Products of Generalised Functions

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Abstract

A new space of generalised functions extending the space D', together with a well defined product, is constructed. The new space of generalized functions is used to prove interesting equalities involving products among elements of D'. A way of multiplying the defined generalised functions with polynomials is also derived.

Key Words: distribution theory, product of distributions.

1 Introduction

Products of distributions are quite common in several fields of both mathematics and physics. Examples arise naturally in quantum field theory, gravitation and in partial differential equation (e.g shock wave solutions in hydrodynamics) see [1]. An important issue, related to product of distributions, is the fact that the product, in the general case, is not well defined in D'. This issue is known as the Schwartz impossibility result (see [1] §1.3). In the Schwartz classical theory, only the product between a smooth function and a distribution is well defined. Historically, products of distributions are addressed by means of algebras of generalised functions developed initially by J. F. Colombeau (see [1] and [2]). In this paper we will propose a new approach for defining products of distributions.

In paragraphs from 2 to 4, we construct a new space of generalised functions, extending the space D'. In paragraph 5, we define a products in the above mentioned space of generalised functions. In paragraphs 6, we use the new developed theory to derive interesting equalities involving products among elements of D'. In paragraphs 7, we derive a method for multiplying the generalised functions, defined in this paper, with polynomials.

2 Preliminary definitions

Generalised functions can be defined by means of the limit of sequences of functions $f_n(x)$. In this paper we will deal only with generalised functions defined by means of the limit of a sequence of the form:

$$h = \lim_{n\to\infty} n^q f(nx) \tag{1}$$

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with $f(x) \in C^{\infty}$. Note that the above sequences are not the most general way to define distributions. For example, there is no sequence of the form (1) converging to $\delta + \delta'$. We will call $f(x)$ the generating function, $n^q f(nx)$ the generating sequence and q the growing index of the generalised function defined by the (1). Moreover, given any function $f \in C^{(0)}$, we say that f is a function of order $p \geq 0$ if:

$$0 < \left| \int_{-\infty}^{+\infty} x^p f(x) dx \right| < +\infty \tag{2}$$

Given any function f, with the notation $f^{(p)}$, where $p \in \mathbb{Z}$, we refer to the derivatives of order p of f, for $p \geq 0$, and the function defined by the following recursive formula:

$$f^{(p-1)} = \int_{-\infty}^{x} f^{(p)}(\tau) d\tau \tag{3}$$

for $p < 0$.

We say that a function f is of order $p < 0$, if for each $k \in \mathbb{N}$ then $f^{(|p|+k)}$ is a function of order k, according to the definition given above.

Most of the work developed in this paper deals with generalised function defined by means of a genearting function having order $p \geq 0$. However, all the content of this paper can be generalised to the case $p < 0$.

Before we proceed, we need to give a couple of definitions which will be intensively used later on:

Definition 1. *We define F to be the set of all the functions $f(x)$ having the following characteristics.*

$$\begin{array}{l} 1)\ f(x) \in C^{\infty} \\ 2)\ \lim_{x \to -\infty} f(x) x^k = 0 \ \ for\ any\ k \in \mathbb{N} \\ 3)\ \lim_{x \to +\infty} f(x) x^k = 0 \ \ for\ any\ k \in \mathbb{N} \end{array} \tag{4}$$

Definition 2. *Given any function $\xi(x) \in F$ then, if $\xi(x)$ verifies the following conditions:*

$$\int_{-\infty}^{+\infty} \xi(x) x^k dx = \begin{cases} 1 & for\ k = 0 \\ 0 & for\ 0 < k \leq s\ \ with\ s \in \mathbb{N} \end{cases} \tag{5}$$

then we call ξ a main generating function of order 0. We also call its derivatives $\xi^{(p)}$ with $p \in \mathbb{N}$ and $p < s$ a main generating function of order p. Finally, for each s, we define X_s to be the set of ξ function relevant to s.

For more details on the X_s sets, see [1] §8.2. In most cases, we will assume that s is large enough for our purpose (i.e. given $\xi^{(p)}$ the main generating function of higher order, we are working with, we have $p < s$) and we will not explicitly mention it in our discussion.

3 Structure of a generalised function

Let us see how to determine all the components, of different order, of a generalised function defined by means of the (1) and having generating function $f(x)$.

We will suppose, for the moment, that it is possible to find a function $\xi(x) \in X_s$ such that it is possible to express the generating function f as follows:

$$f(x) = \sum_{p=0}^{s} a_p \xi^{(p)} + r(x) \tag{6}$$

where $r(x)$ is a function having all momenta, of order lower then s, equal to 0. We will see, further on, that the above ξ function exists. First of all, we note that all the components of the distribution (1) have the same growing index q. We will call this kind of generating functions homogeneous. We have:

$$h = \lim_{n\to\infty} n^q f(nx) = \sum_{p=0}^{s} a_p \frac{\delta^{(p)}}{n^{p-q+1}} + h_r \tag{7}$$

where the a_p, although not explicitly noted, refer and depend from the function ξ. We note explicitly that, if $q \geq 1$, h always contains one (and only one) distribution $a_p\delta^{(q-1)}(x) \in D'$.

For the distribution defined by the (7), we can determine the a_p coefficients by applying the Schwartz theory of distribution to our sequence of functions divided by n^{q-p-1}. Let $\phi \in D$ be a test function and given p, we have:

$$\frac{h - h_r}{n^{q-p-1}} = \lim_{n\to\infty} \int_{-\infty}^{+\infty} n^{p+1} f(nx)\phi(x)dx = \lim_{n\to\infty} \sum_{k=0}^{s} (-1)^k a_k \frac{\phi^{(k)}(0)}{n^{k-p}} \tag{8}$$

In the right side of the above equation we have two problems which make difficult to evaluate the a_k. The coefficients a_k are mixed up by the summation on k and, given a generic ϕ, this test function may pick up in the same coefficient a_k components related to different $\xi^{(p)}$. To better evaluate all a_p we decide to use, as a test function, x^p. In this way we solve both problems mentioned above since x^p has all derivatives of order i equal to 0 for $i \neq p$ (so the summation will not mix various a_k terms) and x^p will filter out all components $\xi^{(i)}$ with $i \neq p$. Of course a test function should vanish outside a compact interval (compact support) and x^p does not. However, the above requirement is needed to ensure integrability which in our case is ensured by the fact that $f \in F$. So the fact that x^i has not compact support it is not a problem. We have:

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} n^{p+1} f(nx) x^p dx = (-1)^p \, a_p \, p! \tag{9}$$

where $p!$ is the value of the p^{th} derivatives of x^p. From the (9) we can easily evaluate the a_p as follows:

$$\begin{aligned} a_p &= \lim_{n\to\infty} \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} n^{p+1} f(nx) x^p dx \\ &= \lim_{n\to\infty} \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} n \, f(nx)(nx)^p dx \end{aligned} \tag{10}$$

We note that the right part of the (10), for n that goes to infinity, in the (x, y) plane, shrinks (along x) and grows (along y) like n, which leaves the integral unchanged. For the above reason, the limit of the (10) is simply the value of the integrals for any n. We may as well evaluate it for n=1. We have:

$$a_p = \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} f(x) x^p dx \tag{11}$$

and therefore the a_p coefficients are related to the momenta of f.

The (11) allow us to evaluate the coefficient a_p of all the components of order lower then s. However, neither a dependency from s nor a dependency from $\xi \in X_s$ is explicitly present in the above equations and therefore we can use it to evaluate components of any order just assuming s is big enough. As a matter of fact, the (11) can be used to evaluate all the infinite components of a generalised function.

So a generalised function can be expressed as:

$$h = \lim_{n \to \infty} n^q f(nx) = \sum_{p=0}^{\infty} a_p \frac{\delta^{(p)}}{n^{p-q+1}} \tag{12}$$

This is what we call the structure of a generalised function (compare with [4]).

When we use the $\delta^{(p)}/n^k$ notation to give the structure of a generalised function, as we did in the (12), this structure is depending from the underling ξ base function and therefore from the generating function f.

As mentioned above, we need to justify our assumption that it is possible to find a $\xi \in X_s$ such that we can write f as in the (6). This can be done by giving a constructive algorithm to evaluate the required ξ, Given f and the a_p, if we evaluate $f_1 = f/a_0$, we have a function with the same base function ξ and with first momentum equal to 1. We evaluate the a^1_p coefficient of f_1. We evaluate $f_2 = f_1 - a^1_1 f'_1$ and we get a function with the same base function ξ, with first momentum equal to 1 and second momentum equal to 0. Iterating the process s times we get eventually our required ξ function in X_s.

We are now ready for the following definition:

Definition 3. *Given two homogeneous generalised functions h_1, h_2, with same growing index and defined by means of generating function having the same base function ξ, then if it is possible to find an integer k such that:*

$$\begin{cases} a_p(h_1) = a_p(h_2) & for\, p = k \\ a_p(h_1) = a_p(h_2) = 0 & for\, p < k \end{cases} \tag{13}$$

we say that h_1 and h_2 are representatives of the same generalised function and we use the notation $h_1 \sim h_2$. Moreover, if:

$$a_p(h_1) = a_p(h_2) \quad for\, each\, p \tag{14}$$

then we say that the two generating function are equal and we use the notation $h_1 = h_2$.

Note that, by using the expression representatives of the same generalised function, we are using a terminology similar to the one used in the Colombeau algebras. Although we are facing a similar situation, the theory developed in this paper is rather different, with respect of the Colombeau algebras, and therefore identical terms do not imply identical meanings in the two theories.

4 Additional remarks on the a_p coefficients

We show now an important fact about the coefficient a_p of the (11). Let $\xi \in X_s$ be any main generating function of order 0 and $h \in \mathbb{G}^\eta$ the related generalised function of growing index q. We have:

$$\lim_{n\to\infty} a_p(\xi) n^q \xi^{(p)}(nx) = h = a_p(\xi) \frac{\delta^{(p)}}{\eta^{p-q+1}} \tag{15}$$

If we choose $\xi_\alpha = \alpha\xi(\alpha x)$, as a different generating function of order 0, we have:

$$\lim_{n\to\infty} a_p(\xi_\alpha) n^q \alpha^{p+1} \xi^{(p)}(n\,\alpha x) = h_\alpha = a_p(\xi_\alpha) \frac{\delta^{(p)}}{\eta^{p-q+1}} \tag{16}$$

If the (15) and (16) are the same generalised function (i.e. $h = h_\alpha$) then we can write:

$$\lim_{n\to\infty} a_p(\xi) n^q \xi^{(p)}(nx) = \lim_{n\to\infty} \frac{a_p(\xi_\alpha)}{\alpha^{q-p-1}} (n\alpha)^q \xi^{(p)}(n\,\alpha x) \tag{17}$$

but, at the same time, we have:

$$\lim_{n\to\infty} n^q \xi^{(p)}(nx) = \lim_{n\to\infty} (n\alpha)^q \xi^{(p)}(n\,\alpha x) \tag{18}$$

because the left and the right side limit of the (18) are the same function growing and shrinking at the same rate with n, and therefore converge to the same generalised function $\eta_\xi^{p,q}$. For example, if α is an integer, the sequence on the right hand side of the equation is a sub-sequence of the one on the left hand side. We conclude that:

$$a_p(\xi_\alpha) = \alpha^{q-p-1} a_p(\xi) \;\;\Rightarrow\;\; h = h_\alpha \tag{19}$$

Note that if $p = q - 1$ then, as expected, the a_p coefficients are independent from ξ.

The key point here is that, even though the coefficients a_p of two separate generalised functions are different, this does not necessarily imply that the two generalised functions are different. From the (19) above we see that the coefficient a_p depends from the base function ξ and therefore from the generation function f. For Example, if we change the scaling factor of f by using a different function $\alpha f(\alpha x)$, the coefficients a_p will change accordingly. We will call the a_p relative coefficients. We will see, in the following paragraph that we can define coefficient b_p which are independent from the function f. We will call the b_p absolute coefficients. We will see that, the b_p coefficients are a rescaling of the a_p coefficients with respect of a reference function depending both on p and q.

If we express some components of a generalised function in terms of the b_p coefficients (e.g the first l components), then we have a representation which is independent from f. In this case we can express the structure of our generalised function as follows:

$$\begin{aligned} h = \lim_{n\to\infty} n^q f(nx) &= \sum_{p=0}^{l} b_p \hat{\eta}^{p\cdot q} + \sum_{p=l+1}^{\infty} a_p \frac{\delta^{(p)}}{\eta^{p-q+1}} \\ &= \sum_{p=0}^{l} b_p \hat{\eta}^{p\cdot q} + R(\eta^{l+1,q}) \end{aligned} \tag{20}$$

where the hat on the η means that we are using η functions that have, a part from the component of order p, components of order lower than $l+1$ that vanish. Moreover, with the notation $R\left(\eta^{l+1,q}\right)$ we means that, to have the above equality exact, we need to add an infinity number of η components of growing index q and order $p \geq l+1$. In the next paragraph we will see that, in addition to the lower vanishing components, we are able to evaluate at most the first two $b_p \neq 0$ coefficients (i.e. at most the first two components are independent from f).

Definition 4. *Let $f(x) \in C^p$ be any function such that $\int_{-\infty}^{+\infty} f(x)dx = 1$. We define the generalised functions $\eta^{p,q}$, with $q > p$ to be the following limit:*

$$\eta^{p,q}(x) = \lim_{n\to\infty} n^q \frac{d^p}{dx^p}(f(nx))^{q-p} \quad with\, p, q \in \mathbb{Z} \tag{21}$$

What kind of generalised function are the $\eta^{p,q}$? If the sequence of distributions $f_n = n^q f^{(p)}(nx)$, in the (21), converges to $\eta^{p,q}$, then $\frac{f_n}{n^{q-p-1}}$ converges to $\delta^{(p)}$. So, with an abuse of notation, we may say that:

$$\eta^{p,q} = A\frac{\delta^{(p)}}{n^{p-q+1}} \quad with\, A\, depending\, on\, f \tag{22}$$

The $\eta^{p,q}$ are therefore the limit of sequences of functions that are shaped like $\delta^{(p)}$ and that, when we take the limit, grow at a lower or faster rate with respect to it (according to the sign of p-q+1).

We are now ready to define our new space of generalised functions.

Definition 5. *We define $\mathbb{G}^\eta$ to be the space of generalised functions which elements are the limits of sequences of the type (21), or a linear combinations of them.*

The (21) tells us what is the real nature of the $\eta^{p,q}$ and that we may rename them as for the following table:

$\eta^{p,q}$	p=-1	p=0	p=1	p=2	p=3
q=5		$\cdots$	$\cdots$	$\cdots$	$\cdots$
q=4		$\cdots$	$\frac{d}{dx}(\delta^3(x))$	$\frac{d^2}{dx^2}(\delta^2(x))$	$\cdots$
q=3		$\delta^3(x)$	$\frac{d}{dx}(\delta^2(x))$	$\delta''(x)$	
q=2	$\cdots$	$\delta^2(x)$	$\delta'(x)$		
q=1	$(\delta^2(x))^{(-1)}$	$\delta(x)$			
q=0	$u(x)$				

Figure 1 : η functions

5 Product of generalised functions

Let us see now, how to use the theory developed in the previous paragraphs to define the product of generalised functions in $\mathbb{G}^\eta$.

Definition 6. *Let $f(x)$ be any function of class C^m with $m \in \mathbb{N}$ and $f(x) \geq 0$ for each $x \in \mathbb{R}$. Given k generalised functions $h_i = \eta^{p_i,q_i} \in \mathbb{G}^\eta$ having generating sequences S_i given by the (21) with orders $p_i < m$ and growing indexes $q_i \in \mathbb{Z}$, we define the product h of the h_i to be the limit of the product of their generating sequences:*

$$h = \lim_{n\to\infty} S_1 S_2 \cdots S_k \tag{23}$$

The reason why we require $f(x) \geq 0$ is to avoid problems when we have situations like $f^n(x)$ with n even where, given f^n, the sign of the original function f is not unique.

If for each possible choice of f we get generalised functions which are always representatives of the same generalised function, then the above product is independent from the choice of f and therefore is a well defined product.

The following proposition applies:

Proposition 1. *Given any function $f \in C^m$ with $m \in \mathbb{N}$, $\int_{-\infty}^{+\infty} f(x)dx = 1$ and $f(x) \geq 0$ for each $x \in \mathbb{R}$, the product of k generalised functions, having generating function $f_i = \frac{d^{p_i}}{dx^{p_i}}(f(x))^{q_i - p_i}$ with orders $p_i < m$ and growing indexes $q_i \in \mathbb{Z}$:*

$$h = \eta^{p_1,q_1}\eta^{p_2,q_2}\cdots\eta^{p_k,q_k} \tag{24}$$

is a representatives of the following generalised function:

$$h \sim \frac{a_p\left(f_*\right)}{a_p\left(\frac{d^p}{dx^p}f^{q-p}\right)}\eta^{p,q} = \frac{\int_{-\infty}^{+\infty} x^p f_* dx}{\int_{-\infty}^{+\infty} x^p \frac{d^p}{dx^p} f^{q-p} dx}\eta^{p,q} \tag{25}$$

where $f_* = f_1 f_2 \cdots f_k$, $p < m$ is the order of the function f_* and $q = q_1 q_2 \cdots q_k$, provided that the condition $q > p$ is verified.
Moreover, the amplitude evaluated above is independent from f.
In particular, if $q = p + 1$, the above product h is an element of D' and it is equal to:

$$h = \frac{\int_{-\infty}^{+\infty} x^p f_* dx}{\int_{-\infty}^{+\infty} x^p \frac{d^p}{dx^p} f dx}\delta^{(p)} \tag{26}$$

We will not prove proposition 1 here. A formal prove will be provided in the next issue of this paper. However, we note that the amplitude of the generalised function h given by the (25) is just a rescaling, of the base function ξ of f, with respect of the reference functions (i.e. the derivatives of f^{q-p}). This rescaling keeps the coefficient b_p, mentioned in the previous paragraph, unchanged because f_* and $\frac{d^p}{dx^p}f^{q-p}$ in the (25) have the same base function ξ and therefore we have:

$$b_p = \frac{a_p(f_*(x))\xi(x)}{a_p(\frac{d^p}{dx^p}f^{q-p}(x))\xi(x)} \tag{27}$$

and if we rescale f and we use $\alpha f(\alpha x)$ with $\alpha > 0$, given the (19) we have:

$$b_p = \frac{\alpha^{q-p-1} a_p(f_*(x))\xi_\alpha(x)}{\alpha^{q-p-1} a_p(\frac{d^p}{dx^p} f^{q-p}(x))\xi_\alpha(x)} \tag{28}$$

which leave b_p unchanged. In the same way, it is possible to see that b_p does not change if we use a new function, that is a linear combinations of the previous f scaled by different values of α.

So the main point here is to show that f_* and f^{q-p}, a part from the order, have the same base function ξ. The above statement can be proved easily for particular cases. For example, $D(f^2) = 2ff'$ and therefore f^2 and ff' have the same ξ base function. This this can be used to evaluate products $\delta\delta'$ which are independent from f.

Given the proposition above, the product defined by definition 6 is a well defined product and it is independent from the function f. Moreover, we note that the above defined product is commutative, associative and compliant with the Leibniz rule for the derivatives. This is because each distribution is associated to a sequence of functions and the product of sequences of functions is commutative, associative and compliant with the Leibniz rule

The above proposition is also applicable to generalised functions of order $p < 0$. In this case, however, we need to extend the definition of the a_p to generating function of negative order. Given any function $f \in C^0$ and an integer $p < 0$ we define the relevant a_p coefficient of f as follows:

$$a_p = \int_{-\infty}^{+\infty} f^{(|p|)} dx \tag{29}$$

For example, for the Heaviside function $u(x)$, if we use the generating function $f^{(-1)}$ with growing index $q = 0$, the above defined product keeps working. For example, we have proven in [4] that:

$$f(u(x))\delta(x) = \left(\int_0^1 f(x)dx \right) \delta(x) \tag{30}$$

Now, for $f = x^k$ and $k \in \mathbb{N}$, the above statement is a particular case of proposition 1.

Since for the Heaviside function $u(x)$ we have generating functions $f^{(-1)}$ and growing index $q = 0$, we may say that:

$$u(x) = \eta^{-1,0} = \delta^{(-1)} \tag{31}$$

6 Equalities and examples of products in D'

By using the above defined product, we can prove interesting equalities involving products among elements of D'. We will see some examples in this paragraph.

Example 6.1: Evaluate the following product:

$$u(x)\delta'(x) \tag{32}$$

We use proposition 1. Before we start we need to choose the function f. In this example we need C^1 class functions, we choose the most simple one which is a triangular window centred in the origin with base 2 and hight 1:

$$f(x) = (x+1)u(x+1) - 2xu(x) + (x-1)u(x-1) \tag{33}$$

we have $q = q_1 + q_2 = 2$ and $f_*(x) = f^{(-1)}(x)f^{(1)}(x)$ and therefore:

$$u(x)\delta'(x) = \lim_{n\to\infty} n^2\, f^{(-1)}(nx)f^{(1)}(nx) \tag{34}$$

We can now evaluate all the coefficients of the structure of our generalised function:

$$\begin{aligned} b_0 &= \frac{\int_{-\infty}^{+\infty} f_*(x)dx}{\int_{-\infty}^{+\infty} f^2(x)dx} = \frac{-\frac{2}{3}}{\frac{2}{3}} = -1 \quad coeff.\, of\, \eta^{0,2} = \delta^2 \\ b_1 &= a_1 = \textstyle\int_{-\infty}^{+\infty} x f_*(x)dx = \frac{1}{2} \quad coeff.\, of\, \eta^{1,2} = \delta' \end{aligned} \tag{35}$$

where $b_1 = a_1$ because for $p = 1$, $p + 1 = q$ and therefore, given the (19), the coefficient a_1 is independent from f. We have:

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) + R\left(\eta^{2,2}\right) \tag{36}$$

We may also express $u(x)\delta'(x)$ as an equality among products of elements of D' (compare with [3]), by ignoring the higher order terms:

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) \tag{37}$$

There is a second way to get to the same result. By using proposition 1 we evaluate the the product of $u(x)\delta(x)$. We have:

$$u(x)\delta(x) \to n\; f^{(-1)}(nx)f(nx) \to q = 1 \tag{38}$$

From which we have:

$$u(x)\delta(x) = \frac{1}{2}\delta(x) + R\left(\eta^{1,1}\right) \tag{39}$$

We use the Leibniz rule, which we know to work with our definition of product. By taking the derivatives of both sides of the above equality we have:

$$\delta^2(x) + u(x)\delta'(x) = \frac{1}{2}\delta'(x) + R\left(\eta^{2,2}\right) \tag{40}$$

as expected.
Finally, there is a third way to get to the same result. First we use the (11) for $u(x)\delta'(x)$. We apply it to $f_* = f(x)^{(-1)}f(x)^{(1)}$ with $q = 2$:

$$a_p = \frac{(-1)^p}{p!}\int_{-\infty}^{+\infty} f_*(x)x^p dx = \begin{cases} \frac{1}{p!}\left(\frac{1}{p+3} - \frac{2}{p+2}\right) & for\, p\, even \\ \frac{1}{p!}\frac{1}{p+1} & for\, p\, odd \end{cases} \tag{41}$$

and therefore, taking into account that $\eta^{1,2} = \delta'$:

$$u(x)\delta'(x) = -\frac{2}{3}\eta^{0,2} + \frac{1}{2}\delta'(x) - \frac{3}{20}\eta^{2,2} + R\left(\eta^{3,2}\right) \tag{42}$$

The a_p coefficients above refer to a ξ_1 base function of f_* which is unknown. Then we use the (11) for $\delta^2(x)$. We apply it to $f_* = f^2(x)$ with $q = 2$:

$$a_p = \frac{(-1)^p}{p!}\int_{-\infty}^{+\infty} f^2(x)x^p dx = \begin{cases} \frac{2}{p!}\left(\frac{1}{p+3} - \frac{2}{p+2} + \frac{1}{p+1}\right) & for\, p\, even \\ \\ 0 & for\, p\, odd \end{cases} \tag{43}$$

and therefore:

$$\delta^2(x) = \frac{2}{3}\eta^{0,2} + \frac{17}{60}\eta^{2,2} + R\left(\eta^{4,2}\right) \tag{44}$$

The a_p coefficients above refer to a ξ_2 base function of $f^2(x)$ which is unknown.

To compare the (42) and the (44) we should transform the two expressions in the b_p notation which is independent from the function f. However, $D(f^2(x)) = 2f(x)f'(x)$ and therefore is easy to see that $\xi_1 = \xi_2$ and the a_p notations of the two generalised functions above are comparable each other. We conclude that we can add and subtract them in the a_p notation. By adding them we have:

$$u(x)\delta'(x) + \delta^2(x) = \frac{1}{2}\,\delta'(x) + R\left(\eta^{2,2}\right) \tag{45}$$

as expected. The last method shows us why, in proposition 1, is so convenient rescaling the a_0 coefficient, of the $\eta^{0,2}$ component, using the a_0 coefficient of f^2. As a matter of fact, the former is propositional to the amplitude of the δ^2 component of our generalised function while the latter has the same proportionality with respect to a δ^2 of amplitude 1.

Example 6.2: Evaluate the following product:

$$u(x)\delta^{''}(x) \tag{46}$$

We use proposition 1. Before we start we need to choose the function f. In this example we need C^1 class functions, we choose again the (33) of the previous example.
We have $q = q_1 + q_2 = 3$ and $f_*(x) = f^{(-1)}(x)f^{(2)}(x)$. and therefore:

$$u(x)\delta^{''}(x) = \lim_{n\to\infty} n^3\; f^{(-1)}(nx)f^{(2)}(nx) \tag{47}$$

We can now evaluate all the coefficients of the structure of our generalised function:

$$\begin{array}{ll} a_0 = \int_{-\infty}^{+\infty} f_*(x)dx = 0 & coeff.\, of\, \eta^{0,3} = \delta^3 \\ \\ b_1 = \frac{\int_{-\infty}^{+\infty} x f_*(x)dx}{\int_{-\infty}^{+\infty} x\frac{d}{dx}f^2(x)dx} = -\frac{3}{2} & coeff.\, of\, \eta^{1,3} = (\delta^2)' \\ \\ b_2 = a_2 = \int_{-\infty}^{+\infty} f_*(x)x^2dx = \frac{1}{2} & coeff.\, of\, \eta^{2,3} = \delta^{''} \end{array} \tag{48}$$

where $b_2 = a_2$ because for $p = 2$, $p+1 = q$ and therefore, given the (19), the coefficient a_2 is independent from f. We have:

$$u(x)\delta^{''}(x) = -\frac{3}{2}\eta^{1,3} + \frac{1}{2}\delta^{''} + R\left(\eta^{3,3}\right) \tag{49}$$

We see that $u(x)\delta^{''}(x) \notin D'$ since its component $\delta^{''}$ is negligible with respect of $\eta^{1,3}$ and therefore $u(x)\delta^{''}(x) \sim -\frac{3}{2}\eta^{1,3}$.

Example 6.3: Evaluate the following product:

$$\delta(x)\delta'(x) \tag{50}$$

We use proposition 1. Before we start we need to choose the function f. In this example we need C^1 class functions, we choose once again the (33) of the previous example.
We have $q = q_1 + q_2 = 3$ and $f_*(x) = f(x)f^{(1)}(x)$. and therefore:

$$\delta(x)\delta'(x) = \lim_{n\to\infty} n^3\ f(nx)f^{(1)}(nx) \tag{51}$$

We can now evaluate all the coefficients of the structure of our generalised function:

$$\begin{array}{ll} a_0 = \int_{-\infty}^{+\infty} f_*(x)dx = 0 & coeff.\,of\ \eta^{0,3} = \delta^3 \\ \\ b_1 = \frac{\int_{-\infty}^{+\infty} f_*(x)xdx}{\int_{-\infty}^{+\infty} \frac{d}{dx} f^2(x)xdx} = \frac{1}{2} & coeff.\,of\ \eta^{1,3} = (\delta^2)' \\ \\ a_2 = \int_{-\infty}^{+\infty} f_*(x)x^2dx = 0 & coeff.\,of\ \eta^{2,3} = \delta^{''} \end{array} \tag{52}$$

we have:

$$\delta(x)\delta'(x) = \frac{1}{2}\eta^{1,3} + R\left(\eta^{3,3}\right) \tag{53}$$

Once again, there is a second way to get the same result. By taking twice the derivative of both sides of the (39), and rearranging the terms we get:

$$\delta(x)\delta'(x) = -\frac{1}{3}u(x)\delta^{''}(x) + \frac{1}{6}\delta^{''}(x) + R\left(\eta^{3,3}\right) \tag{54}$$

We see easily that, taking into account the (49), the (53) and the (54) are in perfect agreement.

Example 6.4: Evaluate the following product:

$$sign^2(x)\delta(x) \tag{55}$$

We use proposition 1. We have:

$$sign^2(x)\delta(x) = \lim_{n\to\infty} n\,(2f^{(-1)}(nx) - 1)^2 f(nx) \to q = 1 \tag{56}$$

which is actually the sum of three products one of which is trivial. We have:

$$sign^2(x)\delta(x) = \frac{1}{3}\delta(x) + R\left(\eta^{1,1}\right) \tag{57}$$

compare with [2] §1.1 ex. iii and with [4].

7 Products with polynomials

Now we want to extend our product of distributions to products involving polynomial and therefore product with any function that can be expanded in a Taylor series.

We note that x^k with $k \in \mathbb{N}$ can be expressed as the limit of the following sequence of functions:

$$x^k = \lim_{n\to\infty} n^{-k}(nx)^k \tag{58}$$

and therefore, it is the limit of a sequence of functions of the kind (1) with generating function $f = x^k$ and growing index $q = -k$.

From the above limit, we see immediately that the product of a generalised function with a monomial of degree k, lowers the growing index of the generalised function by k. Given the generalised function h defined by the (1), then we have that $x^k h$ is the limit of the following sequence:

$$x^k h = x^k \lim_{n\to\infty} x^q f(x) = \lim_{n\to\infty} n^{q-k} x^k f(x) \tag{59}$$

Let us see what happens, to the order and the amplitude of a generalised function, when we multiply it by x. The generalization to multiplication by x^k is trivial.

For $p > 0$, from the (10) is possible to show that given any $\xi_1^{(p)} \in X^p$ we have:

$$a_{p-1}\left(\xi^{(p-1)}(x)\right) = a_p\left(-\frac{x}{p}\xi^{(p)}(x)\right) \;\; for\, p > 0 \tag{60}$$

From which we see clearly that the product of a generalised function of order p, with x, lower the order of the generalised function by 1. To sum up, we have:

$$\eta^{p-1,q-1} = -\frac{x}{p}\eta^{p,q} \;\; for\, p > 0 \tag{61}$$

and in particular:

$$\delta^{(p-1)} = -\frac{x}{p}\delta^{(p)} \;\; for\, p > 0 \tag{62}$$

which is a well known result in literature (compare with [3]).

For $p = 0$, the situation is a bit more complex. It is possible to show that:

$$a_1\left(\xi^{(1)}(x)\right) = a_0\left(-x\,\xi^{(0)}(x)\right) \tag{63}$$

From with we have:

$$\eta^{1,q-1} = -x\,\eta^{0,q} \tag{64}$$

and in particular:

$$\eta^{1,0} = -x\delta(x) \tag{65}$$

we see that the order $p = 0$ cannot be further lowered by multiplying by x. If we keep multiplying a generalised function of order 0 with x, the growing index keep decreasing but the order toggles between 0 and 1.

For $p < 0$ the situation is even more complex since by multiplying an η function of order lower then 0 with x, we do not even get a η function any more. We will develop this part in a further issue of this paper.

We can now define some more reference functions to be used for performing our multiplications:

$\eta^{p,q}$	p=-1	p=0	p=1	p=2	p=3
q=2		$\delta^2(x)$	$\delta'(x)$	$-(x\delta^2(x))'$	$-(x\delta(x))''$
q=1		$\delta(x)$	$-x\delta^2(x)$	$-(x\delta(x))'$	$\cdots$
q=0	$u(x)$	$-(x\delta^2(x))^{(-1)}$	$-x\delta(x)$	$-(x^3\delta^2(x))'$	$\cdots$
q=-1	$-(x\delta^2(x))^{(-2)}$	$-(x\delta(x))^{(-1)}$	$-x^3\delta^2(x)$	$-(x^3\delta(x))'$	
q=-2	$-(x\delta(x))^{(-2)}$	$-(x^3\delta^2(x))^{(-1)}$	$-x^3\delta(x)$	$\cdots$	
q=-3	$\cdots$	$\cdots$	$\cdots$		

Figure 2 : η functions for $q - p \leq 1$

Note that, we have some freedom in choosing the above reference functions as shown by the following example:

Example 7.1: Evaluate the following product:

$$x^2\delta^2(x) \tag{66}$$

We use proposition 1. Once again we need to choose the function f and once again we choose the (33) of the previous examples.
If $q_1 = 2$ is the growing index of $\delta^2(x)$, we have $q = q_1 - 2 = 0$ and $f_*(x) = x^2 f^2(x)$. and therefore:

$$x^2\delta^2(x) = \lim_{n\to\infty} x^2 f^2(nx) \tag{67}$$

We can now evaluate the first coefficient of the structure of our generalised function:

$$b_0 = \frac{\int_{-\infty}^{+\infty} f_*(x)dx}{\int_{-\infty}^{+\infty} -(xf^2(x))^{(-1)}dx} = 1 \ \ coeff.\ of\ \eta^{0,0} = -(x\delta^2(x))^{(-1)} \tag{68}$$

which is independent from f. We have:

$$x^2\delta(x) = -(x\delta^2(x))^{(-1)} + R\left(\eta^{1,0}\right) \tag{69}$$

So, by choosing $x^2\delta^2(x)$ as a reference function for $\eta^{0,0}$, we would get the same result.

We will use now the theory developed above to discuss a well known example in the theory of product of distributions (compare with [2] §1.1 ex. i).

Example 7.2: If $vp\frac{1}{x}$ is the Cauchy principal value of $\frac{1}{x}$ then we have:

$$0 = (\delta(x)\cdot x)\cdot vp\frac{1}{x} = \delta(x)\cdot\left(x\cdot vp\frac{1}{x}\right) = \delta(x) \tag{70}$$

which is absurd.
By using our theory we know that $x\delta(x) = -\eta^{1,0} \neq 0$. We have:

$$0 = (x \cdot \delta(x) + \eta^{1,0}) \cdot \frac{1}{x} = \delta(x) + \frac{1}{x}\eta^{1,0} = \delta(x) - \delta(x) \tag{71}$$

a results that now makes sense.

References

[1] J. F. Colombeau. *Multiplication of Distributions.* Springer-Verlag (1992)

[2] M. Grosser, M. Kunzinger, M. Oberguggenberger, R. Steinbauer. *Geometric Theory of Genralized Functions with Applications to General Relativity.* Mathematics and its Applications, Vol. 537, Kluwer Academic Publishers, Dordrecht (2001).

[3] B. P. Damyanov. *Multiplication of Schwartz Distributions and Colombeau Generalized Functions.* Journal of Applied Analysis Vol. 5, No. 2 (1999), pp. 249-260.

[4] V. Nardozza. *Product of Distributions Applied to Discrete Differential Geometry.* www.vixra.org/abs/1211.0099 (2012) version v9 or most recent.