Octonionic Ternary Gauge Field Theories Revisited

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Abstract

An octonionic ternary gauge field theory is explicitly constructed based on a ternary-bracket defined earlier by Yamazaki. The ternary infinitesimal gauge transformations do obey the key *closure* relations $[\delta_1, \delta_2] = \delta_3$. An invariant action for the octonionic-valued gauge fields is displayed after solving the previous problems in formulating a non-associative octonionic ternary gauge field theory.

1 Introduction

Exceptional, Jordan, Division, Clifford, noncommutative and nonassociative algebras are deeply related and are essential tools in many aspects in Physics, see [1], [2], [3], [4], [5], [6], [7], [8], [8], [12], [13], [14], [25], [26] for references, among many others. A thorough discussion of the relevance of ternary and nonassociative structures in Physics has been provided in [10], [15], [16],[17],[18], [19]. The earliest example of nonassociative structures in Physics can be found in Einstein's special theory of relativity. Only colinear velocities are commutative and associative, but in general, the addition of non-colinear velocities is non-associative and non-commutative.

Great activity was launched by the seminal works of Bagger, Lambert and Gustavsson (BLG) [27], [28], [29] who proposed a Chern-Simons type Lagrangian describing the world-volume theory of multiple $M2$-branes. The original BLG theory requires the algebraic structures of generalized Lie 3-algebras and also of nonassociative algebras. Later developments by [30] provided a $3D$ Chern-Simons matter theory with $\mathcal{N} = 6$ supersymmetry and with gauge groups $U(N) \times U(N)$, $SU(N) \times SU(N)$. The original construction of [30] did not require generalized Lie 3-algebras, but it was later realized that it could be understood as a special class of models based on Hermitian 3-algebras [31], [32].

A Nonassociative Gauge theory based on the Moufang S^7 loop product (not a Lie algebra) has been constructed by [33], [34]. Taking the algebra of octonions with a unit norm as the Moufang S^7-loop, one reproduces a nonassociative octonionic gauge theory which is a generalization of the Maxwell and Yang-Mills gauge theories based on Lie algebras. $BPST$-like instantons solutions in $D = 8$ were also found. These solutions represented the physical degrees of freedom of the transverse 8-dimensions of superstring solitons in $D = 10$ preserving one and two of the 16 spacetime supersymmetries. Nonassociative deformations of Yang-Mills Gauge theories involving the left and right bimodules of the octonionic algebra were presented by [35]. Non-associative generalizations of supersymmetry have been proposed by [36] which is very relevant to hidden variables theory and alternative Quantum Mechanics.

The ternary gauge theory developed in this work differs from the work by [27], [28], [29], in that our 3-Lie algebra-valued gauge field strengths $F_{\mu\nu}$ are explicitly defined in terms of a 3-bracket $[A_\mu, A_\nu, \mathbf{g}]$ involving a 3-Lie algebra-valued coupling $\mathbf{g} = g^a t_a$. Whereas the definition of $F_{\mu\nu}$ by [27], [28], [29] was based on the standard commutator of the matrices $(\tilde{A}_\mu)^a_c(\tilde{A}_\nu)^c_b - (\tilde{A}_\nu)^a_c(\tilde{A}_\mu)^c_b$. These matrices were defined as $\mathbf{A}_\mu = A^{ab}_\mu f_{ab}{}^{cd} = (\tilde{A}_\mu)^{cd}$ and given in terms of the structure constants $f_{ab}{}^{cd}$ of the 3-Lie algebra $[t_a, t_b, t^c] = f_{ab}{}^{cd} t_d$.

In the next section we shall analyze Nonassociative Octonionic Ternary Gauge Field Theories based on a ternary octonionic product with the fundamental difference, besides the nonassociativity, that the structure constants f_{abcd} are *no* longer totally antisymmetric in their indices. Thus the bracket in the octonion case $[[A, B]] \equiv [A, B, \mathbf{g}]$ is *not* effectively a Lie bracket (as it occurs in the 3-Lie algebra case) because the bracket $[[A, B]]$ in the octonion case does *not* obey the Jacobi identity since the structure constants f_{abcd} are *no* longer totally antisymmetric in their indices. This work is quite an improvement of our prior results where we focused solely on the global rigid symmetries and homothecy transformations [38].

It is shown that the octonionic-valued field strength $F_{\mu\nu} = F^a_{\mu\nu} e_a$ transforms homogeneously (covariantly) under gauge transformations and that the Yang-Mills-like action is indeed invariant under *local* gauge transformations involving ternary octonionic brackets and antisymmetric gauge parameters $\Lambda^{ab}(x) = -\Lambda^{ba}(x), a, b = 0, 1, 2, 3, ...7$. Furthermore, there is *closure* of these transformations based on antisymmetric parameters $\Lambda^{ab} = -\Lambda^{ba}$

2 Octonionic Ternary Gauge Field Theories

The nonassociative and noncommutative octonionic ternary gauge field theory is based on a ternary-bracket structure involving the octonion algebra. The ternary bracket obeys the fundamental identity (generalized Jacobi identity) and was developed earlier by Yamazaki [24]. Given an octonion $\mathbf{X}$ it can be expanded in a basis (e_o, e_m) as

$$\mathbf{X} = x^o\ e_o\ +\ x^m\ e_m, \quad m,n,p = 1,2,3,.....7. \tag{2.1}$$

where e_o is the identity element. The Noncommutative and Nonassociative algebra of octonions is determined from the relations

$$e_o^2 = e_o,\ e_o e_i = e_i e_o = e_i,\ e_i e_j = -\delta_{ij} e_o\ + c_{ijk} e_k,\ i,j,k = 1,2,3,....7. \tag{2.2}$$

where the fully antisymmetric structure constants c_{ijk} are taken to be 1 for the combinations $(124),(235),(346),(457),(561),(672),(713)$. The octonion conjugate is defined by $\bar{e}_o = e_o$, $\bar{e}_i = -e_i$

$$\bar{\mathbf{X}}\ =\ x^o\ e_o\ -\ x^k\ e_k. \tag{2.3}$$

and the norm is

$$N(\mathbf{X})\ =\ |\ <\ \mathbf{X}\ \mathbf{X}\ >\ |^{\frac{1}{2}}\ = |\ Real\ (\bar{\mathbf{X}}\ \mathbf{X})\ |^{\frac{1}{2}}\ = |\ (x_o\ x_o\ + x_k\ x_k)\ |^{\frac{1}{2}}. \tag{2.4}$$

The inverse

$$\mathbf{X}^{-1}\ =\ \frac{\bar{\mathbf{X}}}{<\ \mathbf{X}\ \mathbf{X}\ >}, \quad \mathbf{X}^{-1}\mathbf{X} = \mathbf{X}\mathbf{X}^{-1} = 1. \tag{2.5}$$

The non-vanishing associator is defined by

$$(\mathbf{X},\mathbf{Y},\mathbf{Z}) = (\mathbf{X}\mathbf{Y})\mathbf{Z} - \mathbf{X}(\mathbf{Y}\mathbf{Z}) \tag{2.6}$$

In particular, the associator

$$(e_i,e_j,e_k)\ =\ (e_i e_j)e_k - e_i(e_j e_k)\ =\ 2\ d_{ijkl}\ e_l$$
$$d_{ijkl}\ =\ \frac{1}{3!}\ \epsilon_{ijklmnp}\ c^{mnp},\ i,j,k.... = 1,2,3,.....7 \tag{2.7}$$

Yamazaki [24] defined the three-bracket as

$$[\,u,\ v,\ x\,]\ \equiv\ D_{u,v}\,x\ =\ \frac{1}{2}\ (\ u(vx) - v(ux)\ +\ (xv)u\ -\ (xu)v\ +\ u(xv)\ -\ (ux)v\ ). \tag{2.8}$$

After a straightforward calculation when the indices span the imaginary elements $a,b,c,d = 1,2,3,......,7$, and using the relationship [37]

$$c_{abd}\ c_{dcm}\ =\ -\ d_{abcm}\ +\ \delta_{ac}\ \delta_{bm}\ -\ \delta_{bc}\ \delta_{am} \tag{2.9a}$$

the ternary bracket becomes

$$[\ e_a,\ e_b,\ e_c\ ]\ =\ f_{abcd}\ e_d\ =\ [\ d_{abcd}\ + 2\ \delta_{ac}\ \delta_{bd}\ - 2\ \delta_{bc}\ \delta_{ad}\ ]\ e_d \tag{2.9b}$$

whereas e_0 has a vanishing ternary bracket

$$[\ e_a,\ e_b,\ e_0\ ]\ =\ [\ e_a,\ e_0,\ e_b\ ]\ =\ [\ e_0,\ e_a,\ e_b\ ]\ =\ 0 \qquad (2.9c)$$

It is important to emphasize that $f_{abcd} \neq \pm\ c_{abd}\ c_{dcm}$ otherwise one would have been able to *rewrite* the ternary bracket in terms of ordinary 2-brackets as follows $[e_a,\ e_b,\ e_c] \sim \frac{1}{4}[\ [e_a,\ e_b],\ e_c\ ]$ and this would have defeated the whole purpose of studying ternary structures.

The ternary bracket (2.8) obeys the fundamental identity

$$[\ [x,\ u,\ v],\ y,\ z\ ]\ +\ [\ x,\ [y,\ u,\ v],\ z\ ]\ +\ [\ x,\ y,\ [z,\ u,\ v]\ ]\ =\ [\ [x,\ y,\ z],\ u,\ v\ ] \qquad (2.10)$$

A bilinear positive symmetric product $< u, v >=< v, u >$ is required such that that the ternary bracket/derivation obeys what is called the metric compatibility condition

$$<\ [u,\ v,\ x],\ y\ >\ =\ -\ <\ [u,\ v,\ y],\ x\ >\ =\ -\ <\ x,\ [u,\ v,\ y]\ >\ \Rightarrow$$

$$D_{u,v}\ < x, y >\ =\ 0 \qquad (2.11)$$

The symmetric product remains invariant under derivations. There is also the additional symmetry condition required by [24]

$$<\ [u,\ v,\ x],\ y\ >\ =\ <\ [x,\ y,\ u],\ v\ > \qquad (2.12)$$

The ternary product provided by Yamazaki (2.8) *obeys* the key fundamental identity (2.10) and leads to the structure constants f_{abcd} that are *pairwise* antisymmetric but are *not* totally antisymmetric in all of their indices : $f_{abcd} = -f_{bacd} = -f_{abdc} = f_{cdab}$; however : $f_{abcd} \neq f_{cabd}$; and $f_{abcd} \neq -\ f_{dbca}$. The associator ternary operation for octonions $(x, y, z) = (xy)z - x(yz)$ *does not obey* the fundamental identity (2.10) as emphasized by [24]. For this reason we cannot use the associator to construct the 3-bracket.

The physical motivation behind constructing an octonionic-valued field strength in terms of *ternary* brackets is because the ordinary 2-bracket does *not* obey the Jacobi identity

$$[\ e_i,\ [\ e_j,\ e_k\ ]\ ]\ +\ [\ e_j,\ [\ e_k,\ e_i\ ]\ ]\ +\ [\ e_k,\ [\ e_i,\ e_j\ ]\ ]\ =\ 3\ d_{ijkl}\ e_l\ \neq\ 0 \qquad (2.13)$$

If one has the ordinary Yang-Mills expression for the field strength

$$F_{\mu\nu}\ =\ \partial_\mu A_\nu\ -\ \partial_\nu A_\mu\ +\ [\ A_\mu,\ A_\nu\ ] \qquad (2.14)$$

because the 2-bracket does *not* obey the Jacobi identity, one has an extra (spurious) term in the expression for

$$[\ D_\mu,\ D_\nu\ ]\ \Phi\ =\ [\ F_{\mu\nu},\ \Phi\ ]\ +\ (\ A_\mu,\ A_\nu,\ \Phi\ ) \qquad (2.15)$$

given by the crucial contribution of the non-vanishing associator $(A_\mu, A_\nu, \Phi) = (A_\mu A_\nu)\Phi - A_\mu(A_\nu \Phi) \neq 0$. For this reason, due to the non-vanishing condition (2.13), the ordinary Yang-Mills field strength does *not* transform homogeneously under ordinary gauge transformations involving the parameters $\Lambda = \Lambda^a e_a$

$$\delta A_\mu = \partial_\mu \Lambda \ + \ [A_\mu, \Lambda] \tag{2.16}$$

but it yields an extra contribution of the form

$$\delta F_{\mu\nu} \ = \ [F_{\mu\nu}, \Lambda] \ + \ (\ \Lambda, \ A_\mu, \ A_\nu) \tag{2.17}$$

As a result of the additional contribution (Λ, A_μ, A_ν) in eq-(2.17), the ordinary Yang-Mills action $S = \int < F_{\mu\nu} F^{\mu\nu} >$ will *no* longer be gauge invariant. Under infinitesimal variations (2.17), the variation of the action is *no* longer zero but receives spurious contributions of the form $\delta S = -4 F^l_{\mu\nu} \Lambda^i A^{\mu j} A^{\nu k} d_{ijkl} \neq 0$ due to the non-associativity of the octonion algebra. For these reasons we focus our attention on ternary brackets.

We define the field strength in terms of the *ternary* bracket as

$$F_{\mu\nu} \ = \ \partial_\mu A_\nu \ - \ \partial_\nu A_\mu \ + \ [\ A_\mu, \ A_\nu, \ \mathbf{g} \ ] \tag{2.18}$$

where $\mathbf{g} = g^a e_a$ is an octonionic-valued "coupling" function. One finds that under the naive infinitesimal ternary gauge transformations

$$\delta(A^d_\mu e_d) \ = \ - \ \partial_\mu(\Lambda^d e_d) + [\ \Lambda^a e_a, A^b_\mu e_b, g^c e_c \ ] \Rightarrow \delta(F^d_{\mu\nu} \ e_d) = [\Lambda^a(x) e_a, F^b_{\mu\nu} e_b, g^c e_c] \tag{2.19}$$

the ordinary quadratic action

$$S \ = \ - \ \frac{1}{4\kappa^2} \ \int \ d^D x \ < \ F_{\mu\nu} \ F^{\mu\nu} \ > \tag{2.20}$$

is *not* invariant under ternary infinitesimal gauge transformations as we shall see next. κ is a suitable dimensionful constant introduced to render the action dimensionless. The octonionic valued field strength is $F_{\mu\nu} = F^a_{\mu\nu} \ e_a$, and has *real valued* components $F^0_{\mu\nu}, F^i_{\mu\nu}; \ i = 1, 2, 3, ..., 7$. The $< \ >$ operation extracting the e_0 part is defined as $< XY >= Real(\bar{X} Y) =< YX >= Real(\bar{Y} X)$. Under infinitesimal ternary gauge transformations of the ordinary quadratic action one has

$$\delta \ S \ = \ - \ \frac{1}{4\kappa^2} \ \int \ d^D x \ < \ F_{\mu\nu} \ (\delta F^{\mu\nu}) \ + \ (\delta F_{\mu\nu}) \ F^{\mu\nu} \ > \ =$$

$$- \ \frac{1}{4\kappa^2} \ \int \ d^D x \ < \ F^c_{\mu\nu} \ e_c \ [\ \Lambda^a e_a, \ F^{\mu\nu \ b} \ e_b, \ g^n e_n] \ > \ +$$

$$- \ \frac{1}{4\kappa^2} \ \int \ d^D x \ < \ [\Lambda^a e_a, \ F^b_{\mu\nu} \ e_b, \ g^n e_n] \ F^{\mu\nu \ c} \ e_c \ > \ =$$

$$- \frac{1}{4\kappa^2} \ \int \ d^D x \ \Lambda^a \ F^c_{\mu\nu} \ F^{\mu\nu \ b} \ (\ < \ e_c \ f_{abnk} \ e_k \ > \ + \ < \ f_{abnk} \ e_k \ e_c \ > \ ) \ =$$

$$ -\frac{1}{2\kappa^2} \int d^D x \ \Lambda^a \ g^n \ F^c_{\mu\nu} \ F^{\mu\nu \ b} \ f_{abnc} \ = $$

$$ -\frac{1}{\kappa^2} \int d^D x \ \big(\ (\Lambda^a g_a) \ (F^b_{\mu\nu} F^{\mu\nu}_b) \ - \ (\Lambda^a F^{\mu\nu}_a) \ (g_c F^c_{\mu\nu}) \ \big) \ \neq \ 0 \qquad (2.21) $$

Hence, because

$$ f_{abnc} \ = \ (\ d_{abnc} \ + \ 2 \ \delta_{an} \ \delta_{bc} \ - \ 2 \ \delta_{bn} \ \delta_{ac} \ ) \qquad (2.22) $$

is *not* antisymmetric under the exchange of indices $b \leftrightarrow c$: $f_{abnc} \neq -f_{acnb}$, the variation in eq-(3.15) is *not* zero. Had f_{abnc} been fully antisymmetric then the variation δS would have been zero due to the fact that $F^c_{\mu\nu} \ F^{\mu\nu \ b}$ is symmetric under $b \leftrightarrow c$.

Concluding, in the octonionic ternary algebra case, the naive transformations (2.19) do *not* leave the action (2.20) invariant : $\delta S \neq 0$. One alternative would be to find counter terms, if *possible*, to the action (2.20) $S + \Delta S$ so that $\delta(S + \Delta S) = 0$. The authors [35] used counter terms of the form $F \wedge A \wedge A + A \wedge A \wedge A \wedge A$ in the non-associative deformations of ordinary Yang-Mills theories based on the left and right actions by octonions and ordinary brackets. Unfortunately it does not work in our case and for this reason we shall follow a different approach.

Another problem due to the fact that f_{abcd} is *not* totally antisymmetric in all of its indices is that there *is no closure* of the infinitesimal octonionic ternary gauge transformations $\delta A_\mu = -\partial_\mu \Lambda + [\Lambda, A_\mu, \mathbf{g}]$. Furthermore, the bracket in the octonion case $[[A, B]] \equiv [A, B, \mathbf{g}]$ is *not* effectively a Lie bracket (as it is in the 3-Lie algebra case) since the bracket $[[A, B]]$ in the octonion case does *not* obey the Jacobi identity because the structure constants f_{abcd} are *no* longer totally antisymmetric in their indices. As said previously, because the associator ternary operation for octonions $(x, y, z) = (xy)z - x(yz)$ *does not obey* the fundamental identity (3.10) one cannot use the associator to construct the 3-bracket and this rules out the use of the totally antisymmetric d_{abcd}.

Nevertheless, as we shall show below, the quadratic Yang-Mills-like action (2.20) is invariant under the *local* octonionic ternary gauge transformations defined by

$$ \delta(A^d_\mu \ e_d) \ = \ \Lambda^{ab}(x) \ [\ e_a, \ e_b, \ A^c_\mu \ e_c \ ] \qquad (2.23a) $$

and

$$ \delta(g^d \ e_d) \ = \ \Lambda^{ab}(x) \ [\ e_a, \ e_b, \ g^c \ e_c] \qquad (2.23b) $$

where one introduces a local spacetime depedence on the antisymmetric gauge parameters $\Lambda^{ab}(x) = -\Lambda^{ba}(x)$. One may notice now that the coupling $g^c e_c$ is *not* inert under the transformations (2.23b). Only the real part of the coupling g^0 is inert.

After some straightforward algebra one can verify that the ternary field strength $F_{\mu\nu}$ defined in terms of the 3-brackets transforms properly (homogeneously) under the *local* transformations (2.23)

$$ \delta(F^m_{\mu\nu} \ e_m) \ = \ \Lambda^{ab} \, [\, e_a, \ e_b, \ F^c_{\mu\nu} \ e_c \,] \ = \ \Lambda^{ab} \ F^c_{\mu\nu} \ f_{abc}{}^m \ e_m \ \Rightarrow \ \delta F^m_{\mu\nu} \ = \ \Lambda^{ab} \ F^c_{\mu\nu} \ f_{abc}{}^m \qquad (2.24) $$

if the following conditions are satisfied

$$[\ (\partial_\mu \Lambda^{ij})\ A^k_\nu\ -\ (\partial_\nu \Lambda^{ij})\ A^k_\mu\ ]\ f_{ijk}{}^l\ =\ 0 \tag{2.25}$$

Due to the *key* presence of the imaginary parts of the couplings g^i we shall prove below that one can partially gauge the fields by setting $A^i_\mu = g^i(x)\partial_\mu\phi(x)$ (in terms of an auxiliary scalar field $\phi(x)$) and still leave room for *residual* symmetries. One should note that even if the coupling functions g^i are chosen to be constants $g^i = constant$, it must be kept in mind that after gauge transformations the new couplings g'^i will acquire a spacetime dependence via the x^μ-dependence of the $\Lambda^{ab}(x^\mu)$ parameters. For this reason we should *not* set a priori the couplings to constants. Furthermore, the field strength will *not* become trivially zero. It can be rewritten, when the gauge fields are partially gauged as $A^l_\mu = g^l(x)\partial_\mu\phi(x)$, in the following way

$$F^l_{\mu\nu}\ =\ \partial_{[\mu}A^l_{\nu]}\ +\ A^i_\mu\ A^j_\nu\ g^k\ f_{ijk}{}^l\ =\ (\partial_\mu g^l)\ (\partial_\nu\phi)\ -\ (\partial_\nu g^l)\ (\partial_\mu\phi)\ \neq 0 \tag{2.26}$$

after using the conditions $(g^i\partial_\mu\phi)(g^j\partial_\nu\phi)f_{ijk}{}^l = 0$ due to the antisymmetry of $f_{ijk}{}^l = -f_{jik}{}^l$ and the symmetry $g^ig^j = g^jg^i$. Therefore, due to the x-dependence of the imaginary parts of the couplings $g^i(x)$, the field strength components $F^l_{\mu\nu}$ are *not* zero and their contribution to the action (2.20) is not trivially zero.

After this detour let us look for nontrivial solutions to (2.25). $\Lambda^{ij} = constant$ are the trivial solutions leading to global rigid symmetries. A partial gauge fixing provided by $A^k_\mu = g^k\partial_\mu\phi, A^k_\nu = g^k\partial_\nu\phi$ yields in eq-(2.25), after factoring g^k and defining $\tilde{\Lambda}^l_k = f_{ijk}{}^l\Lambda^{ij}$, the following

$$(\partial_\mu\tilde{\Lambda}^l_k)\ (\partial_\nu\phi)\ -\ (\partial_\nu\tilde{\Lambda}^l_k)\ (\partial_\mu\phi)\ =\ 0 \tag{2.27}$$

eq-(2.27) can be rewritten as

$$\partial_\mu(\ \tilde{\Lambda}^l_k\ \partial_\nu\phi\ )\ -\ \partial_\nu(\ \tilde{\Lambda}^l_k\ \partial_\mu\phi\ )\ =\ 0 \tag{2.28}$$

a solution of (2.28) can be obtained in terms of a square 7×7 matrix whose entries are given by a family of integrating functions $\Theta^l_k(x)$ as follows

$$\tilde{\Lambda}^l_k\ \partial_\mu\phi\ =\ 7\ \partial_\mu\Theta^l_k\ \Rightarrow\ \partial_\mu\phi\ =\ \tilde{\Lambda}^k_l\ \partial_\mu\Theta^l_k \tag{2.29a}$$

and similarly

$$\tilde{\Lambda}^l_k\ \partial_\nu\phi\ =\ 7\ \partial_\nu\Theta^l_k\ \Rightarrow\ \partial_\nu\phi\ =\ \tilde{\Lambda}^k_l\ \partial_\nu\Theta^l_k \tag{2.29b}$$

$\tilde{\Lambda}^l_k$ can be represented by the entries of a square 7×7 matrix and admits an inverse $\tilde{\Lambda}^k_l\tilde{\Lambda}^l_j = \delta^k_j$ if the determinant of $\Lambda^k_l \neq 0$. Without loss of generality we can set the integrating functions to be given by $\Theta^l_k \equiv \tilde{\Lambda}^l_k = f_{ijk}{}^l\Lambda^{ij}$ so that

$$\partial_\mu\phi\ =\ \tilde{\Lambda}^k_l\ \partial_\mu\tilde{\Lambda}^l_k\ =\ Trace\ \partial_\mu[ln(\tilde{\Lambda}^l_k)] \tag{2.30a}$$

Therefore, the partially gauged field which solves eq-(2.25) is given by

$$A^i_\mu(x) \ = \ g^i(x) \ \partial_\mu \phi \ = \ g^i(x) \ Trace \ \partial_\mu [ln(\tilde\Lambda^l_k(x))] \qquad (2.30b)$$

The solution (2.31) indicates that the gauge parameters $\tilde\Lambda^l_k = f_{ijk}{}^l \Lambda^{ij}$ are themselves field-dependent such that the gauge transformations (2.23) are *nonlinear* in the fields. From eq-(2.30b) one can infer that

$$\partial_\mu \phi \ = \ \partial_\mu \left(Trace \ ln(\tilde\Lambda^l_k(x)) \right) \ \Rightarrow \ \phi - \phi_o \ = \ Trace \ [ln(\tilde\Lambda^l_k(x))] \ \Rightarrow$$

$$exp\,(\phi - \phi_o) \ = \ exp \ \left(\ Trace \ ln(\tilde\Lambda^l_k(x)) \ \right) \ = \ det\,(\tilde\Lambda^l_k(x)) \ = \ exp \ \left(\ \int_0^{x^\mu} \ \frac{g_i A^i_\mu}{g^2} \ dx^\mu \ \right) \qquad (2.31)$$

The constant $\phi_o \ \equiv \ \phi(x^\mu \ = \ 0)$. It is interesting that the determinant of $\tilde\Lambda^l_k(x)$ is given in terms of the gauge fields by an expression which resembles the Wilson loop expression with the main difference that one has a line integral instead of a loop. To conclude, when one partially gauges the fields as $A^i_\mu = g^i \partial_\mu \phi = g^i(x) Trace[\partial_\mu ln(\tilde\Lambda^l_k(x))]$, there is still a residual symmetry that remains such that the gauge transformations (2.23) become nonlinear in the fields and the nonvanishing field strength $F^k_{\mu\nu}$ given by eq-(2.26) transforms homogeneously (2.24) under the local gauge transformations (2.23) due to the vanishing conditions imposed by eq-(2.25). The reason the field strength $F^k_{\mu\nu}$ transforms homogeneously (2.24) when the inhomogeneous terms (2.25) vanish is a direct consequence of the fundamental identity (2.10) because the 3-bracket (2.8) is defined as a derivation

$$[\ [e_a, \ e_b, \ A_\mu], \ A_\nu, \ \mathbf{g} \ ] \ + \ [\ A_\mu, \ [e_a, \ e_b, \ A_\nu], \ \mathbf{g} \ ] \ + \ [\ A_\mu, \ A_\nu, \ [e_a, \ e_b, \ \mathbf{g}] \ ] \ =$$

$$[\ e_a, \ e_b, \ [A_\mu, \ A_\nu, \ \mathbf{g}] \ ] \qquad (2.32)$$

Another important condition due to the antisymmetry $f_{ijkl} \ = \ -f_{jikl} \ = \ -f_{ijlk}$, and symmetry $g^l g^k \ = \ g^k g^l$, is the invariance of $g^2 \ = \ g_l g^l$ under gauge transformations

$$\delta g^2 = \delta(g_l g^l) \ = \ 2 \ g_l \ \delta g^l \ = \ 2 \ g_l \ \Lambda^{ij} \ g^k \ f_{ijk}{}^l \ = \ 2 \ g^l \ g^k \ \Lambda^{ij} \ f_{ijkl} \ = \ 0 \ (2.33)$$

Therefore, the full octonionic norm-squared $(g^o)^2 + g_i g^i$ of the octonionic-valued coupling function $\mathbf{g} \ = \ g^o e_o + g^i e_i$ is invariant under gauge transformations (2.23b).

An important remark is in order. There is a plausible caveat about the conditions (2.25). One must ensure that such conditions, which do *not* appear to be explicitly gauge covariant, will not break the gauge covariance (invariance) of the theory one is trying to construct. In particular, after performing a gauge variation of the conditions $C^l_{[\mu\nu]} = 0$ in (2.25) one would introduce the secondary conditions $\delta C^l_{[\mu\nu]} = 0$. Performing yet another gauge variation of the secondary conditions $\delta(\delta C^l_{[\mu\nu]}) = 0$ .... , and so forth, one obtains a hierarchy of equations to be satisfied by the gauge parameters $\Lambda^{ij}(x)$. It is clear that the trivial solutions $\Lambda^{ij} = c^{ij} = constants$ will satisfy automatically all the equations suggesting, perhaps, that octonionic ternary field theories cannot be gauged. Nevertheless, as we shall show next, there is a very natural way to bypass this problem such that the gauge variation of the conditions (2.25) remains zero *without* introducing additional constraints on the parameters Λ^{ij} that might have forced them to be constants. From the gauge variations

$$\delta A^l_\mu = \Lambda^{ij}(x)\ A^k_\mu\ f_{ijk}{}^l\ =\ \Lambda^{ij}(x)\ g^k\ \partial_\mu\phi\ f_{ijk}{}^l\ =\ \delta(g^l\partial_\mu\phi)\ =$$

$$\delta(g^l)\ \partial_\mu\phi\ +\ g^l\ \partial_\mu(\delta\phi)\ =\ \Lambda^{ij}(x)\ g^k\ \partial_\mu\phi\ f_{ijk}{}^l\ +\ g^l\ \partial_\mu(\delta\phi)\ \Rightarrow\ \delta\phi = 0 \quad (2.34)$$

one learns that $\delta\phi = 0$, and in turn, we can infer from (2.31) that $\delta\Lambda^{ij} = 0$ so that the variation of the condition (2.25) (when $A^i_\mu = g^i\partial_\mu\phi$, $\delta\Lambda^{ij} = \delta\phi = 0$) remains zero without introducing further constraints on the parameters. A variation of (2.25) gives

$$f_{ijk}{}^l\ \big(\ (\partial_\mu\Lambda^{ij})\ (\delta g^k)\ (\partial_\nu\phi)\ -\ (\partial_\nu\Lambda^{ij})\ (\delta g^k)\ (\partial_\mu\phi)\ \big)\ =\ 0 \qquad (2.35)$$

therefore, a simple factorization of δg^k in (2.35) leads to the exact same equation (2.27) at the beginning obtained from a factorization of g^k in (2.25) and which admits the solutions described above. Hence, the variations of (2.25) do *not* impose additional constraints on the parameters Λ^{ij}.

Finally, given the octonionic valued field strength $F_{\mu\nu} = F^a_{\mu\nu}\ e_a$, with *real valued* components $F^0_{\mu\nu}, F^i_{\mu\nu};\ \ i = 1, 2, 3, ..., 7$, one can verify that the quadratic action (2.20) is indeed *invariant* under the ternary infinitesimal local gauge transformations (2.23) when the field strength transforms as provided by eq-(2.24)

$$\delta\ S\ =\ -\ \frac{1}{4\kappa^2}\int d^Dx\ <\ F_{\mu\nu}\ (\delta F^{\mu\nu})\ +\ (\delta F_{\mu\nu})\ F^{\mu\nu}\ >\ =$$

$$-\ \frac{1}{4\kappa^2}\int d^Dx\ <\ F^c_{\mu\nu}\ e_c\ \Lambda^{ab}\ [e_a,\ e_b,\ F^{\mu\nu\ n}\ e_n]\ >\ +$$

$$-\ \frac{1}{4\kappa^2}\int d^Dx\ <\ \Lambda^{ab}\ [e_a,\ e_b,\ F^c_{\mu\nu}\ e_c]\ F^{\mu\nu\ n}\ e_n\ >\ =$$

$$-\frac{1}{4\kappa^2}\int d^Dx\ \Lambda^{ab}\ F^c_{\mu\nu}\ F^{\mu\nu\ n}\ (\ <\ e_c\ f_{abnk}\ e_k\ >\ +\ <\ f_{abck}\ e_k\ e_n\ >\ )\ =\ 0. \qquad (2.36)$$

this is a direct result of

$$<\ e_c\ f_{abnk}\ e_k\ >\ +\ <\ f_{abck}\ e_k\ e_n\ >\ =\ -(f_{abnk}\ \delta_{ck} + f_{abck}\ \delta_{kn})\ =\ -(f_{abnc} + f_{abcn})\ =\ 0 \qquad (2.37)$$

due to the property $f_{abnc} + f_{abcn} = 0$ which can be explicitly verified as follows

$$[\ d_{abnc}\ + 2\ \delta_{an}\ \delta_{bc}\ - 2\ \delta_{bn}\ \delta_{ac}\ ]\ +\ [\ d_{abcn}\ + 2\ \delta_{ac}\ \delta_{bn}\ - 2\ \delta_{bc}\ \delta_{an}\ ]\ =\ 0 \quad (2.38)$$

because $d_{abnc} + d_{abcn} = 0$; $d_{nabc} + d_{cabn} = 0$, due to the total antisymmetry of the associator structure constant d_{nabc} under the exchange of any pair of indices. A shortcut to prove the invariance $\delta S = 0$ is simply $\delta(F^k_{\mu\nu})^2 = 2\Lambda^{ij}\ f_{ij}{}^{lk}\ F_{\mu\nu l}\ F^{\mu\nu}_k = 0$ due to the antisymmetry $f_{ij}{}^{lk} = -f_{ij}{}^{kl}$ and symmetry $F_{\mu\nu l}\ F^{\mu\nu}_k = F_{\mu\nu k}\ F^{\mu\nu}_l$ under the exchange of indices $k \leftrightarrow l$. The variation of the components associated with the e_0 generator is trivially zero $\delta(F^0_{\mu\nu})^2 = 0$ because $[e_i, e_j, e_0] = 0$.

This work is not complete until we show the *closure* of the infinitesimal transformations (2.23). To achieve this one needs first to recast them as *derivations*

$$\delta_1 A_\mu \ = \ \delta_1 (A^k_\mu\ e_k) \ = \ \Lambda^{ab}_1 [\ e_a,\ e_b,\ A^c_\mu\ e_c\ ] \ = \ \Lambda^{ab}_1\ D_{e_a, e_b}\ A_\mu \qquad (2.39a)$$

$$\delta_2 A_\mu \ = \ \delta_2 (A^k_\mu\ e_k) \ = \ \Lambda^{cd}_2 [\ e_c,\ e_d,\ A^l_\mu\ e_l\ ] \ = \ \Lambda^{cd}_2\ D_{e_c, e_d}\ A_\mu \qquad (2.39b)$$

by recurring to the fundamental identity (2.10) in order to evaluate the commutator of two derivations and after relabeling indices, one arrives at

$$[\delta_1,\ \delta_2]\ A_\mu \ = \ \Lambda^{cd}_2\ \Lambda^{ab}_1\ [\ D_{e_c,e_d},\ D_{e_a,e_b}\ ]\ A_\mu \ = \ \Lambda^{cd}_2\ \Lambda^{ab}_1\ \left(\ D_{[e_c,e_d,e_a],e_b} \ + \ D_{e_a,[e_c,e_d,e_b]}\ \right)\ A_\mu \ =$$
$$\Lambda^{cd}_2\ \Lambda^{ab}_1\ \ (\ [\ [e_c, e_d, e_a],\ e_b,\ A_\mu\ ] \ + \ [\ e_a,\ [e_c, e_d, e_b],\ A_\mu\ ]\ ) \ =$$
$$\Lambda^{cd}_2\ \Lambda^{ab}_1\ f_{cdak}\ [\ e_k,\ e_b,\ A_\mu\ ] \ + \ \Lambda^{cd}_2\ \Lambda^{ab}_1\ f_{cdbk}\ [\ e_a,\ e_k,\ A_\mu\ ] \ =$$
$$-\ (\ \Lambda^{cd}_2\ \Lambda^{ak}_1\ f_{cda}{}^b - \Lambda^{cd}_2\ \Lambda^{ab}_1\ f_{cda}{}^k\ )\ [\ e_k,\ e_b,\ A_\mu\ ] \ = \ \Lambda^{kb}_3\ [\ e_k,\ e_b,\ A_\mu\ ] \ = \ \delta_3 A_\mu \qquad (2.40)$$

Therefore the antisymmetric parameter resulting from the closure of two transformations is given by

$$\Lambda^{kb}_3 \ = \ -\ (\ \Lambda^{cd}_2\ \Lambda^{ak}_1\ f_{cda}{}^b \ - \ \Lambda^{cd}_2\ \Lambda^{ab}_1\ f_{cda}{}^k\ ) \qquad (2.41)$$

We must still enforce the determinant condition (2.31) . Multiplying both sides of (2.41) by $f_{kb}{}^{ij}$ gives

$$\tilde\Lambda^{ij}_3 \ = \ -\ (\ \tilde\Lambda^b_{2a}\ \Lambda^{ak}_1 \ - \ \tilde\Lambda^k_{2a}\ \Lambda^{ab}_1\ )\ f_{kb}{}^{ij} \ = \ -\ (\ \Upsilon^{bk} \ - \ \Upsilon^{kb}\ )\ f_{kb}{}^{ij} \ =$$
$$\Upsilon^{[kb]}\ f_{kb}{}^{ij} \ \equiv \ \tilde\Upsilon^{[ij]} \qquad (2.42)$$

so that the determinant obeys the relations

$$det\ (\tilde\Lambda^{ij}_3) \ = \ det\ (\tilde\Upsilon^{[ij]}) \ = \ det\ (\tilde\Lambda^{ij}_1) \ = \ det\ (\tilde\Lambda^{ij}_2) \ =$$
$$exp\ \left(\ \int_0^{x^\mu}\ \frac{g_i A^i_\mu}{g^2}\ dx^\mu\ \right) \qquad (2.43)$$

Therefore, to attain closure of two gauge transformations eqs-(2.43) must be satisfied. The parameters Λ^{kb}_3 are explicitly determined in terms of $\Lambda^{kb}_1, \Lambda^{kb}_2$ as described by (2.41) and the latter must obey the determinant conditions (2.43). Therefore, as a result of the determinant conditions, the number of independent components of $\Lambda_1^{kb}, \Lambda_2^{kb}$ is reduced from 2×21 to $2\times 21-3$. When the parameters are *constants*, there is no need to impose the determinant conditions. Closure of the global rigid transformations is automatic and the parameters are related as described by eq-(2.41).

The finite ternary transformations can be obtained by "exponentiation" as follows

$$F' \ = \ F \ + \ \delta F \ + \ \frac{1}{2!}\delta(\delta F) \ + \frac{1}{3!}(\delta(\delta(\delta F))) \ + \ .... \tag{2.44}$$

where $\delta(F^m_{\mu\nu} \ e_m) = \Lambda^{ab}[e_a, e_b, F^c_{\mu\nu} e_c]$; $\delta(\delta F) = \Lambda^{mn}[\ e_m, e_n, \Lambda^{ab}[e_a, e_b, F^c_{\mu\nu} e_c]\ ]$; ...... To show that the action is invariant under finite ternary local transformations requires to follow a few steps. Firstly, one defines

$$< x\, y > \ \equiv \ Real\, [\, \bar x\, y\,] \ = \ \frac{1}{2}\, (\, \bar x\, y + \bar y\, x\,) \ \Rightarrow \ < x\, y > \ = \ < y\, x > \tag{2.45}$$

Despite nonassociativity, the *very special conditions*

$$x(\bar x u) \ = \ (x\bar x)u; \quad x(u\bar x) \ = \ (xu)\bar x; \quad x(xu) \ = \ (xx)u; \quad x(ux) \ = \ (xu)x \tag{2.46}$$

are obeyed for octonions resulting from the Moufang identities. Despite that $(xy)z \neq x(yz)$ one has that their real parts obey

$$Real \ [\ (x\ y)\ z\ ] \ = \ Real\ [x\ (y\ z)\ ] \tag{2.47}$$

Due to the nonassociativity of the algebra, in general one has that $(UF)U^{-1} \neq U(FU^{-1})$. However, if and only if $U^{-1} = \bar U \Rightarrow \bar U U = U \bar U = 1$, as a result of the the *very special conditions* (2.46) one has that $F' = (UF)U^{-1} = U(FU^{-1}) = UFU^{-1} = UF\bar U$ is *unambiguously* defined. One can equate the result of the exponentiation procedure in eq-(2.44) to the expression

$$F' \ = \ UFU^{-1} \ = \ UF\bar U \ = \ e^{\Sigma^k(\Lambda^{ab})e_k}\ (F^c\ t_c)\ e^{-\Sigma^k(\Lambda^{ab})e_k}; \quad k = 1, 2, 3, ..., 7. \tag{2.48}$$

where $\Sigma^k(\Lambda^{ab})e_k$ is a complicated function of Λ^{ab}. It yields the finite transformations which agree with the infinitesimal *ternary* ones when Λ^{ab} are $infinitesimals$. For instance, to lowest order in Λ^{ab}, one has that Σ^k satisfies $2\Sigma^k c_{kcd} = \Lambda^{ab} f_{abcd}$ and which follows by comparing the transformations in (2.44) to those in (2.48), to lowest order.

Dropping the spacetime indices for convenience in the expressions for $F^{\mu\nu}, F_{\mu\nu}$, and by repeated use of eqs-(2.45, 2.46, 2.47), when $U^{-1} = \bar U$, the action density is also invariant under (unambiguously defined) transformations of the form $F' = UFU^{-1} = UF\bar U$,

$$< F'\, F' > \ = \ Re\,[\bar F'\, F'] \ = \ Re\,[(U\bar F U^{-1})\,(UFU^{-1})] \ = \ Re\,[(U\bar F)\,(\,U^{-1}\,(UF\,U^{-1})\,)] \ =$$

$$Re\ [(U\ \bar F)\ (U^{-1}\ U)\ (FU^{-1})] \ = \ Re\ [(U\bar F)\ (FU^{-1})] \ = \ Re\ [(FU^{-1})\ (U\bar F)] \ =$$

$$Re\,[F\,(\,U^{-1}\,(U\,\bar{F})\,)]\;=\;Re\,[F\,(U^{-1}U)\,\bar{F}]\;=\;Re\,[F\,\bar{F}]\;=\;Re\,[\bar{F}\,F]\;=\;<\,F\,F\,>. \quad (2.49)$$

The real part of the coupling g^0 is inert under the transformations (2.23b) and it decouples from the definition of the field strength $F_{\mu\nu}$ because e_0 has a vanishing 3-bracket with other elements of the octonion algebra. The coupling $g^0 = constant$ can be incorporated into the field strength in the same fashion as it occurs in ordinary Yang-Mills. One may rewrite the physical coupling g^0 as a prefactor in front of the 3-bracket as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g^0[A_\mu, A_\nu, \mathbf{g}]$, and reabsorb g^0 into the definition of the A_μ field as $F_{\mu\nu} = \frac{1}{g^0}\left(\partial_\mu(g^0 A_\nu) - \partial_\nu(g^0 A_\mu) + [g^0 A_\mu, g^0 A_\nu, \mathbf{g}]\right)$. Thus $F_{\mu\nu} \rightarrow \frac{1}{g^0} F_{\mu\nu}$ and the action is rescaled as $S \rightarrow \frac{1}{(g^0)^2} S$ as it is customary in the Yang-Mills action.

To conclude this work, when $A^i_\mu = g^i \partial_\mu \phi$ we have an action

$$S \;=\; -\;\frac{1}{4\;\kappa^2\;(g^0)^2}\int d^D x\;(\;F^0_{\mu\nu}\;F^{\mu\nu}_0\;-\;F^i_{\mu\nu}\;F^{\mu\nu}_i\;),\quad i = 1, 2, 3, ..., 7 \quad (2.50)$$

with

$$F^i_{\mu\nu}\;F^{\mu\nu}_i\;=\;[\;\partial_{[\mu} A^l_{\nu]}\;+\;A^i_\mu\;A^j_\nu\;g^k\;f_{ijk}{}^l\;]^2\;=\;[\;(\partial_\mu g^i)\;(\partial_\nu \phi) - (\partial_\nu g^i)\;(\partial_\mu \phi)\;]^2\;=$$

$$2\;\left(\;(\partial_\mu g^i)^2\;(\partial_\nu \phi)^2\;-\;(\partial_\mu g^i)\;(\partial^\mu \phi)\;(\partial_\nu g_i)\;(\partial^\nu \phi)\;\right)\;\neq\;0 \quad (2.51)$$

$$F^0_{\mu\nu}\;F^{\mu\nu}_0\;=\;(\partial_\mu A^0_\nu - \partial_\nu A^0_\mu)\;(\partial^\mu A^\nu_0 - \partial^\nu A^\mu_0) \quad (2.52)$$

and which is invariant under the transformations (2.23b) due to eqs-(2.27) and $\delta\phi = 0$ (2.34). The solutions to eqs-(2.27) are provided by eq-(2.31) after following eqs-(2.28-2.30). It would have been desirable to avoid the conditions (2.27) which force the gauge parameters $\Lambda^{ij}(x)$ to be field-dependent in the sense that they must obey the determinant condition (2.31) which has a Wilson-loop-like expression. There is closure of two gauge transformations as indicated by eqs-(2.40, 2.41).

These octonionc ternary gauge field theories deserve further investigation. In particular, to study their relation to Yang-Mills theories based on the G_2 group which is the automorphism group of the Octonions. The nontrivial role of the 7+1 scalars in the action (2.51) given by the coupling functions $g^i(x)$ and $\phi(x)$ warrants to be studied further. The terms (2.52) have the same functional form as the Maxwell action. The inclusion of potential terms for the scalar fields and Chern-Simons actions will be the subject of future investigation.

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