# Heuristic Study of the Concept of *pq*-Radial Functions as a New Class of Potentials

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**Abstract:** The main purpose of the present paper is the heuristic study of the structure, properties and consequences of new class of potential functions results from the concept of *pq*-Radial functions which are fundamental family of solutions of second order *pq*-PDE.

**Keyword:** Potential functions, Laplace equation, *pq*-Radial function, *pq*-PDE, Legendre polynomials

## 1. Introduction

The potential (function) theory [1,2] has a long history and has a large domain of applications particularly in gravitational physics, electrostatics, hydrodynamics, engineering, probability theory [3], and many other branches of science. The term 'potential theory' arises from the fact that, in 19th century physics the known fundamental forces of Nature were believed to be derived from potentials which satisfied Laplace equation. Therefore, potential theory was the study of functions that could serve as potentials. However, nowadays, we know that Nature is more complicated and in some very interesting cases, the equations that describe forces are systems of non linear PDEs, such as Einstein's equations and Yang-Mills equations, and the Laplace equation is only valid as a limiting case. Nevertheless, the term 'potential theory' has remained as a convenient expression for describing the study of functions satisfying the Laplace equation and its generalization. For instance, the well-known classical potential function or harmonic function [4]

$$V(r) = r^{-1}, \tag{1}$$

where $r = (x^2 + y^2 + z^2)^{1/2} > 0$ and $x,y,z \in \mathbf{R}$. (1) is a fundamental solution to the Laplace equation

$$\Delta U = 0\,, \quad U \equiv U(x,y,z)\,, \tag{2}$$

because the Laplace equation (2) is in fact, rotation invariant, *i.e*., it is a radial symmetry that is why the function (1) is also called radial solution. In general, the radial solutions are natural to look for since they reduce a PDE to an ODE, which is generally easier to solve. In this sense, Eq.(2) reduces to ODE *through* $U = V$, and we find

$$\Delta V = \frac{d^2V}{dr^2} + \frac{2}{r} + \frac{dV}{dr} \equiv \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0\,. \tag{3}$$

## 2. Concept of *pq*-Radial Functions

In spite of its abstraction, there is a strong link between mathematics and physics since the major mathematical discoveries are made in theoretical physics. Indeed, during our preliminary investigation on the 'hypothetical' dark matter [5,6] and its gravitodynamical effects on the baryon (luminous/ordinary) matter at large-scale structures and as a direct consequence, we arrived at the concept of *pq*-Radial functions (*pq*-RFs) as a new class of potential functions. Conceptually, the *pq*-RF is defined, for well-determined weight $u$ and characteristic function $v$, and for any two real orders $p$ and $q$, as follows:

$$\eta_{p,q}(r,s,\rho,\theta) = u^p(r,s)v^{-q}(r,\rho,\theta)\,, \tag{4}$$

where the weight function and the characteristic function are, respectively, defined by

$$u \equiv u(r,s) = r^2 - sr\,, \tag{5}$$

$$v \equiv v(r,\rho,\,\theta) = r^2 - 2\rho\, r\cos\theta + \rho^2\,, \tag{6}$$

with $r,s \in \mathbf{R}_+\,;\, \rho\,,\theta \in \mathbf{R}$ and $r$ is the radial distance from the origin of the coordinates, $\rho,\,\theta$ and $s$ are variable parameters. Hence more explicitly, we have

$$\eta_{p,q} \equiv \eta_{p,q}(r,s,\rho,\theta) = \frac{\left(r^2 - sr\right)^p}{\left(r^2 - 2\rho\, r\cos\theta + \rho^2\right)^q}\,. \tag{7}$$

which has the following domain of definition with respect to the *main* radial variable $r$

$$r \in \mathbf{R}_+ \setminus \{0\} : r \neq \begin{cases} \rho \text{ if } \theta = 0,\, \rho \in \mathbf{R}_+ \setminus \{0\} \\ -\rho \text{ if } \theta = \pi,\, \rho \in \mathbf{R}_- \setminus \{0\} \end{cases}. \tag{8}$$

As it said, the *pq*-RF (4) or (7) is a fundamental family of solutions of the following second order *pq*-PDE:

$$\frac{\partial}{\partial r}\left[\frac{\partial w}{\partial r} - \left(\frac{p}{u}\frac{\partial u}{\partial r} - \frac{q}{v}\frac{\partial v}{\partial r}\right)w\right] + \frac{\partial}{\partial s}\left[\frac{\partial w}{\partial s} - \left(\frac{p}{u}\frac{\partial u}{\partial s}\right)w\right] + \frac{\partial}{\partial \rho}\left[\frac{\partial w}{\partial \rho} + \left(\frac{q}{v}\frac{\partial v}{\partial \rho}\right)w\right] +$$

$$\frac{\partial}{\partial \theta}\left[\frac{\partial w}{\partial \theta} + \left(\frac{q}{v}\frac{\partial v}{\partial \theta}\right)w\right] = 0,\quad w = \eta_{p,q}\,,\ u \neq 0,\ v \neq 0\,. \tag{9}$$

Like *pq*-RF, conceptually Eq.(9) is new and has not previously reported in literature since it is characterized by several important properties that will be treated later. The name of *pq*-Radial function or shortly *pq*-RF comes from the specific property of the couple $(p\,,q)$, which plays the role of index in $\eta_{p,q}$ and power in $u^p v^{-q}$. Further, *pq*-RF for well-fixed value of $p$ and $q$ is sometimes called power potential function. Since in our preliminary study the *pq*-RF has graphically studied the evolution of the galactic rotation curves in terms of the general aspect and behavior of the curves themselves, it seems that the same function may be used, under some special conditions, in economic science to investigate the dynamicity or stagnation of markets and the evolution of the investments in the short/medium/long-term, all that with the help of Bayes' theorem and Markov's chain, the economist/investor may be able to evaluate realistically the dynamicity/stagnation and the general evolution of the project. Also, in aerodynamics and fluid mechanics, we can use the *pq*-RF to investigate the evolution of certain dynamic systems that are highly sensitive to any small regular or irregular perturbations particularly when, in the characteristic function (6), the angular parameter $\theta$ becomes function of the form $\theta \equiv \theta(t) = \omega t + \theta_0$.

## 3. Specific Properties of *pq*-RF

In this section we will seek the basic properties of *pq*-RF according to the explicit expression (7). These specific properties may be split into three parts as follows:

I/ *Properties of pq-RF with respect to* $(r\,,\,s,\,\rho,\,\theta)$

1) $\forall p,q,s,\rho \in \mathbf{R}$ and $\rho \neq 0$, we have for $r = 0$: $\eta_{p,q} = 0$.
2) $\forall p,q \in \mathbf{R} \setminus \{0\}$; $\forall \rho,\theta \in \mathbf{R}$ and $\forall s \in \mathbf{R}_+$, we have for $r = s$: $\eta_{p,q} = 0$.

3) $\forall r\in \mathbf{R}_+ \setminus\{0\}$ and $\forall s,\rho,\theta\in\mathbf{R}$, we have for $(p,q)=(0,0)$: $\eta_{0,0}=1$.

4) $\forall p,q,\theta\in\mathbf{R}$, we have for $r=1$, $\rho=0$ and $s=0$: $\eta_{p,q}=1$.

5) *Homogeneity of* $\eta_{p,q}$ *with respect to* $(r,s,\rho)$

$\forall p,q,s,\rho,\theta\in\mathbf{R}$, we have for $\forall r,\sigma\in\mathbf{R}_+\setminus\{0\}$: $\eta_{p,q}(\sigma r,\sigma s,\sigma\rho,\theta)=\sigma^{2(p-q)}\eta_{p,q}(r,s,\rho,\theta)$.

6) *Periodicity with respect to* $\theta$

$\forall p,q,s,\rho,\theta\in\mathbf{R}$ and $\forall r\in\mathbf{R}_+\setminus\{0\}$, we have for $\forall k\in\mathbf{Z}$: $\eta_{p,q}(r,s,\rho,\theta+2k\pi)=\eta_{p,q}(r,s,\rho,\theta)$.

-*Remark*: properties (I.1) and (I.2) are useful particularly for the orthogonality condition of *pq*-RFs as we will see.

II/ *Properties of pq-RF with respect to* $(p,q)$: The following series of specific properties is very important because it shows us how some basic operations performed on *pq*-RFs should reduce to the operations performed on their orders. The demonstration of each property should be exclusively based on the compact expression (4). So we have the subsequent properties for $\forall p,q,s,\rho,\theta\in\mathbf{R}$; $\forall r\in\mathbf{R}_+\setminus\{0\}$; $u\equiv u(r,s)\neq 0$ and $v\equiv v(r,\rho,\theta)\neq 0$:

1) $\eta_{p,q}^{-1}=\eta_{-p,-q}$.

2) $\forall\ell\in\mathbf{N}\setminus\{0\}$: $\eta_{p,q}^{\ell}=\eta_{\ell p,\ell q}=\eta_{\ell(p,q)}$.

3) $\forall\ell,m\in\mathbf{N}\setminus\{0\}$: $\eta_{\ell p,mq}=\eta_{\frac{p}{m},\frac{q}{\ell}}^{\ell\cdot m}$.

4) $\forall\ell,m\in\mathbf{N}\setminus\{0\}$: $\eta_{\frac{p}{\ell},\frac{q}{m}}=\eta_{mp,\ell q}^{(\ell\cdot m)^{-1}}$.

5) $\eta_{-p,q}=u^{-2p}\eta_{p,q}$.

6) $\eta_{p,-q}=v^{2q}\eta_{p,q}$.

7) $\forall p_1,q_1,p_2,q_2\in\mathbf{R}$: $\eta_{p_1,q_1}+\eta_{p_2,q_2}=v^{q_2-q_1}\eta_{p_1,q_2}+v^{q_1-q_2}\eta_{p_2,q_1}$.

8) $\forall p_1,q_1,p_2,q_2\in\mathbf{R}$: $\eta_{p_1,q_1}-\eta_{p_2,q_2}=v^{q_2-q_1}\eta_{p_1,q_2}-v^{q_1-q_2}\eta_{p_2,q_1}$.

9) $\forall p_1,q_1,p_2,q_2\in\mathbf{R}$: $\eta_{p_1,q_1}/\eta_{p_2,q_2}=\eta_{p_1-q_2,q_1-q_2}$.

10) $\forall p_1,q_1,p_2,q_2\in\mathbf{R}$: $\eta_{p_1,q_1}\times\eta_{p_2,q_2}=\eta_{p_1+q_2,q_1+q_2}$.

III/ *Properties of pq-RF with respect to* $(u,v)$: After we have investigated the properties of *pq*-RF with respect to $(r,s,\rho,\theta)$ and $(p,q)$, we now examine the third series of properties relative to the weight and characteristic function of $\eta_{p,q}$. We shall call such properties ‘structural properties’, which are in fact two transformations and one transposition.

1) Transformation $u\to v$ may be performed through the substitution of

$$s=2\rho\cos\theta-\rho^2 r^{-1}$$

in the weight function $u\equiv u(r,s)$ that reduces $\eta_{p,q}$ to $\mu_{p,q}\equiv\mu_{p,q}(r,\rho,\theta)=v^{p-q}$.

2) Transformation $v\to u$ may be performed through the substitution of

$$\rho=r\cos\theta+\sqrt{r^2\cos^2\theta-sr}\quad\text{with}\quad r^2\cos^2\theta-sr>0,$$

in the characteristic function $v \equiv v(r,\rho,\theta)$ that reduces $\eta_{p,q}$ to $\Lambda_{p,q} \equiv \Lambda_{p,q}(r,s) = u^{p-q}$.

3) Transposition $u \leftrightarrow v$ may be performed through the simultaneous substitution of

$$\begin{cases} s = 2\rho\cos\theta - \rho^2 r^{-1} \\ \rho = r\cos\theta + \sqrt{r^2\cos^2\theta - sr} \end{cases},$$

in the weight and characteristic function which reduce $\eta_{p,q}$ to the form $\overline{\eta}_{p,q} = v^p u^{-q}$.

**4. Classification of $\eta_{p,q}$ when $(s,\rho) = (0,0)$**

Here our aim is to illustrate the physical importance of *pq*-RFs through the classification of $\eta_{p,q}$ for the particular case when the two real parameters $s = 0$ and $\rho = 0$. Since $r \in \mathbf{R}_+ \setminus \{0\}$, thus the classification should be split in two parts, that is, when $0 < r < 1$ and when $1 < r < \infty$. It is clear from the explicit expression (7) that for the case when $(s,\rho) = (0,0)$, we get for $\forall p,q,s,\rho,\theta \in \mathbf{R}$:

$$\eta_{p,q}(r,0,0,\theta) = r^{2(p-q)} \quad . \tag{10}$$

This means $\forall\theta \in \mathbf{R}$, $\eta_{p,q}$ becomes independent of $\theta$ when $(s,\rho) = (0,0)$. The specific importance of this property is well reflected by the fact that (10) is a fundamental family of solutions of the two following DEs:

$$\frac{d^2W}{dr^2} - \frac{2(p-q)}{r}\frac{dW}{dr} + \frac{2(p-q)}{r^2}W = 0,\ W = \eta_{p,q}(r,0,0,\theta), \tag{11}$$

Eq.(11) is in reality a special case of *pq*-PDE (9). And the other DE is of the form

$$\frac{d^2H}{dr^2} + \frac{2}{r}\frac{dH}{dr} - \frac{2(p-q)(1+2(p-q))}{r^2}H = 0,\ H = \eta_{p,q}(r,0,0,\theta). \tag{12}$$

Hence from (10), (11) and (12), we can classify $\eta_{p,q}(r,0,0,\theta)$ as follows:

-*First classification*: for the case when $r \in (1,\infty)$, that is when all the points are outside the sphere of unit radius. Therefore $\eta_{p,q}(r,0,0,\theta)$ is
– lower potential if $q > p$,
– higher potential if $q < p$, $\quad p,q \in \mathbf{R}$
– harmonic potential if $q = p + 1/2$.

-*Second classification*: for the case when $r \in (0,1)$, that is when all the points are inside the sphere of unit radius. Therefore $\eta_{p,q}(r,0,0,\theta)$ is
– higher potential if $q > p$,
– lower potential if $q < p$, $\quad p,q \in \mathbf{R}$
– harmonic potential if $q = p + 1/2$.

*Remark*: the two Eqs. (10) and (11) reduce to the same expression only if $q = p + 1$, that is, for the case when $\eta_{p,q}(r,0,0,\theta)$ defines us a lower potential for $r \in (1,\infty)$ or a higher potential for $r \in (0,1)$.

## 5. Orthogonality of *pq*-RFs

The determination of orthogonality condition of *pq*-RFs on the interval $(0,\,s)$ with $s \le 1$, that is when all the points are inside the sphere of unit radius and when *pq*-RFs are independent of $(s,\rho,\theta)$. To this end, we use exclusively the properties (I.1) and (I.2). Let $w_1 = \eta_{p_1,q_1}$ and $w_2 = \eta_{p_2,q_2}$ be two fundamental family of solutions of the following equations

$$\frac{d}{dr}\left[ w_1' - \left( p_1 \frac{u'}{u} - q_1 \frac{v'}{v} \right) w_1 \right] = 0\,, \tag{13}$$

and

$$\frac{d}{dr}\left[ w_2' - \left( p_2 \frac{u'}{u} - q_2 \frac{v'}{v} \right) w_2 \right] = 0\,. \tag{14}$$

With $r \in (0\,,\,s)$; $p_1, q_1, p_2\,, q_2 \in \mathbf{R}$; $(p_1,q_1) \ne (p_2\,,q_2)$; $u \ne 0$ and $v \ne 0$.

Eqs. (13,14) are in fact particular case of *pq*-PDE (9) since, here, as aforementioned *pq*-RFs are independent of $(s,\rho,\theta)$. So integrating Eqs. (13,14) and multiplying them, respectively, by $uw_2$ and $uw_1$, we get

$$u w_2 \left( w_1' - c_1 \right) = \left( p_1 u' - q_1 \frac{v'u}{v} \right) w_1 w_2\,, \tag{15}$$

$$u w_1 \left( w_2' - c_2 \right) = \left( p_2 u' - q_2 \frac{v'u}{v} \right) w_1 w_2\,. \tag{16}$$

Subtracting (16) from (15) and integrating the two sides of the result on $(0\,,s)$, we obtain after omission of integration constants

$$\int_0^s \left[ w_1' \, w_2 - w_2' \, w_1 \right] u\, dr = \left( p_2 - p_1 \right)\left( q_2 - q_1 \right) \int_0^s \left[ \frac{v'}{p_2 - p_1} \, \frac{u}{v} - \frac{u'}{q_2 - q_1} \right] w_1 w_2 dr\,. \tag{17}$$

By taking into account the properties (I.1) and (I.2), and the expression of the weight function (5), the left side in (17) should be equal to zero, consequently, we should have

$$\left( p_2 - p_1 \right)\left( q_2 - q_1 \right) \int_0^s \left[ \frac{v'}{p_2 - p_1} \, \frac{u}{v} - \frac{u'}{q_2 - q_1} \right] w_1 w_2 dr = 0\,. \tag{18}$$

Since $(p_1,q_1) \ne (p_2\,,q_2)$, thus

$$\int_0^s \left[ \frac{v'}{p_2 - p_1} \, \frac{u}{v} - \frac{u'}{q_2 - q_1} \right] w_1 w_2 dr = 0\,. \tag{19}$$

Furthermore, according to property (II.10), we have $w_1 w_2 = \eta_{p_1+p_2,\, q_1+q_2}$, therefore the relation (19) becomes after substitution

$$\int_0^s \left[ \frac{v'}{p_2 - p_1} \, \frac{u}{v} - \frac{u'}{q_2 - q_1} \right] \eta_{p_1+p_2,\, q_1+q_2} dr = 0\,. \tag{20}$$

The relation (20) is exactly the expected orthogonality condition of *pq*-RFs.

## 6. Some physical interpretations of *pq*-RFs

In this section we are interesting in the importance and central role that should be played by *pq*-RFs in physics through the study of the properties of $\eta_{p,q}(r,s,\rho,\theta)$ with respect to $p,q$ and $\rho$. To this aim, it is useful to begin this study by a practical example that is when $(p,q)$ and $\rho$ take, respectively, the following fixed values $(0,1/2)$ and 1. Hence, in such a case, we get after substitution in (7):

$$\eta_{0,1/2}(r,s,1,\theta)=\left(r^2-2r\cos\theta+1\right)^{-1/2}. \tag{21}$$

As we can remark it, *pq*-RF (21) is explicitly independent of the real parameter $s$. Moreover, (21) has two very interesting interpretations, namely, mathematical and physical one. Mathematically (21) is identical to the well known Legendre generating function. Therefore, it follows from (21) that the Legendre generating function is itself a *pq*-RF since it is a particular case of (7). Further, (21) is a special fundamental solution of *pq*-PDE (9), which reduces to

$$\frac{\partial}{\partial r}\left[\frac{\partial w}{\partial r}+\frac{1}{2}\left(\frac{\partial v}{\partial r}\right)\frac{w}{v}\right]+\frac{\partial}{\partial\theta}\left[\frac{\partial w}{\partial\theta}+\frac{1}{2}\left(\frac{\partial v}{\partial\theta}\right)\frac{w}{v}\right]=0\,,\quad w=\eta_{0,1/2}(r,s,1,\theta), \tag{22}$$

Since the Legendre generating function is itself a *pq*-RF of the orders $(0,1/2)$ and $\rho=1$, thus according to the classical theory of special functions, the *pq*-RF (21) may be expanded in the Legendre polynomials as follows

$$\eta_{0,1/2}(r,s,1,\theta)=\left(r^2-2r\cos\theta+1\right)^{-1/2}=\sum_{n=0}^{\infty}P_n(\cos\theta)\,r^n\,,\quad 0<r<1, \tag{23}$$

This expression may be used as a new definition for the Legendre polynomials and in this sense we can affirm that $P_n(\cos\theta)$ is the coefficient of $r^n$ in the expansion of $\eta_{0,1/2}(r,s,1,\theta)$ through Newton's general binomial theorem. Physically, *pq*-RF (21) may be interpreted as the Coulombian potential. Indeed, if for example we put at north pole *N* of the sphere of unit radius a positive charge, and let *M* be a variable point of spherical coordinates $r,\theta,\varphi$. The Coulombian potential of the charge at the point *M* is

$$\frac{1}{d}=\left(r^2-2r\cos\theta+1\right)^{-1/2}. \tag{24}$$

Where $d$ is the relative distance between the charge and the variable point *M*.
*Recall*: the expansion (23) is, of course, valid only for the case when $r\in(0,1)$, that is when all the points are inside the sphere of unit radius. For the points outside this sphere, we would have another expansion. Indeed, when $r\in(1,\infty)$, hence we can rewrite *pq*-RF (23) is as follows

$$\eta_{0,1/2}(r,s,1,\theta)=\frac{1}{r}\left(\left(\frac{1}{r}\right)^2-2\left(\frac{1}{r}\right)\cos\theta+1\right)^{-1/2}. \tag{25}$$

Thus $r^{-1}<1$ such that we can apply the previous expansion, and we find

$$\eta_{0,1/2}(r,s,1,\theta)=\sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{r^{n+1}}\,,\quad 1<r<\infty. \tag{26}$$

*Remark*: Each term of this sum has no any singularity outside the sphere and consequently vanishing at infinity.

Until now, we have only focused our attention on the sphere of unit radius. However, for a sphere of any radius $\rho \in \mathbf{R}_+ \setminus \{0\}$, we are obliged to return to the explicit expression (7) and by applying the transformation $u \to v$, *i.e.*, property (III.1), we get

$$\eta_{p,q}(r,s,\rho,\theta) = \left(r^2 - 2r\cos\theta + \rho^2\right)^{p-q}. \tag{27}$$

From (27) we obtain, for the case when $(p,q) = (0,1/2)$, the following expressions

$$\eta_{0,1/2}(r,s,\rho,\theta) = \sum_{n=0}^{\infty} P_n(\cos\theta)\frac{r^n}{\rho^{n+1}}\ , \quad r < \rho\ , \tag{28}$$

and

$$\eta_{0,1/2}(r,s,\rho,\theta) = \sum_{n=0}^{\infty} P_n(\cos\theta)\frac{\rho^n}{r^{n+1}}\ , \quad r > \rho\ . \tag{29}$$

It is clear that since the beginning our main interest is essentially the investigation of structure, properties and consequences of *pq*-RFs as new class of potential functions that is why, here, we are not particularly concerned with the study of the Legendre polynomials because they are well-known since their introduction in 1784 by the French mathematician A.M. Legendre [7]. Nevertheless, later we will return to these polynomials when we derive the *q*-polynomials $A_n(\cos\theta, q)$. So at present let us show that the Poisson's kernel

$$K(\rho, \phi) = \frac{r^2 - \rho^2}{r^2 - 2r\rho\cos(\varphi - \phi) + \rho^2}\ , \quad \rho < r\ , \tag{30}$$

is a special case of *pq*-RF. To this end, substituting $s = \rho^2 r^{-1}$ and $\theta = \varphi - \phi$ in the explicit expression (7), we get $\eta_{p,q}(r,s,\rho,\theta) \to \overline{K}_{p,q}(\rho,\phi)$:

$$\overline{K}_{p,q}(\rho, \phi) = \frac{\left(r^2 - \rho^2\right)^p}{\left(r^2 - 2r\rho\cos(\varphi - \phi) + \rho^2\right)^q}, \tag{31}$$

which is a generalization of (30), that is, $\overline{K}_{p,q}(\rho, \phi)$ reduces to $K(\rho, \phi)$ when $(p,q) = (1,1)$.

## 7. Consequences of *pq*-RFs

### 7.1. *q-Polynomials*

We have already seen that the Legendre generating function is in fact a special case of *pq*-RFs, now we will show that the Legendre polynomials are also a special case of another kind of polynomials called '*q*-polynomials' which are a direct consequence of *pq*-RFs. The main property that characterizes the *q*-polynomials is that: all *q*-polynomials should reduce to the Legendre polynomials when $q = 1/2$. But before, let us prove more conclusively that the expression (23) is an interesting particular case of another formula more general, namely

$$\eta_{p,q}(r,s,\rho,\theta)=\left(r^2-sr\right)^p\sum_{n=0}^{\infty}\rho^{-(n+2q)}A_n\left(\cos\theta,q\right)r^n\,, \tag{32}$$

with

$$0<r<1;\;\;\theta,p,q\in\mathbf{R};\;\;s,\rho\in\mathbf{R}_+\;\text{ and }\;(s,\rho)\neq(0,0)\,.$$

Returning to the explicit expression (7) and focusing our attention on the characteristic function (6) which may be written as follows

$$v\equiv v(r,\rho,\theta)=r^2-2\rho r\cos\theta+\rho^2=\rho^2\left(1+\xi^2-2\xi\cos\theta\right),\;\;0<\xi=r/\rho<1\,. \tag{33}$$

We have according to the Newton's generalized binomial (theorem) formula

$$\begin{aligned}(1+\varepsilon)^\alpha&=1+\frac{\alpha}{1!}\varepsilon+\frac{\alpha(\alpha-1)}{2!}\varepsilon^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}\varepsilon^3+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}\varepsilon^4+\\&\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{5!}\varepsilon^5+\ldots+\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}\varepsilon^n+\ldots\end{aligned}\;, \tag{34}$$

with $\varepsilon<1$ and $\alpha\in\mathbf{R}$. Putting $\varepsilon=\left(\xi^2-2\xi\cos\theta\right)$ and $\alpha=-q$, we get after substitution in (34):

$$\begin{aligned}\left[1+\left(\xi^2-2\xi\cos\theta\right)\right]^{-q}&=1-\frac{q}{1!}\left(\xi^2-2\xi\cos\theta\right)+\frac{q(q+1)}{2!}\left(\xi^2-2\xi\cos\theta\right)^2+\\&-\frac{q(q+1)(q+2)}{3!}\left(\xi^2-2\xi\cos\theta\right)^3+\frac{q(q+1)(q+2)(q+3)}{4!}\left(\xi^2-2\xi\cos\theta\right)^4+\\&-\frac{q(q+1)(q+2)(q+3)(q+4)}{5!}\left(\xi^2-2\xi\cos\theta\right)^5+\ldots\\&+(-1)^n\frac{q(q+1)\ldots(q+n-1)}{n!}\left(\xi^2-2\xi\cos\theta\right)^n+\ldots\end{aligned}\;. \tag{35}$$

Re-arranging and collecting terms in powers of $\xi$, we have

$$\begin{aligned}\left[1+\left(\xi^2-2\xi\cos\theta\right)\right]^{-q}&=1+\left(\frac{2q}{1!}\cos\theta\right)\xi+\left(\frac{4q(q+1)}{2!}\cos^2\theta-\frac{q}{1!}\right)\xi^2+\\&\left(\frac{8q(q+1)(q+2)}{3!}\cos^3\theta-\frac{4q(q+1)}{2!}\cos\theta\right)\xi^3+\\&\left(\frac{16q(q+1)(q+2)(q+3)}{4!}\cos^4\theta-\frac{12q(q+1)(q+2)}{3!}\cos^2\theta+\frac{q(q+1)}{2!}\right)\xi^4+\\&\left(\frac{32q(q+1)(q+2)(q+3)(q+4)}{5!}\cos^5\theta-\frac{32q(q+1)(q+2)(q+3)}{4!}\cos^3\theta+\frac{6q(q+1)(q+2)}{3!}\cos\theta\right)\xi^5+\ldots\end{aligned} \tag{36}$$

Therefore the coefficients of $\xi$ should take the explicit expressions

$$A_0(\cos\theta, q) = 1;$$
$$A_1(\cos\theta, q) = \frac{2q}{1!}\cos\theta;$$
$$A_2(\cos\theta, q) = \frac{4q(q+1)}{2!}\cos^2\theta - \frac{q}{1!};$$
$$A_3(\cos\theta, q) = \frac{8q(q+1)(q+2)}{3!}\cos^3\theta - \frac{4q(q+1)}{2!}\cos\theta; \qquad (37)$$
$$A_4(\cos\theta, q) = \frac{16q(q+1)(q+2)(q+3)}{4!}\cos^4\theta - \frac{12q(q+1)(q+2)}{3!}\cos^2\theta + \frac{q(q+1)}{2!}$$
$$A_5(\cos\theta, q) = \frac{32q(q+1)(q+2)(q+3)(q+4)}{5!}\cos^5\theta - \frac{32q(q+1)(q+2)(q+3)}{4!}\cos^3\theta + \frac{6q(q+1)(q+2)}{3!}\cos\theta$$
$$\ldots$$

The coefficients $A_n(\cos\theta, q)$ are exactly the expected $q$-polynomials. Further, it is clear that when $q = 1/2$, the $q$-polynomials (37) reduce to those of Legendre, that is

$$A_n(\cos\theta, 1/2) = P_n(\cos\theta). \qquad (38)$$

Thus, now (35) may be written as

$$\left(1 - 2\xi\cos\theta + \xi^2\right)^{-q} = \sum_{n=0}^{\infty} A_n(\cos\theta, q)\xi^n . \qquad (39)$$

Or equivalently

$$\left(r^2 - 2r\rho\cos\theta + \rho^2\right)^{-q} = \sum_{n=0}^{\infty} \rho^{-(n+2q)} A_n(\cos\theta, q) r^n . \qquad (40)$$

In order to arrive at the expected general formula (32), it suffices to multiply the two sides in (40) by $\left(r^2 - sr\right)^p$. We will return to (32) later on.

7.2. *Properties of q-Polynomials*

7.2.1. *Expressions of q-Polynomials for* $\theta = 0$ *and* $\theta = \pi$

Many important properties of $q$-polynomials can be obtained from (39). Here, we derive immediately a few ones as follows. Let $\theta = 0$ in (39), then the left-hand side is

$$(1-\xi)^{-2q} = 1 + \frac{2q}{1!}\xi + \frac{2q(2q+1)}{2!}\xi^2 + \frac{2q(2q+1)(2q+2)}{3!}\xi^3 + \ldots + \frac{2q(2q+1)\ldots(2q+n-1)}{n!}\xi^n + \ldots$$

The right-hand side is

$$A_0(1,q) + A_1(1,q)\xi + A_2(1,q)\xi^2 + A_3(1,q)\xi^3 + \ldots + A_n(1,q)\xi^n + \ldots$$

Comparing the coefficients of $\xi^n$ on both sides we get

$$A_n(1,q)=\frac{2q(2q+1)\ldots(2q+n-1)}{n!}\,. \tag{41}$$

And when we substitute $\theta=\pi$ in (39), we can derive

$$A_n(-1,q)=(-1)^n\,\frac{2q(2q+1)\ldots(2q+n-1)}{n!}\,. \tag{42}$$

7.2.2. *q-Bonnet's recursion formula*

To obtain the recurrence relation or more precisely $q$-Bonnet's recursion formula, first we put $t=\cos\theta$ in (39), we get

$$\left(1-2\xi t+\xi^2\right)^{-q}=\sum_{n=0}^{\infty}A_n(t,q)\xi^n\,. \tag{43}$$

Differentiating (43) with respect to $\zeta$ on both sides and rearranged to obtain

$$\frac{2q(t-\xi)}{\left(1-2\xi t+\xi^2\right)^q}=\left(1-2\xi t+\xi^2\right)\sum_{n=0}^{\infty}n\,A_n(t,q)\xi^{n-1}\,. \tag{44}$$

Replacing the dominator with its definition (43), and equating the coefficients of powers of $\zeta$ in the resulting expansion gives the expected $q$-Bonnet's recursion formula:

$$(n+1)A_{n+1}(t,q)=2(n+q)t\,A_n(t,q)\;-\;(n+2q-1)A_{n-1}(t,q), \tag{45}$$

with

$$A_0(t,q)=1\;\text{ and }\;A_1(t,q)=2qt$$

This relation, along with the first two polynomials $A_0(t,q)$ and $A_1(t,q)$, allows the Legendre Polynomials to be generalized recursively.

7.2.3. *Associated q-functions*

Our purpose here is to show the existence of $q$-functions. As we will see, this kind of functions is a direct consequence of $p$-polynomials. Since we have previously found that the Legendre polynomials $P_n(\cos\theta)$ are special case of $A_n(\cos\theta,q)$ when $q=1/2$; therefore, the $q$-polynomials should conserve all the principal properties of the Legendre polynomials that is why if for example $P_n(\cos\theta)$ are a fundamental solution of the equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\,\frac{d\mathscr{L}}{d\theta}\right]+n(n+1)\mathscr{L}=0\,. \tag{46}$$

Or by substituting $t=\cos\theta$, we get

$$\frac{d}{dt}\left[\left(1-t^2\right)\frac{d\mathscr{L}}{dt}\right]+n(n+1)\mathscr{L}=0\,,\quad \mathscr{L}=P_n(t)\,. \tag{47}$$

Accordingly this implies the $q$-polynomials $A_n(\cos\theta,q) \equiv A_n(t,q)$ should be also a fundamental solution of equation

$$\frac{d}{dt}\left[\left(1-t^2\right)\frac{d\mathscr{H}}{dt}\right]+n(n+1)\mathscr{H} = f_n(t,q), \quad \mathscr{H} = A_n(t,q) \tag{48}$$

It is worthwhile to note that according to the relation (38), Eq.(48) should reduce to the Legendre Eq.(47) when $q = 1/2$, this implies

$$f_n(t,1/2) = 0, \quad \forall n \in \mathbf{N}, \tag{49}$$

-*Result*: It follows from all that the $q$-functions $f_n(t,q)$ are associated to $q$-polynomials $A_n(t,q)$ through Eq.(48) that is why are called '*associated q-functions*'. To illustrate this association, the Table1 below gives us the first few $q$-polynomials and their associated $q$-functions

| $q$-polynomial | associated $q$-function |
|---|---|
| $A_1(t,q)$ | $f_1(t,q) = 0$ |
| $A_2(t,q)$ | $f_2(t,q) = 2q(2q-1)$ |
| $A_3(t,q)$ | $f_3(t,q) = 4q(q+1)(2q-1)t$ |
| $A_4(t,q)$ | $f_4(t,q) = 4q(q+1)(q+2)(2q-1)t^2 - 2q(q+1)(2q-1)$ |
| $A_5(t,q)$ | $f_5(t,q) = \frac{8}{3}q(q+1)(q+2)(q+3)(2q-1)t^3 - 4q(q+1)(q+2)(2q-1)t$ |

**Table 1**: Expressions of the associated $q$-function $f_n(t,q)$, $n = 1, 2 \ldots 5$

7.2.4 *Orthogonality of q-Polynomials*

We have already seen the orthogonality of the $pq$-RFs on the interval $(0, s)$, now we will show the orthogonality of q-polynomials on the interval $(-1, 1)$. With this aim, let $g = A_m(t,q)$ and $h = A_n(t,q)$ then by Eq.(48), we have

$$\frac{d}{dt}\left[\left(1-t^2\right)g'\right] + k_m\, g = f_m, \tag{50}$$

$$\frac{d}{dt}\left[\left(1-t^2\right)h'\right] + k_n\, h = f_n, \tag{51}$$

with

$$f_m \equiv f_m(t,q); \; f_n \equiv f_n(t,q); \; k_m = m(m+1); \; k_n = n(n+1) \text{ and } m \neq n.$$

Multiplying (50) by $h$ and integrating from $t = -1$ to $t = 1$ to obtain

$$\int_{-1}^{1}\frac{d}{dt}\left[\left(1-t^2\right)g'\right]h\,dt + k_m\int_{-1}^{1} g\,h\,dt = \int_{-1}^{1} h\, f_m dt.$$

Integrating the first integral by parts we get

$$\left[\left(1-t^2\right)g'h\right]_{-1}^{1}-\int_{-1}^{1}\left(1-t^2\right)g'h'\,dt+k_m\int_{-1}^{1}g\,h\,dt=\int_{-1}^{1}h\,f_m dt.$$

But since $\left(1-t^2\right)$ is zero both at $t=-1$ and $t=1$ this becomes

$$-\int_{-1}^{1}\left(1-t^2\right)g'h'\,dt+k_m\int_{-1}^{1}g\,h\,dt=\int_{-1}^{1}h\,f_m dt. \tag{52}$$

In exactly the same way we can multiply (51) by $g$ and integrating from $t=-1$ to $t=1$ to obtain

$$-\int_{-1}^{1}\left(1-t^2\right)g'h'\,dt+k_n\int_{-1}^{1}g\,h\,dt=\int_{-1}^{1}g\,f_n dt. \tag{53}$$

Subtracting (53) from (52), we get

$$\left(k_m-k_n\right)\int_{-1}^{1}g\,h\,dt=\int_{-1}^{1}h\,f_m dt-\int_{-1}^{1}g\,f_n dt.$$

Or since $g=A_m(t,q)$; $h=A_n(t,q)$; $f_m\equiv f_m(t,q)$ and $f_n\equiv f_n(t,q)$, hence we find after substitution

$$\left(k_m-k_n\right)\int_{-1}^{1}A_m(t,q)A_n(t,q)dt=\int_{-1}^{1}\left[A_n(t,q)\,f_m(t,q)-A_m(t,q)\,f_n(t,q)\right]dt,$$

this gives us the following expected orthogonality condition

$$\int_{-1}^{1}A_m(t,q)A_n(t,q)\left[1-\frac{1}{k_m-k_n}\left(\frac{f_m(t,q)}{A_m(t,q)}-\frac{f_n(t,q)}{A_n(t,q)}\right)\right]dt=0,\quad m\neq n. \tag{54}$$

According to (38) and (49), we should have $f_m(t,1/2)=f_n(t,1/2)=0$, thus the relation (54) reduces to

$$\int_{-1}^{1}A_m(t,1/2)A_n(t,1/2)dt=0,\quad m\neq n. \tag{55}$$

This coincides with the well-known orthogonality condition for the Legendre polynomials. Besides the important property (54), there is another, namely $\int_{-1}^{1}A_n^2(t,q)dt$, which may be determined as follows: first squaring and integrating (43) from $t=-1$ to $t=1$. Due to orthogonality only the integrals of terms having $A_n^2(t,q)$ survive on the right-hand side. So we have

$$\int_{-1}^{1}\left(1-2\xi t+\xi^2\right)^{-2q}dt=\sum_{n=0}^{\infty}\xi^{2n}\int_{-1}^{1}A_n^2(t,q)dt. \tag{56}$$

For the special case when $q = 1/2$, we have from (56)

$$\frac{1}{\xi}\ln\left(\frac{1+\xi}{1-\xi}\right) = \sum_{n=0}^{\infty}\frac{2\xi^{2n}}{2n+1} = \sum_{n=0}^{\infty}\xi^{2n}\int_{-1}^{1}A_n^2(t,1/2)dt \ . \tag{57}$$

Comparing coefficient of $\xi^{2n}$ we get the important relation

$$\int_{-1}^{1}A_n^2(t,1/2)dt = \frac{2}{2n+1} \, . \tag{58}$$

Relation (58) coincides with the well-known property of the Legendre polynomials. Hence, what we need for the general case is only to put

$$B_n(q) = \int_{-1}^{1}A_n^2(t,q)dt, \ \ q \in \mathbf{R} \, . \tag{59}$$

The formula (59) defines us the polynomials $B_n(q)$ that exclusively depend on the real parameter $q$ . As we will see, $B_n(q)$ are characterized by the following properties:

$$B_0(q) = 2 \, , \quad \forall q \in \mathbf{R} \, , \tag{60}$$

and

$$B_n(0) = 0 \, , \quad \forall n \in \mathbf{N} \, , \ n \neq 0 \ , \tag{61}$$

Expressions of $B_n(q)$ for $n = 0,1,2,3$ :

$$B_0(q) = \int_{-1}^{1}A_0^2(t,q)dt = 2$$

$$B_1(q) = \int_{-1}^{1}A_1^2(t,q)dt = \frac{8}{3}q^2$$

$$B_2(q) = \int_{-1}^{1}A_2^2(t,q)dt = \frac{8}{5}q^2(q+1)^2 - \frac{8}{3}q^2(q+1) + 2q^2$$

$$B_3(q) = \int_{-1}^{1}A_3^2(t,q)dt = \frac{32}{63}q^2(q+1)^2(q+2)^2 - \frac{32}{15}q^2(q+1)^2(q+2) + \frac{8}{3}q^2(q+1)^2$$

7.2.5. *Series of q-Polynomials*

As a direct consequence of the existence of *q*-polynomials we can refer to the series of *q*-polynomials; that is to say any continuous function $g(t)$ such that $-1 < t < 1$, may be expanded in series of *q*-polynomials. More precisely, let us prove that if

$$g(t)=\sum_{k=0}^{\infty} c_k A_k(t,q)\,, \quad -1<t<1\,, \quad \forall q\in \mathbf{R}\,, \tag{62}$$

this implies

$$c_k = B_k^{-1}(q)\int_{-1}^{1} A_k(t,q)g(t)dt. \tag{63}$$

To this end, multiplying the series (62) by $A_n(t,q)$ and integrating from $t=-1$ to $t=1$, and taking into account the previous result, namely formula (59), we get

$$\int_{-1}^{1} A_n(t,q)g(t)dt=\sum_{k=0}^{\infty} c_k \int_{-1}^{1} A_n(t,q)A_k(t,q)g(t)dt,$$

for the case when $n=k$, we have

$$\int_{-1}^{1} A_n(t,q)g(t)dt=c_n \int_{-1}^{1} A_n^2(t,q)dt=c_n\, B_n(q),$$

from where we obtain the very expected formula (63). Furthermore, if we consider the important special case that is when $q=1/2$, we get according to (58), (62) and (63):

$$g(t)=\sum_{k=0}^{\infty} c_k A_k(t,1/2)\,, \quad -1<t<1\,, \tag{64}$$

and

$$c_k = \frac{2k+1}{2}\int_{-1}^{1} A_k(t,1/2)g(t)dt\,. \tag{65}$$

Again, the formulae (64) and (65) coincide with those of Legendre.

## 8. *pq*-Series

Now returning to the general formula (32), which allows us to establish the notion of *pq*-series that may be used to expand any *pq*-RF when $r\in(0,1)$; $p,q,\theta\in \mathbf{R}$; $s,\rho\in \mathbf{R}_+$ and $(s,\rho)\neq(0,0)$. With this aim, let us expand $\left(r^2-sr\right)^p$ using the binomial formula (34) for the case when $r/s<1$, and we find:

$$\left(r^2-sr\right)^p = \sum_{n=0}^{\infty}(-1)^{n+p} s^{p-n}\tbinom{p}{n} r^{n+p}\,, \tag{66}$$

for $n\geq 0$, the symbol $\binom{p}{n}$ is defined by

$$\tbinom{p}{n}=\frac{p(p-1)\ldots(p-n+1)}{n!}\,. \tag{67}$$

Finally, substituting (66) in (32), we obtain the desired *pq*-series

$$\eta_{p,q}(r,s,\rho,\theta)=\sum_{n=0}^{\infty}(-1)^{n+p}\frac{s^{p-n}}{\rho^{n+2q}}\tbinom{p}{n}A_n(\cos\theta,q)r^{2n+p}\,, \tag{68}$$

with

$$r\in(0,1)\,;\ p,q,\theta\in \mathbf{R}\,;\ \ s,\rho\in \mathbf{R}_+ \text{ and } (s,\rho)\neq(0,0)\,.$$

## 9. Properties of *pq*-PDE

In this section, we would examine the properties of *pq*-PDE or more shortly Eq.(9). As it said repetitively, the *pq*-RF (4) or (7) is a fundamental family of solutions of Eq.(9), which conceptually is new and has not previously reported in literature since it is characterized by several properties among them we have:

-*Decomposition*: The structure and intern symmetry of Eq.(9) allows us to split it into a system of four PDEs without using the usual method of separation of variables, the system conserve the same fundamental family of solutions, namely *pq*-RF (4).

-*Homogeneity*: Eq.(9) is homogenous with respect to *pq*-RF. Let $W$ be a certain *pq*-RF whose domain of definition is comparable to that of (4), and let $\zeta$ be any positive real number, such that $\zeta \in \mathbf{R}_+ \setminus \{0\}$ we have $w = \zeta W$. Thus after substitution and simplification, we get

$$\frac{\partial}{\partial r}\left[\frac{\partial W}{\partial r} - \left(\frac{p}{u}\frac{\partial u}{\partial r} - \frac{q}{v}\frac{\partial v}{\partial r}\right)W\right] + \frac{\partial}{\partial s}\left[\frac{\partial W}{\partial s} - \left(\frac{p}{u}\frac{\partial u}{\partial s}\right)W\right] + \frac{\partial}{\partial \rho}\left[\frac{\partial W}{\partial \rho} + \left(\frac{q}{v}\frac{\partial v}{\partial \rho}\right)W\right] +$$

$$\frac{\partial}{\partial \theta}\left[\frac{\partial W}{\partial \theta} + \left(\frac{q}{v}\frac{\partial v}{\partial \theta}\right)W\right] = 0. \tag{69}$$

-*Permutability*: If we perform the permutation of the orders $p$ and $q$ in Eq.(9), we should have $w = \eta_{q,p}$ as a fundamental family of solutions, furthermore since the general form and structure of the resulting equation after permutation $(p,q) \to (q,p)$ are not basically different from those of Eq.(9), except of course the permutation of the orders, hence in this sense the *permutability* may be defined as a sort of symmetry through permutation.

## 10. Structure of *pq*-PDE

Finally, we shall focus our attention exclusively on the structure of Eq.(9) by considering the very important case when *pq*-RF is independent of the real parameters $s, \rho$ and $\theta$. Hence, for such an independence, Eq.(9) becomes

$$\frac{d}{dr}\left[w' - \left(p\frac{u'}{u} - q\frac{v'}{v}\right)w\right] = 0 \,, w = \eta_{p,q} \,,\ u \neq 0 \text{ and } v \neq 0. \tag{70}$$

In the context of the present work, we call Eq.(70) '*pq*-Radial Differential Equation' or shortly *pq*-RDE. We remark from Eq.(70) that the orders, the weight function, the characteristic function and their derivatives are essential elements that entering in the structure of this equation. This allows us to say that the investigation of Eq.(70) is completely depending on those mentioned elements as we shall see.

### 10.1. *Relationship between pq-RDE and Fuchs' class*

Our aim here is to prove that under some conditions relative to very interesting particular cases, Eq.(70) belongs to Fuchs' class. For this purpose considering the following cases:

1) When $p \neq 0$ and $q = 0$, Eq.(70) takes the form

$$\frac{d}{dr}\left[w' - p\left(\frac{u'}{u}\right)w\right] = 0\,,\ w = \eta_{p,0}\ . \tag{71}$$

Or more explicitly

$$w'' - p\left(\frac{u'}{u}\right)w' - p\left(\frac{u''}{u} - \frac{u'^2}{u^2}\right)w = 0\,. \tag{72}$$

Anyone familiarized with the equations of Fuchs' class can immediately affirm that Eq.(72) is really belonging to Fuchsian class since its variable coefficients satisfying Fuchs' condition, and according to the explicit expression of the weight function (5), namely $u \equiv u(r,s) = r^2 - sr$, Eq.(72) has three regular singular points: $r = 0,\ r = s$ and $r = \infty$ .

2) When $p = 0$ and $q \neq 0$, Eq.(70) takes the form

$$w'' + q\left(\frac{v'}{v}\right)w' + q\left(\frac{v''}{v} - \frac{v'^2}{v^2}\right)w = 0\,. \tag{73}$$

Also, the variable coefficients of Eq.(73) satisfying Fuchs' condition, and according to the explicit expression of the characteristic function (6), namely $v \equiv v(r,\rho,\theta) = r^2 - 2\rho r\cos\theta + \rho^2$, Eq.(73) has three regular singular points: $r = \rho$ for $\theta = 0$, $\rho \in \mathbf{R}_+ \setminus \{0\}$; $r = -\rho$ for $\theta = \pi$, $\rho \in \mathbf{R}_- \setminus \{0\}$ and $r = \infty$.

3) When $(p,q) \neq (0,0)$ and $u \to v$. To arrive at this important particular case, we must take into consideration the property (III.1) explicitly $u \to v \Rightarrow \eta_{p,q} \to \mu_{p,q} = v^{p-q}$. Obviously, this means $u \equiv v$. Now supposing $\mu_{p,q}$ is independent of $\rho$ and $\theta$, thus when $w = \mu_{p,q}$, Eq.(70) reduces to

$$w'' + (q-p)\left(\frac{v'}{v}\right)w' + (q-p)\left(\frac{v''}{v} - \frac{v'^2}{v^2}\right)w = 0\,. \tag{74}$$

As we can remark it easily, Eq.(74) has three regular singular points similar to those of Eq.(73).

4) when $(p,q) \neq (0,0)$ and $v \to u$. To arrive at this important particular case, we must take into account the property (III.2) explicitly $v \to u \Rightarrow \eta_{p,q} \to \Lambda_{p,q} = u^{p-q}$. Clearly, this means $v \equiv u$. Now supposing $\Lambda_{p,q}$ is independent of $s$ thus when $w = \mu_{p,q}$, Eq.(70) reduces to

$$w'' + (q-p)\left(\frac{u'}{u}\right)w' + (q-p)\left(\frac{u''}{u} - \frac{u'^2}{u^2}\right)w = 0\,, \tag{75}$$

which is manifestly comparable to Eq.(72), thus Eq.(75) has three regular singular points: $r = 0,\ r = s$ and $r = \infty$ .

10.2. *Relationship between pq-RDE and DE of Sturm-Liouville form*

After we have proven that *pq*-RDE (70) belongs to Fuchsian class under some well-established conditions, at present we will show that the same equation may be written in classical form of Sturm-Liouville DE, particularly, when its spectral (eigenvalue) $\lambda = 1$, and when the orders $(p,q) = (-1,1)$ for Eq.(70). First, let us write the classical form of Sturm-Liouville DE for the real radial variable $r \in (0,1)$:

$$\frac{d}{dr}\left[\alpha(r)R'\ \right]+\left[\lambda\beta(r)-\gamma(r)\right]R=0\,,\quad R\equiv R(r),\ \ \alpha(r)>0\,. \tag{76}$$

Considering the very important case when $\lambda=1$ and $R\equiv R(r)$ is supposed a radial function in the classical sense. Hence, we have after substitution, differentiation and rearrangement:

$$R''+\frac{\alpha'(r)}{\alpha(r)}R'+\frac{1}{\alpha(r)}\left[\beta(r)-\gamma(r)\right]R=0\,. \tag{77}$$

Concerning Eq.(70), we have for the case $(p,q)=(-1,1)$:

$$w''+\left(\frac{u'}{u}+\frac{v'}{v}\right)w'+\left[\left(\frac{u''}{u}-\frac{u'^2}{u^2}\right)+\left(\frac{v''}{v}-\frac{v'^2}{v^2}\right)\right]w=0\,. \tag{78}$$

Or equivalently

$$w''+\left(\frac{u'v+v'u}{uv}\right)w'+\frac{1}{uv}\left[\left(u''v+v''u\right)-\left(\frac{u'^2v}{u}-\frac{v'^2u}{v}\right)\right]w=0\,,\ \ uv\neq 0\,. \tag{79}$$

A simple comparison between (77) and (79) allows us to write the latter in the following form

$$\frac{d}{dr}\left[(uv)w'\right]+\left[\left(u''v+v''u\right)-\left(\frac{u'^2v}{u}-\frac{v'^2u}{v}\right)\right]w=0\,, \tag{80}$$

which is exactly the expected classical form of Sturm-Liouville DE for the case when $\lambda=1$. Moreover, if we take into account the previous result we find that the variable coefficients of Eq.(80) do not justify Fuchs' condition therefore Eq.(80) does not belong to Fuchsian class. In this sense, we call Eq.(80) '*pq*-RDE in Sturm-Liouville form for the case $\lambda=1$ and $(p,q)=(-1,1)$'.

-*Question*: From all that we arrive at the central question that arises in the context of *pq*-RDEs is how we can prove if there is some relationship between the DEs of Fuchsian class and the DEs of Sturm-Liouville form in spite of their quite distinct structures. From the previous result concerning the structure of *pq*-RDEs that belonging to Fuchsian class and Eq.(80), we begin to answer this question as follows: the above-mentioned relationship may be really exist through *pq*-RDEs only if $(p,q)=(-1,0)$ or $(p,q)=(0,1)$ and $\lambda=1$. Indeed, for the case when $(p,q)=(-1,0)$, Eq.(70) reduces to

$$\frac{d}{dr}\left[w'+\left(\frac{u'}{u}\right)w\right]=0\,,\ \ w=\eta_{-1,0}\,. \tag{81}$$

Or more explicitly

$$w''+\left(\frac{u'}{u}\right)w'+\left(\frac{u''}{u}-\frac{u'^2}{u^2}\right)w=0\,. \tag{82}$$

It is clear from the expression of Eq.(82), which is also an important special case of Eq.(72) when $p=-1$, therefore it follows that the variable coefficients of Eq.(82) satisfying Fuchs' condition and the equation has three regular singular points similar to those of Eq.(72). Furthermore, the structure of Eq.(82) allows us to write in Sturm-Liouville form for the case when $\lambda=1$:

$$\frac{d}{dr}(u\,w')+\left(u''-\frac{u'^2}{u}\right)w=0\,. \tag{83}$$

Eq.(83) is precisely the first answer to our question relating to relationship between the DEs of Fuchsian class and the DEs of Sturm-Liouville form. The second answer comes from the case when $(p,q)=(0,1)$, thus Eq.(70) reduces to

$$\frac{d}{dr}\left[w'+\left(\frac{v'}{v}\right)w\right]=0\,,\, w=\eta_{0,1}\,. \tag{84}$$

Or more explicitly

$$w''+\left(\frac{v'}{v}\right)w'+\left(\frac{v''}{v}-\frac{v'^2}{v^2}\right)w=0\,. \tag{85}$$

Eq.(85) is also an important special case of Eq.(73) when $q=1$. Hence, it follows that the variable coefficients of (85) satisfying Fuchs' condition and consequently the equation has three regular singular points similar to those of Eq.(73). Moreover, the structure of (85) permits us to write in Sturm-Liouville form for the case when $\lambda=1$:

$$\frac{d}{dr}(v\,w')+\left(v''-\frac{v'^2}{v}\right)w=0\,. \tag{86}$$

Eq.(86) is exactly the second answer to our question. So, after we have found the relationship between the DEs of Fuchsian class and the DEs of Sturm-Liouville form in spite of their quite distinct structures, we end this section with the investigation of the structural properties of *pq*-RDE.

10.3. *Structural properties of pq-RDE*

The main purpose behind the study of the structural properties of *pq*-RDE is to show the existence of some reciprocal properties that characterize, at the same time, the structure of *pq*-RF and its *pq*-RDE. Hence, we shall return to the Eq.(70), which has in reality three families of solutions, namely:

$$w_1=\eta_{p,q}\,,\;\; w_2=w_1\int w_1^{-1}dr\,,\; w_3=c_1w_1+c_2w_2\,,\;\; c_1,c_2\in\mathbf{R}\,. \tag{87}$$

In order to make the understanding of the study more easy let us, first, show that these solutions (87) and their Eq.(70) are themselves special case. With this aim, let $\ell\in\mathbf{N}\setminus\{0\}$, the specific property (II.2) allows us to write

$$w_1^\ell=\eta_{\ell p,\ell q}\,,\; w_2^\ell=w_1^\ell\left(\int w_1^{-1}dr\right)^\ell\,,\; w_3^\ell=(c_1w_1+c_2w_2)^\ell\,. \tag{88}$$

Since the families of solution (87) are special case of (88) when $\ell=1$, it follows from this that the solutions (88) themselves should be families of solutions of the following *pq*-RDE

$$\frac{d}{dr}\left[w'-\ell\left(p\frac{u'}{u}-q\frac{v'}{v}\right)w\right]=0\,. \tag{89}$$

The mutual presence of the parameter $\ell$ in the solutions (88) and their Eq.(89) defines us, in this sense, the structural properties of *pq*-RDE. Indeed, like its solutions, Eq.(89) reduce to (70) when $\ell=1$.
If presently we suppose that $\ell$ is not *fixed* in such a case $w$ is not simply a fundamental family of solutions but it is more compactly a system of fundamental families of solutions defined by finite summation

$$w = \sum_{\ell=1}^{n} H_\ell\,, \quad H_\ell \equiv \mu_{\ell p,\ell q} = \mu_{\ell(p,q)}\,, \tag{90}$$

therefore, Eq.(89) becomes, after substitution and rearrangement, a system of *pq*-RDEs

$$\frac{d}{dr}\left\{ \sum_{\ell=1}^{n}\left[ H'_\ell - \ell\left( p\frac{u'}{u} - q\frac{v'}{v} \right) H_\ell \right] \right\} = 0. \tag{91}$$

Recall that until now the orders $(p,q)$ are always considered as fixed real numbers, however, if hereafter are supposed to be non fixed positive integers that is $p,q \in \mathbf{N}$, in such a case we can distinguish two systems of fundamental families of solutions defined as a finite sum.

Case 1: $\ell \in \mathbf{N}\setminus\{0\};\ p,q \in \mathbf{N}$ and $p > q$:

$$w = \sum_{\ell=1}^{n}\sum_{p>q}^{n} \eta_{\ell(p,q)} = \eta_{1(1,0)} + \eta_{2(2,1)} + \eta_{3(3,2)} + \ldots + \eta_{n(n,n-1)}\,, \tag{92}$$

and its system of *pq*-RDEs takes the form

$$\frac{d}{dr}\left\{ \sum_{\ell=1}^{n}\sum_{p>q}^{n}\left[ \eta'_{\ell(p,q)} - \ell\left( p\frac{u'}{u} - q\frac{v'}{v} \right) \eta_{\ell(p,q)} \right] \right\} = 0\,. \tag{93}$$

Case 2: $\ell \in \mathbf{N}\setminus\{0\};\ p,q \in \mathbf{N}$ and $p < q$:

$$w = \sum_{\ell=1}^{n}\sum_{p<q}^{n} \eta_{\ell(p,q)} = \eta_{1(0,1)} + \eta_{2(1,2)} + \eta_{3(2,3)} + \ldots + \eta_{n(n-1,n)}\,, \tag{94}$$

and its corresponding system of *pq*-RDEs takes the form

$$\frac{d}{dr}\left\{ \sum_{\ell=1}^{n}\sum_{p<q}^{n}\left[ \eta'_{\ell(p,q)} - \ell\left( p\frac{u'}{u} - q\frac{v'}{v} \right) \eta_{\ell(p,q)} \right] \right\} = 0\,. \tag{95}$$

Finally, let us examine the following two structural properties, that is, when the orders $p$ and $q$ do not appear explicitly in the resulting system of *p*q-RDEs but instead we shall find $n(n+1)/2$ and $n(n-1)/2$, respectively. This property should, of course, occur when $w$ is supposed to be a system of fundamental families of solutions defined as a finite product of the form

$$w = \begin{cases} \prod_{p>q}^{n} \eta_{p,q}\,, & q = 0 \\ \prod_{p<q}^{n} \eta_{p,q}\,, & p = 0 \end{cases}\,. \tag{96}$$

To this end, focusing our attention slightly on (96), and noting that with the help of specific property (II.10), we can prove that for the case when $p > q$, we get

$$w = \prod_{p>q}^{n} \eta_{p,q} = \eta_{\frac{n}{2}(n+1,n-1)} \ , \tag{97}$$

this solves

$$\frac{d}{dr}\left[ w' - \frac{n}{2}\left( (n+1)\frac{u'}{u} - (n-1)\frac{v'}{v} \right) w \right] = 0 , \quad w = \eta_{\frac{n}{2}(n+1,n-1)} \ , \tag{98}$$

and for the case when $p < q$, we obtain

$$w = \prod_{p<q}^{n} \eta_{p,q} = \eta_{\frac{n}{2}(n-1,n+1)} \ , \tag{99}$$

which solves

$$\frac{d}{dr}\left[ w' - \frac{n}{2}\left( (n-1)\frac{u'}{u} - (n+1)\frac{v'}{v} \right) w \right] = 0 , \quad w = \eta_{\frac{n}{2}(n-1,n+1)} \ . \tag{100}$$

Hence, Eqs.(98) and (100) define us two systems of *pq*-RDEs when $p$ and $q$ are non fixed positive integers and $w$ is defined by (96). Furthermore, as it was already mentioned, the different structural properties of Eq.(70) as a system of *pq*-RDEs depend exclusively on the expressions of *pq*-RF and *vice versa*. Also, the specific properties (II.2) and (II.10) of *pq*-RF have played a central role.

## 11. Conclusion:

In this paper, we have heuristically developed a theory based exclusively on the concept of *pq*-RFs which is in fact a direct consequence of our preliminary investigation on the ‘hypothetical’ dark matter and its gravitodynamical effects on the ordinary matter. We have studied the specific properties of *pq*-RFs and the structural properties of their *pq*-PDE, which, to our knowledge have not previously been reported in the literature.

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