The Extended Born's Reciprocal Relativity Theory : Division, Jordan, N-ary Algebras, and Higher Order Finsler spaces

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Abstract

We extend the construction of Born's Reciprocal Relativity theory in ordinary phase spaces to an extended phase space based on Quaternions. The invariance symmetry group is the (pseudo) unitary quaternionic group $U(N_+, N_-, \mathbf{H})$ which is isomorphic to the unitary symplectic group $USp(2N_+, 2N_-, \mathbf{C})$. It is explicitly shown that the quaternionic group $U(N_+, N_-, \mathbf{H})$ leaves invariant both the quadratic norm (corresponding to the generalized Born-Green interval in the extended phase space) and the tri-symplectic 2-form. The study of Octonionic, Jordan and ternary algebraic structures associated with generalized spacetimes (and their phase spaces) described by Gunaydin and collaborators is reviewed. A brief discussion on n-plectic manifolds whose Lie n-algebra involves multi-brackets and n-ary algebraic structures follows. We conclude with an analysis on the role of higher-order Finsler geometry in the construction of extended relativity theories with an upper and lower bound to the higher order accelerations (associated with the higher order tangent and cotangent spaces).

1 Introduction : Born's Reciprocal Relativity in Phase Spaces

Born's Reciprocal Relativity [1] was an extension of Einstein's special relativity where in addition to a maximal light speed (derivative of the position coordinates), by reciprocity ("duality") , there was a maximal bound to the derivatives of the momentum (maximal force). Born's Reciprocal Relativity incorporates the principle of maximal proper force (related also to acceleration [2]) from the perspective of Phase Spaces. In the case of four spacetime dimensions one has an $8D$ phase space and the invariance $U(1,3)$ Group. The $U(1,3) = SU(1,3) \times U(1)$ group transformations leave invariant the phase-space intervals under rotations, velocity and acceleration boosts as shown by Low [3]. These transformations can be simplified drastically when the velocity/acceleration boosts are taken to lie in the z-direction, leaving the transverse directions x, y, p_x, p_y intact ; i.e., the $U(1,1) = SU(1,1) \times U(1)$ subgroup transformations leave invariant the phase-space interval given by (in units of $\hbar = c = 1$)

$$(d\omega)^2 = (dT)^2 - (dX)^2 + \frac{(dE)^2 - (dP)^2}{b^2} =$$

$$(d\tau)^2 [1 + \frac{(dE/d\tau)^2 - (dP/d\tau)^2}{b^2}] = (d\tau)^2 [1 - \frac{m^2 g^2(\tau)}{m_P^2 A_{max}^2}]. \quad (1.1)$$

where we have factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in eq-(1.1) and the maximal proper-force is set to be $b \equiv m_P A_{max}$. m_P is the Planck mass $1/L_P$ so that $b = (1/L_P)^2$, may also be interpreted as the maximal string tension when L_P is the Planck scale.

The quantity $g(\tau)$ is the proper four-acceleration of a particle of mass m in the z-direction which we take to be defined by the X coordinate. The interval $(d\omega)^2$ described by Low [3] is $U(1,3)$-invariant for the most general transformations in the $8D$ phase-space. The analog of the Lorentz relativistic factor in eq-(1.1) involves the ratios of two proper *forces*. One variable force is given by $mg(\tau)$ and the maximal proper force sustained by an *elementary* particle of mass m_P is assumed to be $F_{max} = m_{Planck} c^2 / L_P$.

The transformations laws of the coordinates in that leave invariant the interval (1.1) were given by [3]:

$$T' = T cosh\xi + (\frac{\xi_v X}{c^2} + \frac{\xi_a P}{b^2}) \frac{sinh\xi}{\xi}. \quad (1.2a)$$

$$E' = E cosh\xi + (-\xi_a X + \xi_v P) \frac{sinh\xi}{\xi}. \quad (1.2b)$$

$$X' = X cosh\xi + (\xi_v T - \frac{\xi_a E}{b^2}) \frac{sinh\xi}{\xi}. \quad (1.2c)$$

$$P' = P cosh\xi + (\frac{\xi_v E}{c^2} + \xi_a T) \frac{sinh\xi}{\xi}. \quad (1.2d)$$

The ξ_v is velocity-boost rapidity parameter and the ξ_a is the force/acceleration-boost rapidity parameter of the primed-reference frame. They are defined respectively :

$$tanh(\frac{\xi_v}{c}) = \frac{v}{c}. \quad tanh(\frac{\xi_a}{b}) = \frac{ma}{m_P A_{max}}. \quad (1.3)$$

The *effective* boost parameter ξ of the $U(1,1)$ subgroup transformations appearing in eqs-(2-2a, 2-2d) is defined in terms of the velocity and acceleration boosts parameters ξ_v, ξ_a respectively as:

$$\xi \equiv \sqrt{\frac{\xi_v^2}{c^2} + \frac{\xi_a^2}{b^2}}. \tag{1.4}$$

Straightforward algebra allows us to verify that these transformations leave the interval of eq- (1.1) in classical phase space invariant. They are are fully consistent with Born's duality Relativity symmetry principle [1] $(X, P) \rightarrow (P, -X)$. By inspection we can see that under Born reciprocity, the transformations in eqs-(1.2a-1.2d) are *rotated* into each other, up to numerical b factors in order to match units. When on sets $\xi_a = 0$ in (1.2a-1.2d) one recovers automatically the standard Lorentz transformations for the X, T and E, P variables *separately*, leaving invariant the intervals $dT^2 - dX^2 = (d\tau)^2$ and $(dE^2 - dP^2)/b^2$ separately.

Also the transformations leave invariant the symplectic two-form

$$dT' \wedge dE' \ - \ dX' \wedge dP' \ = \ dT \wedge dE \ - \ dX \wedge dP \tag{1.5}$$

For simplicity, unless otherwise indicated, we shall choose the natural units $\hbar = c = G = 1$ so that $b = m_P = L_P = 1$.

The most general $U(D-1, 1)$ transformations leaving invariant the quadratic interval in phase space and the symplectic 2-form were given by [3], in units $\hbar = G = c = b = 1$

$$T' \ = \ T \ cosh\xi \ + \ (\xi_v^i X_i + \xi_a^i P_i) \ \frac{sinh\xi}{\xi}. \tag{1.6a}$$

$$E' \ = \ E \ cosh\xi \ + \ (\xi_v^i P_i - \xi_a^i X_i) \ \frac{sinh\xi}{\xi}. \tag{1.6b}$$

$$X'^i \ = \ X^i \ + \ X_j \ (\xi_v^i \xi_v^j + \xi_a^i \xi_a^j) \ \frac{cosh\xi - 1}{\xi^2} \ + \ (\ \xi_v^i T \ - \ \xi_a^i E \ ) \ \frac{sinh\xi}{\xi}. \tag{1.6c}$$

$$P'^i \ = \ P^i \ + \ P_j \ (\xi_v^i \xi_v^j + \xi_a^i \xi_a^j) \ \frac{cosh\xi - 1}{\xi^2} \ + \ (\ \xi_v^i E \ + \ \xi_a^i T \ ) \ \frac{sinh\xi}{\xi}. \tag{1.6d}$$

where the *effective* boost parameter ξ is defined in terms of the velocity and acceleration boosts parameters ξ_v^i, ξ_a^i, respectively, as

$$\xi \ \equiv \ \sqrt{(\xi_v^i)^2 + (\xi_a^i)^2}, \quad i = 1, 2, 3, ..., D-1 \tag{1.7}$$

The Eddington-Dirac large numbers coincidence (and an ultraviolet/infrared entanglement) can be easily implemented if one equates the upper bound on the proper-four force sustained by a fundamental particle , $(mg)_{bound} = m_P(c^2/L_P)$, with the proper-four force associated with the mass of the (observed) universe M_U, and whose *minimal* acceleration c^2/R is given in terms of an infrared-cutoff R (the Hubble horizon radius). Equating these proper-four forces gives

$$\frac{m_P c^2}{L_P} = \frac{M_U c^2}{R} \Rightarrow \frac{M_U}{m_P} = \frac{R}{L_P} \sim 10^{61}. \tag{1.8}$$

from this equality of proper-four forces associated with a maximal/minimal acceleration one infers $M_U \sim 10^{61} m_{Planck} \sim 10^{61} 10^{19} m_{proton} = 10^{80} m_{proton}$ which agrees with observations and with the Eddington-Dirac number 10^{80} [4]

$$N = 10^{80} = (10^{40})^2 \sim (\frac{F_e}{F_G})^2 \sim (\frac{R}{r_e})^2. \tag{1.9}$$

where $F_e = e^2/r^2$ is the electrostatic force between an electron and a proton ; $F_G = G m_e m_{proton}/r^2$ is the corresponding gravitational force and $r_e = e^2/m_e \sim 10^{-13} cm$ is the classical electron radius (in units $\hbar = c = 1$).

One may notice that the above equation (1.8) is also consistent with the Machian postulate [4] that the rest mass of a particle is determined via the gravitational potential energy due to the other masses in the universe. In particular, by equating

$$m_i c^2 = G m_i \sum_j \frac{m_j}{|r_i - r_j|} = \frac{G m_i M_U}{R} \Rightarrow \frac{c^2}{G} = \frac{M_U}{R}. \tag{1.10}$$

Due to the negative binding energy, the composite mass m_{12} of a system of two objects of mass m_1, m_2 is not equal to the sum $m_1 + m_2 > m_{12}$. We can now arrive at the conclusion that the *minimal* acceleration c^2/R is also the same acceleration induced on a test particle of mass m by a spherical mass distribution M_U inside a radius R . The acceleration felt by a test particle of mass m sitting at the edge of the observable Universe (at the Hubble horizon radius R) is

$$|a| \ = \ \frac{G M_U}{R^2} \tag{1.11}$$

From the last two equations one gets the same expression for the *minimal* acceleration $a = a_{minimal} = \frac{c^2}{R}$ and which is of the same order of magnitude as the anomalous acceleration of the Pioneer and Galileo spacecrafts $a \sim 10^{-8} cm/s^2$. Let us examine closer the equality between the proper-four forces

$$\frac{m_P c^2}{L_P} = \frac{M_U c^2}{R} \Rightarrow \frac{m_P}{L_P} = \frac{M_U}{R} = \frac{c^2}{G}. \tag{1.12}$$

The last term in eq-(1.12) is directly obtained after implementing the Machian principle. Thus, one concludes from eq-(1.12) that as the universe evolves in time one must have the conserved ratio of the quantities $M_U/R = c^2/G = m_P/L_P$. This interesting possibility, advocated by Dirac long ago, for the fundamental constants $\hbar, c, G, ...$ to vary over cosmological time is a plausible idea with the provision that the above ratios satisfy the relations in eq-(1.12) at any given moment of cosmological time. If the fundamental constants do not vary over time then the ratio $M_U/R = c^2/G$ must refer then to the *asymptotic* values of the Hubble horizon radius $R = R_{asymptotic}$.

We provided in [5] six specific results stemming from Born's reciprocal Relativity and which are not present in Special Relativity. These were : momentum-dependent time delay in the emission and detection of photons; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations. One of the most interesting conclusions was that there are null hypersurfaces in a flat phase-spaces where points can have superluminal $v > c$ behavior in ordinary spacetime, despite corresponding to a null hypersurface in a flat phase-space. Superluminal behavior in spacetime can occur without having superluminal behavior in C-spaces [6].

In [8] we extend the construction of Born's Reciprocal Phase Space Relativity to the case of Clifford Spaces and which involve the use of *polyvectors* and a *lower/upper* length scale. A Clifford Phase-Space Gravitational Theory based in gauging the generalization of the Quaplectic group and invoking Born's reciprocity principle between coordinates and momenta (maximal speed of light velocity and maximal force) was provided. The purpose of this work is to continue this line of research and explore further generalizations.

2 Quaternions algebras and Extended Born's Reciprocal Relativity

Let us begin with the quaternionic-valued variable

$$Z_\mu \;=\; Z_\mu^{(0)}\, e_0 + Z_\mu^{(1)}\, e_1 + Z_\mu^{(2)}\, e_2 + Z_\mu^{(3)}\, e_3,\;\; e_i e_j = -\delta_{ij} e_o + \epsilon_{ijk} e_k,\; i,j,k = 1,2,3 \tag{2.1}$$

Upon using the units $\hbar = G = c = b = 1$, the spacetime coordinates X_μ can be regrouped with a triplet of momenta given by the triad of variables P_μ, U_μ, V_μ (the imaginary quaternionic components) as follows

$$Z_\mu^{(0)} = X_\mu,\; Z_\mu^{(1)} = P_\mu,\; Z_\mu^{(2)} = U_\mu,\; Z_\mu^{(3)} = V_\mu \tag{2.2}$$

the indices μ, ν span the values $1, 2, 3, ..., D$. If one has a different choice of units one needs to introduce physical constants (length/mass scales) in order to ensure that all quantities in eq-(2.1) have the same physical units.

The quaternionic conjugate is

$$\bar{Z}_\mu \;=\; Z_\mu^{(0)}\, e_0 \;-\; Z_\mu^{(1)}\, e_1 \;-\; Z_\mu^{(2)}\, e_2 \;-\; Z_\mu^{(3)}\, e_3 \tag{2.3}$$

and the norm squared is

$$|Z|^2 \;\equiv\; \bar{Z}^\mu Z_\mu \;=\; Z_\mu\, \bar{Z}^\mu \;=\; X^\mu X_\mu \;+\; P^\mu P_\mu \;+\; U^\mu U_\mu \;+\; V^\mu V_\mu \tag{2.4}$$

for Minkowski signature one has

$$X^\mu X_\mu \;=\; (X_0)^2 \;-\; (X_1)^2 \;-\; (X_3)^2 \;-\; ...... \;-\; (X_{D-1})^2,\; etc\; .... \tag{2.5}$$

To simplify the calculations, and without loss of generality, let us take the underlying spacetime to have $D = 2$ dimensions, one spatial and one temporal. The generalized boost parameter involving velocity and acceleration boosts along the spatial direction is encoded in the quaternionic-valued parameter

$$\xi \ \equiv \ \xi^{(m)} e_m \ = \ \xi_v \ e_0 \ + \ \xi_a^{(1)} \ e_1 \ + \ \xi_a^{(2)} \ e_2 \ + \ \xi_a^{(3)} \ e_3 \tag{2.6}$$

its conjugate is

$$\xi \ \equiv \ \xi^{(m)} \bar{e}_m \ = \ \xi_v \ e_0 \ - \ \xi_a^{(1)} \ e_1 \ - \ \xi_a^{(2)} \ e_2 \ - \ \xi_a^{(3)} \ e_3 \tag{2.7}$$

and its norm is

$$|\xi| \ \equiv \ \sqrt{\xi\bar{\xi}} \ = \ \sqrt{(\xi_v)^2 + (\xi_a^{(1)})^2 + (\xi_a^{(2)})^2 + (\xi_a^{(3)})^2} \tag{2.8}$$

Let us propose the following transformations

$$(Z_0)' \ = \ Z_0 \ cosh|\xi| \ + \ Z_1 \ \xi^{(m)} e_m \ \frac{sinh|\xi|}{|\xi|} \tag{2.9a}$$

$$(Z_1)' \ = \ Z_1 \ cosh|\xi| \ + \ Z_0 \ \xi^{(m)} \bar{e}_m \ \frac{sinh|\xi|}{|\xi|} \tag{2.9b}$$

their quaternionic conjugates are

$$\bar{Z}_0' \ = \ \bar{Z}_0 \ cosh|\xi| \ + \ \xi^{(m)} \bar{e}_m \ \bar{Z}_1 \ \frac{sinh|\xi|}{|\xi|} \tag{2.10a}$$

$$\bar{Z}_1' \ = \ \bar{Z}_1 \ cosh|\xi| \ + \ \xi^{(m)} e_m \ \bar{Z}_0 \ \ \frac{sinh|\xi|}{|\xi|} \tag{2.10b}$$

where one has recurred to the relation $(AB)^* = B^* A^*$ under quaternionic conjugation. Since quaternions are noncommutative one has to be very careful with the ordering of factors. In the Appendix one can show, after some straightforward algebra, that

$$[\ \xi^{(m)} e_m, \ dZ_1 \ d\bar{Z}_0 \ ] \ + \ [\ dZ_0 \ d\bar{Z}_1, \ \xi^{(m)} \bar{e}_m \ ] \ = \ 0 \tag{2.10c}$$

Therefore, from eqs-(2.9a, 2.9b, 2.10a, 2.10b, 2.10c), after using the condition $\bar{W} Z \bar{Z} W = W Z \bar{Z} \bar{W} = |Z|^2 |W|^2 = |W|^2 |Z|^2$ for two quaternions Z, W due to the associativity property, and to the identity $cosh^2|\xi| - sinh^2|\xi| = 1$, one can finally deduce that the quadratic form below is indeed *invariant* under the above transformations (2.9, 2.10)

$$(Z_0)' \ (\bar{Z_0})' \ - \ (Z_1)' \ (\bar{Z_1})' \ = \ Z_0 \ \bar{Z}_0 \ - \ Z_1 \ \bar{Z}_1 \ \Rightarrow$$

$$(X_0')^2 \ + \ (P_0')^2 \ + \ (U_0')^2 \ + \ (V_0')^2 \ - \ (X_1')^2 \ - \ (P_1')^2 \ - \ (U_1')^2 \ - \ (V_1')^2 \ =$$

$$(X_0)^2 \ + \ (P_0)^2 \ + \ (U_0)^2 \ + \ (V_0)^2 \ - \ (X_1)^2 \ - \ (P_1)^2 \ - \ (U_1)^2 \ - \ (V_1)^2 \tag{2.11}$$

In component form the transformations (2.9a, 2.9b, 2.10a, 2.10b) which leave invariant the quadratic form (2.11) are given by

$$(X_0)' = X_0 \ cosh|\xi| \ + \ \left(\xi_v X_1 - \xi_a^{(1)} P_1 - \xi_a^{(2)} U_1 - \xi_a^{(3)} V_1\right) \ \frac{sinh|\xi|}{|\xi|} \quad (2.12a)$$

$$(P_0)' = P_0 \ cosh|\xi| \ + \ \left(\xi_a^{(1)} X_1 + \xi_v P_1 + \xi_a^{(3)} U_1 - \xi_a^{(2)} V_1\right) \ \frac{sinh|\xi|}{|\xi|} \quad (2.12b)$$

$$(U_0)' = U_0 \ cosh|\xi| \ + \ \left(\xi_a^{(2)} X_1 - \xi_a^{(3)} P_1 + \xi_v U_1 + \xi_a^{(1)} V_1\right) \ \frac{sinh|\xi|}{|\xi|} \quad (2.12c)$$

$$(V_0)' = V_0 \ cosh|\xi| \ + \ \left(\xi_a^{(3)} X_1 + \xi_a^{(2)} P_1 - \xi_a^{(1)} U_1 + \xi_v V_1\right) \ \frac{sinh|\xi|}{|\xi|} \quad (2.12d)$$

$$(X_1)' = X_1 \ cosh|\xi| \ + \ \left(\xi_v X_0 + \xi_a^{(1)} P_0 + \xi_a^{(2)} U_0 + \xi_a^{(3)} V_0\right) \ \frac{sinh|\xi|}{|\xi|} \quad (2.13a)$$

$$(P_1)' = P_1 \ cosh|\xi| \ + \ \left(-\xi_a^{(1)} X_0 + \xi_v P_0 - \xi_a^{(3)} U_0 + \xi_a^{(2)} V_0\right) \ \frac{sinh|\xi|}{|\xi|} \quad (2.13b)$$

$$(U_1)' = U_1 \ cosh|\xi| \ + \ \left(-\xi_a^{(2)} X_0 + \xi_a^{(3)} P_0 + \xi_v U_0 - \xi_a^{(1)} V_0\right) \ \frac{sinh|\xi|}{|\xi|} \quad (2.13c)$$

$$(V_1)' = V_1 \ cosh|\xi| \ + \ \left(-\xi_a^{(3)} X_0 - \xi_a^{(2)} P_0 + \xi_a^{(1)} U_0 + \xi_v V_0\right) \ \frac{sinh|\xi|}{|\xi|} \quad (2.13d)$$

Hence one can construct an extended relativity theory where the coordinates X and the triad of momenta P, U, V all become *mixed* under the transformations (2.12, 2.13) which leave invariant the interval (omitting indices for convenience) $(ds)^2 = (dX)^2 + (dP)^2 + (dU)^2 + (dV)^2$. In this respect we have constructed an extension of Born's reciprocal relativity to an extended phase space which can be identified with a quaternionic space.

Using the units $\hbar = G = c = b = 1$, we can see that the transformations for the X_0, P_0, X_1, P_1 coordinates in eqs-(2.12, 2.13) *coincide* with those given by eqs-(1.2) after setting $\xi_a = -\xi_a^{(1)}$, and $\xi_a^{(2)} = \xi_a^{(3)} = 0$. Therefore, one can conclude that the transformations (2.12, 2.13) are a natural *extension* of the transformations (1.2) involving the coordinates X and momenta P . The transformations (2.12, 2.13) involve both a quaternionic extension of the coordinates *and* the rapidity parameters $\xi = \xi^m e_m$. One must *not* interpret the components of the quaternionic coordinates, respectively, with the spacetime coordinates, momenta, second and third order momenta. And one must *not* interpret also the components of the quaternionic rapidity parameters, respectively, with the velocity boosts, acceleration boosts, second and third order acceleration boosts along the spatial direction X_1.

Instead of higher order momenta and higher order accelerations typical of higher order (co)tangent spaces and Jet Spaces, rigorously speaking, one has a *triad* of momenta P_μ, U_μ, V_μ, and a *triad* of acceleration boosts $\xi_a^{(i)}, i = 1,2,3$ which is more closely related to a tri-holomorphic and tri-symplectic structure associated to hyper Kahler geometry. For references on higher order symplectic geometry, multisymplectic, polysymplectic, and n-plectic geometry geometry see [17], [16].

It is important also to emphasize that the transformations (2.12, 2.13) are *not* obtained by naively replacing the coordinates X_0, P_0, X_1, P_1 in eqs-(1.2) for their quaternionic extensions. A quaternionic extension of the rapidity parameters $\xi = \xi^m e_m$ is also required in the transformations (2.12, 2.13).

Therefore, to conclude, one can be view the transformations (2.12, 2.13) as an extension of the (pseudo) unitary group $U(2), U(1,1)$ symmetry transformations in eqs-(1.2) to the quaternionic group $U(2,H), U(1,1,H)$ case. In general, for D spacetime dimensions, one will have the quaternionic group $U(D,H), U(D-1,1,H)$ extension of the (pseudo) unitary $U(D), U(D-1,1)$ group of symmetry transformations in the $2D$-dim phase space (cotangent space) given by eqs-(1.6).

Let us study now what happens to the tri-symplectic forms under the quaternionic $U(1,1,H)$ transformations. For ordinary complex numbers $z_\mu = X_\mu + iP_\mu$, the pseudo-unitary $U(1,1)$ transformations are equivalent to those given by eqs-(1.2) when the complex rapidity parameter is $\xi = \frac{\xi_v}{c} - i\frac{\xi_a}{b}$ $(b = c = 1)$. In the complex numbers case one does have for $\mu = 1,2$

$$(dz_0)' \wedge (d\bar z_0)' \ - \ (dz_1)' \wedge (d\bar z_1)' \ = \ dz_0 \wedge d\bar z_0 \ - \ dz_1 \wedge d\bar z_1 \ \Rightarrow$$

$$(dX_0)' \wedge (dP_0)' \ - \ (dX_1)' \wedge (dP_1)' \ = \ dX_0 \wedge dP_0 \ - \ dX_1 \wedge dP_1 \qquad (2.14a)$$

To simplify matters, and without loss of generality, one can verify that when the rapidity parameters $\xi_v = \xi_a^{(2)} = \xi_a^{(3)} = 0$ are set to zero, *except* $\xi_a^{(1)}$ such that $|\xi_a^{(1)}| = |\xi|$, one has an invariance of the following 2-form

$$(dZ_0)' \wedge (d\bar Z_0)' \ - \ (dZ_1)' \wedge (d\bar Z_1)' = \ dZ_0 \wedge d\bar Z_0 \ - \ dZ_1 \wedge d\bar Z_1 \qquad (2.14b)$$

under the quaternionic $U(1,1,H)$ transformations (2.9, 2.10). To verify this one may first use the component-wise transformations (2.12, 2.13) to show that when $\xi_v = \xi_a^{(2)} = \xi_a^{(3)} = 0$ are set to zero, *except* $\xi_a^{(1)}$, one has

$$(\ (dX_0)' \wedge (dP_0)' \ - \ (dX_1)' \wedge (dP_1)' \ ) \ e_1 \ = \ (\ dX_0 \wedge dP_0 \ - \ dX_1 \wedge dP_1 \ ) \ e_1 \qquad (2.15a)$$

$$((dU_0)' \wedge (dV_0)' - (dU_1)' \wedge (dV_1)') e_1 = (dU_0 \wedge dV_0 - dU_1 \wedge dV_1) e_1 \tag{2.15b}$$

due to the identity $cosh^2|\xi| - sinh^2|\xi| = 1$ and the antisymmetry property of the wedge product $dX \wedge dP = -dP \wedge dX$, $dU \wedge dV = -dV \wedge dU$. We also have the additional conditions on the other imaginary components, after lengthy but straightforward algebra

$$(- (dX_0)' \wedge (dU_0)' + (dX_1)' \wedge (dU_1)' + (dP_0)' \wedge (dV_0)' - (dP_1)' \wedge (dV_1)') e_2 =$$

$$(- dX_0 \wedge dU_0 + dX_1 \wedge dU_1 + dP_0 \wedge dV_0 - dP_1 \wedge dV_1) e_2 \tag{2.16}$$

$$(- (dX_0)' \wedge (dV_0)' + (dX_1)' \wedge (dV_1)' - (dP_0)' \wedge (dU_0)' + (dP_1)' \wedge (dU_1)') e_3 =$$

$$(- dX_0 \wedge dV_0 + dX_1 \wedge dV_1 - dP_0 \wedge dU_0 + dP_1 \wedge dU_1) e_3 \tag{2.17}$$

finally, one can verify that upon adding eqs-(2.15a, 2.15b, 2.16, 2.17) one arrives precisely at the equality

$$\Omega' = \frac{1}{2} (dZ_0)' \wedge (d\bar{Z}_0)' - \frac{1}{2} (dZ_1)' \wedge (d\bar{Z}_1)' = \Omega = \frac{1}{2} dZ_0 \wedge d\bar{Z}_0 - \frac{1}{2} dZ_1 \wedge d\bar{Z}_1 \tag{2.18}$$

after decomposing the quaternionic-valued variables into their respective components. Therefore, one has invariance of the quaternionic-valued 2-form $\Omega' = \Omega$ under pure acceleration boosts along the X_1 directions such that $\xi_a^{(1)} \neq 0$; $\xi_v = \xi_a^{(2)} = \xi_a^{(3)} = 0$. Under the most general $U(1, 1, H)$ transformations (2.9, 2.10, 2.12, 2.13), when *all* the rapidity parameters are included without setting some of them to zero, one also has that $\Omega' = \Omega$. In order to avoid the extremely tedious and cumbersome algebra one may recur to group theory [9]. The unitary-symplectic groups $USp(2N_+, 2N_-)$ are the *intersection* of the (pseudo) unitary groups $U(2N_+, 2N_-, C)$ with the symplectic group $Sp(2N, C)$, where $N = N_+ + N_-$. All these $USp(2N)$ groups are isomorphic to the unitary groups in N-dimensional quaternionic space $USp(2N_+, 2N_-) \simeq U(N_+, N_-, H)$. Therefore one may ascertain that eq-(2.18) is satisfied under the $U(1, 1, H)$ transformations (2.9, 2.10).

3 On Octonions, Jordan and N-ary Algebraic Extensions of Born's Reciprocal Relativity

Based on the quaternionic extension of Born's Reciprocal Relativity presented in the prior section we would like to explore the extensions to octonions, Jordan algebras and n-ary algebras. The symmetries of generalized spacetimes and their corresponding phase spaces defined by Jordan algebras of degree three were studied by [14]. These generalized spacetimes were coordinatized by elements of general Jordan algebras J whose rotation $Rot(J)$, Lorentz $Lor(J)$ and conformal $Conf(J)$ groups are identified with the automorphism $Aut(J)$, reduced structure $Str_o(J)$ and Mobius $Mob(J)$ groups of the Jordan algebra J, respectively.

In particular, the automorphism, reduced structure and Mobius groups of the Jordan algebras $\Gamma_{(1,d)}$ of gamma matrices (Clifford algebras) are simply the rotation, Lorentz and conformal groups of $d+1$-dim Minkowski spacetime [14]. Furthermore, there exist the special isomorphisms between the Jordan algebras of 2×2 Hermitian matrices over the four division algebras and the Jordan algebras of gamma matrices [14]

$$J_2^R \simeq \Gamma_{(1,2)}, \quad J_2^C \simeq \Gamma_{(1,3)}, \quad J_2^H \simeq \Gamma_{(1,5)}, \quad J_2^O \simeq \Gamma_{(1,9)} \tag{3.1}$$

The corresponding Minkwoski spacetimes have dimensions $3, 4, 6, 10$ which are precisely the dimensions for the existence of super-Yang-Mills and classical superstring theories; i.e. the transverse dimensions to the world sheet of the superstring are $1, 2, 4, 8$, respectively, which coincide with the dimensions of the four division algebras.

These Jordan algebras are all quadratic and their norm forms are the quadratic invariants constructed from the Minkowski metric. For Jordan algebras J_n^A of $n \times n$ matrices defined over a division algebra $\mathbf{A}$, the norm $N(J)$ is an n-th form which is given by the determinant form, or its generalization to the quaternionic and octonionic matrices [14]. For example, in the case of the Jordan algebra of degree three over the octonions J_3^O, the analog of determinant of an element $M \subset J_3^O$ is given by the Freudenthal determinant (cubic form)

$$det\ M \ = \ \frac{1}{3}\ Trace\ (\ M *_J\ (M \times_F M)\ ) \tag{3.2}$$

where the commutative but nonassociative Jordan product of two Jordan matrices is

$$X\ *_J\ Y\ =\ \frac{1}{2}\ (X\ Y\ +\ Y\ X\ ) \tag{3.3}$$

and the symmetric Freudenthal product is

$$X\ \times_F\ Y\ =\ X\ *_J\ Y\ - \frac{1}{2}\ [\ YTrX\ +\ XTrY\ ]\ + \frac{1}{2}\ [\ TrX\ TrY\ -\ Tr(X *_J Y)\ ] \tag{3.4}$$

The *cubic* norms in Jordan algebras have been classified by Schafer [13] and there are three cases. In particular the four "magical" cases consisting of 3×3 Hermitian matrices whose components take values in the four division algebras, real, complex, quaternions and octonions. The four magical Jordan algebras were key ingredients in the construction of magical supergravities [15].

The conformal group of the Jordan algebra J is generated by translations $T_{\mathbf{a}}$, special conformal generators $K_{\mathbf{a}}$, dilatations and Lorentz $M_{\mathbf{ab}}$ where $\mathbf{a}, \mathbf{b}$ are

Jordan algebra elements. Now is where the role of *ternary* structures becomes relevant. The action of the conformal algebra $conf(J)$ on the elements $\mathbf{x}$ of a Jordan algebra J is [14]

$$T_{\mathbf{a}}\,\mathbf{x} \;=\; \mathbf{x},\;\; M_{\mathbf{ab}}\,\mathbf{x} \;=\; \{\,\mathbf{a}\,\mathbf{b}\,\mathbf{x}\,\},\;\; K_{\mathbf{a}}\,\mathbf{x} \;=\; -\frac{1}{2}\,\{\,\mathbf{x}\,\mathbf{a}\,\mathbf{x}\,\} \tag{3.5}$$

where $\{\,\mathbf{a}\,\mathbf{b}\,\mathbf{x}\,\}$ is the Jordan triple product given by

$$\{\,\mathbf{a}\,\mathbf{b}\,\mathbf{x}\,\} \;=\; \mathbf{a}\,*_J\,(\mathbf{b}\,*_J\,\mathbf{x})\;-\;\mathbf{b}\,*_J\,(\mathbf{a}\,*_J\,\mathbf{x})\;+\;(\mathbf{a}\,*_J\,\mathbf{b})\,*_J\,\mathbf{x} \tag{3.6}$$

The commutation relations of those generators are given in [14]. The Freudenthal determinant expression given by the *cubic* norm in (3.3) is invariant under the action of the reduced structure group $Lor(J_3^O)$ and which is generated by $M_{\mathbf{ab}}$. The analog of the exponentiation procedure of the adjoint action of $M_{\mathbf{ab}}$ on $\mathbf{x}$ and associated with the *ternary* structures is

$$exp(\alpha^{ab}M_{\mathbf{ab}})\,\mathbf{x} \;=\; \mathbf{x}\;+\;\alpha^{ab}\{\,\mathbf{a}\,\mathbf{b}\,\mathbf{x}\,\}\;+\frac{1}{2!}\,\alpha^{ab}\,\{\mathbf{a},\,\mathbf{b},\,\alpha^{ab}\{\,\mathbf{a}\,\mathbf{b}\,\mathbf{x}\,\}\,\}\;+\;\ldots\ldots \tag{3.7}$$

where α^{ab} are the suitable parameters associated with $M_{\mathbf{ab}}$.

The construction of the covariant phase spaces associated with these generalized spacetimes defined by cubic forms and based on Jordan algebras (and ternary structures) requires the use of Freudenthal triple systems defined over these Jordan algebras [14]. The conformal groups are now extended to quasi-conformal groups like $E_{8(-25)}, E_{7(-5)}, E_{6(2)}, F_{4(4)}$ and $SO(d{+}2,4)$ [14]. The case of $d = 10$ gives $SO(12,4)$ which is interesting because $SO(12,4)$ is the symmetry group of the following quadratic form associated with a 16-dim extended phase space given by

$$(ds)^2 \;=\; dX_\mu dX^\mu \;+\; dP^{(1)}_\mu dP^{(1)\mu} \;+\; dP^{(2)}_\mu dP^{(2)\mu} \;+\; dP^{(3)}_\mu dP^{(3)\mu},\;\; \mu = 1,2,3,4 \tag{3.8}$$

where we have omitted the numerical constants to adjust physical units for simplicity. $P^{(1)}_\mu$ is the first order momentum, $P^{(2)}_\mu, P^{(3)}_\mu$ are the second and third order momentum, respectively. A Born's reciprocal relativity based on the invariance of the quadratic form (3.8) requires the quasi-conformal $SO(12,4)$ group.

If one wishes, further, to have an invariant four-form

$$\omega_{\mu\nu\rho\sigma}\; dX^\mu \wedge dP^{(1)\nu} \wedge dP^{(2)\rho} \wedge dP^{(3)\sigma} \tag{3.9}$$

one requires to introduce multi-symplectic (3-plectic) transformations leaving invariant the 4-form (3.9).

Born's reciprocal relativity in an $8D$ phase space required the invariance group to be provided by the *intersection* of $SO(6,2)$ with the symplectic group

$Sp(8)$. The intersection contains the pseudo-unitary group $U(3,1)$ and which allowed Low [3] to write down the symmetry transformations under velocity and force/acceleration boosts in (1.6). In general, the intersection of $SO(2n)$ with $Sp(2n)$ contains the $U(n)$ algebra. Therefore, the relevant group in question is now given by the intersection of the group $SO(12,4)$ with the 3-plectic group and which leaves invariant the four-form (3.9) and the quadratic interval (3.8). In general n-spacetime dimensions, corresponding to an extended phase space of $4n$ dimensions associated to $X_\mu, P^{(i)}_\mu; i = 1,2,3$, we should have the intersection of $SO(4n-4,4)$ and the 3-plectic group.

Just as a symplectic manifold gives a Poisson algebra of functions, any 2-plectic manifold gives a Lie 2-algebra of 1-forms and functions. n-plectic manifolds give Lie n-algebras and which are examples of strong homotopy algebras L_∞ equipped with a collection of skew-symmetric multi-brackets that satisfy a generalized Jacobi identity [17]. This is where n-plectic geometry will become relevant in the extensions of Born's reciprocal relativity associated with the extended phase spaces of the form described above. One must not confuse n-plectic geometry with polysymplectic geometry. For an introduction of the latter see [16].

Instead of writing the quadratic form (3.8) one could begin instead with a *cubic* norm involving only the first and second order momentum

$$(ds)^3 \ = \ d_{IJK}\, dZ^I\, dZ^J\, dZ^K, \quad dZ^I \equiv (dX^\mu, dP^{(1)\nu}, dP^{(2)\rho}), \ \ \mu,\nu,\rho = 1,2,3,...,D \qquad (3.10)$$

where d_{IJK} is a symmetric rank three tensor whose indices span the values of $I,J,K = 1,2,3,.....,3D$. The first D values of Z^I correspond to the X^μ coordinates. The second set of D values correspond to the $P^{(1)\nu}$ coordinates, and the last set of D values correspond to the $P^{(2)\rho}$ coordinates.

The relevant form is now the 3-form associated with a 2-plectic geometry

$$\omega_{\mu\nu\rho}\ dX^\mu \wedge dP^{(1)\nu} \wedge dP^{(2)\rho} \qquad (3.11)$$

Now one can ask the question : what is the *ternary* algebra resulting from the intersection of the ternary algebras which leave invariant the 3-form (3.11) and the cubic norm (3.10) ? . One can then extend this construction to the n-ary algebra case by having the n-norm

$$(ds)^n \ = \ d_{I_1 I_2 .... I_n}\ dZ^{I_1}\ dZ^{I_2}\ ....\ dZ^{I_n} \qquad (3.12a)$$

and the n-form

$$\omega^{(n)} \ = \ \omega_{\mu_1\mu_2......\mu_n}\ dX^{\mu_1} \wedge dP^{(1)\mu_2} \wedge \ ....\ \wedge dP^{(n-1)\mu_n} \qquad (3.12b)$$

and then finding the intersection of the n-ary algebras which leave invariant the n-norm (3.12a) and the n-form (3.12b). The indices $I_1, I_2, ...$ in (3.12a) span the values $1,2,3,.....,nD$ of the extended phase space, where D is the underlying spacetime dimension.

This intersection procedure of the n-ary algebras will allow us to find the n-ary analog of the group transformations which will now *mix* all the $X, P^{(1)}, P^{(2)}, ...$ coordinates of the higher order tangent (cotangent) spaces in this extended relativity theory based on Born's reciprocal gravity and n-ary algebraic structures. A proper place to begin is in the pioneering work on n-ary groups by [18].

4 Higher Order Finsler Spaces and Extended Born Reciprocal Relativity

In this concluding section we shall examine extensions of Born's reciprocal relativity theory based on the geometry of higher-order Lagrange-Finsler and Hamilton-Cartan Spaces [11], [10] and see whether or not one can set bounds to the higher order accelerations, and secondly, to study the symmetry transformation laws. The bundle of accelerations $T^k M$ of order $k \geq 1$ involves the coordinates

$$x^i,\ y^{(1)i} = \frac{dx^i}{d\tau},\ y^{(2)i} = \frac{1}{2!}\frac{d^2x^i}{d\tau^2},\ ......,\ y^{(k)i} = \frac{1}{k!}\frac{d^kx^i}{d\tau^k},\ \ i = 1, 2, 3, ..., n \quad (4.1)$$

In (4.1) we use the proper time τ instead of t [11]. A change of local coordinates compatible with the differentiable atlas on the manifold M and the higher order tangent bundle $T^k M$ is of the form [11]

$$\tilde{x}^i \ = \ \tilde{x}^i(x^1, x^2, ..., x^n),\ rank\ of\ Jacobian(\frac{\partial \tilde{x}^i}{\partial x^j}) \ = n$$

$$\tilde{y}^{(1)i} \ = \ \frac{\partial \tilde{x}^i}{\partial x^j}\ y^{(1)j}$$

$$2\tilde{y}^{(2)i} \ = \ \frac{\partial \tilde{y}^{(1)i}}{\partial x^j}\ y^{(1)j} \ + \ 2\frac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}}\ y^{(2)j}$$

....

$$k\tilde{y}^{(k)i} \ = \ \frac{\partial \tilde{y}^{(k-1)i}}{\partial x^j}\ y^{(1)j} \ + \ 2\frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(1)j}}\ y^{(2)j} \ + \ ....\ k\frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(k-1)j}}\ y^{(k)j} \quad (4.2)$$

A prolongation to the higher order tangent bundle $T^k M$ of the Riemannian and Finsler structures was attained by Miron [11]. This allowed the introduction of a metric on the space of total dimension $(k+1)n$. The coordinate transformations (4.2) preserve the quadratic form defined by the metric. We may notice that the coordinate transformations (4.2) mix the base manifold coordinates x^i with the vertical coordinates $y^{(1)i}, y^{(2)i}, ..., y^{(k)i}$. However these transformations differ from the ones described by the $U(D-1,1)$ and $U(D-1,1,H)$ group transformations in this work.

A Hamiltonian space of higher order $k \geq 1$ was also studied by [11] and references therein. It consists of local coordinates $x^i, p_i, y^{(1)i}, y^{(2)i}, ..., y^{(k-1)i}$. The change of coordinates is similar to (2.20) with the addition of

$$\tilde{p}_i \;=\; \frac{\partial x^j}{\partial \tilde{x}^i}\; p_j \qquad (4.3)$$

Let us work in flat space and set *all* the nonlinear connection **N** coefficients to *zero* so that the dual basis elements are in this special case given by

$$\delta x^i = dx^i,\; \delta y^{(1)i} = dy^{(1)i},\; \delta y^{(2)i} = dy^{(2)i},\; ....,\; \delta y^{(k)i} = dy^{(k)i} \qquad (4.4)$$

The metric on $T^k M$ is given by the lift of the metric on M [11]

$$(ds)^2 \;=\; g_{ij}(x) dx^i dx^j \;+\; g_{ij}(x)\delta y^{(1)i}\delta y^{(1)j} \;+\; ......\; g_{ij}(x)\delta y^{(k)i}\delta y^{(k)j} \;\rightarrow$$

$$(ds)^2 \;=\; g_{ij}(x) dx^i dx^j \;+\; g_{ij}(x) dy^{(1)i} dy^{(1)j} \;+\; ......\; g_{ij}(x) dy^{(k)i} dy^{(k)j} \qquad (4.5)$$

Setting $g_{ij}(x) = \eta_{ij}$ and factoring out $\eta_{ij} dx^i dx^j = d\tau^2$ in (4.5) gives

$$(ds)^2 \;=\; (d\tau)^2 \left(1 \;+\; \frac{dy^{(1)i}}{d\tau}\;\frac{dy_i^{(1)}}{d\tau} \;+\; ......\; +\; \frac{dy^{(k)i}}{d\tau}\;\frac{dy_i^{(k)}}{d\tau} \right) \qquad (4.6)$$

Because now there are *alternating* signs , $\pm$, in the values of the terms inside the bracket in (4.6) there are *no* bounds on the first order and higher order accelerations with the provision that $(ds)^2 \geq 0$ when $(d\tau)^2 \geq 0$ (non-tachyonic intervals). If the velocity is timelike $\eta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 1$ (positive), then by taking derivatives it gives $2\eta_{ij}\frac{d^2x^i}{d\tau^2}\frac{dx^j}{d\tau} = 0$, and one can infer that the first order acceleration is spacelike $\eta_{ij}\frac{dy^{(1)i}}{d\tau}\;\frac{dy^{(1)j}}{d\tau} < 0$, because it is orthogonal to the timelike velocity. Iterating this procedure by taking further derivatives one infers that the second order acceleration is timelike $\eta_{ij}\frac{dy^{(2)i}}{d\tau}\;\frac{dy^{(2)j}}{d\tau} > 0$, because its inner product with the timelike velocity $y^{(1)}$ is positive definite. And so forth. Therefore, to conclude, there are no bounds to the accelerations in this case. This was the underlying mechanism behind the possibility of having superluminal velocities [5], [6], [7] in Phase and Clifford spaces.

APPENDIX

In this Appendix we will show that the relation in eq-(2.10c) is obeyed. The latter equation has the same functional form of a commutator $[,]$ plus its quaternionic conjugate $[,]^*$, as follows

$$[\; A,\; BC\;] \;+\; [\;\bar{C}\bar{B},\; \bar{A}\;] \;=\; ABC \;-\; BCA \;+\; \bar{C}\bar{B}\bar{A} \;-\; \bar{A}\bar{C}\bar{B} \qquad (A.1)$$

Given the quaternionic valued quantities

$$A = A^o e_o + A^i e_i, \; B = B^o e_o + B^j e_j, \; C = C^o e_o + C^k e_k, \quad i, j, k = 1, 2, 3 \quad (A.2)$$

one has

$$ABC \;=\; \Big(\; (A^o B^o - A^i B_i) C^o - (A^o B^i + A^i B^o) C_i - (A^i B^j C^l)\epsilon_{ijl} \; \Big) \; e_o \; +$$
$$\Big(\; (A^o B^o - A^i B_i) C^k + (A^o B^k + A^k B^o) C^o + (A^o B^i + A^i B^o) C^j \epsilon_{ijk} + A^i B^j C^o \epsilon_{ijk} \; \Big) \; e_k \; +$$
$$(A^i B^j C^l) \; \epsilon_{ijk} \; \epsilon_{klm} \; e_m \qquad (A.3)$$

$$BCA \;=\; \Big(\; (B^o C^o - B^i C_i) A^o - (B^o C^i + B^i C^o) A_i - (B^i C^j A^l)\epsilon_{ijl} \; \Big) \; e_o \; +$$
$$\Big(\; (B^o C^o - B^i C_i) A^k + (B^o C^k + B^k C^o) A^o + (B^o C^i + B^i C^o) A^j \epsilon_{ijk} + B^i C^j A^o \epsilon_{ijk} \; \Big) \; e_k \; +$$
$$(B^i C^j A^l) \; \epsilon_{ijk} \; \epsilon_{klm} \; e_m \qquad (A.4)$$

Under quaternionic conjugation one has $e_o \to e_o$ and $e_i \to -e_i$, so only the real parts of eq-(A.1) will contribute because the imaginary parts will *cancel* each other. From eqs-(A.3), (A.4) one can verify by simple inspection [1] that the real part of ABC is equal to the real part of BCA, therefore the real part of $(ABC - BCA) = 0$. Taking the quaternionic conjugate of the latter expression gives that the real part of $(\bar{C}\bar{B}\bar{A} - \bar{A}\bar{C}\bar{B}) = 0$. Therefore eq-(A.1) is identically *zero* and eq-(2.10c) is satisfied, as announced.

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References

[1] M. Born, Proc. Royal Society **A 165** 291 (1938).
M. Born, Rev. Mod. Physics **21** 463 (1949).

[2] E. Caianiello, Lett. Nuovo Cimento **32** 65 (1981).

[3] S. Low, Jour. Phys **A 35** 5711 (2002).
S. Low, J. Math. Phys. **38** 2197 (1997).

[4] C. Castro, Foundations of Physics **35**, no.6 (2005) 971.

[5] C. Castro, Int. J. of Mod. Phys **A 26**, no. 21 (2011) 3653.
C. Castro, Foundations of Physics, vol **42**, no. 9 (2012) 1135.

[1] It helps to write $B^i C^j A^l \epsilon_{ijl} = A^l B^i C^j \epsilon_{ijl} = A^l B^i C^j \epsilon_{lij}$ so that $B^i C^j A^l \epsilon_{ijl} - A^i B^j C^l \epsilon_{ijl} = 0$ after relabeling indices

[6] M. Pavsic, Found. of Phys. 33 (2003) 1277.

[7] C. Castro and M. Pavsic, C. Castro and M. Pavsic, Progress in Physics **vol 1** (April 2005) 31.

[8] C. Castro, "The Extended Relativity Theory in Clifford Phase Spaces and Modifications of Gravity at the Planck/Hubble scales" submitted to Foundations of Physics.

[9] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Dover, 2005).

[10] S. Vacaru, S. Vacaru, P. Stavrinos, E. Gaburov and D. Gonta, *Clifford and Riemann- Finsler Structures in Geometric Mechanics and Gravity* (Balkan Press, 2006).

S. Vacaru, "Non-holonomic Clifford-Structures and Noncommutative Riemann-Finsler Geometry " arXiv.org : math.DG/0408121.

S. Vacaru, "(Non) Commutative Finsler Geometry from String/M Theory " arXiv.org: hep-th/0211068.

[11] R. Miron, "Lagrangian and Hamiltonian geometries. Applications to Analytical Mechanics" arXiv.org : 1203.4101;

R. Miron, *The Geometry of Higher-Order Lagrange Spaces. Applications to Mechanics and Physics*, (Kluwer Acad. Publ. FTPH no. 82, 1997);

R. Miron, *The Geometry of Higher-Order Finsler Spaces*, (Hadronic Press Inc., SUA, 1998);

R. Miron, D. Hrimiuc, H. Shimada and S. Sabau, *The Geometry of Hamilton and Lagrange Spaces* (Kluwer Academic Publishers, Dordrecht, Boston, 2001).

[12] Carlos Castro, Phys Letts **B 668** (2008) 442.

[13] R. Schafer, J. Math. Mech. **10**, no. 1 (1969) 159.

[14] M. Gunaydin and O. Pavlyk, " Generalized spacetimes defined by cubic forms and the minimal unitary realizations of their quasiconformal groups" arXiv : hep-th/0506010.

M. Gunaydin, " Lectures on Spectrum Generating Symmetries and U-duality in Supergravity, Extremal Black Holes, Quantum Attractors and Harmonic Superspace" arXiv : 0908.0374.

[15] M. Gunaydin, G. Sierra and P. Townsend, Nucl. Phys. **B 242** (1984) 244.

[16] I. Kanatchikov, "Novel algebraic structures from the polysymplectic form in field theory" arXiv : hep-th/9612255.

M. Forger and M. Salles, " Hamiltonian Vector Fields on Multiphase Spaces of Classical Field Theory" arXiv : 1010.0337.

[17] C. Rogers, "Higher Symplectic Geometry" arXiv : 1106.4068.

C. Rogers, " L_∞ Algebras from Multisymplectic Geometry" arXiv : 1005.2230.

[18] A. Galmark, "N-ary Groups" arXiv : 1111.7191 [math.DG] .