Products of Generalised Functions

V. Nardozza*

April 2013†

Abstract

An elementary algebra of products of distributions is constructed. An equivalent relation between products of distributions is given and a space of generalised functions is constructed as a partition space of the elementary algebra with respect to the equivalent relation. The new space of generalized functions is used to prove interesting equalities involving products among elements of D'. A way of multiplying the defined generalised functions with polynomials is also derived.

Key Words: distribution theory, product of distributions.

1 Introduction

Products of distributions are quite common in several fields of both mathematics and physics. Examples arise naturally in quantum field theory, gravitation and in partial differential equation (e.g shock wave solutions in hydrodynamics) see [1]. An important issue, related to product of distributions, is the fact that the product, in the general case, is not well defined in D'. This issue is known as the Schwartz impossibility result (see [1] §1.3). In the Schwartz classical theory, only the product between a smooth function and a distribution is well defined. Historically, products of distributions are addressed by means of algebras of generalised functions developed initially by J. F. Colombeau (see [1] and [2]). In this paper we will propose a new approach for defining products of distributions.

In paragraphs 2 and 3, we construct a new space of generalised functions. In paragraphs 4, we use the new developed theory to derive interesting equalities involving products among elements of D'. In paragraphs 5, we derive a method for multiplying the generalised functions, defined in this paper, with polynomials.

Preliminary Definitions. Let $g \in C^\infty(\mathbb{R})$ be any function, we say that g is a function of order $p \geq 0$ if:

$$0 < \left| \int_{-\infty}^{+\infty} x^p g(x) dx \right| < +\infty \tag{1}$$

We say that a function g is of order $p < 0$ if, for each $k \in \mathbb{N}$, $g^{(|p|+k)}$ is a function of order k according to the definition given above.

*Electronic Engineer (MSc). Turin, Italy. mailto: vinardo@nardozza.eu

†Posted at: www.vixra.org/abs/1304.0158 - Current version: v5 Dic. 2013

Given any function $g \in C^{\infty}$, with the notation $g^{(p)}$, where $p \in \mathbb{Z}$, we refer to the derivatives of order p of g for $p \geq 0$ and to the function defined by the following recursive formula:

$$g^{(p-1)} = \int_{-\infty}^{x} g^{(p)}(\tau)d\tau \tag{2}$$

for $p < 0$.

2 An algebra of generalised functions

Definition 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be any function such that $f \in C^{\infty}(\mathbb{R})$, $f(x) \geq 0$ for each $x \in \mathbb{R}$ and $\int_{-\infty}^{+\infty} f(x)dx = 1$. We define $\mathbb{F}^{\eta}$ to be the vector space of infinite dimension spanned by the elements:*

$$\eta^{q,p}_{x^k f^{(p_1)} \cdot \ldots \cdot f^{(p_m)}} \tag{3}$$

where the elements $\eta^{q,p}_{x^k f^{(p_1)} \cdot \ldots \cdot f^{(p_m)}}$ are the sequences $S_n(x) = n^q g(nx)$ where:

- $q \in \mathbb{Z}$
- $p_1, \cdots, p_m \in \mathbb{Z}$
- $k \in \mathbb{N} \cup \{0\}$
- $g(x) = x^k f^{(p_1)}(x) \cdot \ldots \cdot f^{(p_m)}(x)$
- *p is the order of $g(x)$*

and where the operation + and the multiplication by a scalar are the usual ones for series.

We will call the above defined elements and their linear combinations η functions. Note that the p in the above notation is redundant since the information on the order of the function g is already given by the function g itself (i.e by the integers m and p_i) and it is independent from f. The p may therefore be omitted in the notation.

Definition 2. *We define the following operation on the elements $\eta^{q,p}_{g}$ introduced above:*

$$\begin{aligned} &1)\ \eta^{q_1}_{g_1} \cdot \eta^{q_2}_{g_2} = \eta^{q_1+q_2}_{g_1 g_2}, \\ &2)\ \tfrac{d}{dx}\eta^{q}_{g} = \eta^{q+1}_{g'} \qquad \textit{derivative term by terms on the sequence} \end{aligned} \tag{4}$$

Note that the operations $(\cdot\,, +)$ defined above are commutative, associative and that the product is distributive with respect of the sum. Note finally that the above definition of derivative of an η function is compliant with the Leibniz rule.
It is easy to see that $\mathbb{F}^{\eta}$, with the operations $(\cdot, +)$, is an associative and derivative algebra (with the definition of derivative given above).

The elements of the set $\mathbb{F}^\eta$ are dependent from the function f of definition 1, however the structure of $\mathbb{F}^\eta$ is not. So we may see the $\mathbb{F}^\eta$ set as the sets of all possible forms of their elements when we make f vary in C^∞ or, which is the same we may, see $\mathbb{F}^\eta$ the structure of the vector space wich is independent from f.

We want to show now that it is possible to map a subset of elements of $\mathbb{F}^\eta$ to a subset of elements of D'. Given $\eta = (\eta_{g_1}^{q_1} + \cdots + \eta_{g_m}^{q_m}) \in \mathbb{F}^\eta$, we define the following linear transformation:

$$T : \eta \in \mathbb{F}^\eta \to h \in D' \Longleftrightarrow h = \lim_{n\to\infty} n^{q_1} g_1(nx) + \cdots + n^{q_m} g_m(nx) \qquad (5)$$

where T is defined provided that the above limit exists in D'. Moreover, it is easy to see that, if the limit exists, it is independent from the function f of definition 1. For example we have:

$$T(\eta_f^1 + \eta_{f'}^2) = \delta + \delta^{'} \in D' \qquad (6)$$

Given the transformation T defined above, we want to identify some elements of $\mathbb{F}^\eta$ with some elements of D' (the ones for which T converges with n). Moreover we want to identify all elements of $\mathbb{F}^\eta$ with linear combinations of products of $\delta^{(p)}$ with $p \in \mathbb{Z}$, whatever they are (to us they are just elements of $\mathbb{F}^\eta$), even if T does not converge. We will use the symbol $\rightleftarrows$ for the above identification and we will perform this identification in the obvious way. For example we have:

$$\begin{aligned} &\eta_f^1 + \eta_{f'}^2 && \rightleftarrows \quad \delta + \delta^{'} \\ &\eta_{ff'}^3 + \eta_{f'f''}^5 && \rightleftarrows \quad \delta\delta^{'} + \delta^{'}\delta^{''} \\ &\eta_{f^{(-1)}f}^1 && \rightleftarrows \quad u(x)\delta(x) \qquad u(x) \text{ being the Heaviside function} \end{aligned} \qquad (7)$$

where the coefficient q of the η functions are evaluated by using the definition of product given above.

Once we have identified elements of D' with elements of $\mathbb{F}^\eta$ (or $\mathbb{G}^\eta$), we will use sometimes the notation $=$ instead of $\rightleftarrows$ to refer to this identification. So, for example, we may write $\delta = \eta_f^1 \in \mathbb{G}^\eta$ or $\delta\delta' = \eta_{ff'}^3 \in \mathbb{G}^\eta$ although the symbol $\rightleftarrows$ should be used instead.

We introduce now a new terminology we will use later in the paper. Given an η function $\eta_g^{q,p}$, we will call $g(x)$ the generating function, q the growing index and p the order of η. Moreover, we will call the elements of $\mathbb{F}^\eta$,which are sequences, the generating sequence of the relevant η function. Note that, even thought the generating sequence of a specific η function does not converge with n, this is not a problem for us since we are not interested in its limit but rather in the sequence itself.

We define the following equivalence relation between elements of $\mathbb{F}^\eta$:

Definition 3. *Given the two elements η_1 and η_2 of $\mathbb{F}^\eta$ with equal growing indexes q and equal orders p and with generating sequences $S_n(x)$ and $R_n(x)$ we say that the two η functions are two representative of the same element of $\mathbb{F}^\eta$ if, given any test function $\phi \in D$, we have:*

$$\lim_{n\to\infty} \frac{\int_{-\infty}^{+\infty} S_n(x)\phi(x)dx}{\int_{-\infty}^{+\infty} R_n(x)\phi(x)dx} = 1 \qquad (8)$$

and if the above limit is independent from ϕ and from the function f of definition 1. In this case we use the notation $\eta_1 \approx \eta_2$

Note that, by using the expression representatives of the same element of $\mathbb{F}^\eta$, we are using a terminology similar to the one used in the Colombeau algebras. Although we are facing a similar situation, the theory developed in this paper is rather different with respect of the Colombeau algebras, and therefore identical terms do not imply identical meanings in the two theories.

Note finally that, in this case, the criteria given by the above definition is used on the elements of $\mathbb{F}^\eta$ but it is quite a general criteria which can be used to tell if two generalised functions are equivalent.

Example 2.1. Given $\eta_1, \eta_2 \in \mathbb{F}^\eta$, with $\eta_1 = \delta^2(x)$ and $\eta_2 = u(x)\delta'(x)$, we have:

$$\begin{aligned} \lim_{n\to\infty} \frac{\int_{-\infty}^{+\infty} n^2 f^2(nx)\phi(x)dx}{\int_{-\infty}^{+\infty} n^2 f^{(-1)}(nx)f'(nx)\phi(x)dx} &= \lim_{n\to\infty} \frac{n\int_{-\infty}^{+\infty} nf^2(nx)\phi(x)dx}{n\int_{-\infty}^{+\infty} nf^{(-1)}(nx)f'(nx)\phi(x)dx} \\ &= \frac{\int_{-\infty}^{+\infty} f^2(x)dx}{\int_{-\infty}^{+\infty} f^{(-1)}(x)f'(x)} \frac{\phi(0)}{\phi(0)} \qquad (9) \\ &= \frac{\int_{-\infty}^{+\infty} f^2(x)dx}{\int_{-\infty}^{+\infty} f^{(-1)}(x)f'(x)} = -1 \end{aligned}$$

which is independent from f as it is easy to show by integrating by parts $\int_{-\infty}^{+\infty} f^2(x)dx$ and using the fact that f vanishes for f that goes to infinity. We conclude that $\delta^2(x) \approx -u(x)\delta'(x)$.

Definition 4. *We define $\mathbb{G}^\eta$ to be the set of partitions of $\mathbb{F}^\eta$ with respect of the equivalence relation given by definition 3.*

The set $\mathbb{G}^\eta$ is not an algebra. The set $\mathbb{G}^\eta$ can be effectively seen as a set of generalised function for witch an associative and commutative product is well defined and with a definition of derivative (see definition 2) that is compliant with the Leibniz rule and that gives the right answer when applied to elements of $\mathbb{G}^\eta$ that can be mapped to elements of D'.

Before we finish this paragraphs, we give below some examples of η functions. The η functions given below are only examples since, for each element $n^q g(x)$, there are infinitely many number of elements of $\mathbb{F}^\eta$ that are the same representative of the element reported in the table below as well as infinitely many elements which are not.

$\eta^{q,p}$	p=-1	p=0	p=1	p=2	p=3
q=5		$\cdots$	$\cdots$	$\cdots$	$\cdots$
q=4		$\cdots$	$\frac{d}{dx}(\delta^3(x))$	$\frac{d^2}{dx^2}(\delta^2(x))$	$\cdots$
q=3		$\delta^3(x)$	$\frac{d}{dx}(\delta^2(x))$	$\delta''(x)$	
q=2	$\cdots$	$\delta^2(x)$	$\delta'(x)$		
q=1	$(\delta^2(x))^{(-1)}$	$\delta(x)$			
q=0	$u(x)$				

Figure 1 : Examples of η functions

Where, for example, the element $\frac{d}{dx}(\delta^2(x))$ has to be read as $2\delta(x)\delta'(x) \in \mathbb{F}^\eta$ and so on.

3 Structure of a generalised function

We have seen in the previous paragraph that the simplest element $h = \eta^q_{g(x)} \in \mathbb{G}^\eta$ can be defined by means of its generating function S_n of the type:

$$S_n(x) = n^q g(nx) \tag{10}$$

which most of the times does not converge in D'. We want so determine the structure of such a generalised function. Before we proceed, we need to give a couple of definitions which will be used later on:

Definition 5. *We define $S(\mathbb{R})$ to be the set of all the functions $f(x)$ having the following characteristics.*

$$\begin{array}{l} 1)\ f(x) \in C^\infty \\ 2)\ \lim_{x\to-\infty} f(x)x^k = 0\ \ for\ any\ k \in \mathbb{N} \\ 3)\ \lim_{x\to+\infty} f(x)x^k = 0\ \ for\ any\ k \in \mathbb{N} \end{array} \tag{11}$$

This functions are known in literature as rapidly decreasing functions on $\mathbb{R}$.

Definition 6. *Given any function $\xi(x) \in S(\mathbb{R})$ then, if $\xi(x)$ verifies the following conditions:*

$$\int_{-\infty}^{+\infty} \xi(x)x^k dx = \begin{cases} 1 & for\ k = 0 \\ 0 & for\ 0 < k \leq s\ \ with\ s \in \mathbb{N} \end{cases} \tag{12}$$

then we call ξ a main generating function of order 0. We also call its derivatives $\xi^{(p)}$ with $p \in \mathbb{N}$ and $p < s$ a main generating function of order p. Finally, for each s, we define A_s to be the set of ξ function relevant to s.

For more details on the A_s sets, see [1] §8.2. In most cases, we will assume that s is large enough for our purpose (i.e. given $\xi^{(p)}$ the main generating function of higher order, we are working with, we have $p < s$) and we will not explicitly mention it in our discussion.

Let us see how to determine all the components, of different order, of a generalised function defined by means of the (10) and having generating function $f(x)$. We will suppose, for the moment, that it is possible to find a function $\xi(x) \in A_s$ such that it is possible to express the generating function f as follows:

$$g(x) = \sum_{p=0}^{s} a_p \xi^{(p)} + r(x) \tag{13}$$

where $r(x)$ is a function having all momenta, of order lower then s, equal to 0. We will see, further on, that the above ξ function exists. By multiplying the right hand side of the above equation by x^p, integrating by parts p times and taking into account that $\xi(x)$ vanishes at infinity, we find easily that:

$$a_p = \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} f(x) x^p dx \tag{14}$$

and therefore the a_p coefficients are related to the momenta of f. The (14) allow us to evaluate the coefficient a_p of all the components of order lower then s. However, neither a dependency from s nor a dependency from $\xi \in A_s$ is explicitly present in the above equations and therefore we can use it to evaluate components of any order just assuming s is big enough.

As mentioned above, we need to justify our assumption that it is possible to find a $\xi \in A_s$ such that we can write f as in the (13). This can be done by giving a constructive algorithm to evaluate the required ξ, Given f and the a_p, if we evaluate $f_1 = f/a_0$, we have a function with the same base function ξ and with first momentum equal to 1. We evaluate the a_p^1 coefficient of f_1. We evaluate $f_2 = f_1 - a_1^1 f_1'$ and we get a function with the same base function ξ, with first momentum equal to 1 and second momentum equal to 0. Iterating the process s times we get eventually our required ξ function in A_s.

Now, given the (13), it is easy to see the distribution h associated to the sequence (10), it is actually the sum of infinite components in $\mathbb{F}$ of same growing index q and increasing order p as follows:

$$h \rightleftarrows n^q g(nx) = n^q \sum_{p=0}^{s} a_p \xi^p(nx) + n^q r(nx) \tag{15}$$

We will call this kind of generating functions homogeneous. We note that a_p, although not explicitly noted, refer and depend from the function ξ.

Since a_p do not depend on s, we may say that the above generalised function h is composed of infinite components which are η functions of same growing index and increasing order and, whereas we are interested in the component of order p, we can evaluate it by choosing a ξ function in A_s with $s > p$. In a few words, h can be expressed as:

$$h \rightleftarrows n^q \sum_{p=0}^{s} \eta^{q,p}_{\xi^{(p)}} + R(\eta^{q,s}) \tag{16}$$

Where, with the notation $R(\eta^{q,s})$ we mean that, to have the above equality exact, we need to add an infinity number of η components of growing index q and order $p \geq s$. The (16) is what we call the structure of a generalised function (compare with [4]).

We show now an important fact about the coefficient a_p of the (14). Let $\xi \in A_s$ be any main generating function of order 0 and $h \in \mathbb{G}^{\eta}$ the related generalised function of growing index q. We have:

$$h \rightleftarrows a_p(\xi)n^q\xi^{(p)}(nx) \tag{17}$$

If we choose $\xi_\alpha = \alpha\xi(\alpha x)$, as a different generating function of order 0, we have:

$$h_\alpha \rightleftarrows a_p(\xi_\alpha)n^q\alpha^{p+1}\xi^{(p)}(n\,\alpha x) \tag{18}$$

If the (17) and (18) are the same generalised function (i.e. $h = h_\alpha$) then we can write:

$$a_p(\xi)n^q\xi^{(p)}(nx) \equiv \frac{a_p(\xi_\alpha)}{\alpha^{q-p-1}}(n\alpha)^q\xi^{(p)}(n\,\alpha x) \tag{19}$$

but, at the same time, we have:

$$n^q\xi^{(p)}(nx) \equiv (n\alpha)^q\xi^{(p)}(n\,\alpha x) \tag{20}$$

because the left and the right side of the (20) are the same function growing and shrinking at the same rate with n, and therefore can be associated to the same generalised function $\eta_\xi^{q,p}$. For example, if α is an integer, the sequence on the right hand side of the equation is a sub-sequence of the one on the left hand side. We conclude that:

$$a_p(\xi_\alpha) = \alpha^{q-p-1}a_p(\xi) \;\;\Rightarrow\;\; h = h_\alpha \tag{21}$$

Note that if $p = q - 1$ then, as expected, the a_p coefficients are independent from ξ.

The key point here is that, even though the coefficients a_p of two separate generalised functions are different, this does not necessarily imply that the two generalised functions are different. From the (21) above we see that the coefficient a_p depends from the base function ξ and therefore from the generation function f. For Example, if we change the scaling factor of f by using a different function $\alpha f(\alpha x)$, the coefficients a_p will change accordingly.

We are now ready for the following definition:

Definition 7. *Given two homogeneous generalised functions h_1, h_2, with same growing index and defined by means of generating function having the same base function ξ, then if it is possible to find an integer k such that:*

$$\begin{cases} a_p(h_1) = a_p(h_2) & for\, p = k \\ a_p(h_1) = a_p(h_2) = 0 & for\, p < k \end{cases} \tag{22}$$

we say that h_1 and h_2 are representatives of the same generalised function and we use the notation $h_1 \approx h_2$. Moreover, if:

$$a_p(h_1) = a_p(h_2) \quad for\, each\, p \tag{23}$$

then we say that the two generating function are equal and we use the notation $h_1 = h_2$.

Note that the criteria given by the above definition is equivalent to the Criteria given by the (8).

We have seen above that the a_p coefficients depend on the underlying f function of definition 1. We will call the a_p relative coefficients. We may want to express the structure of a distributions by means of b_p coefficients which are independent from the function f. In this case we would have b_p which are absolute coefficients. Such b_p coefficients exists at least for the first lower order component of our generalised function h and they can bean be evaluated by taking the ratio $a_p(h)/a_p(h_r)$ where h_r is a reference function, for all generalised function which are proportional to the same representative of h (i.e. proportional to the same element of $\mathbb{G}^\eta$ defined in the above paragraph). If $h \rightleftarrows n^q g(nx)$ and $h_r \rightleftarrows n^q g_r(nx)$ We have:

$$b_p = \frac{a_p(g(x))}{a_p(g_r(x)} = \frac{\int_{-\infty}^{+\infty} x^p g(x)}{\int_{-\infty}^{+\infty} x^p g_r(x)} \tag{24}$$

which is equivalent to the (8), and

$$\eta_g^{q,p} = b_p \eta_{g_r}^{q,p} \tag{25}$$

If we express the generalised function in terms of the b_p coefficients then we have a representation which is independent from f. In this case we can express the structure of our generalised function as follows:

$$h = \sum_{p=0}^{l} b_p \hat{\eta}^{p.q} + R(\eta^{l+1,q})$$

where the hat on the η means that we are using η functions that have, a part from the component of order p, components of order lower than $l+1$ that vanish. With the equation above we have given the way to evaluate the first b_p non vanishing component of h. We will see with an example, in the next paragraph, that in addition to the lower vanishing components, we are able to evaluate at most the first two $b_p \neq 0$ coefficients (i.e. at most the first two components are independent from f).

It is worth here to point out explicitly that The above theory is also applicable to generalised functions of order $p < 0$. In this case, however, we need to extend the definition of the a_p to generating function of negative order. Given any function $g \in C^0$ and an integer $p < 0$ we define the relevant a_p coefficient of g as follows:

$$a_p = \int_{-\infty}^{+\infty} g^{(|p|)} dx \tag{26}$$

For example, for the Heaviside function $u(x)$, if we use the generating function $f^{(-1)}$ with growing index $q = 0$, the above defined product keeps working. For example, we have proven in [4] that:

$$g(u(x))\delta(x) = \left(\int_0^1 g(x)dx \right) \delta(x) \tag{27}$$

Now, for $g = x^k$ and $k \in \mathbb{N}$, the above statement is a particular case of (24).

Since for the Heaviside function $u(x)$ we have generating functions $f^{(-1)}$ and growing index $q = 0$, we may say that:

$$u(x) = \eta^{0,-1}_{f^{(-1)}} = \delta^{(-1)} \tag{28}$$

4 Equalities and examples of products in D'

By using the above defined product, we can prove interesting equalities involving products among elements of D'. We will see some examples in this paragraph.

Example 6.1: Evaluate the following product:

$$u(x)\delta'(x) \tag{29}$$

Before we start we need to choose the function f of definition 1. Although the theory has been developed with $f \in C^{\infty}$, for practical calculations we need a much less smooth function. In this example we need just C^1 class functions, we choose the most simple one which is a triangular window centred in the origin with base 2 and hight 1:

$$f(x) = (x+1)u(x+1) - 2xu(x) + (x-1)u(x-1) \tag{30}$$

we have $q = q_1 + q_2 = 2$ and $g(x) = f^{(-1)}(x) f^{(1)}(x)$ and therefore:

$$u(x)\delta'(x) \rightleftarrows n^2\, f^{(-1)}(nx) f^{(1)}(nx) \tag{31}$$

We can now evaluate all the coefficients of the structure of our generalised function:

$$\begin{aligned} b_0 &= \frac{\int_{-\infty}^{+\infty} g(x)dx}{\int_{-\infty}^{+\infty} f^2(x)dx} = \frac{-\frac{2}{3}}{\frac{2}{3}} = -1 \quad \text{coeff. of } \eta^{2,0} = \delta^2 \\ b_1 &= a_1 = \int_{-\infty}^{+\infty} x g(x) dx = \tfrac{1}{2} \quad \text{coeff. of } \eta^{2,1} = \delta' \end{aligned} \tag{32}$$

where $b_1 = a_1$ because for $p = 1$, $p + 1 = q$ and therefore, given the (21), the coefficient a_1 is independent from f. We have:

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) + R\left(\eta^{2,2}\right) \tag{33}$$

We may also express $u(x)\delta'(x)$ as an equality among products of elements of D' (compare with [3]), by ignoring the higher order terms:

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) \tag{34}$$

There is a second way to get to the same result. By using (24) we evaluate the the product of $u(x)\delta(x)$. We have:

$$u(x)\delta(x) \rightleftarrows n\, f^{(-1)}(nx) f(nx) \to q = 1 \tag{35}$$

From which we have:

$$u(x)\delta(x) = \frac{1}{2}\delta(x) + R\left(\eta^{1,1}\right) \tag{36}$$

We use the Leibniz rule, which we know to work with our definition of product. By taking the derivatives of both sides of the above equality we have:

$$\delta^2(x) + u(x)\delta'(x) = \frac{1}{2}\delta'(x) + R\left(\eta^{2,2}\right) \tag{37}$$

as expected.
Finally, there is a third way to get to the same result. First we use the (14) for $u(x)\delta'(x)$. We apply it to $g = f(x)^{(-1)}f(x)^{(1)}$ with $q = 2$:

$$a_p = \frac{(-1)^p}{p!}\int_{-\infty}^{+\infty} g(x)x^p dx = \begin{cases} \frac{1}{p!}\left(\frac{1}{p+3} - \frac{2}{p+2}\right) & for\, p\, even \\ \\ \frac{1}{p!}\frac{1}{p+1} & for\, p\, odd \end{cases} \tag{38}$$

and therefore, taking into account that $\eta^{2,1} = \delta'$:

$$u(x)\delta'(x) = -\frac{2}{3}\eta^{2,0}_{\xi_1} + \frac{1}{2}\delta'(x) - \frac{3}{20}\eta^{2,2}_{\xi_1''} + R\left(\eta^{2,3}\right) \tag{39}$$

The a_p coefficients above refer to a ξ_1 base function of g which is unknown.
Then we use the (14) for $\delta^2(x)$. We apply it to $g = f^2(x)$ with $q = 2$:

$$a_p = \frac{(-1)^p}{p!}\int_{-\infty}^{+\infty} f^2(x)x^p dx = \begin{cases} \frac{2}{p!}\left(\frac{1}{p+3} - \frac{2}{p+2} + \frac{1}{p+1}\right) & for\, p\, even \\ \\ 0 & for\, p\, odd \end{cases} \tag{40}$$

and therefore:

$$\delta^2(x) = \frac{2}{3}\eta^{2,0}_{\xi_2} + \frac{17}{60}\eta^{2,2}_{\xi_2''} + R\left(\eta^{2,4}\right) \tag{41}$$

The a_p coefficients above refer to a ξ_2 base function of $f^2(x)$ which is unknown.

To compare the (39) and the (41) we should transform the two expressions in the b_p notation which is independent from the function f. However, $D(f^2(x)) = 2f(x)f'(x)$ and therefore is easy to see that $\xi_1 = \xi_2$ and the a_p notations of the two generalised functions above are comparable each other. We conclude that we can add and subtract them in the a_p notation. By adding them we have:

$$u(x)\delta'(x) + \delta^2(x) = \frac{1}{2}\,\delta'(x) + R\left(\eta^{2,2}\right) \tag{42}$$

as expected.

Example 6.2: Evaluate the following product:

$$u(x)\delta''(x) \tag{43}$$

Before we start we need to choose the function f of definition 1. In this example we need a C^2 class functions, we choose the following function:

$$f(x) = \frac{3}{2}\left((x+1)^2u(x+1) - 4xu(x) - (x-1)^2u(x-1)\right) \tag{44}$$

We have $q = q_1 + q_2 = 3$ and $g(x) = f^{(-1)}(x)f^{(2)}(x)$. and therefore:

$$u(x)\delta''(x) \rightleftarrows n^3\, f^{(-1)}(nx)f^{(2)}(nx) \tag{45}$$

We can now evaluate all the coefficients of the structure of our generalised function:

$$
\begin{aligned}
&a_0 = \int_{-\infty}^{+\infty} g(x)dx = 0 && \text{coeff. of } \eta^{3,0} = \delta^3 \\
&b_1 = \frac{\int_{-\infty}^{+\infty} xg(x)dx}{\int_{-\infty}^{+\infty} x\frac{d}{dx}f^2(x)dx} = -\frac{3}{2} && \text{coeff. of } \eta^{3,1} = (\delta')^2 \\
&b_2 = a_2 = \int_{-\infty}^{+\infty} g(x)x^2dx = \frac{1}{2} && \text{coeff. of } \eta^{3,2} = \delta''
\end{aligned}
\tag{46}
$$

where $b_2 = a_2$ because for $p = 2$, $p + 1 = q$ and therefore, given the (21), the coefficient a_2 is independent from f. We have:

$$
u(x)\delta''(x) = -\frac{3}{2}\eta^{3,1}_{(f^2)'} + \frac{1}{2}\delta'' + R\left(\eta^{3,3}\right) \tag{47}
$$

We see that $u(x)\delta''(x) \notin D'$ since its component δ'' is negligible with respect of $\eta^{3,1}_{(f^2)'}$ and therefore $u(x)\delta''(x) \approx -\frac{3}{2}\eta^{3,1}_{(f^2)'}$.

Example 6.3: Evaluate the following product:

$$
\delta(x)\delta'(x) \tag{48}
$$

Before we start we need to choose the function f of definition 1. In this example we need just C^1 class functions, we choose once again the (30) of the previous example.
We have $q = q_1 + q_2 = 3$ and $g(x) = f(x)f^{(1)}(x)$. and therefore:

$$
\delta(x)\delta'(x) \rightleftarrows n^3 \; f(nx)f^{(1)}(nx) \tag{49}
$$

We can now evaluate all the coefficients of the structure of our generalised function:

$$
\begin{aligned}
&a_0 = \int_{-\infty}^{+\infty} g(x)dx = 0 && \text{coeff. of } \eta^{3,0} = \delta^3 \\
&b_1 = \frac{\int_{-\infty}^{+\infty} g(x)xdx}{\int_{-\infty}^{+\infty} \frac{d}{dx}f^2(x)xdx} = \frac{1}{2} && \text{coeff. of } \eta^{3,1} = (\delta^2)' \\
&a_2 = \int_{-\infty}^{+\infty} g(x)x^2dx = 0 && \text{coeff. of } \eta^{3,2} = \delta''
\end{aligned}
\tag{50}
$$

we have:

$$
\delta(x)\delta'(x) = \frac{1}{2}\eta^{3,1}_{(f^2)'} + R\left(\eta^{3,3}\right) \tag{51}
$$

Once again, there is a second way to get the same result. By taking twice the derivative of both sides of the (36), and rearranging the terms we get:

$$
\delta(x)\delta'(x) = -\frac{1}{3}u(x)\delta''(x) + \frac{1}{6}\delta''(x) + R\left(\eta^{3,3}\right) \tag{52}
$$

We see easily that, taking into account the (47), the (51) and the (52) are in perfect agreement.

Example 6.4: Evaluate the following product:

$$sign^2(x)\delta(x) \tag{53}$$

We have:

$$sign^2(x)\delta(x) \rightleftarrows n\,(2f^{(-1)}(nx)-1)^2 f(nx) \to q=1 \tag{54}$$

which is actually the sum of three products one of which is trivial. We have:

$$sign^2(x)\delta(x) = \frac{1}{3}\delta(x) + R\left(\eta^{1,1}\right) \tag{55}$$

compare with [2] §1.1 ex. iii and with [4].

5 Products with polynomials

Now we want to extend our product of distributions to products involving polynomial and therefore product with any function that can be expanded in a Taylor series.

We note that x^k with $k \in \mathbb{N}$ can be expressed as the limit of the following sequence of functions:

$$x^k = \lim_{n\to\infty} n^{-k}(nx)^k \tag{56}$$

and therefore, it is the limit of a sequence of functions of the kind (10) with generating function $g = x^k$ and growing index $q = -k$.

From the above limit, we see immediately that the product of a generalised function with a monomial of degree k, lowers the growing index of the generalised function by k. Given the generalised function h defined by the (10), then we have that $x^k h$ can be associated to the following sequence:

$$x^k h \rightleftarrows n^{q-k} x^k g(x) \tag{57}$$

Let us see what happens, to the order and the amplitude of a generalised function, when we multiply it by x. The generalization to multiplication by x^k is trivial.

For $p > 0$, from the (14) is possible to show that given any $\xi_1^{(p)} \in A^p$ we have:

$$a_{p-1}\left(\xi^{(p-1)}(x)\right) = a_p\left(-\frac{x}{p}\xi^{(p)}(x)\right) \;\; for\, p>0 \tag{58}$$

From which we see clearly that the product of a generalised function of order p, with x, lower the order of the generalised function by 1. To sum up, we have:

$$\eta^{q-1,p-1} = -\frac{x}{p}\eta^{q,p} \;\; for\, p>0 \tag{59}$$

and in particular:

$$\delta^{(p-1)} = -\frac{x}{p}\delta^{(p)} \;\; for\, p>0 \tag{60}$$

which is a well known result in literature (compare with [3]).

For $p = 0$, the situation is a bit more complex. It is possible to show that:

$$a_1\left(\xi^{(1)}(x)\right) = a_0\left(-x\,\xi^{(0)}(x)\right) \tag{61}$$

From with we have:

$$\eta^{q-1,1} = -x\,\eta^{q,0} \tag{62}$$

and in particular:

$$\eta^{0,1} = -x\delta(x) \tag{63}$$

we see that the order $p = 0$ cannot be further lowered by multiplying by x. If we keep multiplying a generalised function of order 0 with x, the growing index keep decreasing but the order toggles between 0 and 1.

For $p < 0$ the situation is even more complex since by multiplying an η function of order lower then 0 with x, we do not even get a η function any more. We will develop this part in a further issue of this paper.

We can now define some more examples of reference functions to be used for performing our multiplications:

$\eta^{q,p}$	p=-1	p=0	p=1	p=2	p=3
q=2		$\delta^2(x)$	$\delta'(x)$	$-(x\delta^2(x))'$	$-(x\delta(x))''$
q=1		$\delta(x)$	$-x\delta^2(x)$	$-(x\delta(x))'$	$\cdots$
q=0	$u(x)$	$-(x\delta^2(x))^{(-1)}$	$-x\delta(x)$	$-(x^3\delta^2(x))'$	$\cdots$
q=-1	$-(x\delta^2(x))^{(-2)}$	$-(x\delta(x))^{(-1)}$	$-x^3\delta^2(x)$	$-(x^3\delta(x))'$	
q=-2	$-(x\delta(x))^{(-2)}$	$-(x^3\delta^2(x))^{(-1)}$	$-x^3\delta(x)$	$\cdots$	
q=-3	$\cdots$	$\cdots$	$\cdots$		

Figure 2 : Examples of η functions for $q - p \leq 1$

Example 7.1: Evaluate the following product:

$$x^2\delta^2(x) \tag{64}$$

We use (24). Once again we need to choose the function f and once again we choose the (30) of the previous examples.

If $q_1 = 2$ is the growing index of $\delta^2(x)$, we have $q = q_1 - 2 = 0$ and $g(x) = x^2 f^2(x)$. and therefore:

$$x^2\delta^2(x) = \lim_{n\to\infty} x^2 f^2(nx) \tag{65}$$

We can now evaluate the first coefficient of the structure of our generalised function:

$$b_0 = \frac{\int_{-\infty}^{+\infty} g(x)dx}{\int_{-\infty}^{+\infty} -(xf^2(x))^{(-1)}dx} = 1\text{coeff. of } \eta^{0,0} = -(x\delta^2(x))^{(-1)} \tag{66}$$

which is independent from f. We have:

$$x^2\delta(x) = -(x\delta^2(x))^{(-1)} + R\left(\eta^{1,0}\right) \tag{67}$$

So, by choosing $x^2\delta^2(x)$ as a reference function for $\eta^{0,0}$, we would get the same result.

We will use now the theory developed above to discuss a well known example in the theory of product of distributions (compare with [2] §1.1 ex. i).

Example 7.2: If $vp\frac{1}{x}$ is the Cauchy principal value of $\frac{1}{x}$ then we have:

$$0 = (\delta(x) \cdot x) \cdot vp\frac{1}{x} = \delta(x) \cdot \left(x \cdot vp\frac{1}{x}\right) = \delta(x) \tag{68}$$

which is absurd.
By using our theory we know that $x\delta(x) = -\eta^{1,0} \neq 0$. We have:

$$0 = (x \cdot \delta(x) + \eta^{1,0}) \cdot \frac{1}{x} = \delta(x) + \frac{1}{x}\eta^{1,0} = \delta(x) - \delta(x) \tag{69}$$

a results that now makes sense.

6 η functions Vs Colombeau algebras

The Colombeau algebras are a beautiful theory as well as very general since, in this algebras, C^∞ and D' are automatically embedded. However, working with the Colombeau algebras is, most of the times, very difficult. The η function are much less general but they have the great advantage to be very convenient for practical calculations. We believe that, once the η functions are properly formalised, most of the problems that can be addressed by means of Colombeau algebras, can also be addressed by means of the η functions. Moreover, the η functions can be used also to perform calculations with products of delta functions and polynomials. In the Colombian algebras this kind of products (e.g. $x\delta$) falls in the category of the null functions and therefore are removed from the theory. We note finally that the set $\mathbb{G}^\eta$ is not an algebras and this make the Colombeau algebras and the η functions very different. We believe that this difference will possibly make the η functions suitable fro solving some of the problem related to the Colombeau algebras and that have lead to the definition of many difference variants of them.

References

[1] J. F. Colombeau. *Multiplication of Distributions.* Springer-Verlag (1992)

[2] M. Grosser, M. Kunzinger, M. Oberguggenberger, R. Steinbauer. *Geometric Theory of Genralized Functions with Applications to General Relativity.* Mathematics and its Applications, Vol. 537, Kluwer Academic Publishers, Dordrecht (2001).

[3] B. P. Damyanov. *Multiplication of Schwartz Distributions and Colombeau Generalized Functions*. Journal of Applied Analysis Vol. 5, No. 2 (1999), pp. 249-260.

[4] V. Nardozza. *Product of Distributions Applied to Discrete Differential Geometry*. www.vixra.org/abs/1211.0099 (2012) version v9 or most recent.