slice also by the Trapezoidal Rule and in the end we sum the result of all the integrations.As higher the number of slices as accurate the integration becomes.Following pg 14 and 15 in [18] if we want to divide the region between a and b in n slices each slice have a width given by:

$$h = \frac{b-a}{n} \tag{36}$$

And the final integration is the sum of the integration of all the slices by the Trapezoidal Rule given by:

$$\int_a^b n'(rs)^2 drs = \int_a^{a+h} n'(rs)^2 drs + \int_{a+h}^{a+2h} n'(rs)^2 drs + ... + \int_{a+(n-2)h}^{a+(n-1)h} n'(rs)^2 drs + \int_{a+(n-1)h}^{b} n'(rs)^2 drs \tag{37}$$

Note that each slice above have a width h.

Writing the integral using sums we get the following expressions(pg 15 in [18]):

$$\int_a^b n'(rs)^2 drs = \frac{1}{2n}(b-a)[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \cong 0 \dashrightarrow b-a \cong 0 \dashrightarrow b \cong a \dashrightarrow n >> 1 \tag{38}$$

$$\int_a^b n'(rs)^2 drs = \frac{b-a}{2n}[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \cong 0 \dashrightarrow b-a \cong 0 \dashrightarrow b \cong a \dashrightarrow n >> 1 \tag{39}$$

Inserting these expressions in the integral of the negative energy density we get:

$$E = \int_a^b \rho drs = \int_a^b -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3n'(rs))^2\right] drs = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\int_a^b \left[n'(rs)^2\right] drs \tag{40}$$

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{1}{2n}(b-a)[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \tag{41}$$

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{b-a}{2n}[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \tag{42}$$

Look again to the equations of the total negative energy needed to sustain a Natario warp bubble:

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{1}{2n}(b-a)[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \quad (43)$$

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{b-a}{2n}[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \quad (44)$$

Now note an interesting thing:

- 1)-The number of slices raises the accuracy of the integration method and we have the factor $\frac{c^2}{G} \times \frac{vs^2}{8\pi}$ generating the huge factor 10^{48} for a bubble speed $vs = 200$ times light speed constraining the negative energy densities in the Natario warp bubble.
- 2)-How about to make the numerical integration by the Trapezoidal Rule using 10^{48} slices??.How about to make $n = 10^{48}$???

If $n = 10^{48}$ then n in the denominator of the fraction $\frac{b-a}{2n}$ will completely obliterate the factor $\frac{c^2}{G} \times \frac{vs^2}{8\pi}$ in the equations of the total energy integral lowering the negative energy density requirements for the Natario warp bubble.

Again a reader can see the power of the White idea.Compare the distance between the points a (beginning of the negative energy warped region) and b(end of the negative energy warped region) in the Appendices B and C.The points a and b are more close to each other in Appendix C than in the Appendix B.The thickness parameter passed from $@ = 900$ to $@ = 5000$.If we make $@ = 500.000$ or $@ = 5.000.000$ the points a and b will be so close that the difference between b and a will approach zero for real obliterating the value of 10^{48} or any other value.$(b \simeq a)(b - a \simeq 0)$

If we want a rigorous integration of the Natario negative energy density warped region shown in the Appendix C which starts at $rs = 100$ meters from the center of the bubble(bubble radius R) and ends up at $rs = 100,03$ meters then we must divide this region in 10^{48} slices each one with a width h of:

$$h = \frac{b-a}{n} = \frac{100,03 - 100}{10^{48}} = \frac{0,03}{10^{48}} = \frac{3}{10^{49}} = 3 \times 10^{-49} \quad (45)$$

Inserting these values in the total energy integral equations we should expect for:

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{3}{2 \times 10^{49}}[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \quad (46)$$

We choose to divide the Natario negative energy warped region in 10^{48} slices due to the factor $\frac{c^2}{G}\frac{v_s^2}{8\pi}$ which is 10^{48} for a bubble speed $vs = 200$ times faster than light .In the equation above keeping the bubble radius $R = 100$ meters and the thickness parameter $@ = 5000$ and the warp factor $WF = 200$ all constants and since c and G are constants if we use a different bubble velocity vs higher than 200 times faster than light giving a factor $\frac{c^2}{G}\frac{v_s^2}{8\pi} > 10^{48}$ then the number of slices n needed to integrate accurately by the Trapezoidal Rule the energy density in the Natario warp bubble must be equal to this new factor in order to reduce the total energy integral.(see also Appendix E)

5 Horizon and Infinite Doppler Blueshifts in both Alcubierre and Natario Warp Drive Spacetimes

According to pg 6 in [2] warp drives suffers from the pathology of the Horizons and according to pg 8 in [2] warp drive suffer from the pathology of the infinite Doppler Blueshifts that happens when a photon sent by an Eulerian observer to the front of the warp bubble reaches the Horizon.This would render the warp drive impossible to be physically feasible.

For a complete mathematical demonstration of the Horizon and Doppler Blueshift Problems see pg 20 section 6 in [7](basic) and pg 4 section 2 in [6](advanced).The Horizon occurs in both spacetimes.This means to say that the Eulerian observer cannot signal the front of the warp bubble wether in Alcubierre or Natario warp drive because the photon sent to signal will stop in the Horizon..The solution for the Horizon problem must be postponed until the arrival of a Quantum Gravity theory that encompasses both General Relativity and Non-Local Quantum Entanglements of Quantum Mechanics however in the next section we will present a possible solution for this problem that only encompasses General Relativity.

The infinite Doppler Blueshift happens in the Alcubierre warp drive but not in the Natario one.This means to say that Alcubierre warp drive is physically impossible to be achieved but the Natario warp drive is perfectly physically possible to be achieved.

Consider again the negative energy density distribution in the Alcubierre warp drive spacetime(see eq 8 pg 6 in [3])[12][13]:

$$\langle T^{\mu\nu}u_\mu u_\nu\rangle = \langle T^{00}\rangle = \frac{1}{8\pi}G^{00} = -\frac{1}{8\pi}\frac{v_s^2(t)[y^2+z^2]}{4r_s^2(t)}\left(\frac{df(r_s)}{dr_s}\right)^2, \tag{47}$$

And considering again the negative energy density in the Natario warp drive spacetime(see pg 5 in [2])[14][15]:

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{48}$$

In pg 6 in [2] a warp drive with a x-axis only is considered.In this case for the Alcubierre warp drive $[y^2+z^2]=0$

$$\langle T^{\mu\nu}u_\mu u_\nu\rangle = \langle T^{00}\rangle = \frac{1}{8\pi}G^{00} = -\frac{1}{8\pi}\frac{v_s^2(t)[y^2+z^2]}{4r_s^2(t)}\left(\frac{df(r_s)}{dr_s}\right)^2, = 0 \tag{49}$$

And the negative energy density is zero but the Natario energy density is not zero and given by:.

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2\right] \tag{50}$$

[12] $f(rs)$ is the Alcubierre shape function.Equation written in the Geometrized System of Units $c = G = 1$
[13] Equation written in Cartezian Coordinates
[14] $n(rs)$ is the Natario shape function.Equation written the Geometrized System of Units $c = G = 1$
[15] Equation written in Cartezian Coordinates.See Appendix A

Note that in front of the ship in the Alcubierre case the spacetime is empty but in the Natario case there exists negative energy density in the front of the ship.

According with Natario in pg 7 before section 5.2 in [14] negative energy density means a negative mass density and a negative mass generates a repulsive gravitational field. This repulsive gravitational field in front of the ship in the Natario warp drive spacetime protects the ship from impacts with the interstellar matter.The objections raised by Clark,Hiscock,Larson,McMonigal,Lewis,O'Byrne,Barcelo,Finazzi and Liberati in the introduction of this work are not valid for the Natario warp drive spacetime.

Since the Alcubierre warp drive dont have negative energy in front of the ship but only empty spacetime it doest not have protection against the interstellar medium making valid the objections raised by Clark,Hiscock,Larson,McMonigal,Lewis,O'Byrne,Barcelo,Finazzi and Liberati in the introduction of this work.

The Alcubierre shape function $f(rs)$ is defined as being 1 inside the warp bubble and 0 outside the warp bubble while being $1 > f(rs) > 0$ in the Alcubierre warped region according to eq 7 pg 4 in [1] or top of pg 4 in [2].

Expanding the quadratic term in eq 8 pg 4 in [1] and solving eq 8 for a null-like interval $ds^2 = 0$ we will have the following equation for the motion of the photon sent to the front(see pg 3 in [12] and pg 22 eqs 146 and 147 in [7])[16]:

$$\frac{dx}{dt} = vsf(rs) - 1 \tag{51}$$

Inside the Alcubierre warp bubble $f(rs) = 1$ and $vsf(rs) = vs$.Outside the warp bubble $f(rs) = 0$ and $vsf(rs) = 0$.

Somewhere inside the Alcubierre warped region when $f(rs)$ starts to decrease from 1 to 0 making the term $vsf(rs)$ decreases from vs to 0 and assuming a continuous behavior then in a given point $vsf(rs) = 1$ and $\frac{dx}{dt} = 0$.The photon stops,A Horizon is established and in the Horizon the Doppler Blueshift occurs rendering the Alcubierre warp drive impossible.This due to the fact that there are no negative energy density in the front of the Alcubierre warp drive in the x-axis to deflect the photon.

Now taking the components of the Natario vector defined in the top of pg 5 in [2] and inserting these components in the first equation of pg 2 in [2] and solving for the same null-like interval $ds^2 = 0$ considering only radial motion we will get the following equation for the motion of the photon sent to the front(see eqs 16 and 17 pg 5 in [6])[17]:

$$\frac{dx}{dt} = 2vsn(rs) - 1 \tag{52}$$

The Natario shape function $n(rs)$ is defined as being 0 inside the warp bubble and $\frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region according to pg 5 in [2].

[16]The coordinate frame for the Alcubierre warp drive as in [1] is the remote observer outside the ship

[17]The coordinate frame for the Natario warp drive as in [2] is the ship frame observer in the center of the warp bubble $xs = 0$

Inside the Natario warp bubble $n(rs) = 0$ and $2vsn(rs) = 0$.Outside the warp bubble $n(rs) = \frac{1}{2}$ and $2vsn(rs) = vs$. Somewhere inside the Natario warped region $n(rs)$ starts to increase from 0 to $\frac{1}{2}$ making the term $2vsn(rs)$ increase from 0 to vs and assuming a continuous behavior then in a given point we would have a $2vsn(rs) = 1$ and a $\frac{dx}{dt} = 0$ The photon would stops.A Horizon would be established.

However when the photon reaches the beginning of the Natario warped region it suffers a deflection by the negative energy density in front of the Natario warp drive because this negative energy is not null.So in the case of the Natario warp drive the photon never reaches the Horizon and the Natario warp drive never suffer from the pathology of the infinite Doppler Blueshift due to a different distribution of energy density when compared to its Alcubierre counterpart.This negative energy with repulsive gravitational behavior deflects the photon from inside avoiding it to reach the Horizon and protects the Natario warp drive from the dangers of collisions with the interstellar medium at superluminal speeds.

Adapted from the negative energy in Wikipedia:The free Encyclopedia:

"if we have a small object with equal inertial and passive gravitational masses falling in the gravitational field of an object with negative active gravitational mass (a small mass dropped above a negative-mass planet, say), then the acceleration of the small object is proportional to the negative active gravitational mass creating a negative gravitational field and the small object would actually accelerate away from the negative-mass object rather than towards it."

The Natario warp drive as a solution of the Einstein Field Equations of General Relativity that allows faster than light motion is the first valid candidate for interstellar space travel.

6 A causally connected superluminal Natario warp drive spacetime using micro warp bubbles

In 2002 Gauthier,Gravel and Melanson appeared with the idea of the micro warp bubbles.([15],[16])

According to them,microscopical particle-sized warp bubbles may have formed spontaneously immediately after the Big Bang and these warp bubbles could be used to transmit information at superluminal speeds.These micro warp bubbles may exist today.(see abs of [16])

A micro warp bubble with a radius of 10^{-10} meters could be used to transport an elementary particle like the electron whose Compton wavelength is 2.43×10^{-12} meters at a superluminal speed.These micro warp bubbles may have formed when the Universe had an age between the Planck time and the time we assume that Inflation started.(see pg 306 of [15])

Following the ideas of Gauthier,Gravel and Melanson ([15],[16]) a micro warp bubble can send information or particles at superluminal speeds.(abs of [16],pg 306 in [15]).Since the infinite Doppler Blueshift affect the Alcubierre warp drive but not the Natario one and a superluminal micro warp bubble can only exists without Infinite Doppler Blueshifts[18] we consider in this section only the Natario warp drive spacetime.

The idea of Gauthier,Gravel and Melanson ([15],[16]) to send information at superluminal speeds using micro warp bubbles is very interesting and as a matter of fact shows to us how to solve the Horizon problem. Imagine that we are inside a large superluminal warp bubble and we want to send information to the front.Photons sent from inside the bubble to the front would stop in the Horizon but we also know that incoming photons from outside would reach the bubble.[19] The external observer outside the bubble have all the bubble causally connected while the internal observer is causally connected to the point before the Horizon.Then the external observer can create the bubble while the internal observer cannot. This was also outlined by Everett-Roman in pg 3 in [12]. Unless we find a way to overcome the Horizon problem. We inside the large warp bubble could create and send one of these micro warp bubbles to the front of the large warp bubble but with a superluminal speed $vs2$ larger than the large bubble speed $X = 2vsn(rs)$.Then $vs2 >> X$ or $vs2 >> 2vsn(rs)$ and this would allow ourselves to keep all the warp bubble causally connected from inside overcoming the Horizon problem without the need of the "tachyonic" matter.

- 1)- Superluminal micro warp bubble sent towards the front of the large superluminal warp bubble $vs2 = \frac{dx}{dt} > X - 1 > vs - 1 \dashrightarrow X = 2vsn(rs)$

From above it easy to see that a micro warp bubble with a superluminal speed $vs2$ maintains a large superluminal warp bubble with speed vs causally connected from inside if $vs2 > vs$

[18] assuming a continuous growth of the warp bubble speed vs from zero to a superluminal speed at a given time the speed will be equal to c and the Infinite Doppler Blueshift crashes the bubble.The Alcubierre warp drive can only exists for $vs < c$ so it cannot sustain a micro warp bubble able to shelter particles or information at superluminal speeds

[19] true for the Alcubierre warp drive but not for the Natario one because the negative energy density in the front with repulsive gravitational behavior would deflect all the photons sent from inside and outside the bubble effectively shielding the Horizon from the photon avoiding the catastrophical Infinite Doppler Blueshift

From the point of view of the astronaut inside the large warp bubble he is the internal observer with respect to the large warp bubble but he is the external observer from the point of view of the micro warp bubble so he keeps all the light-cone of the micro warp bubble causally connected to him so he can use it to send superluminal signals to the large warp bubble from inside.(Everett-Roman in pg 3 in [12]).

Gauthier,Gravel and Melanson developed the concept of the micro warp bubble but the idea is at least 5 years younger.The first time micro warp bubbles were mentioned appeared in the work of Ford-Pfenning pg 10 and 11 in [3]

7 Conclusion

In this work we demonstrated that the analysis of Harold White can be applied also to the Natario warp drive spacetime. From 10 times the mass of the Universe manipulating the parameter @ we lowered the negative energy density requirements to arbitrary low levels.

White point of view is entirely correct:by manipulating the parameter @ in the Alcubierre equations he lowered the negative energy density requirements to arbitrary low levels and we matched his results.

However the objections raised by Clark,Hiscock,Larson,McMonigal,Lewis,O'Byrne,Barcelo,Finazzi and Liberati in the introduction of this work are valid for the Alcubierre warp drive so it can be regarded as physically impossible independently from the arbitrary lower levels of negative energy White can obtain for it.

On another way these objections do not affect the Natario warp drive which is perfectly possible to be achieved.This was the main reason behind our interest in the reproduction of the White analysis for the Natario warp drive spacetime.

The Natario warp drive once created can survive against all the obstacles pointed as physical impossibilities that rules out the warp drive as a dynamical spacetime.

Lastly and in order to terminate this work:There exists another problem not covered here:the fact that we still dont know how to generate the negative energy density and negative mass and above everything else we dont know how to generate the shape function that distorts the spacetime geometry creating the warp drive effect.So unfortunately all the discussions about warp drives are still under the domain of the mathematical conjectures.

However we are confident to affirm that the Natario warp drive will survive the passage of the Century XXI and will arrive to the Future.The Natario warp drive as the first Human candidate for faster than light interstellar space travel will arrive to the the Century $XXIV$ on-board the future starships up there in the middle of the stars helping the human race to give his first steps in the exploration of our Galaxy

Live Long And Prosper

8 Appendix A:The Natario Warp Drive Negative Energy Density in Cartezian Coordinates

The negative energy density according to Natario is given by(see pg 5 in [2])[20]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{53}$$

In the bottom of pg 4 in [2] Natario defined the x-axis as the polar axis.In the top of page 5 we can see that $x = rs\cos(\theta)$ implying in $\cos(\theta) = \frac{x}{rs}$ and in $\sin(\theta) = \frac{y}{rs}$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{54}$$

Considering motion in the equatorial plane of the Natario warp bubble (x-axis only) then $[y^2 + z^2] = 0$ and $rs^2 = [(x - xs)^2]$ and making $xs = 0$ the center of the bubble as the origin of the coordinate frame for the motion of the Eulerian observer then $rs^2 = x^2$ because in the equatorial plane $y = z = 0$.

Rewriting the Natario negative energy density in cartezian coordinates in the equatorial plane we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{55}$$

[20] $n(rs)$ is the Natario shape function.Equation written in the Geometrized System of Units $c = G = 1$

9 Appendix B:Alcubierre and Natario warped regions with @ = 900 $R = 100$ meters $\delta = 0,01$ meters WF=200

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$99,81$	1	0	$2,838699289278E-291$	0
$99,82$	1	0	$1,223828992894E-275$	0
$99,83$	1	0	$5,276210162534E-260$	0
$99,84$	1	0	$2,274696370235E-244$	0
$99,85$	1	0	$9,806742751644E-229$	0
$99,86$	1	0	$4,227913872611E-213$	0
$99,87$	1	0	$1,822751566642E-197$	0
$99,88$	1	0	$7,858304056807E-182$	0
$99,89$	1	0	$3,387896835715E-166$	0
$99,90$	1	0	$1,460600771678E-150$	0
$99,91$	1	0	$6,296988124713E-135$	0
$99,92$	1	0	$2,714777385557E-119$	0
$99,93$	1	0	$1,170403390823E-103$	0
$99,94$	1	0	$5,045880021458E-088$	0
$99,95$	1	0	$2,175395713187E-072$	0
$99,96$	1	0	$9,378634626322E-057$	0
$99,97$	1	0	$4,043346547061E-041$	0
$99,98$	1	0	$1,743180318993E-025$	0
$99,99$	$9,9E-001$	0	$7,515253525863E-010$	0
100	$4,9E-001$	$3,111507925458E-061$	$2,025000000000E+005$	$3,136800025852E-111$
$100,01$	$1,5E-008$	$4,999984770044E-001$	$7,515253498181E-010$	$7,515207944370E-006$
$100,02$	0	$0,5$	$1,743180312572E-025$	$1,743180312572E-021$
$100,03$	0	$0,5$	$4,043346532168E-041$	$4,043346532168E-037$
$100,04$	0	$0,5$	$9,378634591777E-057$	$9,378634591777E-053$
$100,05$	0	$0,5$	$2,175395705174E-072$	$2,175395705174E-068$
$100,06$	0	$0,5$	$5,045880002871E-088$	$5,045880002871E-084$
$100,07$	0	$0,5$	$1,170403386511E-103$	$1,170403386511E-099$
$100,08$	0	$0,5$	$2,714777375558E-119$	$2,714777375558E-115$
$100,09$	0	$0,5$	$6,296988101519E-135$	$6,296988101519E-131$
$100,1$	0	$0,5$	$1,460600766298E-150$	$1,460600766298E-146$
$100,11$	0	$0,5$	$3,387896823236E-166$	$3,387896823236E-162$
$100,12$	0	$0,5$	$7,858304027861E-182$	$7,858304027861E-178$
$100,13$	0	$0,5$	$1,822751559928E-197$	$1,822751559928E-193$
$100,14$	0	$0,5$	$4,227913857038E-213$	$4,227913857038E-209$
$100,15$	0	$0,5$	$9,806742715522E-229$	$9,806742715522E-225$
$100,16$	0	$0,5$	$2,274696361856E-244$	$2,274696361856E-240$
$100,17$	0	$0,5$	$5,276210143100E-260$	$5,276210143100E-256$
$100,18$	0	$0,5$	$1,223828988386E-275$	$1,223828988386E-271$
$100,19$	0	$0,5$	$2,838699278822E-291$	$2,838699278822E-287$

According with Alcubierre any function $f(rs)$ that gives 1 inside the bubble and 0 outside the bubble while being $1 > f(rs) > 0$ in the Alcubierre warped region is a valid shape function for the Alcubierre warp drive.(see eqs 6 and 7 pg 4 in [1] or top of pg 4 in [2]).

According with Natario(pg 5 in [2]) any function $n(rs)$ that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive. (see eq 38 pg 9 and eqs 39 and 40 pg 10 in [4]).

The table in the previous page depicts two warp bubbles(one for Alcubierre and another for Natario) placed side by side in the same plot each one with 100 meters of radius $R = 100$ meters and a dimensionless thickness parameter $@ = 900$ and not $@ = 900\ meters$ or $@ = 900\ meters^{-1}$.The Natario bubble have a dimensionless warp factor $WF = 200$.

For two Eulerian observers one in the center of the Alcubierre bubble and the other in the center of the Natario bubble we verify that $rs = 0$ and $x = xs$ being xs the center of the bubble in both cases.The expression for rs is given by:

$$rs = \sqrt{(x - xs)^2 + y^2 + z^2} \tag{56}$$

Inside the Alcubierre bubble we can see that $f(rs) = 1$ and inside the Natario bubble we can see that $n(rs) = 0$.The derivative of each shape function is zero in the neighborhoods of each bubble center and at a faraway distance from each bubble radius meaning flat spacetime.Eulerian observers moving themselves inside each bubble at a distance rs from the center of each bubble remains in flat spacetime if rs is less than the bubble radius R or $rs << R$.The expressions for the Alcubierre and Natario shape functions and their derivatives are given by:

- Alcubierre shape function and its derivatives(see eqs 6 and 7 pg 4 in [1] or top of pg 4 in [2])[21].

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{57}$$

$$f'(rs) = \frac{1}{2}[-\frac{@}{cosh^2[@(rs - R)]}] \tag{58}$$

$$f'(rs)^2 = \frac{1}{4}[\frac{@^2}{cosh^4[@(rs - R)]}] \tag{59}$$

- Natario shape function and its derivatives(see eq 38 pg 9 and eqs 39 and 40 pg 10 in [4]).

$$n(rs) = [\frac{1}{2}][1 - f(rs)]^{WF} \tag{60}$$

$$n'(rs) = -[\frac{1}{2}]WF[1 - f(rs)]^{WF-1}f'(rs) \tag{61}$$

$$n'(rs)^2 = [\frac{1}{4}]WF^2[1 - f(rs)]^{2(WF-1)}f'(rs)^2 \tag{62}$$

[21] $tanh[@(rs * R)] = 1$,$tanh(@R) = 1$.See section 2

The negative energy density distribution in the Alcubierre warp drive spacetime is given by the following expression(see eq 9 pg 3 in [4])[22][23]:

$$\langle T^{\mu\nu} u_\mu u_\nu \rangle = -\frac{c^2}{G}\frac{1}{8\pi}\frac{v_s^2(t)[y^2+z^2]}{4r_s^2(t)}\left(\frac{df(r_s)}{dr_s}\right)^2, \tag{63}$$

The negative energy density distribution in the Natario warp drive spacetime is given by the following expression[24][25](see eqs 21 and 23 pg 6 in [4]):

$$\rho = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{64}$$

Note that in both equations the term given below appears:

$$\rho = -\frac{c^2}{G}\frac{v_s^2}{8\pi} \tag{65}$$

Considering a speed of 200 times faster than light $3 \times 10^8 \times 2 \times 10^2 = 6 \times 10^{10}$ for both bubbles the value of this term[26] is:

$c^2 = (3 \times 10^8)^2 = 9 \times 10^{16}$ being divided by $6,67 \times 10^{-11}$ giving $1,35 \times 10^{27}$ and this is multiplied by $(6 \times 10^{10})^2 = 36 \times 10^{20}$ giving $1,35 \times 10^{27} \times 36 \times 10^{20} = 1,35 \times 10^{27} \times 3,6 \times 10^{21} = 4,86 \times 10^{48}$!!!

A number with 48 zeros!!!The planet Earth have a mass[27] of about $6 \times 10^{24} kg$.

This term is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!or better:The amount of negative energy density needed to sustain a warp bubble at a speed of 200 times faster than light requires the magnitude of the masses of 1.000.000.000.000.000.000.000.000 planet Earths for both Alcubierre and Natario cases!!!!

Unless we can use very low derivatives close to zero $\simeq 0$ in the respective expressions for the negative energy density in both cases in order to obliterate this term.

Now lets back to our Eulerian observers in the center of each bubble.In a given time they leave their centers and starts to move inside each bubble towards each bubble radius R at 100 meters from the center($rs = 0$).The "surface" of the warp bubble lies at $R = 100$ meters.At 99 meters from the center of each bubble $rs = 99$ meters and $rs < R$ both observers are still in flat spacetime because the derivatives of the respective shape functions are zero in this region.

But what happens when both Eulerian observers rs approaches respectively each bubble radius R??What happens when $rs \simeq R$??

[22] $f(rs)$ is the Alcubierre shape function.Equation written in the International System of Units
[23] Equation written in Cartezian Coordinates
[24] $n(rs)$ is the Natario shape function.Equation written in the International System of Units
[25] Equation written in Cartezian Coordinates
[26] in units of the International System
[27] see Wikipedia:The free Encyclopedia

Since we have two different Eulerian observers in different warp bubbles(one for Alcubierre and another for Natario) we must analyze the physical situation of each observer separately:

- Eulerian observer rs inside the Alcubierre warp bubble approaching the bubble radius $R(rs \simeq R)$. $R = 100$ meters.

From the center of the warp bubble ($rs = 0$) to a distance of 99,80 meters from the center $rs = 99,80$ the spacetime is flat.The derivatives of the Alcubierre shape function are zero and the observer lies well inside the bubble($f(rs) = 1$).

However at 99,81 meters from the center $rs = 99,81$ and still well inside the bubble ($f(rs) = 1$) the derivative square of the Alcubierre shape function ceases to be zero.It have a very small value but it is not zero meaning that this region is no longer flat spacetime.Then we have negative energy density still inside the bubble.This derivative square grows from 10^{-291} when $rs = 99,81$ meters to 10^{-25} when $rs = 99,98$ meters and still inside the bubble ($f(rs) = 1$).Remember that the negative energy density means that this derivative square must be multiplied by 10^{48} and $10^{48} \times 10^{-25} = 10^{23}$ giving a negative energy density of about 10^{23} when $rs = 99,98$ meters almost the magnitude of the mass of the Earth and still inside the bubble($f(rs) = 1$).!!!!

This negative energy density of 10^{23} inside the bubble and the other non-null negative energy densities also inside the bubble resulting from the multiplication of the other derivative squares by 10^{48} must be taken into account when computing the total energy requirements needed to sustain a warp bubble(total energy integral).

From $rs = 99,99$ meters to $rs = 100,01$ meters we can see the Alcubierre warped region where $1 > f(rs) > 0$.Exactly in the center of the Alcubierre warped region when $rs = R$ the derivative square reaches its maximum value of 10^5.Note that 100 is not only the value of the bubble radius R.It is exactly the midpoint between $rs = 99,99$ and $rs = 100,01$ the delimiters of the Alcubierre warped region.Note that the values of the derivative squares when $rs = 99,99$ and $rs = 100,01$ are almost equal. Almost symmetric with respect to the value of the derivative square when $rs = R = 100$.

From $rs = 100,02$ meters to $rs = 100,19$ meters we have a region outside the bubble where the derivative square is not zero meaning a non-flat spacetime and a region of non-null negative energy density outside the bubble($f(rs) = 0$). As far as we move away from the bubble radius the derivative squares decreases almost by the same way it increased when we approached the bubble radius.

Beyond 100,19 meters the square of the derivatives are zero again and we recover flat spacetime.

These non-null negative energy densities outside the bubble must also be taken into account when computing the total energy integral.

Then we can see that the Alcubierre warp drive have two warped regions:

- 1)-The region where $1 > f(rs) > 0$ separating the interior of the bubble($f(rs) = 1$) from the exterior of the bubble($f(rs) = 0$)
- 2)-The regions around the bubble radius from inside the bubble($f(rs) = 1$) to outside the bubble ($f(rs) = 0$) where the negative energy is not null.The total energy integral must encompass these negative energy densities inside and outside the bubble and not only the ones when $1 > f(rs) > 0$.

Since we have two different Eulerian observers in different warp bubbles(one for Alcubierre and another for Natario) we must analyze the physical situation of each observer separately.Now it is the time for the Natario case:

- Eulerian observer rs inside the Natario warp bubble approaching the bubble radius $R(rs \simeq R)$. $R = 100$ meters.

In the Natario bubble there are no negative energy densities inside the bubble where $n(rs) = 0$ because the derivatives of the negative energy density are zero inside the bubble.So an Eulerian observer can go from the center of the bubble($rs = 0$) to the end of the region inside the bubble still with $n(rs) = 0$ at $rs = 99,99$ meters always in flat spacetime.

The Natario warped region where $0 < n(rs) < \frac{1}{2}$ begins at $rs = 100$ meters exactly the bubble radius and ends up at $rs = 100,01$ meters.Note that the Natario warped region is thinner than its Alcubierre counterpart.

At $rs = 100,02$ meters the region outside the bubble where $n(rs) = \frac{1}{2}$ begins.

The geometric form of this Natario bubble is due to the choice we made for the Natario shape function.The details are explained in pg 9 to 13 in [4].

Now we must examine the derivative squares of the Natario shape function:The derivative squares starts to be non-null exactly in the bubble radius:the beginning of the Natario warped region with a low value of 10^{-111} and terminates with a value of 10^{-6} in the end of the Natario warped region at $rs = 100,01$ meters.So as far as we approach the end of the Natario warped region from the bubble radius the derivative squares grows from 10^{-111} to 10^{-6} .

Note that when $rs = 100,02$ meters where the region outside the bubble begins the derivative square is 10^{-21} and as far as we move away from the bubble the derivative squares decreases and beyond $rs = 100,19$ the derivative squares are zero again and we recover flat spacetime.

Note also that when we multiply 10^{48} by 10^{-21} we will get 10^{27} a magnitude of the masses of 1000 Earths!!!. So we have huge concentrations of negative energy densities also in the Natario warp drive like its Alcubierre counterpart but in this case these concentrations are outside the bubble and not in the middle of the warped region.

These negative energy densities outside the bubble must be taken into account when computing the total energy integral.

Then we can see that the Natario warp drive like its Alcubierre counterpart have two warped regions:

- 1)-The region where $0 < n(rs) < \frac{1}{2}$ separating the interior of the bubble($n(rs) = 0$) from the exterior of the bubble($n(rs) = \frac{1}{2}$)
- 2)-The region outside the bubble where the negative energy is not null.The total energy integral must encompass these negative energy densities outside the bubble and not only the ones when $0 < n(rs) < \frac{1}{2}$

Between $rs = 100,01$ meters where the derivative square is 10^{-6} the end of the Natario warped region and $rs = 100,02$ meters where the derivative square is 10^{-21} the beginning of the region outside the bubble the derivative square do not decreases directly from 10^{-6} to 10^{-21}.As far as we move away from the end of the Natario warped region in the transition region between $rs = 100,01$ to $rs = 100,02$ the square derivative continues to grow to reach irs maximum value of 10^{5} in an inflexion point located somewhere between $rs = 100,01$ meters to $rs = 100,02$ meters and passing this inflection point the derivative square decreases to 10^{-21} when $rs = 100,02$ meters.This will be addressed in the next Appendices.

10 Appendix C:Alcubierre and Natario warped regions with $@ = 5000$ $R = 100$ meters $\delta = 0,01$ meters (WF=200)

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$99,97$	1	0	$2,650396579309E-253$	0
$99,98$	1	0	$1,915169615918E-166$	0
$99,99$	1	0	$1,383896540755E-079$	0
100	$4,9E-001$	$3,111507925458E-061$	$6,250000000000E+006$	$9,681489643510E-110$
$100,01$	0	$0,5$	$1,383896512435E-079$	$1,383896512435E-075$
$100,02$	0	$0,5$	$1,915169576726E-166$	$1,915169576726E-162$
$100,03$	0	$0,5$	$2,650396525072E-253$	$2,650396525072E-249$

The plots in the Appendices B and C illustrate the power of White idea.By manipulation of the dimensionless thickness parameter @ in the Alcubierre and Natario shape functions we can reduce the negative energy density requirements to sustain a warp bubble.

In Appendix B a plot for two warp bubbles one for Alcubierre and another for Natario with a bubble radius $R = 100$ meters using a dimensionless thickness parameter $@ = 900$ and not $@ = 900$ meters was presented.The warped region for negative energy density starts at $rs = 99,81$ meters and ends up at $rs = 100,19$ meters for Alcubierre and for Natario the negative energy density warped region starts at $rs = 100$ meters and ends up at $rs = 100,19$ meters.

Now for a plot for the same warp bubbles with a dimensionless thickness parameter $@ = 5000$ and not $@ = 5000$ meters the warped region for negative energy density is more thicker or thinner.It starts at $rs = 99,97$ meters and ends up at $rs = 100,03$ meters for Alcubierre and starts at $rs = 100$ meters and ends up at $100,03$ meters for Natario.

Note that the points a and b where the warped region for negative energy begins(a) and ends(b) are now much more close from each other in the Appendix C when compared to the same points in the Appendix B.The difference between a and b is much reduced when the dimensionless thickness parameter is $@ = 5000$.

As higher the thickness parameter @ is as smaller the difference between the points of the beginning and the end of the warped region for negative energy becomes.As higher the thickness parameter @ is as thicker or thinner the warped region for negative energy becomes.As higher the thickness parameter @ is as close the points a (beginning of the negative energy warped region) and b(end of the negative energy warped region) are from each other.($@ >> 1 \dashrightarrow b \simeq a \dashrightarrow (b-a) \simeq 0$).

Consider a function:

$$y = F(x) \tag{66}$$

its derivative

$$f(x) = \frac{dF(x)}{dx} \tag{67}$$

The primitive

$$\int f(x)dx = F(x) + C \tag{68}$$

And the definite integral between the regions a and b

$$\int_a^b f(x)dx = [F(b) - F(a)] \tag{69}$$

Independently of the values of $f(x)$ the derivative of $F(x)$ in the points a and b if the point b is close to a ($b \cong a$) then $F(b)$ is close to $F(a)$ ($F(b) \simeq F(a)$) and the difference between $F(b)$ and $F(a)$ is close to zero ($F(b) - F(a) \simeq 0$) making the result of the definite integral between a and b also close to zero.

$$\int_a^b f(x)dx \simeq 0 \tag{70}$$

This illustrates how the White idea works.By manipulation of the thickness parameter @ he approaches the points a and b the beginning and the end of the negative energy warped region and the total energy integral(the integration of all non-null negative energy densities-squares of the derivatives of the shape functions in the negative energy warped region that starts at point a and ends at point b) is also close to zero.

The negative energy requirements to sustain a warp bubble with a thickness parameter @ = 5000 are much smaller than the ones to sustain a warp bubble with a thickness parameter @ = 900.

By making @ = 50.000 or @ = 500.000 or @ = 5.000.000 we can reduce the negative energy requirements even further.[28]

For our particular choice of the Natario shape function no primitive is known so we must integrate the derivative squares of the Natario shape function numerically using the Trapezoidal Rule.(see pg 4 in [18])[29]

$$\int_a^b f(x)dx = (b-a)\frac{f(b)+f(a)}{2} \cong 0 \dashrightarrow b-a \cong 0 \dashrightarrow b \cong a \dashrightarrow f(b) \cong f(a) \tag{71}$$

Note that when b(end of the negative energy warped region) is close to a(beginning of the negative energy warped region)($b \simeq a$) the thickness of the negative energy warped region is very small and the difference between b and a is close to zero ($b - a \simeq 0$) independently of the values of $f(b)$ and $f(a)$.

[28] we are limited by the floating point precision of Microsoft Excel or Open Office

[29] see Wikipedia the Free Encyclopedia

11 Appendix D:Natario warped region with $@ = 5000$ $R = 100$ meters $\delta = 1,000000000005E-003$ meters.End of the warp bubble and beginning of the region outside the bubble (WF=200)

.

rs	$n(rs)$	$n'(rs)^2$	Tpz
$1,00000000000E+002$	$3,111507638931E-061$	$9,681479787123E-110$	NA
$1,00001000000E+002$	$4,954806584405E-001$	$2,023878013030E+003$	$1,011939006520E+000$
$1,00002000000E+002$	$4,999997938847E-001$	$4,248350734364E-006$	$1,011939008644E+000$
$1,00003000000E+002$	$4,999999999906E-001$	$8,756510759858E-015$	$2,124175371571E-009$
$1,00004000000E+002$	$5,000000000000E-001$	$1,804851387156E-023$	$4,378255388974E-018$
$1,00005000000E+002$	$5,000000000000E-001$	$3,720075974245E-032$	$9,024256954423E-027$
$1,00006000000E+002$	$5,000000000000E-001$	$7,667648069329E-041$	$1,860037990965E-035$
$1,00007000000E+002$	$5,000000000000E-001$	$1,580420062475E-049$	$3,833824042585E-044$
$1,00008000000E+002$	$5,000000000000E-001$	$3,257488529719E-058$	$7,902100312411E-053$
$1,00009000000E+002$	$5,000000000000E-001$	$6,714184282441E-067$	$1,628744268224E-061$
$1,00010000000E+002$	$5,000000000000E-001$	$1,383896525415E-075$	$3,357092148156E-070$

The plot above presents the regions of the Natario warp drive from the end of the warp bubble[30] to the beginning of the regions outside the bubble.

Like in the previous Appendices B and C the Natario warped region starts at $rs = 1,00000000000E+002$ meters the bubble radius and ends at $rs = 1,00003000000E+002$ meters.The region starting at the point $rs = 1,00004000000E+002$ meters and beyond depicts the regions outside the bubble.

Note that the region outside the bubble depicted here lies in the neighborhoods of the bubble radius at short distances so the negative energy density warped region(the region where the squares of the derivatives of the Natario shape function are non-null) encompasses these regions outside the bubble.

Note also that as far as we move away from the bubble radius in the region outside the bubble the derivative squares decreases as the distance to the bubble radius increases.

Note also that from $rs = 1,00000000000E+002$ meters to $rs = 1,00001000000E+002$ meters the derivative square increases from $n'(rs)^2 = 9,681479787123E-110$ to $n'(rs)^2 = 2,023878013030E+003$

When $rs = 1,00002000000E+002$ meters the square of the derivative is $n'(rs)^2 = 4,248350734364E-006$. This do not means that the derivative square decreases linearly from 10^3 to 10^{-6} in the region between $rs = 1,00001000000E+002$ to $rs = 1,00002000000E+002$ .As a matter of fact somewhere between these points the derivative square reaches the maximum value of 10^6 in an inflexion point and after this point starts to decrease to reach 10^{-6}.

[30] from the point where rs the distance travelled by an Eulerian observer starting in the center of the bubble $rs = 0$ reaches the bubble radius $rs = R$ and leaves the bubble "surface" located at R and enter in the regions outside the bubble where $rs > R$.

If we want to integrate the derivative squares of the Natario shape function in the regions depicted by the plot shown[31] which means to say the regions between $rs = 1,00000000000E + 002$ to $rs = 1,00010000000E + 002$ where the derivative squares are respectively $n'(rs)^2 = 9,681479787123E - 110$ and $n'(rs)^2 = 1,383896525415E - 075$ we can apply the direct formula of numerical integration by the Trapezoidal Rule.(see pg 4 in [18]).The point b corresponds to $rs = 1,00010000000E + 002$ and the point a corresponds to $rs = 1,00000000000E + 002$ and the functions $f(b)$ and $f(a)$ corresponds to the squares of the derivatives in these points.

If the point b(end of the warped region) is too much close to the point a(the beginning of the warped region) and this happens in White approach when the value of the thickness parameter @ is high(see Appendices B and C and compare the negative energy warped regions) then the Trapezoidal Rule can be written as:

$$\int_a^b f(x)dx = (b-a)\frac{f(b)+f(a)}{2} \cong 0 \dashrightarrow b-a \cong 0 \dashrightarrow b \cong a \dashrightarrow f(b) \cong f(a) \tag{72}$$

And the value of the definite integral-the total amount of negative energy needed to sustain a warp bubble is close to zero.

However there exists an error margin in the integration by the Trapezoidal Rule.The correct procedure is to decompose the area to be integrated in slices and integrate numerically still by the Trapezoidal Rule separately each slice and in the end we sum the result of all the slices integrated.In the plot we have 11 rows.Each row is a slice.This method is known as the Composite Trapezoidal Rule.The formula for the Composite Trapezoidal Rule is given by:(see pg 15 in [18])

$$\int_a^b f(x)dx = \int_a^{a_1} f(x)dx + \int_{a_1}^{a_2} f(x)dx + \int_{a_2}^{a_3} f(x)dx + \int_{a_3}^{a_4} f(x)dx + \int_{a_4}^{b} f(x)dx \tag{73}$$

In the formula above we divided the region between a and b in 5 slices:(a to $a1$)($a1$ to $a2$)($a2$ to $a3$)($a3$ to $a4$) and ($a4$ to b).If a is close to b then $b \cong a$ and each slice boundary limits are close to the ones of the next slice.Then we have:

$$\int_a^{a_1} f(x)dx = (a_1-a)\frac{f(a_1)+f(a)}{2} \cong 0 \dashrightarrow a_1-a \cong 0 \dashrightarrow a_1 \cong a \dashrightarrow f(a_1) \cong f(a) \tag{74}$$

$$\int_{a_1}^{a_2} f(x)dx = (a_2-a_1)\frac{f(a_2)+f(a_1)}{2} \cong 0 \dashrightarrow a_2-a_1 \cong 0 \dashrightarrow a_2 \cong a_1 \dashrightarrow f(a_2) \cong f(a_1) \tag{75}$$

$$\int_{a_2}^{a_3} f(x)dx = (a_3-a_2)\frac{f(a_3)+f(a_2)}{2} \cong 0 \dashrightarrow a_3-a_2 \cong 0 \dashrightarrow a_3 \cong a_2 \dashrightarrow f(a_3) \cong f(a_2) \tag{76}$$

$$\int_{a_3}^{a_4} f(x)dx = (a_4-a_3)\frac{f(a_4)+f(a_3)}{2} \cong 0 \dashrightarrow a_4-a_3 \cong 0 \dashrightarrow a_4 \cong a_3 \dashrightarrow f(a_4) \cong f(a_3) \tag{77}$$

$$\int_{a_4}^{b} f(x)dx = (b-a_4)\frac{f(b)+f(a_4)}{2} \cong 0 \dashrightarrow b-a_4 \cong 0 \dashrightarrow b \cong a_4 \dashrightarrow f(b) \cong f(a_4) \tag{78}$$

[31]Although this is not the complete Natario negative energy warped region depicted in the Appendix C,This is only a slice for illustrative purposes

As higher the number of slices is as more precise the result of the integral by the Trapezoidal Rule is.(see pgs 14 and 15 in [18]).Remember from the Appendix B that a warp bubble wether in the Alcubierre or Natario cases with a radius of 100 meters moving at 200 times light speed have the total amount of negative energy equal to the product of 10^{48} by the integral of the square derivatives of the shape function in the region between the point b(end of the warped region) and the point a(beginning of the warped region)

The total energy required to sustain the warp bubble in the Natario case of the Equatorial plane[32] for a Natario negative energy warped region that starts at point a and end at point b is:

$$E = \int_a^b \rho drs \tag{79}$$

ρ is the negative energy density written in Cartezian Coordinates in the International System(SI) of units:

$$\rho = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3n'(rs)^2\right] \tag{80}$$

Then the total energy is given by:

$$E = \int_a^b \rho drs = \int_a^b -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3n'(rs))^2\right] drs = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\int_a^b \left[n'(rs)^2\right] drs \tag{81}$$

Since c G and vs are constants[33]

Applying the Trapezoidal Rule to the integral of the derivative squares of the Natario shape function we have:

$$\int_a^b \left[(n'(rs)^2\right] drs = (b-a)\frac{n'(b)^2 + n'(a)^2}{2} \tag{82}$$

And the total energy integral needed to sustain a warp bubble becomes:

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}(b-a)\frac{n'(b)^2 + n'(a)^2}{2} \simeq 0 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \tag{83}$$

Again a reader can see the power of the White idea.Compare the distance between the points a (beginning of the negative energy warped region) and b(end of the negative energy warped region) in the Appendices B and C.The points a and b are more close to each other in Appendix C than in the Appendix B.The thickness parameter passed from $@ = 900$ to $@ = 5000$.If we make $@ = 500.000$ or $@ = 5.000.000$ the points a and b will be so close that the difference between b and a will approach zero for real obliterating the value of 10^{48} or any other value.$(b \simeq a)(b-a \simeq 0)$

We could theoretically have a thickness of $@ = 5.000.000.000$ and a speed $vs = 6000$ times light speed enough to reach Kepler-22 at 600 light-years in months not in years using an incredible small amount of negative energy because the difference between b and a would obliterate everything since a and b would be infinitely close to each other.[34]

[32] see Appendix A

[33] both Alcubierre and Natario warp drives are spacetimes of constsant bubble speed vs.A real warp drive must accelerate or de-accelerate meaning a variable vs

[34] we are limited by the floating point precision of Microsoft Excel the only program we have.

Back again to the Trapezoidal Rule to the integral of the derivative squares of the Natario shape function(we did not finished yet):

$$\int_a^b \left[(n'(rs)^2\right] drs = (b-a)\frac{n'(b)^2 + n'(a)^2}{2} \tag{84}$$

As pointed out before if we want to integrate from the point a to point b using the Trapezoidal Rule reducing the error margin we must divide the region between a and b in slices and integrate separately each slice also by the Trapezoidal Rule and in the end we sum the result of all the integrations.As higher the number of slices as accurate the integration becomes.

Following pg 14 and 15 in [18] if we want to divide the region between a and b in n slices each slice have a width given by:

$$h = \frac{b-a}{n} \tag{85}$$

And the final integration is the sum of the integration of all the slices by the Trapezoidal Rule given by:

$$\int_a^b f(x)dx = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \int_{a+2h}^{a+3h} f(x)dx + ... + \int_{a+(n-2)h}^{a+(n-1)h} f(x)dx + \int_{a+(n-1)h}^{b} f(x)dx \tag{86}$$

Note that each slice above have a width h.

Writing the integral using sums we get the following expressions(pg 15 in [18]):

$$\int_a^b f(x)dx = \frac{1}{2n}(b-a)[f(b) + f(a + \sum_{i=1}^{n-1} f(a+ih)] \cong 0 \dashrightarrow b-a \cong 0 \dashrightarrow b \cong a \dashrightarrow n >> 1 \tag{87}$$

$$\int_a^b f(x)dx = \frac{b-a}{2n}[f(b) + f(a + \sum_{i=1}^{n-1} f(a+ih)] \cong 0 \dashrightarrow b-a \cong 0 \dashrightarrow b \cong a \dashrightarrow n >> 1 \tag{88}$$

For the integral of the square derivative of the Natario shape function by the Trapezoidal Rule in the region between a and b divided by n slices of width h we get:

$$\int_a^b n'(rs)^2 drs = \int_a^{a+h} n'(rs)^2 drs + \int_{a+h}^{a+2h} n'(rs)^2 drs + ... + \int_{a+(n-2)h}^{a+(n-1)h} n'(rs)^2 drs + \int_{a+(n-1)h}^{b} n'(rs)^2 drs \tag{89}$$

$$\int_a^b n'(rs)^2 drs = \frac{1}{2n}(b-a)[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \cong 0 \dashrightarrow b-a \cong 0 \dashrightarrow b \cong a \dashrightarrow n >> 1 \tag{90}$$

$$\int_a^b n'(rs)^2 drs = \frac{b-a}{2n}[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \cong 0 \dashrightarrow b-a \cong 0 \dashrightarrow b \cong a \dashrightarrow n >> 1 \tag{91}$$

Rewriting the equations of the total energy integral in the Natario warp drive spacetime using the Composite Trapezoidal Rule we get:

$$E = \int_a^b \rho drs = \int_a^b -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3n'(rs))^2\right] drs = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\int_a^b \left[n'(rs)^2\right] drs \tag{92}$$

The integration is being taken between the point a(beginning of the negative energy density warped region) and b(end of the negative energy density warped region).Dividing the region between a and b in n slices we get:

$$h = \frac{b-a}{n} \tag{93}$$

$$\int_a^b n'(rs)^2 drs = \int_a^{a+h} n'(rs)^2 drs + \int_{a+h}^{a+2h} n'(rs)^2 drs + ... + \int_{a+(n-2)h}^{a+(n-1)h} n'(rs)^2 drs + \int_{a+(n-1)h}^{b} n'(rs)^2 drs \tag{94}$$

$$\int_a^b n'(rs)^2 drs = \frac{1}{2n}(b-a)[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \cong 0 \dashrightarrow b-a \cong 0 \dashrightarrow b \cong a \dashrightarrow n >> 1 \tag{95}$$

$$\int_a^b n'(rs)^2 drs = \frac{b-a}{2n}[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \cong 0 \dashrightarrow b-a \cong 0 \dashrightarrow b \cong a \dashrightarrow n >> 1 \tag{96}$$

Inserting the Composite Trapezoidal Rules in the total energy integral we get:

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{1}{2n}(b-a)[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \tag{97}$$

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{b-a}{2n}[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \tag{98}$$

Now note an interesting thing:

- 1)-The number of slices raises the accuracy of the integration method and we have the factor $\frac{c^2}{G} \times \frac{vs^2}{8\pi}$ generating the huge factor 10^{48} for a bubble speed $vs = 200$ times light speed constraining the negative energy densities in the Natario warp bubble.
- 2)-How about to make the numerical integration by the Trapezoidal Rule using 10^{48} slices??.How about to make $n = 10^{48}$???

If $n = 10^{48}$ then n in the denominator of the fraction $\frac{b-a}{2n}$ will completely obliterate the factor $\frac{c^2}{G} \times \frac{vs^2}{8\pi}$ in the equations of the total energy integral lowering the negative energy density requirements for the Natario warp bubble.

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{1}{2n}(b-a)[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b - a \simeq 0 \quad (99)$$

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{b-a}{2n}[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b - a \simeq 0 \quad (100)$$

Again a reader can see the power of the White idea.Compare the distance between the points a (beginning of the negative energy warped region) and b(end of the negative energy warped region) in the Appendices B and C.The points a and b are more close to each other in Appendix C than in the Appendix B.The thickness parameter passed from $@ = 900$ to $@ = 5000$.If we make $@ = 500.000$ or $@ = 5.000.000$ the points a and b will be so close that the difference between b and a will approach zero for real obliterating the value of 10^{48} or any other value.$(b \simeq a)(b - a \simeq 0)$

We could theoretically have a thickness of $@ = 5.000.000.000$ and a speed $vs = 6000$ times light speed enough to reach Kepler-22 at 600 light-years in months not in years using an incredible small amount of negative energy because the difference between b and a would obliterate everything since a and b would be infinitely close to each other.[35]

[35] we are limited by the floating point precision of Microsoft Excel the only program we have.

12 Appendix E:Natario warped region with $@ = 5000$ $R = 100$ meters $\delta = 1,009999550661E-008$ meters.Slightly beyond the end of the warp bubble but within the neighborhoods of the end of the warp bubble and faraway from the distant regions outside the bubble (WF=200)

.

rs	$n(rs)$	$n'(rs)^2$	Tpz
$1,000004370000E+002$	$4,045667505366E-002$	$1,021849168291E+006$	NA
$1,000004370101E+002$	$4,046688557033E-002$	$1,022161107678E+006$	$1,032224730137E-002$
$1,000004370202E+002$	$4,047709764524E-002$	$1,022473090568E+006$	$1,032539810747E-002$
$1,000004370303E+002$	$4,048731127836E-002$	$1,022785116946E+006$	$1,032854935287E-002$
$1,000004370404E+002$	$4,049752646967E-002$	$1,023097186799E+006$	$1,033170103743E-002$

The plot above depicts the region outside the bubble but within the bubble radius neighborhoods where the inflexion point mentioned earlier occurs for the square of the derivatives of the Natario shape function. Note that each derivative square possesses the maximum value for our Natario shape function which is 10^6.

The column Tpz is an integration by the Trapezoidal Rule of 4 slices of this region each slice with a width $\delta = 1,009999550661E-008$.Note that all the square derivatives of the shape function have values around $1,2 \times 10^6$ and the sum of the boundary points(the values of the square derivatives of the shape function in the boundary points)of each slice divided by 2 gives a value of about $1,2 \times 10^6$.Multiplying $1,2 \times 10^6$ by $\delta = 1,009999550661E-008$ we get values of about $1,2 \times 10^{-2}$.As smaller the value of the slice width is as smaller the result of the Trapezoidal Rule becomes.

For a slice width of $\delta = 1,009999550661E-020$ multiplied by $1,2 \times 10^6$ the final result would be $1,2 \times 10^{-14}$.

This is important to illustrate how the Trapezoidal Rule integral must deal with the factor of 10^{48} in order to reduce the total energy integral[36].Slices with smaller width are better than the ones with larger widths.

Remember that the plot above depicts only a small fraction of the Natario warp bubble and the Trapezoidal Rule presented above with 4 slices is not accurate.

The plot that represents the whole Natario bubble is the Appendix C

[36]Trapezoidal Rule integral multiplied by 10^{48}

Plotting again the table of Appendix C

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$99,97$	1	0	$2,650396579309E-253$	0
$99,98$	1	0	$1,915169615918E-166$	0
$99,99$	1	0	$1,383896540755E-079$	0
100	$4,9E-001$	$3,111507925458E-061$	$6,250000000000E+006$	$9,681489643510E-110$
$100,01$	0	$0,5$	$1,383896512435E-079$	$1,383896512435E-075$
$100,02$	0	$0,5$	$1,915169576726E-166$	$1,915169576726E-162$
$100,03$	0	$0,5$	$2,650396525072E-253$	$2,650396525072E-249$

If we want a rigorous integration of the Natario negative energy density warped region which starts at $rs = 100$ meters from the center of the bubble(bubble radius R) and ends up at $rs = 100,03$ meters then we must divide this region in 10^{48} slices each one with a width h of:

$$h = \frac{b-a}{n} = \frac{100,03-100}{10^{48}} = \frac{0,03}{10^{48}} = \frac{3}{10^{49}} = 3 \times 10^{-49} \tag{101}$$

Inserting these values in the total energy integral equations:

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{1}{2n}(b-a)[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \tag{102}$$

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{b-a}{2n}[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \tag{103}$$

we should expect for:

$$E = -3\frac{c^2}{G}\frac{v_s^2}{8\pi}\frac{3}{2 \times 10^{49}}[n'(b)^2 + n'(a)^2 + \sum_{i=1}^{n-1} n'(a+ih)^2] \simeq 0 \dashrightarrow n >> 1 \dashrightarrow b \simeq a \dashrightarrow b-a \simeq 0 \tag{104}$$

We choose to divide the Natario negative energy warped region in 10^{48} slices due to the factor $\frac{c^2}{G}\frac{v_s^2}{8\pi}$ which is 10^{48} for a bubble speed $vs = 200$ times faster than light .In the equation above keeping the bubble radius $R = 100$ meters and the thickness parameter $@ = 5000$ and the warp factor $WF = 200$ all constants and since c and G are constants if we use a different bubble velocity vs higher than 200 times faster than light giving a factor $\frac{c^2}{G}\frac{v_s^2}{8\pi} > 10^{48}$ then the number of slices n needed to integrate accurately by the Trapezoidal Rule the energy density in the Natario warp bubble must be equal to this new factor in order to reduce the total energy integral.

13 Appendix F:Natario warped region with $@ = 5000$ $R = 100$ meters $\delta = 9,999999974752E-007$ meters.From the end of the warp bubble to a region slightly beyond the end of the warp bubble but within the neighborhoods of the end of the warp bubble and faraway from the distant regions outside the bubble (WF=200)

.

rs	$n(rs)$	$n'(rs)^2$	Tpz
$1,000000000000E+002$	$3,111507638931E-061$	$9,681479787123E-110$	NA
$1,000000010000E+002$	$8,436836263117E-061$	$7,047018947824E-109$	$4,007583453150E-115$
$1,000000020000E+002$	$2,276234286411E-060$	$5,078139220097E-108$	$2,891420550140E-114$
$1,000000030000E+002$	$6,110588727772E-060$	$3,622759985185E-107$	$2,065286948383E-113$
$1,000000040000E+002$	$1,632218040042E-059$	$2,558649877926E-106$	$1,460462934535E-112$

The plot above depicts what happens in the beginning of the Natario warp drive negative energy warped region which starts at the end of the bubble $R = rs = 100$ meters.

This region lies between the bubble radius and the inflexion point mentioned earlier

The Trapezoidal Rule Tpz depicted here is not accurate

14 Epilogue

- "The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C.Clarke[37]
- "The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein[38][39]

15 Acknowledgements

We would like to express the most profound and sincere gratitude towards Malcolm A. Shore.He called our attention to the new work of Harold "Sonny" White reference [17]

16 Remarks

References [8] "Warp Field Mechanics 101" and [17] "Warp Field Mechanics 102" by Harold "Sonny" White of NASA Lyndon B.Johnson Space Center Houston Texas are available at NASA Technical Reports Server (NTRS)[40] however we can provide a copy in PDF Acrobat reader of these references for those interested.

Reference [18] "Numerical Integration" from Autar Kaw and Charlie Barker is available at http://numericalmethods.eng.usf.edu however we can provide a copy in PDF Acrobat reader of this reference for those interested.

Reference [15] was online at the time we picked it up for our records.It ceased to be online but we can provide a copy in PDF Acrobat reader of this reference for those interested.

Reference [16] we only have access to the abstract.

We performed all the numerical calculus of our simulations for both Alcubierre and Natario warp drive spacetimes using Microsoft Excel[41].We can provide our Excel files to those interested and although Excel is a licensed program there exists another program that can read Excel files available in the Internet as a freeware for those that perhaps may want to examine our files:the OpenOffice[42] at http://www.openoffice.org

[37] special thanks to Maria Matreno from Residencia de Estudantes Universitas Lisboa Portugal for providing the Second Law Of Arthur C.Clarke

[38] "Ideas And Opinions" Einstein compilation, ISBN $0-517-88440-2$, on page 226."Principles of Research" ([Ideas and Opinions],pp.224-227), described as "Address delivered in celebration of Max Planck's sixtieth birthday (1918) before the Physical Society in Berlin"

[39] appears also in the Eric Baird book Relativity in Curved Spacetime ISBN $978-0-9557068-0-6$

[40] browse Google for "Warp Field Mechanics 101" and "Warp Field Mechanics 102"

[41] Copyright(R) by Microsoft Corporation

[42] Copyright(R) by Oracle Corporation

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