Inconsistent countable set in second order ZFC and Nonexistence of the Strongly Inaccessible Cardinals. Consistency Results in Topology.

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Abstract: In this article we proved so-called strong reflection principles

corresponding to formal theories Th which have omega-models. An posible generalization of the Lob's theorem is considered.Main results are:
(i) $\neg Con(ZFC_2)$, (ii) let k be an inaccessible cardinal then $\neg Con(ZFC + \exists\kappa)$.

Keywords: Gödel encoding, Completion of ZFC_2, Russell's paradox, ω-model, Henkin semantics, full second-order semantic,strongly inaccessible cardinal

1.Introduction.

Let us remind that accordingly to naive set theory, any definable collection is a set. Let R be the set of all sets that are not members of themselves. If R qualifies as a member of itself, it would contradict its own definition as a set containing all sets that are not members of themselves. On the other hand, if such a set is not a member of itself, it would qualify as a member of itself by the same definition. This contradiction is Russell's paradox. In 1908, two ways of avoiding the paradox were proposed, Russell's type theory and the Zermelo set theory, the first constructed axiomatic set theory. Zermelo's axioms went well beyond Frege's axioms of extensionality and unlimited set abstraction, and evolved into the now-canonical Zermelo–Fraenkel set theory ZFC. *"But how do we know that ZFC is a consistent theory, free of contradictions? The short answer is that we don't; it is a matter of faith (or of skepticism)"*— E.Nelson wrote in his not published paper [1]. However, it is deemed unlikely that even ZFC_2 which is a very stronger than ZFC harbors an unsuspected contradiction; it is widely believed that if ZFC_2 were inconsistent, that fact would have been uncovered by now. This much is certain — ZFC_2 is immune to the classic paradoxes of naive set theory: Russell's paradox, the Burali-Forti paradox, and Cantor's paradox.

Remark 1.1. Note that in this paper we view the second order set theory ZFC_2 under the Henkin semantics [2],[3] and under the full second-order semantics [4],[5].Thus we interpret the wff's of ZFC_2 language with the full second-order semantics as required in [4],[5].

Designation 1.1. We will be denote by ZFC_2^{Hs} set theory ZFC_2 with the Henkin semantics and we will be denote by ZFC_2^{fss} set theory ZFC_2 with the full second-order semantics.

Remark 1.2.There is no completeness theorem for second-order logic with the full second-order semantics. Nor do the axioms of ZFC_2^{fss} imply a reflection principle which ensures that if a sentence Z of second-order set theory is true, then it is true in some (standard or nonstandard) model $M^{ZFC_2^{fss}}$ of ZFC_2^{fss} [5]. Let Z be the conjunction of all the axioms of ZFC_2^{fss}. We assume now that: Z is true,i.e. $Con\left(ZFC_2^{fss}\right)$. It is known that the existence of a model for Z requires the existence of strongly inaccessible cardinals, i.e. under ZFC it can be shown that κ is a strongly inaccessible if and only if $(H_\kappa, \in)$ is a model of ZFC_2^{fss}. Thus $\neg Con(ZFC_2^{fss}) \Rightarrow \neg Con(ZFC + \exists\kappa))$. In this paper we prove that $ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$ and ZFC_2^{fss} is inconsistent.

Remark 1.3.We remind that in Henkin semantics, each sort of second-order variable has a particular domain of its own to range over, which may be a proper subset of all sets or functions of that sort. Leon Henkin (1950) defined these semantics and proved that Gödel's completeness theorem and compactness theorem, which hold for first-order logic, carry over to second-order logic with Henkin semantics. This is because Henkin semantics are almost identical to many-sorted first-order semantics, where additional sorts of variables are added to simulate the new variables of second-order logic. Second-order logic with Henkin semantics is not more expressive than first-order logic. Henkin semantics are commonly used in the study of second-order arithmetic.Väänänen [11] argued that the choice between Henkin models and full models for second-order logic is analogous to the choice between ZFC and V as a basis for set theory: "As with second-order logic, we cannot really choose whether we axiomatize mathematics using V or ZFC. The result is the same in both cases, as ZFC is the best attempt so far to use V as an axiomatization of mathematics."

We will start from a simple naive consideration. Let $\Im$ be the countable collection of all sets X such that $ZFC_2^{Hs} \vdash \exists! X\Psi(X)$,where $\Psi(X)$ is an 1-place open wff i.e.,

$$\forall Y\{Y \in \Im \leftrightarrow ZFC_2^{Hs} \vdash \exists\Psi(\cdot)\exists! X[\Psi(X) \wedge Y = X]\}. \tag{1.1}$$

Let $X \notin_{\vdash_{ZFC_2^{Hs}}} Y$ be a predicate such that $X \notin_{\vdash_{ZFC_2^{Hs}}} Y \leftrightarrow ZFC_2^{Hs} \vdash X \notin Y$. Let $\Re$ be the countable collection of all sets such that

$$\forall X\left[X \in \Re \leftrightarrow X \notin_{\vdash_{ZFC_2^{Hs}}} X\right]. \tag{1.2}$$

From (1.2) one obtain

$$\mathfrak{R} \in \mathfrak{R} \leftrightarrow \mathfrak{R} \notin_{\vdash_{ZFC_2^{Hs}}} \mathfrak{R}. \tag{1.3}$$

But obviously this is a contradiction. However contradiction (1.3) it is not a contradiction inside ZFC_2^{Hs} for the reason that predicate $X \notin_{\vdash_{ZFC_2^{Hs}}} Y$ not is a predicate of ZFC_2^{Hs} and therefore countable collections $\mathfrak{I}$ and $\mathfrak{R}$ not is a sets of ZFC_2^{Hs}. Nevertheless by using Gödel encoding the above stated contradiction can be shipped in special consistent completion of ZFC_2^{Hs}.

Remark 1.4.More formally I can to explain the gist of the deriveded in this paper contradiction (see Proposition 2.5) is as follows. Let M be a full model of ZFC_2^{Hs}. Let $\mathfrak{I}$ be the set of the all sets of M provably definable in ZFC_2^{Hs}, and let $\mathfrak{R}$ $= \{x \in \mathfrak{I} : \Box(x \notin x)\}$ where $\Box A$ means 'sentence A derivable in ZFC_2^{Hs}', or some appropriate modification thereof. We replace now (1.1) by

$$\forall Y\{Y \in \mathfrak{I} \leftrightarrow \Box\exists\Psi(\bullet)\exists!X[\Psi(X) \wedge Y = X]\}. \tag{1.4}$$

Assume that $ZFC_2^{Hs} \vdash \mathfrak{R} \in \mathfrak{I}$. Then, we have that $\mathfrak{R} \in \mathfrak{R}$ if and only if $\Box(\mathfrak{R} \notin \mathfrak{R})$, which immediately gives us $\mathfrak{R} \in \mathfrak{R}$ if and only if $\mathfrak{R} \notin \mathfrak{R}$. We choose now $\Box A$ in the following form

$$\Box A \triangleq Bew(\#A) \wedge [Bew(\#A) \Rightarrow A]. \tag{1.5}$$

Here $Bew(\#A)$ is a canonycal Gödel formula which says to us that there exist proof in ZFC_2^{Hs} of the formula A with Gödel number $\#A$.

Remark 1.5. Notice that definition (1.5) holds as definition of predicate really asserting
provability in ZFC_2^{Hs}.

Remark 1.6.In additional ander assumption $Con(\mathbf{Th}_1^{\#})$, we establish an countable sequence $ZFC_2^{Hs} = \mathbf{Th}_1^{\#} \subsetneqq \ldots \subsetneqq \mathbf{Th}_i^{\#} \subsetneqq \mathbf{Th}_{i+1}^{\#} \subsetneqq \ldots \mathbf{Th}_{\infty}^{\#}$, where:

(i) $\mathbf{Th}_{i+1}^{\#}$is an finite consistent extension of the $\mathbf{Th}_i^{\#}$,

(ii) $\mathbf{Th}_{\infty}^{\#} = \bigcup_{i \in \mathbb{N}} \mathbf{Th}_i^{\#}$

(iii) $\mathbf{Th}_{\infty}^{\#}$ proves the all sentences of the $\mathbf{Th}_1^{\#}$, which valid in M,i.e.,$M \models A \Rightarrow \mathbf{Th}_{\infty}^{\#} \vdash A$,

see Proposition 2.1.

Remark 1.7.Let $\mathfrak{I}_i, i = 1,2,\ldots$ be the set of the all sets of M provably definable in $\mathbf{Th}_i^{\#}$,

$$\forall Y\{Y \in \mathfrak{I}_i \leftrightarrow \Box_i\exists\Psi(\bullet)\exists!X[\Psi(X) \wedge Y = X]\}. \tag{1.6}$$

and let $\mathfrak{R}_i = \{x \in \mathfrak{I}_i : \Box_i(x \notin x)\}$ where $\Box_i A$ means 'sentence A derivable in $\mathbf{Th}_i^{\#}$.Then, we have that $\mathfrak{R}_i \in \mathfrak{R}_i$ if and only if $\Box_i(\mathfrak{R}_i \notin \mathfrak{R}_i)$, which immediately gives us $\mathfrak{R}_i \in \mathfrak{R}_i$ if and only if $\mathfrak{R}_i \notin \mathfrak{R}_i$.We choose now $\Box_i A, i = 1,2,\ldots$ in the following form

$$\Box_i A \triangleq Bew_i(\#A) \wedge [Bew_i(\#A) \Rightarrow A]. \tag{1.7}$$

Here $Bew_i(\#A), i = 1,2,\ldots$ is a canonycal Gödel formulae which says to us that there exist proof in $\mathbf{Th}_i^{\#}, i = 1,2,\ldots$ of the formula A with Gödel number $\#A$.

Remark 1.8. Notice that definitions (1.7) holds as definitions of predicates really asserting provability in $\mathbf{Th}_i^{\#}, i = 1,2,\ldots$.

Remark 1.9.Of course the all theories $\mathbf{Th}_i^{\#}, i = 1,2,\ldots$ are inconsistent,see Proposition 2.10.

Remark 1.10.Let $\Im_\infty$ be the set of the all sets of M provably definable in $\mathbf{Th}_\infty^{\#}$,

$$\forall Y\{Y \in \Im_\infty \leftrightarrow \Box_\infty \exists \Psi(\bullet) \exists! X[\Psi(X) \wedge Y = X]\}. \tag{1.8}$$

and let $\Re_\infty = \{x \in \Im_\infty : \Box_\infty(x \notin x)\}$ where $\Box_\infty A$ means ‘sentence A derivable in $\mathbf{Th}_\infty^{\#}$. Then, we have that $\Re_\infty \in \Re_\infty$ if and only if $\Box_\infty(\Re_\infty \notin \Re_\infty)$, which immediately gives us $\Re_\infty \in \Re_\infty$ if and only if $\Re_\infty \notin \Re_\infty$. We choose now $\Box_\infty A, i = 1,2,\ldots$ in the following form

$$\Box_\infty A \triangleq \exists i[Bew_i(\#A) \wedge [Bew_i(\#A) \Rightarrow A]]. \tag{1.9}$$

Remark 1.11.Notice that definition (1.9) holds as definition of an predicate really asserting provability in $\mathbf{Th}_\infty^{\#}$. Of course theory $\mathbf{Th}_\infty^{\#}$ also is inconsistent,see Proposition 2.14.

Remark 1.12.Notice that under intuitive and naive consideration the set $\Im_\infty$ can be defined directly using a truth predicate,which of couse is not available in the language of ZFC_2^{Hs} by well-known Tarski’s undefinability theorem:Let $\mathbf{Th}_{\mathcal{L}}^{Hs}$ be second order theory with Henkin semantics and formal language $\mathcal{L}$, which includes negation and has a Gödel numbering $g(x)$ such that for every $\mathcal{L}$-formula $A(x)$ there is a formula B such that $B \leftrightarrow A(g(B))$ holds.

Assume that $\mathbf{Th}_{\mathcal{L}}^{Hs}$ has an standard Model M. Let T^* be the set of Gödel numbers of $\mathcal{L}$-sentences true in M. Then there is no $\mathcal{L}$-formula $\mathbf{True}(n)$ (truth predicate) which defines T^*. That is, there is no $\mathcal{L}$-formula $\mathbf{True}(n)$ such that for every $\mathcal{L}$-formula A,

$$\mathbf{True}(g(A)) \iff A \tag{1.10}$$

holds.Thus under naive definition of the set $\Im_\infty$ Tarski’s undefinability theorem blocking the biconditional $\Re_\infty \in \Re_\infty \iff \Re_\infty \notin \Re_\infty$.

Remark 1.12.In this paper we define the set $\Im_\infty$ using generalized truth predicate $\mathbf{True}_\infty^{\#}(g(A),A)$ such that

$$\begin{gathered}\mathbf{True}_\infty(g(A),A) \iff \exists i[Bew_i(\#A) \wedge [Bew_i(\#A) \Rightarrow A]] \iff \\ \mathbf{True}_\infty(g(A)) \wedge [\mathbf{True}_\infty(g(A)) \Rightarrow A] \iff A, \\ \mathbf{True}_\infty(g(A)) \iff \exists i Bew_i(\#A).\end{gathered} \tag{1.11}$$

holds.Thus in contrast with naive definition of the sets $\Im_\infty$ and $\Re_\infty$ there is no any problem

which arises from Tarski's undefinability theorem.

Remark 1.13.In order to prove that set theory $ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$ is inconsistent without any

refference to the set $\Im_\infty$,notice that by the properties of the extension $\mathbf{Th}_\infty^\#$ follows that

definition given by (1.11) is correct, i.e.,for every ZFC_2^{Hs}-formula Φ such that $M^{ZFC_2^{Hs}} \models \Phi$

the following equivalence $A \Leftrightarrow \mathbf{True}_\infty(g(A),A)$ holds.

Proposition 1.1.(Generalized Tarski's undefinability theorem) (see Proposition2.30).Let

$\mathbf{Th}_{\mathcal{L}}^{Hs}$ be second order theory with Henkin semantics and with formal language $\mathcal{L}$, which

includes negation and has a Gödel encoding $g(\cdot)$ such that for every $\mathcal{L}$-formula $A(x)$ there

is a formula B such that $B \Leftrightarrow A(g(B)) \wedge [A(g(B) \Rightarrow B]$ holds. Assume that $\mathbf{Th}_{\mathcal{L}}^{Hs}$ has an

standard Model M. Then there is no $\mathcal{L}$-formula $\mathbf{True}(n)$ such that for every $\mathcal{L}$-formula A

such that $M \models A$, the following equivalence

$$A \Leftrightarrow \mathbf{True}(g(A)) \wedge [\mathbf{True}(g(A)) \Rightarrow A] \tag{1.12}$$

holds.

Proposition 1.2.Set theory $\mathbf{Th}_{\mathbf{1}}^\# = ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$ is inconsistent,(see Proposition 2.31).

Proof.Notice that by the properties of the extension $\mathbf{Th}_\infty^\#$ of the theory$\mathbf{Th}_{\mathbf{1}}^\#$ follows that

$$M^{ZFC_2^{Hs}} \models \Phi \Rightarrow \mathbf{Th}_\infty^\# \vdash \Phi. \tag{1.13}$$

Therefore (1.11) gives generalized "truth predicate" for set theory $\mathbf{Th}_{\mathbf{1}}^\#$.By Proposition

1.1 one obtains a contradiction.

Remark 1.14.We note that in order to deduce $\sim Con(ZFC_2^{Hs})$ from $Con(ZFC_2^{Hs})$ by using Gödel encoding, one needs something more than the consistency of ZFC_2^{Hs}, e.g., that ZFC_2^{Hs} has an omega-model $M_\omega^{ZFC_2^{Hs}}$ or an standard model $M_{\mathbf{st}}^{ZFC_2^{Hs}}$ i.e., a model in which the *integers are the standard integers* [6].To put it another way, why should we believe a statement just because there's a ZFC_2^{Hs}-proof of it? It's clear that if ZFC_2^{Hs} is inconsistent, then we won't believe ZFC_2^{Hs}-proofs. What's slightly more subtle is that the mere consistency of ZFC_2 isn't quite enough to get us to believe arithmetical theorems of ZFC_2^{Hs}; we must also believe that these arithmetical theorems are asserting something about the standard naturals. It is "conceivable" that ZFC_2^{Hs} might be consistent but that the only nonstandard models $M_{\mathbf{Nst}}^{ZFC_2^{Hs}}$ it has are those in which the integers are nonstandard, in which case we might not "believe" an arithmetical statement such as "ZFC_2^{Hs} is inconsistent" even if there is a ZFC_2^{Hs}-proof of it.

Remark 1.5. However assumption $\exists M_{\mathbf{st}}^{ZFC_2^{Hs}}$ is not necessary. Note thatin any nonstandard model $M_{\mathbf{Nst}}^{Z_2^{Hs}}$ of the second-order arithmetic Z_2^{Hs} the terms $\overline{0}$, $\mathbf{S}\overline{0} = \overline{1}, \mathbf{SS}\overline{0} = \overline{2}, \ldots$ comprise the initial segment isomorphic to $M_{\mathbf{st}}^{Z_2^{Hs}} \subset M_{\mathbf{Nst}}^{Z_2^{Hs}}$. This initial segment is called the standard cut of the $M_{\mathbf{Nst}}^{Z_2^{Hs}}$. The order type of any nonstandard model of $M_{\mathbf{Nst}}^{Z_2^{Hs}}$ is equal to $\mathbb{N} + A \times \mathbb{Z}$ for some linear order A [6],[7]. Thus one can to choose Gödel encoding inside $M_{\mathbf{st}}^{Z_2^{Hs}}$.

Remark 1.6. However there is no any problem as mentioned above in second order set theory ZFC_2 with the full second-order semantics becouse corresponding second order arithmetic Z_2^{fss} is categorical.

Remark 1.7. Note if we view second-order arithmetic Z_2 as a theory in first-order predicate calculus. Thus a model M^{Z_2} of the language of second-order arithmetic Z_2 consists of a set M (which forms the range of individual variables) together with a constant 0 (an element of M), a function S from M to M, two binary operations $+$ and $\times$ on M, a binary relation $<$ on M, and a collection D of subsets of M, which is the range of the set variables. When D is the full powerset of M, the model M^{Z_2} is called a full model. The use of full second-order semantics is equivalent to limiting the models of second-order arithmetic to the full models. In fact, the axioms of second-order arithmetic have only one full model. This follows from the fact that the axioms of Peano arithmetic with the second-order induction axiom have only one model under second-order semantics, i.e. Z_2, with the full semantics, is categorical by Dedekind's argument, so has only one model up to isomorphism. When M is the usual set of natural numbers with its usual operations, M^{Z_2} is called an ω-model. In this case we may identify the model with D, its collection of sets of naturals, because this set is enough to completely determine an ω-model. The unique full omega-model $M_{\omega}^{Z_2^{fss}}$, which is the usual set of natural numbers with its usual structure and all its subsets, is called the intended or standard model of second-order arithmetic.

Main results are: $\neg Con(ZFC_2^{Hs} + \exists(\omega\text{-model of } ZFC_2^{Hs})), \neg Con(ZFC_2^{fss})$.

2.Derivation inconsistent countable set in $ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$.

Remark 2.1.In this section we use second-order arithmetic Z_2^{Hs} with first-order semantics. Notice that any standard model $M_{st}^{Z_2^{Hs}}$ of second-order arithmetic Z_2^{Hs} consists of a set $\mathbb{N}$ of usual natural numbers (which forms the range of individual variables) together with a constant 0 (an element of $\mathbb{N}$), a function S from $\mathbb{N}$ to $\mathbb{N}$, two binary operations $+$ and $\cdot$ on $\mathbb{N}$, a binary relation $<$ on $\mathbb{N}$, and a collection $D \subseteq 2^{\mathbb{N}}$ of subsets of $\mathbb{N}$, which is the range of the set variables. Omitting D produces a model of first order Peano arithmetic.

When $D = 2^{\mathbb{N}}$ is the full powerset of $\mathbb{N}$, the model $M_{st}^{Z_2^{Hs}}$ is called a full model. The use of full second-order semantics is equivalent to limiting the models of second-order arithmetic to the full models. In fact, the axioms of second-order arithmetic Z_2^{fss} have only one full model. This follows from the fact that the axioms of Peano arithmetic with the second-order induction axiom have only one model under second-order semantics, see section 3.

Let **Th** be some fixed, but unspecified, consistent formal theory. For later convenience, we assume that the encoding is done in some fixed formal second order theory **S** and that **Th** contains **S**.We assume throuht this paper that formal second order theory **S** has an ω-model $M_\omega^{\mathbf{S}}$.The sense in which **S** is contained in **Th** is better exemplified than explained: if **S** is a formal system of a second order arithmetic Z_2^{Hs} and **Th** is, say, ZFC_2^{Hs}, then **Th** contains **S** in the sense that there is a well-known embedding, or interpretation, of **S** in **Th**. Since encoding is to take place in $M_\omega^{\mathbf{S}}$, it will have to have a large supply of constants and closed terms to be used as codes. (e.g. in formal arithmetic, one has $\overline{0}, \overline{1}, \ldots$.) **S** will also have certain function symbols to be described shortly.To each formula, Φ, of the language of **Th** is assigned a closed term, $[\Phi]^c$, called the code of Φ. We note that if $\Phi(x)$ is a formula with free variable x, then $[\Phi(x)]^c$ is a closed term encoding the formula $\Phi(x)$ with x viewed as a syntactic object and not as a parameter. Corresponding to the logical connectives and quantifiers are function symbols, $neg(\bullet)$, $imp(\bullet)$, etc., such that, for all formulae $\Phi, \Psi : \mathbf{S} \vdash$ $neg([\Phi]^c) = [\neg\Phi]^c$, $\mathbf{S} \vdash imp([\Phi]^c, [\Psi]^c) = [\Phi \to \Psi]^c$ etc. Of particular importance is the substitution operator, represented by the function symbol $sub(\bullet, \bullet)$. For formulae $\Phi(x)$, terms t with codes $[t]^c$:

$$\mathbf{S} \ \vdash sub([\Phi(x)]^c, [t]^c) = [\Phi(t)]^c. \tag{2.1}$$

It well known [8] that one can also encode derivations and have a binary relation $\mathbf{Prov_{Th}}(x,y)$ (read "x proves y " or "x is a proof of y") such that for closed t_1, t_2 : $\mathbf{S}$ $\vdash \mathbf{Prov_{Th}}(t_1, t_2)$ iff t_1 is the code of a derivation in **Th** of the formula with code t_2 . It follows that

$$\mathbf{Th} \vdash \Phi \text{ iff } \mathbf{S} \vdash \mathbf{Prov_{Th}}(t, [\Phi]^c) \tag{2.2}$$

for some closed term t. Thus one can define

$$\mathbf{Pr_{Th}}(y) \leftrightarrow \exists x \mathbf{Prov_{Th}}(x,y), \tag{2.3}$$

and therefore one obtain a predicate asserting provability. We note that is not always the case that [8]:

$$\mathbf{Th} \vdash \Phi \text{ iff } \mathbf{S} \vdash \mathbf{Pr_{Th}}([\Phi]^c), \tag{2.4}$$

unless $\mathbf{S}$ is fairly sound,e.g. this is a case when $\mathbf{S}$ and $\mathbf{Th}$ replaced by $\mathbf{S}_\omega = \mathbf{S} \upharpoonright M_\omega^{\mathbf{Th}}$ and $\mathbf{Th}_\omega = \mathbf{Th} \upharpoonright M_\omega^{\mathbf{Th}}$ correspondingly (see Designation 2.1).

Remark 2.2.Noticee that is always the case that:

$$\mathbf{Th}_{\omega} \vdash \Phi_{\omega} \text{ iff } \mathbf{S}_{\omega} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega}}([\Phi_{\omega}]^{c}), \tag{2.5}$$

i.e. that is the case when predicate $\mathbf{Pr}_{\mathbf{Th}_{\omega}}(y), y \in M_{\omega}^{\mathbf{Th}}$:

$$\mathbf{Pr}_{\mathbf{Th}_{\omega}}(y) \leftrightarrow \exists x(x \in M_{\omega}^{\mathbf{Th}})\mathbf{Prov}_{\mathbf{Th}_{\omega}}(x,y) \tag{2.6}$$

really asserts provability.

It well known [8] that the above encoding can be carried out in such a way that the following important conditions $\mathbf{D}1, \mathbf{D}2$ and $\mathbf{D}3$ are meet for all sentences [8]:

$$\begin{gathered}\mathbf{D}1.\, \mathbf{Th} \vdash \Phi \text{ implies } \mathbf{S} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^{c}),\\ \mathbf{D}2.\, \mathbf{S} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^{c}) \to \mathbf{Pr}_{\mathbf{Th}}([\mathbf{Pr}_{\mathbf{Th}}([\Phi]^{c})]^{c}),\\ \mathbf{D}3.\, \mathbf{S} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^{c}) \wedge \mathbf{Pr}_{\mathbf{Th}}([\Phi \to \Psi]^{c}) \to \mathbf{Pr}_{\mathbf{Th}}([\Psi]^{c}).\end{gathered} \tag{2.7}$$

Conditions $\mathbf{D}1, \mathbf{D}2$ and $\mathbf{D}3$ are called the Derivability Conditions.

Remark 2.3.From (2.5)-(2.6) follows that

$$\begin{gathered}\mathbf{D}4.\, \mathbf{Th}_{\omega} \vdash \Phi \text{ iff } \mathbf{S}_{\omega} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega}}([\Phi_{\omega}]^{c}),\\ \mathbf{D}5.\, \mathbf{S}_{\omega} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega}}([\Phi_{\omega}]^{c}) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\omega}}([\mathbf{Pr}_{\mathbf{Th}_{\omega}}([\Phi_{\omega}]^{c})]^{c}),\\ \mathbf{D}6.\, \mathbf{S}_{\omega} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega}}([\Phi_{\omega}]^{c}) \wedge \mathbf{Pr}_{\mathbf{Th}_{\omega}}([\Phi_{\omega} \to \Psi_{\omega}]^{c}) \to \mathbf{Pr}_{\mathbf{Th}_{\omega}}([\Psi_{\omega}]^{c}).\end{gathered} \tag{2.8}$$

Conditions $\mathbf{D}4, \mathbf{D}5$ and $\mathbf{D}6$ are called the Strong Derivability Conditions.

Definition 2.1. Let Φ be well formed formula (wff) of $\mathbf{Th}$. Ten wff Φ is called $\mathbf{Th}$-sentence iff it has no free variables.

Designation 2.1.(i) Assume that a theory $\mathbf{Th}$ has an ω-model $M_{\omega}^{\mathbf{Th}}$ and Φ is an $\mathbf{Th}$-sentence, then:

$\Phi_{M_{\omega}^{\mathbf{Th}}} \triangleq \Phi \upharpoonright M_{\omega}^{\mathbf{Th}}$ (we will write Φ_{ω} instead $\Phi_{M_{\omega}^{\mathbf{Th}}}$) is a $\mathbf{Th}$-sentence Φ with all quantifiers relativized to ω-model $M_{\omega}^{\mathbf{Th}}$ [13],[23] and

$\mathbf{Th}_{\omega} \triangleq \mathbf{Th} \upharpoonright M_{\omega}^{\mathbf{Th}}$ is a theory $\mathbf{Th}$ relativized to model $M_{\omega}^{\mathbf{Th}}$, i.e., any $\mathbf{Th}_{\omega}$-sentence has the form Φ_{ω} for some $\mathbf{Th}$-sentence Φ.

(ii) Assume that a theory $\mathbf{Th}$ has an non-standard model $M_{Nst}^{\mathbf{Th}}$ and Φ is an $\mathbf{Th}$-sentence, then:

$\Phi_{M_{Nst}^{\mathbf{Th}}} \triangleq \Phi \upharpoonright M_{Nst}^{\mathbf{Th}}$ (we will write Φ_{Nst} instead $\Phi_{M_{Nst}^{\mathbf{Th}}}$) is a $\mathbf{Th}$-sentence with all quantifiers relativized to non-standard model $M_{Nst}^{\mathbf{Th}}$, and

$\mathbf{Th}_{Nst} \triangleq \mathbf{Th} \upharpoonright M_{Nst}^{\mathbf{Th}}$ is a theory $\mathbf{Th}$ relativized to model $M_{Nst}^{\mathbf{Th}}$, i.e., any $\mathbf{Th}_{Nst}$-sentence has a form Φ_{Nst} for some $\mathbf{Th}$-sentence Φ.

(iii) Assume that a theory $\mathbf{Th}$ has an model $M^{\mathbf{Th}}$ and Φ is an and Φ is an $\mathbf{Th}$-sentence, then:

$\Phi_{M^{\mathbf{Th}}}$ is a $\mathbf{Th}$-sentence with all quantifiers relativized to model $M^{\mathbf{Th}}$, and

$\mathbf{Th}_{M}$ is a theory $\mathbf{Th}$ relativized to model $M_{M}^{\mathbf{Th}}$, i.e., any $\mathbf{Th}_{M}$-sentence has a form Φ_{M}

for

some $\mathbf{Th}$-sentence Φ.

Designation 2.2. (i) Assume that a theory $\mathbf{Th}$ has an ω-model $M_\omega^{\mathbf{Th}}$ and there exist $\mathbf{Th}$-sentence denoted by $Con(\mathbf{Th};M_\omega^{\mathbf{Th}})$ asserting that $\mathbf{Th}$ has a model $M_\omega^{\mathbf{Th}}$;

(ii) Assume that a theory $\mathbf{Th}$ has an non-standard model $M_{Nst}^{\mathbf{Th}}$ and there exist $\mathbf{Th}$-sentence denoted by $Con(\mathbf{Th};M_{Nst}^{\mathbf{Th}})$ asserting that $\mathbf{Th}$ has a non-standard model $M_{Nst}^{\mathbf{Th}}$;

(iii) Assume that a theory $\mathbf{Th}$ has an model $M^{\mathbf{Th}}$ and there exist $\mathbf{Th}$-sentence denoted by $Con(\mathbf{Th};M^{\mathbf{Th}})$ asserting that $\mathbf{Th}$ has a model $M^{\mathbf{Th}}$;

Remark 2.4. It is well known that there exist an ZFC-sentence $Con(ZFC;M^{ZFC})$ [21],[22].

Obviously there exist an ZFC_2^{Hs}-sentence $Con\left(ZFC_2^{Hs};M^{ZFC_2^{Hs}}\right)$ and there exist an Z_2^{Hs}-sentence $Con\left(Z_2^{Hs};M^{Z_2^{Hs}}\right)$.

Designation 2.3. Let $Con(\mathbf{Th})$ be the formula:

$$\begin{cases} Con(\mathbf{Th}) \triangleq \\ \forall t_1(t_1 \in M_\omega^{\mathbf{Th}})\forall t_1'(t_1' \in M_\omega^{\mathbf{Th}})\forall t_2(t_2 \in M_\omega^{\mathbf{Th}})\forall t_2'(t_2' \in M_\omega^{\mathbf{Th}}) \\ \neg[\mathbf{Prov_{Th}}(t_1,[\Phi]^c) \wedge \mathbf{Prov_{Th}}(t_2,neg([\Phi]^c))], \\ \\ t_1' = [\Phi]^c, t_2' = neg([\Phi]^c) \\ \text{or} \\ Con(\mathbf{Th}) \triangleq \\ \forall\Phi\forall t_1(t_1 \in M_\omega^{\mathbf{Th}})\forall t_2(t_2 \in M_\omega^{\mathbf{Th}})\neg[\mathbf{Prov_{Th}}(t_1,[\Phi]^c) \wedge \mathbf{Prov_{Th}}(t_2,neg([\Phi]^c))] \end{cases} \quad (2.9)$$

and where t_1,t_1',t_2,t_2' is a closed term.

Lemma 2.1. (**I**) Assume that: (i) $Con(\mathbf{Th};M^{\mathbf{Th}})$,(ii) $M^{\mathbf{Th}} \vDash Con(\mathbf{Th})$ and (iii) $\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\Phi]^c)$,where Φ is a closed formula.Then $\mathbf{Th} \nvdash \mathbf{Pr_{Th}}([\neg\Phi]^c)$,

(**II**) Assume that: (i) $Con(\mathbf{Th};M_\omega^{\mathbf{Th}})$ (ii) $M_\omega^{\mathbf{Th}} \vDash Con(\mathbf{Th})$ and (iii) $\mathbf{Th}_\omega \vdash \mathbf{Pr}_{\mathbf{Th}_\omega}([\Phi_\omega]^c)$, where

Φ_ω is a closed formula.Then $\mathbf{Th}_\omega \nvdash \mathbf{Pr}_{\mathbf{Th}_\omega}([\neg\Phi_\omega]^c)$.

Proof. (**I**) Let $Con_{\mathbf{Th}}(\Phi)$ be the formula :

$$\begin{cases} Con_{\mathbf{Th}}(\Phi) \triangleq \\ \forall t_1(t_1 \in M_\omega^{\mathbf{Th}})\forall t_2(t_2 \in M_\omega^{\mathbf{Th}})\neg[\mathbf{Prov_{Th}}(t_1,[\Phi]^c) \wedge \mathbf{Prov_{Th}}(t_2,neg([\Phi]^c))], \\ \forall t_1(t_1 \in M_\omega^{\mathbf{Th}})\forall t_2(t_2 \in M_\omega^{\mathbf{Th}})\neg[\mathbf{Prov_{Th}}(t_1,[\Phi]^c) \wedge \mathbf{Prov_{Th}}(t_2,neg([\Phi]^c))]\leftrightarrow \\ \leftrightarrow\{\neg\exists t_1(t_1 \in M_\omega^{\mathbf{Th}})\neg\exists t_2(t_2 \in M_\omega^{\mathbf{Th}})[\mathbf{Prov_{Th}}(t_1,[\Phi]^c) \wedge \mathbf{Prov_{Th}}(t_2,neg([\Phi]^c))]\}. \end{cases} \quad (2.10)$$

where t_1,t_2 is a closed term. From (i)-(ii) follows that theory $\mathbf{Th} + Con(\mathbf{Th})$ is consistent. We note that $\mathbf{Th} + Con(\mathbf{Th}) \vdash Con_{\mathbf{Th}}(\Phi)$ for any closed Φ. Suppose that

$\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\neg\Phi]^c)$, then (iii) gives

$$\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\Phi]^c) \wedge \mathbf{Pr_{Th}}([\neg\Phi]^c). \tag{2.11}$$

From (2.3) and (2.11) we obtain

$$\exists t_1 \exists t_2 [\mathbf{Prov_{Th}}(t_1, [\Phi]^c) \wedge \mathbf{Prov_{Th}}(t_2, neg([\Phi]^c))]. \tag{2.12}$$

But the formula (2.10) contradicts the formula (2.12). Therefore $\mathbf{Th} \nvdash \mathbf{Pr_{Th}}([\neg\Phi]^c)$.

(**II**) This case is trivial becourse formula $\mathbf{Pr}_{\mathbf{Th}_\omega}([\neg\Phi_\omega]^c)$ by the Strong Derivability Condition $\mathbf{D}4$,see formulae (2.8), really asserts provability of the $\mathbf{Th}_\omega$-sentence $\neg\Phi_\omega$. But this is a contradiction.

Lemma 2.2. (**I**) Assume that: (i) $Con(\mathbf{Th}; M^{\mathbf{Th}})$, (ii) $M^{\mathbf{Th}} \models Con(\mathbf{Th})$ and (iii) $\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\neg\Phi]^c)$, where Φ is a closed formula.Then $\mathbf{Th} \nvdash \mathbf{Pr_{Th}}([\Phi]^c)$,

(**II**) Assume that: (i) $Con(\mathbf{Th}; M_\omega^{\mathbf{Th}})$ (ii) $M_\omega^{\mathbf{Th}} \models Con(\mathbf{Th})$ and (iii) $\mathbf{Th}_\omega \vdash \mathbf{Pr}_{\mathbf{Th}_\omega}([\neg\Phi_\omega]^c)$, where Φ_ω is a closed formula.Then $\mathbf{Th}_\omega \nvdash \mathbf{Pr}_{\mathbf{Th}_\omega}([\Phi_\omega]^c)$.

Proof. Similarly as Lemma 2.1 above.

Example 2.1. (i) Let $\mathbf{Th} = \mathbf{PA}$ be Peano arithmetic and $\Phi \Leftrightarrow 0 = 1$. Then obviously by Löbs theorem $\mathbf{PA} \vdash \mathbf{Pr_{PA}}(0 \neq 1)$, and therefore $\mathbf{PA} \nvdash \mathbf{Pr_{PA}}(0 = 1)$.

(ii) Let $\mathbf{PA}^{\clubsuit} = \mathbf{PA} + \neg Con(\mathbf{PA})$ and $\Phi \Leftrightarrow 0 = 1$. Then obviously by Löbs theorem

$$\mathbf{PA}^{\clubsuit} \vdash \mathbf{Pr}_{\mathbf{PA}^{\clubsuit}}(0 \neq 1),$$

and therefore

$$\mathbf{PA}^{\clubsuit} \nvdash \mathbf{Pr}_{\mathbf{PA}^{\clubsuit}}(0 = 1).$$

However

$$\mathbf{PA}^{\clubsuit} \vdash [\mathbf{Pr_{PA}}(0 \neq 1)] \wedge [\mathbf{Pr_{PA}}(0 = 1)].$$

Remark 2.5.Notice that there is no standard model of $\mathbf{PA}^{\clubsuit}$.

Assumption 2.1. Let $\mathbf{Th}$ be an second order theory with the Henkin semantics. We assume now that:

(i) the language of $\mathbf{Th}$ consists of:

numerals $\bar{0}, \bar{1}, ...$

countable set of the numerical variables: $\{v_0, v_1, \ldots\}$

countable set $\mathcal{F}$ of the set variables: $\mathcal{F} = \{x, y, z, X, Y, Z, \Re, \ldots\}$

countable set of the n-ary function symbols: $f_0^n, f_1^n, \ldots$

countable set of the n-ary relation symbols: $R_0^n, R_1^n, \ldots$

connectives: $\neg, \rightarrow$

quantifier: $\forall$.

(ii) $\mathbf{Th}$ contains ZFC_2,

(iii) $\mathbf{Th}$ has an an ω-model $M_\omega^{\mathbf{Th}}$ or

(iv) $\mathbf{Th}$ has an nonstandard model $M_{Nst}^{\mathbf{Th}}$.

Definition 2.1. An $\mathbf{Th}$-wff Φ (well-formed formula Φ) is closed - i.e. Φ is a sentence - if it

has no free variables; a wff is open if it has free variables.We'll use the slang

'k-place
open wff ' to mean a wff with k distinct free variables.

Definition 2.2.We will say that,$\mathbf{Th}_{\infty}^{\#}$ is a nice theory or a nice extension of the $\mathbf{Th}$ if:
(i) $\mathbf{Th}_{\infty}^{\#}$ contains $\mathbf{Th}$;
(ii) Let Φ be any closed formula of $\mathbf{Th}$, then $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$ implies $\mathbf{Th}_{\infty}^{\#} \vdash \Phi$;
(iii) Let Φ_{∞} be any closed formula of $\mathbf{Th}_{\infty}^{\#}$, then $M_{\omega}^{\mathbf{Th}} \models \Phi_{\infty}$ implies $\mathbf{Th}_{\infty}^{\#} \vdash \Phi_{\infty}$,i.e.
$Con(\mathbf{Th} + \Phi_{\infty}; M_{\omega}^{\mathbf{Th}})$ implies $\mathbf{Th}_{\infty}^{\#} \vdash \Phi_{\infty}$.

Remark 2.6.Notice that formulas $Con(\mathbf{Th} + \Phi_{\infty}; M_{\omega}^{\mathbf{Th}})$ and $Con(\mathbf{Th}_{\infty}^{\#} + \Phi_{\infty}; M_{\omega}^{\mathbf{Th}})$ is expressible in $\mathbf{Th}_{\infty}^{\#}$.

Definition 2.3.Fix an classical propositional logic L. Recall that a set Δ of wff's is said to
be L -consistent, or consistent for short, if $\Delta \nvdash \bot$ and there are other equivalent formulations of consistency:(1) Δ is consistent, (2) $\mathbf{Ded}(\Delta) := \{A \mid \Delta \vdash A\}$ is not the set
of all wff's,(3) there is a formula such that $\Delta \nvdash A$.(4) there are no formula A such that $\Delta \vdash A$ and $\Delta \vdash \neg A$.

We will say that,$\mathbf{Th}_{\infty}^{\#}$ is a maximally nice theory or a maximally nice extension of the $\mathbf{Th}$ iff
$\mathbf{Th}_{\infty}^{\#}$ is consistent and for any consistent nice extension $\mathbf{Th}_{\infty}^{\#'}$ of the $\mathbf{Th}$:
$\mathbf{Ded}(\mathbf{Th}_{\infty}^{\#}) \subseteq \mathbf{Ded}(\mathbf{Th}_{\infty}^{\#'})$ implies $\mathbf{Ded}(\mathbf{Th}_{\infty}^{\#}) = \mathbf{Ded}(\mathbf{Th}_{\infty}^{\#'})$.

Remark 2.7. We note that a theory $\mathbf{Th}_{\infty}^{\#}$ depend on model $M_{\omega}^{\mathbf{Th}}$ or $M_{Nst}^{\mathbf{Th}}$, i.e.
$\mathbf{Th}_{\infty}^{\#} = \mathbf{Th}_{\infty}^{\#}[M_{\omega}^{\mathbf{Th}}]$ or $\mathbf{Th}_{\infty}^{\#} = \mathbf{Th}_{\infty}^{\#}[M_{Nst}^{\mathbf{Th}}]$ correspondingly. We will consider now the case
$\mathbf{Th}_{\infty}^{\#} \triangleq \mathbf{Th}_{\infty}^{\#}[M_{\omega}^{\mathbf{Th}}]$ without loss of generality.

Remark 2.8. Notice that in order to prove the statement:$\neg Con(ZFC_2^{Hs}; M_{\omega}^{\mathbf{Th}})$,
Proposition 2.1 is not necessary, see Proposition 2.18.

Proposition 2.1.(**Generalized Lobs Theorem**) (**I**) Assume that (i) $Con(\mathbf{Th})$ (see 2.9) and
(ii) $\mathbf{Th}$ has an ω-model $M_{\omega}^{\mathbf{Th}}$.Then theory $\mathbf{Th}$ can be extended to a maximally consistent
nice theory $\mathbf{Th}_{\infty}^{\#} \triangleq \mathbf{Th}_{\infty}^{\#}[M_{\omega}^{\mathbf{Th}}]$.
(**II**) Assume that (i) $Con(\mathbf{Th})$ and (ii) $\mathbf{Th}$ has an ω-model $M_{\omega}^{\mathbf{Th}}$.Then theory
$\mathbf{Th}_{\omega}$ can be extended to a maximally consistent nice theory $\mathbf{Th}_{\omega}^{\#} \triangleq \mathbf{Th}_{\omega}^{\#}[M_{\omega}^{\mathbf{Th}}]$.

Proof.(**I**) Let $\Phi_1 \ldots \Phi_i \ldots$ be an enumeration of all closed wff's of the theory $\mathbf{Th}$ (this can be achieved if the set of propositional variables can be enumerated). Define a chain $\wp = \{\mathbf{Th}_i^{\#} | i \in \mathbb{N}\}, \mathbf{Th}_1^{\#} = \mathbf{Th}$ of consistent theories inductively as follows: assume that theory $\mathbf{Th}_i^{\#}$ is defined.

(i) Suppose that the statement (2.13) is satisfied

$$\left[\mathbf{Th}_i^{\#} \nvdash \mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\Phi_i]^c)\right] \wedge [\mathbf{Th}_i^{\#} \nvdash \Phi_i] \text{ and } M_{\omega}^{\mathbf{Th}} \models \Phi_i. \tag{2.13}$$

Then we define a theory $\mathbf{Th}^{\#}_{i+1}$ as follows $\mathbf{Th}^{\#}_{i+1} \triangleq \mathbf{Th}^{\#}_{i} \cup \{\Phi_i\}$. We will rewrite the condition
(2.13) using predicate $\mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{i+1}}(\cdot)$ symbolically as follows:

$$\begin{cases} \mathbf{Th}^{\#}_{i+1} \vdash \mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{i+1}}([\Phi_i]^c), \\ \mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{i+1}}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{i}}([\Phi_i]^c) \wedge [M^{\mathbf{Th}}_{\omega} \models \Phi_i], \\ M^{\mathbf{Th}}_{\omega} \models \Phi_i \Leftrightarrow Con(\mathbf{Th}^{\#}_{\mathbf{i}} + \Phi_i; M^{\mathbf{Th}}_{\omega}), \\ \text{i.e.} \\ \mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{i+1}}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{i}}([\Phi_i]^c) \wedge Con(\mathbf{Th}_i + \Phi_i; M^{\mathbf{Th}}_{\omega}), \\ \mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{i+1}}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{i+1}}([\Phi_i]^c), \\ \mathbf{Pr}_{\mathbf{Th}^{\#}_{i+1}}([\Phi_i]^c) \Rightarrow \Phi_i, \\ \mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{i+1}}([\Phi_i]^c) \Rightarrow \Phi_i. \end{cases} \tag{2.14}$$

(ii) Suppose that the statement (2.15) is satisfied

$$\left[\mathbf{Th}^{\#}_{i} \nvdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{i}}([\neg\Phi_i]^c)\right] \wedge [\mathbf{Th}^{\#}_{i} \nvdash \neg\Phi_i] \text{ and } M^{\mathbf{Th}}_{\omega} \models \neg\Phi_i. \tag{2.15}$$

Then we define a theory $\mathbf{Th}^{\#}_{i+1}$ as follows $\mathbf{Th}^{\#}_{i+1} \triangleq \mathbf{Th}^{\#}_{i} \cup \{\Phi_i\}$. We will rewrite the condition
(2.15) using predicate $\mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{i+1}}(\cdot)$, symbolically as follows:

$$\begin{cases} \mathbf{Th}^{\#}_{i+1} \vdash \mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{i+1}}([\neg\Phi_i]^c), \\ \mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{i+1}}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{i}}([\neg\Phi_i]^c) \wedge [M^{\mathbf{Th}}_{\omega} \models \neg\Phi_i], \\ M^{\mathbf{Th}}_{\omega} \models \neg\Phi_i \Leftrightarrow Con(\mathbf{Th}^{\#}_{i} + \neg\Phi_i; M^{\mathbf{Th}}_{\omega}), \\ \text{i.e.} \\ \mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{i+1}}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{i}}([\neg\Phi_i]^c) \wedge Con(\mathbf{Th}_i + \neg\Phi_i; M^{\mathbf{Th}}_{\omega}), \\ \mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{i+1}}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{i+1}}([\neg\Phi_i]^c), \\ \mathbf{Pr}_{\mathbf{Th}^{\#}_{i+1}}([\neg\Phi_i]^c) \Rightarrow \neg\Phi_i, \\ \mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{i+1}}([\Phi_i]^c) \Rightarrow \neg\Phi_i. \end{cases} \tag{2.16}$$

(iii) Suppose that the statement (2.17) is satisfied

$$\mathbf{Th}^{\#}_{i} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{i}}([\Phi_i]^c) \text{ and } [\mathbf{Th}^{\#}_{i} \nvdash \Phi_i] \wedge [M^{\mathbf{Th}}_{\omega} \models \Phi_i]. \tag{2.17}$$

Then we define a theory $\mathbf{Th}^{\#}_{i+1}$ as follows $\mathbf{Th}^{\#}_{i+1} \triangleq \mathbf{Th}^{\#}_{i} \cup \{\Phi_i\}$. Using Lemma 2.1and predicate $\mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{i+1}}(\cdot)$, we will rewrite the condition (2.17) symbolically as follows:

$$
\begin{cases}
\mathbf{Th}_{i+1}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{i+1}^{\#}}^{\#}([\Phi_i]^c), \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^{\#}}^{\#}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\Phi_i]^c) \wedge [M_\omega^{\mathbf{Th}} \models \Phi_i], \\
M_\omega^{\mathbf{Th}} \models \Phi_i \Leftrightarrow Con(\mathbf{Th}_i^{\#}+\Phi_i; M_\omega^{\mathbf{Th}}), \\
\text{i.e.} \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^{\#}}^{\#}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\Phi_i]^c) \wedge Con(\mathbf{Th}_i+\Phi_i; M_\omega^{\mathbf{Th}}), \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^{\#}}^{\#}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{i+1}^{\#}}([\Phi_i]^c), \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^{\#}}([\Phi_i]^c) \Rightarrow \Phi_i, \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^{\#}}^{\#}([\Phi_i]^c) \Rightarrow \Phi_i.
\end{cases}
\qquad (2.18)
$$

Remark 2.9.Notice that predicate $\mathbf{Pr}_{\mathbf{Th}_{i+1}^{\#}}^{\#}([\Phi_i]^c)$ is expressible in $\mathbf{Th}_i^{\#}$ because $\mathbf{Th}_i^{\#}$ is a finite extension of the recursive theory $\mathbf{Th}$ and $Con(\mathbf{Th}_\mathbf{i}^{\#}+\Phi_i; M^{\mathbf{Th}}) \in \mathbf{Th}_i^{\#}$.

(iv) Suppose that a statement (2.19) is satisfied

$$
\mathbf{Th}_i^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\neg\Phi_i]^c) \text{ and } [\mathbf{Th}_i^{\#} \nvdash \neg\Phi_i] \wedge [M_\omega^{\mathbf{Th}} \models \neg\Phi_i]. \qquad (2.19)
$$

Then we define theory $\mathbf{Th}_{i+1}^{\#}$ as follows: $\mathbf{Th}_{i+1}^{\#} \triangleq \mathbf{Th}_i^{\#} \cup \{\neg\Phi_i\}$. Using Lemma 2.2 and predicate $\mathbf{Pr}_{\mathbf{Th}_{i+1}^{\#}}^{\#}(\bullet)$, we will rewrite the condition (2.15) symbolically as follows

$$
\begin{cases}
\mathbf{Th}_i^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_i^{\#}}^{\#}([\neg\Phi_i]^c), \\
\mathbf{Pr}_{\mathbf{Th}_i^{\#}}^{\#}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\neg\Phi_i]^c) \wedge [M_\omega^{\mathbf{Th}} \models \neg\Phi_i], \\
M_\omega^{\mathbf{Th}} \models \neg\Phi_i \Leftrightarrow Con(\mathbf{Th}_i^{\#}+\neg\Phi_i; M_\omega^{\mathbf{Th}}), \\
\text{i.e.} \\
\mathbf{Pr}_{\mathbf{Th}_i^{\#}}^{\#}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\neg\Phi_i]^c) \wedge Con(\mathbf{Th}_i^{\#}+\neg\Phi_i; M_\omega^{\mathbf{Th}}), \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^{\#}}^{\#}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{i+1}^{\#}}([\Phi_i]^c), \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^{\#}}([\Phi_i]^c) \Rightarrow \Phi_i, \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^{\#}}^{\#}([\Phi_i]^c) \Rightarrow \Phi_i.
\end{cases}
\qquad (2.20)
$$

Remark 2.10. Notice that predicate $\mathbf{Pr}_{\mathbf{Th}_i^{\#}}^{\#}([\neg\Phi_i]^c)$ is expressible in $\mathbf{Th}_i^{\#}$ because $\mathbf{Th}_i^{\#}$ is a finite extension of the recursive theory $\mathbf{Th}$ and $Con(\mathbf{Th}_\mathbf{i}^{\#}+\neg\Phi_i; M_\omega^{\mathbf{Th}}) \in \mathbf{Th}_i^{\#}$.

(v) Suppose that the statement (2.21) is satisfied

$$
\mathbf{Th}_i^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\Phi_i]^c) \text{ and } \mathbf{Th}_i^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\Phi_i]^c) \Rightarrow \Phi_i. \qquad (2.21)
$$

We will rewrite now the conditions (2.21) symbolically as follows

$$\left\{\begin{array}{c}\mathbf{Th}_i^{\#}\vdash\mathbf{Pr}^*_{\mathbf{Th}_i^{\#}}([\Phi_i]^c)\\ \mathbf{Pr}^*_{\mathbf{Th}_i^{\#}}([\Phi_i]^c)\Leftrightarrow\mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\Phi_i]^c)\wedge\left[\mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\Phi_i]^c)\Rightarrow\Phi_i\right]\end{array}\right. \tag{2.22}$$

Then we define a theory $\mathbf{Th}_{i+1}^{\#}$ as follows: $\mathbf{Th}_{i+1}^{\#}\triangleq\mathbf{Th}_i^{\#}$.

(iv) Suppose that the statement (2.23) is satisfied

$$\mathbf{Th}_i^{\#}\vdash\mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\neg\Phi_i]^c)\text{ and }\mathbf{Th}_i^{\#}\vdash\mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\neg\Phi_i]^c)\Rightarrow\neg\Phi_i. \tag{2.23}$$

We will rewrite now the condition (2.23) symbolically as follows

$$\left\{\begin{array}{c}\mathbf{Th}_i^{\#}\vdash\mathbf{Pr}^*_{\mathbf{Th}_i^{\#}}([\neg\Phi_i]^c)\\ \mathbf{Pr}^*_{\mathbf{Th}_i^{\#}}([\neg\Phi_i]^c)\Leftrightarrow\mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\neg\Phi_i]^c)\wedge\left[\mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\neg\Phi_i]^c)\Rightarrow\neg\Phi_i\right]\end{array}\right. \tag{2.24}$$

Then we define a theory $\mathbf{Th}_{i+1}^{\#}$ as follows: $\mathbf{Th}_{i+1}^{\#}\triangleq\mathbf{Th}_i^{\#}$. We define now a theory $\mathbf{Th}_{\infty}^{\#}$ as follows:

$$\mathbf{Th}_{\infty}^{\#}\triangleq\bigcup_{i\in\mathbb{N}}\mathbf{Th}_i^{\#}. \tag{2.25}$$

First, notice that each $\mathbf{Th}_i^{\#}$ is consistent. This is done by induction on i and by Lemmas 2.1-2.2. By assumption, the case is true when $i=1$. Now, suppose $\mathbf{Th}_i^{\#}$ is consistent. Then its deductive closure $\mathbf{Ded}(\mathbf{Th}_i^{\#})$ is also consistent. If the statement (2.14) is satisfied,i.e. $\mathbf{Th}_{i+1}^{\#}\vdash\mathbf{Pr}^{\#}_{\mathbf{Th}_{i+1}^{\#}}([\Phi_i]^c)$ and $\mathbf{Th}_{i+1}^{\#}\vdash\Phi_i$, then clearly $\mathbf{Th}_{i+1}^{\#}\triangleq\mathbf{Th}_i^{\#}\cup\{\Phi_i\}$ is consistent since it is a subset of closure $\mathbf{Ded}(\mathbf{Th}_{i+1}^{\#})$. If a statement (2.16) is satisfied,i.e. $\mathbf{Th}_{i+1}^{\#}\vdash\mathbf{Pr}^{\#}_{\mathbf{Th}_{i+1}^{\#}}([\neg\Phi_i]^c)$ and $\mathbf{Th}_{i+1}^{\#}\vdash\neg\Phi_i$, then clearly $\mathbf{Th}_{i+1}^{\#}\triangleq\mathbf{Th}_i^{\#}\cup\{\neg\Phi_i\}$ is consistent since it is a subset of closure $\mathbf{Ded}(\mathbf{Th}_{i+1}^{\#})$. If the statement (2.18) is satisfied,i.e. $\mathbf{Th}_i^{\#}\vdash\mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\Phi_i]^c)$ and $[\mathbf{Th}_i^{\#}\nvdash\Phi_i]\wedge[M_{\omega}^{\mathbf{Th}}\models\Phi_i]$ then clearly $\mathbf{Th}_{i+1}^{\#}\triangleq\mathbf{Th}_i^{\#}\cup\{\Phi_i\}$ is consistent by Lemma 2.1 and by one of the standard properties of consistency: $\Delta\cup\{A\}$ is consistent iff $\Delta\nvdash\neg A$. If the statement (2.20) is satisfied,i.e. $\mathbf{Th}_i^{\#}\vdash\mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\neg\Phi_i]^c)$ and $[\mathbf{Th}_i^{\#}\nvdash\neg\Phi_i]\wedge[M_{\omega}^{\mathbf{Th}}\models\neg\Phi_i]$ then clearly $\mathbf{Th}_{i+1}^{\#}\triangleq\mathbf{Th}_i^{\#}\cup\{\neg\Phi_i\}$ is consistent by Lemma 2.2 and by one of the standard properties of consistency: $\Delta\cup\{\neg A\}$ is consistent iff $\Delta\nvdash A$. Next, notice $\mathbf{Ded}(\mathbf{Th}_{\infty}^{\#})$ is maximally consistent nice extension of the $\mathbf{Ded}(\mathbf{Th})$. $\mathbf{Ded}(\mathbf{Th}_{\infty}^{\#})$ is consistent because, by the standard Lemma 2.3 belov, it is the union of a chain of consistent sets. To see that $\mathbf{Ded}(\mathbf{Th}_{\infty}^{\#})$ is maximal, pick any wff Φ. Then Φ is some Φ_i in the enumerated list of all wff's. Therefore for any Φ such that $\mathbf{Th}_i\vdash\mathbf{Pr}_{\mathbf{Th}_i}([\Phi]^c)$ or $\mathbf{Th}_i^{\#}\vdash\mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\neg\Phi]^c)$, either $\Phi\in\mathbf{Th}_{\infty}^{\#}$ or $\neg\Phi\in\mathbf{Th}_{\infty}^{\#}$. Since $\mathbf{Ded}(\mathbf{Th}_{i+1}^{\#})\subseteqq\mathbf{Ded}(\mathbf{Th}_{\infty}^{\#})$, we have $\Phi\in\mathbf{Ded}(\mathbf{Th}_{\infty}^{\#})$ or $\neg\Phi\in\mathbf{Ded}(\mathbf{Th}_{\infty}^{\#})$, which implies that $\mathbf{Ded}(\mathbf{Th}_{\infty}^{\#})$ is maximally consistent nice extension of the $\mathbf{Ded}(\mathbf{Th})$.

Proof.(**II**) Let $\Phi_{\omega,1}\ldots\ \Phi_{\omega,i}\ldots$ be an enumeration of all closed wff's of the theory $\mathbf{Th}_{\omega}$ (this can be achieved if the set of propositional variables can be enumerated). Define a chain $\wp = \{\mathbf{Th}^{\#}_{\omega,i} | i \in \mathbb{N}\}, \mathbf{Th}^{\#}_{\omega,1} = \mathbf{Th}_{\omega}$ of consistent theories inductively as follows: assume that theory $\mathbf{Th}^{\#}_{\omega,i}$ is defined.

(i) Suppose that a statement (2.26) is satisfied

$$\mathbf{Th}^{\#}_{\omega,i} \nvdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,i}}([\Phi_{\omega,i}]^c) \text{ and } M^{\mathbf{Th}}_{\omega} \models \Phi_i. \tag{2.26}$$

Then we define a theory $\mathbf{Th}^{\#}_{\omega,i+1}$ as follows

$$\mathbf{Th}^{\#}_{\omega,i+1} \triangleq \mathbf{Th}^{\#}_{\omega,i} \cup \{\Phi_{\omega,i}\}. \tag{2.27}$$

We will rewrite now the conditions (2.26) and (2.27) symbolically as follows

$$\begin{cases} \mathbf{Th}^{\#}_{\omega,i+1} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,i+1}}([\Phi_{\omega,i}]^c) \Leftrightarrow \mathbf{Th}^{\#}_{\omega,i+1} \vdash \Phi_{\omega,i}, \\ \\ \mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{\omega,i+1}}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,i+1}}([\Phi_i]^c) \wedge \Phi_{\omega,i}. \end{cases} \tag{2.28}$$

(ii) Suppose that a statement (2.29) is satisfied

$$\mathbf{Th}^{\#}_{\omega,i} \nvdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,i}}([\neg\Phi_{\omega,i}]^c) \text{ and } M^{\mathbf{Th}}_{\omega} \models \neg\Phi_i. \tag{2.29}$$

Then we define theory $\mathbf{Th}^{\#}_{\omega,i+1}$ as follows:

$$\mathbf{Th}^{\#}_{\omega,i+1} \triangleq \mathbf{Th}^{\#}_{\omega,i} \cup \{\neg\Phi_{\omega,i}\}. \tag{2.30}$$

We will rewrite the conditions (2.25) and (2.26) symbolically as follows

$$\begin{cases} \mathbf{Th}_{\omega,i+1} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i+1}}([\neg\Phi_{\omega,i}]^c) \Leftrightarrow \mathbf{Th}_{\omega,i+1} \vdash \neg\Phi_{\omega,i}, \\ \\ \mathbf{Pr}^{\#}_{\mathbf{Th}_{\omega,i+1}}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\omega,i+1}}([\neg\Phi_i]^c). \end{cases} \tag{2.27}$$

(iii) Suppose that the following statement (2.28) is satisfied

$$\mathbf{Th}_{\omega,i} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\Phi_{\omega,i}]^c), \tag{2.28}$$

and therefore by Derivability Conditions (2.8)

$$\mathbf{Th}_{\omega,i} \vdash \Phi_{\omega,i}. \tag{2.29}$$

We will rewrite now the conditions (2.28) and (2.29) symbolically as follows

$$\mathbf{Pr}^{*}_{\mathbf{Th}_{\omega,i}}([\Phi_{\omega,i}]^c) \Leftrightarrow \mathbf{Th}_{\omega,i} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\Phi_{\omega,i}]^c) \tag{2.30}$$

Then we define a theory $\mathbf{Th}_{\omega,i+1}$ as follows: $\mathbf{Th}_{\omega,i+1} \triangleq \mathbf{Th}_{\omega,i}$.

(iv) Suppose that the following statement (2.31) is satisfied

$$\mathbf{Th}_{\omega,i} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\neg\Phi_{\omega,i}]^c), \tag{2.31}$$

and therefore by Derivability Conditions (2.8)

$$\mathbf{Th}_{\omega,i} \vdash \neg\Phi_{\omega,i}. \tag{2.32}$$

We will rewrite now the conditions (2.31) and (2.32) symbolically as follows

$$\mathbf{Pr}^{*}_{\mathbf{Th}_{\omega,i}}([\neg\Phi_{\omega,i}]^{c}) \Leftrightarrow \mathbf{Th}_{\omega,i} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\neg\Phi_{\omega,i}]^{c}) \tag{2.33}$$

Then we define a theory $\mathbf{Th}_{\omega,i+1}$ as follows: $\mathbf{Th}_{\omega,i+1} \triangleq \mathbf{Th}_{\omega,i}$. We define now a theory $\mathbf{Th}^{\#}_{\infty;\omega}$ as follows:

$$\mathbf{Th}^{\#}_{\infty;\omega} \triangleq \bigcup_{i\in\mathbb{N}} \mathbf{Th}_{\omega,i}. \tag{2.34}$$

First, notice that each $\mathbf{Th}_{\omega,i}$ is consistent. This is done by induction on i. Now, suppose $\mathbf{Th}_{\omega,i}$ is consistent. Then its deductive closure $\mathbf{Ded}(\mathbf{Th}_{\omega,i})$ is also consistent. If statement (2.22) is satisfied,i.e. $\mathbf{Th}_{\omega,i} \nvdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\Phi_{\omega,i}]^{c})$ and $M^{\mathbf{Th}}_{\omega} \models \Phi_i$ then clearly $\mathbf{Th}_{\omega,i+1} \triangleq \mathbf{Th}_{\omega,i} \cup \{\Phi_{\omega,i}\}$ is consistent.If statement (2.25) is satisfied,i.e. $\mathbf{Th}_{\omega,i} \nvdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\neg\Phi_{\omega,i}]^{c})$ and $M^{\mathbf{Th}}_{\omega} \models \neg\Phi_i$, then clearly $\mathbf{Th}_{\omega,i+1} \triangleq \mathbf{Th}_{\omega,i} \cup \{\neg\Phi_{\omega,i}\}$ is consistent. If the statement (2.28) is satisfied,i.e. $\mathbf{Th}_{\omega,i} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\Phi_{\omega,i}]^{c})$, then clearly $\mathbf{Th}_{\omega,i+1} \triangleq \mathbf{Th}_{\omega,i}$ is also consistent. If the statement (2.31) is satisfied,i.e. $\mathbf{Th}_{\omega,i} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\neg\Phi_{\omega,i}]^{c})$, then clearly $\mathbf{Th}_{\omega,i+1} \triangleq \mathbf{Th}_{\omega,i}$ is also consistent.Next, notice $\mathbf{Ded}(\mathbf{Th}^{\#}_{\infty;\omega})$ is maximally consistent nice extension of the $\mathbf{Ded}(\mathbf{Th}_{\infty;\omega})$. The set $\mathbf{Ded}(\mathbf{Th}^{\#}_{\infty;\omega})$ is consistent because, by the standard Lemma 2.3 belov, it is the union of a chain of consistent sets.

Lemma 2.3. The union of a chain $\wp = \{\mathbf{\Gamma}_i | i \in \mathbb{N}\}$ of consistent sets $\mathbf{\Gamma}_i$, ordered by $\subseteq$, is
consistent.

Definition 2.4. (**I**) We define now predicate $\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty}}([\Phi]^{c})$ and predicate $\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty}}([\neg\Phi]^{c})$ asserting provability in $\mathbf{Th}^{\#}_{\infty}$ by the following formulae

$$\begin{cases} \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty}}([\Phi]^{c}) \Leftrightarrow \left\{\exists i(\Phi \in \mathbf{Th}^{\#}_{i})\left[\mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{i}}([\Phi]^{c})\right] \vee \left[\mathbf{Pr}^{*}_{\mathbf{Th}^{\#}_{i}}([\Phi]^{c})\right]\right\} \vee \\ \vee[(\Phi \in \mathbf{Th}^{\#}_{\infty}) \wedge Con(\mathbf{Th}^{\#}_{\infty} + \Phi; M^{\mathbf{Th}}_{\omega})], \\ \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty}}([\neg\Phi]^{c}) \Leftrightarrow \left\{\exists i(\Phi \in \mathbf{Th}^{\#}_{i})\left[\mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{i}}([\neg\Phi]^{c})\right] \vee \left[\mathbf{Pr}^{*}_{\mathbf{Th}^{\#}_{i}}([\neg\Phi]^{c})\right]\right\} \vee \\ \vee[(\Phi \in \mathbf{Th}^{\#}_{\infty}) \wedge Con(\mathbf{Th}^{\#}_{\infty} + \neg\Phi; M^{\mathbf{Th}}_{\omega})]. \end{cases} \tag{2.35}$$

(**II**) We define now predicate $\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}([\Phi_{\omega}]^{c})$ and predicate $\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}([\neg\Phi_{\omega}]^{c})$ asserting provability in $\mathbf{Th}^{\#}_{\infty;\omega}$ by following formulae

$$\begin{cases} \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}([\Phi_{\omega}]^{c}) \Leftrightarrow \\ \left\{\exists i(\Phi_{\omega} \in \mathbf{Th}^{\#}_{\omega,i})\left[\mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{\omega,i}}([\Phi_{\omega}]^{c})\right] \vee \left[\mathbf{Pr}^{*}_{\mathbf{Th}^{\#}_{\omega,i}}([\Phi_{\omega}]^{c})\right]\right\} \vee \\ \vee[(\Phi_{\omega} \in \mathbf{Th}^{\#}_{\infty;\omega}) \wedge Con(\mathbf{Th}^{\#}_{\infty;\omega} + \Phi_{\omega}; M^{\mathbf{Th}}_{\omega})], \\ \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}([\neg\Phi_{\omega}]^{c}) \Leftrightarrow \\ \left\{\exists i(\Phi_{\omega} \in \mathbf{Th}^{\#}_{\omega,i})\left[\mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{\omega,i}}([\neg\Phi_{\omega}]^{c})\right] \vee \left[\mathbf{Pr}^{*}_{\mathbf{Th}^{\#}_{\omega,i}}([\neg\Phi_{\omega}]^{c})\right]\right\} \vee \\ \vee[(\Phi_{\omega} \in \mathbf{Th}^{\#}_{\infty;\omega}) \wedge Con(\mathbf{Th}^{\#}_{\infty;\boldsymbol{\omega}} + \neg\Phi_{\omega}; M^{\mathbf{Th}}_{\omega})]. \end{cases} \tag{2.36}$$

Remark 2.11.(**I**) Notice that both predicate $\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty}}([\Phi]^{c})$ and predicate $\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty}}([\neg\Phi]^{c})$ are
expressible in $\mathbf{Th}^{\#}_{\infty}$ because for any i, $\mathbf{Th}^{\#}_{i}$ is an finite extension of the recursive theory $\mathbf{Th}$
and $Con(\mathbf{Th}^{\#}_{i}+\Phi;M^{\mathbf{Th}}) \in \mathbf{Th}_{i}, Con(\mathbf{Th}^{\#}_{i}+\neg\Phi;M^{\mathbf{Th}}) \in \mathbf{Th}_{i}$.

(**II**) Notice that both predicate $\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}([\Phi_{\omega}]^{c})$ and predicate $\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}([\neg\Phi_{\omega}]^{c})$ are expressible
in $\mathbf{Th}^{\#}_{\infty;\omega}$ because for any i, $\mathbf{Th}^{\#}_{\omega,i}$ is an finite extension of the recursive theory $\mathbf{Th}_{\omega}$ and
$Con(\mathbf{Th}^{\#}_{\omega,i}+\Phi_{\omega};M^{\mathbf{Th}}) \in \mathbf{Th}^{\#}_{\omega,i}, Con(\mathbf{Th}^{\#}_{\omega,i}+\neg\Phi_{\omega};M^{\mathbf{Th}}) \in \mathbf{Th}^{\#}_{\omega,i}$.

Definition 2.5.Let $\Psi = \Psi(x)$ be one-place open $\mathbf{Th}$-wff such that the following condition:

$$\mathbf{Th} \triangleq \mathbf{Th}^{\#}_{1} \vdash \exists! x_{\Psi}[\Psi(x_{\Psi})] \tag{2.37}$$

is satisfied.

Remark 2.12.We rewrite now the condition (2.37) using only lenguage of the theory $\mathbf{Th}^{\#}_{1}$:

$$\begin{gathered}\{\mathbf{Th}^{\#}_{1} \vdash \exists! x_{\Psi}[\Psi(x_{\Psi})]\} \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{1}}([\exists! x_{\Psi}[\Psi(x_{\Psi})]]^{c}) \wedge \\ \wedge\left\{\mathbf{Pr}_{\mathbf{Th}^{\#}_{1}}([\exists! x_{\Psi}[\Psi(x_{\Psi})]]^{c}) \Rightarrow \exists! x_{\Psi}[\Psi(x_{\Psi})]\right\}.\end{gathered} \tag{2.38}$$

Definition 2.6. We will say that, a set y is a $\mathbf{Th}^{\#}_{1}$-set if there exist one-place open wff $\Psi(x)$
such that $y = x_{\Psi}$. We write $y[\mathbf{Th}^{\#}_{1}]$ iff y is a $\mathbf{Th}^{\#}_{1}$-set.

Remark 2.13. Note that

$$\begin{gathered}y[\mathbf{Th}^{\#}_{1}] \Leftrightarrow \exists\Psi\left\{(y = x_{\Psi}) \wedge \mathbf{Pr}_{\mathbf{Th}^{\#}_{1}}([\exists! x_{\Psi}[\Psi(x_{\Psi})]]^{c}) \\ \left\{\mathbf{Pr}_{\mathbf{Th}^{\#}_{1}}([\exists! x_{\Psi}[\Psi(x_{\Psi})]]^{c}) \Rightarrow \exists! x_{\Psi}[\Psi(x_{\Psi})]\right\}\right\}.\end{gathered} \tag{2.39}$$

Definition 2.7.Let $\Im_{1}$ be a collection such that :

$$\forall x\left[x \in \Im_{1} \leftrightarrow x \text{ is a } \mathbf{Th}^{\#}_{1}\text{-set}\right]. \tag{2.40}$$

Proposition 2.2. Collection $\Im_{1}$ is a $\mathbf{Th}^{\#}_{1}$-set.

Proof. Let us consider an one-place open wff $\Psi(x)$ such that conditions (2.37) is satisfied, i.e. $\mathbf{Th}^{\#}_{1} \vdash \exists! x_{\Psi}[\Psi(x_{\Psi})]$.We note that there exists countable collection $\mathcal{F}_{\Psi}$ of the one-place open wff's $\mathcal{F}_{\Psi} = \{\Psi_{n}(x)\}_{n\in\mathbb{N}}$ such that: (i) $\Psi(x) \in \mathcal{F}_{\Psi}$ and (ii)

$$
\left\{\begin{array}{c}
\mathbf{Th} \triangleq \mathbf{Th}_1^{\#} \vdash \exists! x_\Psi[[\Psi(x_\Psi)] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]\}] \\
\text{or in the equivalent form} \\
\mathbf{Th} \triangleq \mathbf{Th}_1^{\#} \vdash \\
\mathbf{Pr}_{\mathbf{Th}_1^{\#}}([\exists! x_\Psi[\Psi(x_\Psi)]]^c) \wedge \\
\left\{\mathbf{Pr}_{\mathbf{Th}_1^{\#}}([\exists! x_\Psi[\Psi(x_\Psi)]]^c) \Rightarrow \exists! x_\Psi[\Psi(x_\Psi)]\right\} \wedge \\
\left[\mathbf{Pr}_{\mathbf{Th}_1^{\#}}([\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]^c)\right] \wedge \\
\mathbf{Pr}_{\mathbf{Th}_1^{\#}}([\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]^c) \Rightarrow \forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]
\end{array}\right. \quad (2.41)
$$

or in the following equivalent form

$$
\left\{\begin{array}{c}
\mathbf{Th}_1^{\#} \vdash \exists! x_1[[\Psi_1(x_1)] \wedge \{\forall n(n \in \mathbb{N})[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]\}] \\
\text{or} \\
\mathbf{Th}_1^{\#} \vdash \\
\mathbf{Pr}_{\mathbf{Th}_1^{\#}}([\exists! x_1 \Psi(x_1)]^c) \wedge \\
\left\{\mathbf{Pr}_{\mathbf{Th}_1^{\#}}([\exists! x_1 \Psi(x_1)]^c) \Rightarrow \exists! x_1 \Psi(x_1)\right\} \wedge \\
\left[\mathbf{Pr}_{\mathbf{Th}_1^{\#}}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^c)\right] \wedge \\
\mathbf{Pr}_{\mathbf{Th}_1^{\#}}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^c) \Rightarrow \forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)],
\end{array}\right. \quad (2.42)
$$

where we have set $\Psi(x) = \Psi_1(x_1), \Psi_n(x_1) = \Psi_{n,1}(x_1)$ and $x_\Psi = x_1$. We note that any collection $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n \in \mathbb{N}}, k = 1,2,\ldots$ such mentioned above, defines an unique set x_{Ψ_k}, i.e. $\mathcal{F}_{\Psi_{k_1}} \cap \mathcal{F}_{\Psi_{k_2}} = \varnothing$ iff $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$. We note that collections $\mathcal{F}_{\Psi_k}, k = 1,2,..$ is not a part of the ZFC_2, i.e. collection $\mathcal{F}_{\Psi_k}$ there is no set in sense of ZFC_2. However that is no problem, because by using Gödel numbering one can to replace any collection $\mathcal{F}_{\Psi_k}, k = 1,2,..$ by collection $\Theta_k = g(\mathcal{F}_{\Psi_k})$ of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n \in \mathbb{N}}, k = 1,2,\ldots\ . \quad (2.43)$$

It is easy to prove that any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}_1^{\#}$-set. This is done by Gödel encoding [8],[10] (2.43), by the statament (2.41) and by axiom schema of separation [9]. Let $g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1,2,..$ be a Gödel number of the wff $\Psi_{n,k}(x_k)$. Therefore $g(\mathcal{F}_k) = \{g_{n,k}\}_{n \in \mathbb{N}}$, where we have set $\mathcal{F}_k = \mathcal{F}_{\Psi_k},\ k = 1,2,..$ and

$$\forall k_1 \forall k_2[\{g_{n,k_1}\}_{n \in \mathbb{N}} \cap \{g_{n,k_2}\}_{n \in \mathbb{N}} = \varnothing \leftrightarrow x_{k_1} \neq x_{k_2}]. \quad (2.44)$$

Let $\{\{g_{n,k}\}_{n \in \mathbb{N}}\}_{k \in \mathbb{N}}$ be a family of the all sets $\{g_{n,k}\}_{n \in \mathbb{N}}$. By axiom of choice [9] one obtains unique set $\mathfrak{I}_1' = \{g_k\}_{k \in \mathbb{N}}$ such that $\forall k[g_k \in \{g_{n,k}\}_{n \in \mathbb{N}}]$. Finally one obtains a set $\mathfrak{I}_1$ from the set $\mathfrak{I}_1'$ by axiom schema of replacement [9].

Proposition 2.3. Any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}_1^{\#}$-set.

Proof. We define $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$. Therefore $g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$ (see [10]). Let us define now predicate $\Pi(g_{n,k}, v_k)$

$$\begin{gathered}\Pi(g_{n,k}, v_k) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_1^{\#}}([\exists! x_k[\Psi_{1,k}(x_1)]]^c) \wedge \\ \wedge \exists! x_k(v_k = [x_k]^c)\Big[\forall n(n \in \mathbb{N})\Big[\mathbf{Pr}_{\mathbf{Th}_1^{\#}}([[\Psi_{1,k}(x_k)]]^c) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_1^{\#}}(\mathbf{Fr}(g_{n,k}, v_k))\Big]\Big].\end{gathered} \tag{2.45}$$

We define now a set Θ_k such that

$$\begin{cases} \Theta_k = \Theta_k' \cup \{g_k\}, \\ \forall n(n \in \mathbb{N})[g_{n,k} \in \Theta_k' \leftrightarrow \Pi(g_{n,k}, v_k)] \end{cases} \tag{2.46}$$

Obviously definitions (2.41) and (2.46) is equivalent.

Definition 2.7.We define now the following $\mathbf{Th}_1^{\#}$-set $\mathfrak{R}_1 \subsetneqq \mathfrak{I}_1$:

$$\forall x\Big[x \in \mathfrak{R}_1 \Leftrightarrow (x \in \mathfrak{I}_1) \wedge \mathbf{Pr}_{\mathbf{Th}_1^{\#}}([x \notin x]^c) \wedge \Big[\mathbf{Pr}_{\mathbf{Th}_1^{\#}}([x \notin x]^c) \Rightarrow x \notin x\Big]\Big]. \tag{2.47}$$

Proposition 2.4. (i) $\mathbf{Th}_1^{\#} \vdash \exists \mathfrak{R}_1$,(ii) $\mathfrak{R}_1$ is a countable $\mathbf{Th}_1^{\#}$-set.

Proof.(i) Statement $\mathbf{Th}_1^{\#} \vdash \exists \mathfrak{R}_1$ follows immediately from the statement $\exists \mathfrak{I}_1$ and axiom

schema of separation [4], (ii) follows immediately from countability of a set $\mathfrak{I}_1$.Notice that

$\mathfrak{R}_1$ is nonempty countable set such that $\mathbb{N} \subset \mathfrak{R}_1$, becouse for any $n \in \mathbb{N}$: $\mathbf{Th}_1^{\#} \vdash n \notin n$.

Proposition 2.5. A set $\mathfrak{R}_1$ is inconsistent.

Proof.From formla (2.47) we obtain

$$\mathbf{Th}_1^{\#} \vdash \mathfrak{R}_1 \in \mathfrak{R}_1 \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_1^{\#}}([\mathfrak{R}_1 \notin \mathfrak{R}_1]^c) \wedge \Big[\mathbf{Pr}_{\mathbf{Th}_1^{\#}}([\mathfrak{R}_1 \notin \mathfrak{R}_1]^c) \Rightarrow \mathfrak{R}_1 \notin \mathfrak{R}_1\Big]. \tag{2.48}$$

From (2.48) we obtain

$$\mathbf{Th}_1^{\#} \vdash \mathfrak{R}_1 \in \mathfrak{R}_1 \Leftrightarrow \mathfrak{R}_1 \notin \mathfrak{R}_1 \tag{2.49}$$

and therefore

$$\mathbf{Th}_1^{\#} \vdash (\mathfrak{R}_1 \in \mathfrak{R}_1) \wedge (\mathfrak{R}_1 \notin \mathfrak{R}_1). \tag{2.50}$$

But this is a contradiction.

Definition 2.8. Let $\Psi = \Psi(x)$ be one-place open $\mathbf{Th}$-wff such that the following

condition:

$$\mathbf{Th}_i^{\#} \vdash \exists! x_\Psi[\Psi(x_\Psi)] \tag{2.51}$$

is satisfied.

Remark 2.14.We rewrite now the condition (2.51) using only lenguage of the theory $\mathbf{Th}_i^{\#}$:

$$\{\mathbf{Th}_i^\# \vdash \exists! x_\Psi[\Psi(x_\Psi)]\} \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists! x_\Psi[\Psi(x_\Psi)]]^c) \wedge \\ \wedge\left\{\mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists! x_\Psi[\Psi(x_\Psi)]]^c) \Rightarrow \exists! x_\Psi[\Psi(x_\Psi)]\right\}. \tag{2.52}$$

Definition 2.9. We will say that, a set y is a $\mathbf{Th}_i^\#$-set if there exist one-place open wff $\Psi(x)$ such that $y = x_\Psi$. We write $y[\mathbf{Th}_i^\#]$ iff y is a $\mathbf{Th}_i^\#$-set.

Remark 2.15. Note that

$$y[\mathbf{Th}_i^\#] \Leftrightarrow \exists\Psi\left\{(y = x_\Psi) \wedge \mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists! x_\Psi[\Psi(x_\Psi)]]^c) \\ \left\{\mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists! x_\Psi[\Psi(x_\Psi)]]^c) \Rightarrow \exists! x_\Psi[\Psi(x_\Psi)]\right\}\right\}. \tag{2.53}$$

Definition 2.10.Let $\Im_i$ be a collection such that :

$$\forall x\left[x \in \Im_i \leftrightarrow x \text{ is a } \mathbf{Th}_i^\#\text{-set}\right]. \tag{2.54}$$

Proposition 2.6. Collection $\Im_i$ is a $\mathbf{Th}_i^\#$-set.

Proof. Let us consider an one-place open wff $\Psi(x)$ such that conditions (2.51) is satisfied, i.e. $\mathbf{Th}_i^\# \vdash \exists! x_\Psi[\Psi(x_\Psi)]$.We note that there exists countable collection $\mathcal{F}_\Psi$ of the one-place open wff's $\mathcal{F}_\Psi = \{\Psi_n(x)\}_{n\in\mathbb{N}}$ such that: (i) $\Psi(x) \in \mathcal{F}_\Psi$ and (ii)

$$\begin{cases} \mathbf{Th}_i^\# \vdash \exists! x_\Psi[[\Psi(x_\Psi)] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]\}] \\ \text{or in the equivalent form} \\ \mathbf{Th}_i^\# \vdash \mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists! x_\Psi[\Psi(x_\Psi)]]^c) \wedge \\ \left\{\mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists! x_\Psi[\Psi(x_\Psi)]]^c) \Rightarrow \exists! x_\Psi[\Psi(x_\Psi)]\right\} \wedge \\ \left[\mathbf{Pr}_{\mathbf{Th}_i^\#}([\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]^c)\right] \wedge \\ \mathbf{Pr}_{\mathbf{Th}_i^\#}([\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]^c) \Rightarrow \forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)] \end{cases} \tag{2.55}$$

or in the following equivalent form

$$\begin{cases} \mathbf{Th}_i^\# \vdash \exists! x_1[[\Psi_1(x_1)] \wedge \{\forall n(n \in \mathbb{N})[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]\}] \\ \text{or} \\ \mathbf{Th}_i^\# \vdash \\ \mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists! x_1\Psi(x_1)]^c) \wedge \\ \left\{\mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists! x_1\Psi(x_1)]^c) \Rightarrow \exists! x_1\Psi(x_1)\right\} \wedge \\ \left[\mathbf{Pr}_{\mathbf{Th}_i^\#}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^c)\right] \wedge \\ \mathbf{Pr}_{\mathbf{Th}_i^\#}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^c) \Rightarrow \forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]. \end{cases} \tag{2.56}$$

where we have set $\Psi(x) = \Psi_1(x_1), \Psi_n(x_1) = \Psi_{n,1}(x_1)$ and $x_\Psi = x_1$. We note that any collection $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n\in\mathbb{N}}, k = 1,2,\ldots$ such mentioned above, defines an unique set x_{Ψ_k},i.e. $\mathcal{F}_{\Psi_{k_1}} \cap \mathcal{F}_{\Psi_{k_2}} = \varnothing$ iff $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$.We note that collections $\mathcal{F}_{\Psi_k}, k = 1,2,..$ is not a part of the ZFC_2, i.e. collection $\mathcal{F}_{\Psi_k}$ there is no set in sense of ZFC_2. However that is no problem, because by using Gödel numbering one can to replace any collection $\mathcal{F}_{\Psi_k}, k = 1,2,..$ by collection $\Theta_k = g(\mathcal{F}_{\Psi_k})$ of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n\in\mathbb{N}}, k = 1,2,\ldots . \tag{2.57}$$

It is easy to prove that any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}_i^{\#}$-set.This is done by Gödel encoding [8],[10] (2.57), by the statament (2.51) and by axiom schema of separation [9]. Let $g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1,2,..$ be a Gödel number of the wff $\Psi_{n,k}(x_k)$. Therefore $g(\mathcal{F}_k) = \{g_{n,k}\}_{n\in\mathbb{N}}$, where we have set $\mathcal{F}_k = \mathcal{F}_{\Psi_k},\ k = 1,2,..$ and

$$\forall k_1 \forall k_2 [\{g_{n,k_1}\}_{n\in\mathbb{N}} \cap \{g_{n,k_2}\}_{n\in\mathbb{N}} = \varnothing \leftrightarrow x_{k_1} \neq x_{k_2}]. \tag{2.58}$$

Let $\{\{g_{n,k}\}_{n\in\mathbb{N}}\}_{k\in\mathbb{N}}$ be a family of the all sets $\{g_{n,k}\}_{n\in\mathbb{N}}$. By axiom of choice [9] one obtains unique set $\Im_i' = \{g_k\}_{k\in\mathbb{N}}$ such that $\forall k[g_k \in \{g_{n,k}\}_{n\in\mathbb{N}}]$. Finally one obtains a set $\Im_i$ from the set $\Im_i'$ by axiom schema of replacement [9].

Proposition 2.8. Any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}_i^{\#}$-set.

Proof. We define $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$. Therefore $g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$ (see [10]). Let us define now predicate $\Pi_i(g_{n,k}, v_k)$

$$\begin{gathered}\Pi_i(g_{n,k}, v_k) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\exists! x_k[\Psi_{1,k}(x_1)]]^c) \wedge \\ \wedge \exists! x_k(v_k = [x_k]^c)\Big[\forall n(n \in \mathbb{N})\Big[\mathbf{Pr}_{\mathbf{Th}_i^{\#}}([[\Psi_{1,k}(x_k)]]^c) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^{\#}}(\mathbf{Fr}(g_{n,k}, v_k))\Big]\Big].\end{gathered} \tag{2.59}$$

We define now a set Θ_k such that

$$\begin{cases} \Theta_k = \Theta_k' \cup \{g_k\}, \\ \forall n(n \in \mathbb{N})[g_{n,k} \in \Theta_k' \leftrightarrow \Pi_i(g_{n,k}, v_k)]. \end{cases} \tag{2.60}$$

Obviously definitions (2.55) and (2.60) is equivalent.

Definition 2.11.We define now the following $\mathbf{Th}_i^{\#}$-set $\Re_i \subsetneq \Im_i$:

$$\forall x\Big[x \in \Re_i \Longleftrightarrow (x \in \Im_i) \wedge \mathbf{Pr}_{\mathbf{Th}_i^{\#}}([x \notin x]^c) \wedge \Big[\mathbf{Pr}_{\mathbf{Th}_i^{\#}}([x \notin x]^c) \Rightarrow x \notin x\Big]\Big]. \tag{2.61}$$

Proposition 2.9. (i) $\mathbf{Th}_i^{\#} \vdash \exists \Re_i$,(ii) $\Re_i$ is a countable $\mathbf{Th}_i^{\#}$-set,$i \in \mathbb{N}$.

Proof.(i) Statement $\mathbf{Th}_i^{\#} \vdash \exists \Re_i$ follows immediately by using statement $\exists \Im_i$ and axiom

schema of separation [4]. (ii) follows immediately from countability of a set $\Im_i$.

Proposition 2.10. Any set $\Re_i, i \in \mathbb{N}$ is inconsistent.

Proof.From formla (2.61) we obtain

$$\mathbf{Th}_i^{\#} \vdash \Re_i \in \Re_i \Longleftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\Re_i \notin \Re_i]^c) \wedge \Big[\mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\Re_i \notin \Re_i]^c) \Rightarrow \Re_i \notin \Re_i\Big]. \tag{2.62}$$

From (2.62) we obtain

$$\mathbf{Th}_i^{\#} \vdash \Re_i \in \Re_i \Longleftrightarrow \Re_i \notin \Re_i \tag{2.63}$$

and therefore

$$\mathbf{Th}_i^{\#} \vdash (\Re_i \in \Re_i) \wedge (\Re_i \notin \Re_i). \tag{2.64}$$

But this is a contradiction.

Definition 2.12.An $\mathbf{Th}_\infty^{\#}$-wff Φ_∞ that is: (i) $\mathbf{Th}$-wff Φ or (ii) well-formed formula Φ_∞ which
contains predicate $\mathbf{Pr}_{\mathbf{Th}_\infty^{\#}}([\Phi]^c)$ given by formula (2.35).An $\mathbf{Th}_\infty^{\#}$-wff Φ_∞ (well-formed formula Φ_∞) is closed - i.e. Φ is a sentence - if it has no free variables; a wff is open if it
has free variables.

Definition 2.13.Let $\Psi = \Psi(x)$ be one-place open $\mathbf{Th}_\infty^{\#}$-wff such that the following condition:

$$\mathbf{Th}_\infty^{\#} \vdash \exists! x_\Psi [\Psi(x_\Psi)] \tag{2.65}$$

is satisfied.

Remark 2.16.We rewrite now the condition (2.65) using only lenguage of the theory $\mathbf{Th}_\infty^{\#}$:

$$\begin{gathered}\{\mathbf{Th}_\infty^{\#} \vdash \exists! x_\Psi [\Psi(x_\Psi)]\} \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_\infty^{\#}}([\exists! x_\Psi [\Psi(x_\Psi)]]^c) \wedge \\ \wedge \{\mathbf{Pr}_{\mathbf{Th}_\infty^{\#}}([\exists! x_\Psi [\Psi(x_\Psi)]]^c) \Rightarrow \exists! x_\Psi [\Psi(x_\Psi)]\}.\end{gathered} \tag{2.66}$$

Definition 2.14.We will say that, a set y is a $\mathbf{Th}_\infty^{\#}$-set if there exist one-place open wff
$\Psi(x)$ such that $y = x_\Psi$. We write $y[\mathbf{Th}_\infty^{\#}]$ iff y is a $\mathbf{Th}_\infty^{\#}$-set.

Definition 2.15. Let $\Im_\infty$ be a collection such that : $\forall x\big[x \in \Im_\infty \leftrightarrow x \text{ is a } \mathbf{Th}_\infty^{\#}\text{-set}\big]$.

Proposition 2.11. Collection $\Im_\infty$ is a $\mathbf{Th}_\infty^{\#}$-set.

Proof. Let us consider an one-place open wff $\Psi(x)$ such that condition (2.65) is satisfied, i.e. $\mathbf{Th}_\infty^{\#} \vdash \exists! x_\Psi [\Psi(x_\Psi)]$.We note that there exists countable collection $\mathcal{F}_\Psi$ of the one-place open wff's $\mathcal{F}_\Psi = \{\Psi_n(x)\}_{n\in\mathbb{N}}$ such that: (i) $\Psi(x) \in \mathcal{F}_\Psi$ and (ii)

$$\left\{\begin{array}{c}\mathbf{Th}_\infty^{\#} \vdash \exists! x_\Psi [[\Psi(x_\Psi)] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]\}] \\ \text{or in the equivalent form} \\ \mathbf{Th}_\infty^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_\infty^{\#}}([\exists! x_\Psi [\Psi(x_\Psi)]]^c) \wedge \\ \{\mathbf{Pr}_{\mathbf{Th}_\infty^{\#}}([\exists! x_\Psi [\Psi(x_\Psi)]]^c) \Rightarrow \exists! x_\Psi [\Psi(x_\Psi)]\} \wedge \\ [\mathbf{Pr}_{\mathbf{Th}_\infty^{\#}}([\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]^c)] \wedge \\ \mathbf{Pr}_{\mathbf{Th}_\infty^{\#}}([\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]^c) \Rightarrow \forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]\end{array}\right. \tag{2.67}$$

or in the following equivalent form

$$\begin{cases} \mathbf{Th}_{\infty}^{\#} \vdash \exists! x_1[[\Psi_1(x_1)] \wedge \{\forall n(n \in \mathbb{N})[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]\}] \\ \text{or} \\ \mathbf{Th}_{\infty}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\exists! x_1 \Psi(x_1)]^c) \wedge \\ \{\mathbf{Pr}_{\mathbf{Th}_{\infty}^{\#}}([\exists! x_1 \Psi(x_1)]^c) \Rightarrow \exists! x_1 \Psi(x_1)\} \wedge \\ \left[\mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^c)\right] \wedge \\ \mathbf{Pr}_{\mathbf{Th}_i^{\#}}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^c) \Rightarrow \forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]. \end{cases} \tag{2.68}$$

where we set $\Psi(x) = \Psi_1(x_1), \Psi_n(x_1) = \Psi_{n,1}(x_1)$ and $x_{\Psi} = x_1$. We note that any collection $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n \in \mathbb{N}}, k = 1,2,\ldots$ such above defines an unique set x_{Ψ_k}, i.e. $\mathcal{F}_{\Psi_{k_1}} \bigcap \mathcal{F}_{\Psi_{k_2}} = \varnothing$ iff $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$. We note that collections $\mathcal{F}_{\Psi_k}, k = 1,2,..$ is no part of the ZFC_2, i.e. collection $\mathcal{F}_{\Psi_k}$ there is no set in sense of ZFC_2. However that is no problem, because by using Gödel numbering one can to replace any collection $\mathcal{F}_{\Psi_k}, k = 1,2,..$ by collection $\Theta_k = g(\mathcal{F}_{\Psi_k})$ of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n \in \mathbb{N}}, k = 1,2,\ldots . \tag{2.69}$$

It is easy to prove that any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}^{\#}$-set.This is done by Gödel encoding [8],[10] by the statament (2.66) and by axiom schema of separation [9]. Let $g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1,2,..$ be a Gödel number of the wff $\Psi_{n,k}(x_k)$. Therefore $g(\mathcal{F}_k) = \{g_{n,k}\}_{n \in \mathbb{N}}$, where we have set $\mathcal{F}_k = \mathcal{F}_{\Psi_k},\ k = 1,2,..$ and

$$\forall k_1 \forall k_2[\{g_{n,k_1}\}_{n \in \mathbb{N}} \bigcap \{g_{n,k_2}\}_{n \in \mathbb{N}} = \varnothing \leftrightarrow x_{k_1} \neq x_{k_2}]. \tag{2.70}$$

Let $\{\{g_{n,k}\}_{n \in \mathbb{N}}\}_{k \in \mathbb{N}}$ be a family of the all sets $\{g_{n,k}\}_{n \in \mathbb{N}}$. By axiom of choice [9] one obtains unique set $\mathfrak{I}' = \{g_k\}_{k \in \mathbb{N}}$ such that $\forall k[g_k \in \{g_{n,k}\}_{n \in \mathbb{N}}]$. Finally one obtains a set $\mathfrak{I}_{\infty}$ from the set $\mathfrak{I}'_{\infty}$ by axiom schema of replacement [9].Thus one can define $\mathbf{Th}_{\infty}^{\#}$-set

$\mathfrak{R}_{\infty} \subsetneqq \mathfrak{I}_{\infty}$:

$$\forall x[x \in \mathfrak{R}_{\infty} \leftrightarrow (x \in \mathfrak{I}_{\infty}) \wedge [\mathbf{Pr}_{\mathbf{Th}_{\infty}^{\#}}([x \notin x]^c) \wedge \{\mathbf{Pr}_{\mathbf{Th}_{\infty}^{\#}}([x \notin x]^c) \Rightarrow x \notin x\}]]. \tag{2.71}$$

Proposition 2.12. Any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}_{\infty}^{\#}$-set.

Proof. We define $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$. Therefore $g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$ (see [10]). Let us define now predicate $\Pi_{\infty}(g_{n,k}, v_k)$

$$\begin{cases} \Pi_{\infty}(g_{n,k}, v_k) \Leftrightarrow \\ \mathbf{Pr}_{\mathbf{Th}_{\infty}^{\#}}([\exists! x_k[\Psi_{1,k}(x_1)]]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}_{\infty}^{\#}}([\exists! x_k[\Psi_{1,k}(x_1)]]^c) \Rightarrow \exists! x_1 \Psi(x_1)] \\ \wedge \exists! x_k(v_k = [x_k]^c)[\forall n(n \in \mathbb{N})[\mathbf{Pr}_{\mathbf{Th}_{\infty}^{\#}}([[\Psi_{1,k}(x_k)]]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\infty}^{\#}}(\mathbf{Fr}(g_{n,k}, v_k))]]. \end{cases} \tag{2.72}$$

We define now a set Θ_k such that

$$\begin{cases} \Theta_k = \Theta'_k \cup \{g_k\}, \\ \forall n(n \in \mathbb{N})[g_{n,k} \in \Theta'_k \Leftrightarrow \Pi(g_{n,k}, v_k)] \end{cases} \tag{2.73}$$

Obviously definitions (2.66) and (2.73) is equivalent by Proposition 2.1.

Proposition 2.13. (i) $\mathbf{Th}_{\infty}^{\#} \vdash \exists \mathfrak{R}_{\infty}$, (ii) $\mathfrak{R}_{\infty}$ is a countable $\mathbf{Th}_{\infty}^{\#}$-set.

Proof.(i) Statement $\mathbf{Th}_{\infty}^{\#} \vdash \exists \mathfrak{R}_{\infty}$ follows immediately from the statement $\exists \mathfrak{I}_{\infty}$ and axiom schema of separation [9], (ii) follows immediately from countability of the set $\mathfrak{I}_{\infty}$.

Proposition 2.14. Set $\mathfrak{R}_{\infty}$ is inconsistent.

Proof.From the formula (2.71) we obtain

$$\mathbf{Th}_{\infty}^{\#} \vdash \mathfrak{R}_{\infty} \in \mathfrak{R}_{\infty} \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\infty}^{\#}}([\mathfrak{R}_{\infty} \notin \mathfrak{R}_{\infty}]^c) \wedge \{\mathbf{Pr}_{\mathbf{Th}_{\infty}^{\#}}([\mathfrak{R}_{\infty} \notin \mathfrak{R}_{\infty}]^c) \Rightarrow \mathfrak{R}_{\infty} \notin \mathfrak{R}_{\infty}\}. \tag{2.74}$$

From (2.74) one obtains

$$\mathbf{Th}_{\infty}^{\#} \vdash \mathfrak{R}_{\infty} \in \mathfrak{R}_{\infty} \Leftrightarrow \mathfrak{R}_{\infty} \notin \mathfrak{R}_{\infty} \tag{2.75}$$

and therefore

$$\mathbf{Th}_{\infty}^{\#} \vdash (\mathfrak{R}_{\infty} \in \mathfrak{R}_{\infty}) \wedge (\mathfrak{R}_{\infty} \notin \mathfrak{R}_{\infty}). \tag{2.76}$$

But this is a contradiction.

Definition 2.16.An $\mathbf{Th}_{\infty;\omega}^{\#}$-wff $\Phi_{\infty;\omega}$ that is: (i) $\mathbf{Th}_{\omega}$-wff Φ_{ω} or (ii) well-formed formula $\Phi_{\infty;\omega}$ which contains predicate $\mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^{\#}}([\Phi]^c)$ given by formula (2.36).An $\mathbf{Th}_{\infty;\omega}^{\#}$-wff $\Phi_{\infty;\omega}$ (well-formed formula $\Phi_{\infty;\omega}$) is closed - i.e. $\Phi_{\infty;\omega}$ is a sentence - if it has no free variables; a wff is open if it has free variables.

Definition 2.17.Let $\Psi = \Psi(x)$ be one-place open $\mathbf{Th}$-wff such that the following condition:

$$\mathbf{Th}_{\omega} \triangleq \mathbf{Th}_{\omega,1}^{\#} \vdash \exists! x_{\Psi}[\Psi(x_{\Psi})] \tag{2.77}$$

is satisfied.

Remark 2.17.We rewrite now the condition (2.77) using only lenguage of the theory $\mathbf{Th}_{\omega,1}^{\#}$:

$$\{\mathbf{Th}_{\omega,1}^{\#} \vdash \exists! x_{\Psi}[\Psi(x_{\Psi})]\} \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\omega,1}^{\#}}([\exists! x_{\Psi}[\Psi(x_{\Psi})]]^c). \tag{2.78}$$

Definition 2.18. We will say that, a set y is a $\mathbf{Th}_{\omega,1}^{\#}$-set if there exist one-place open wff $\Psi(x)$ such that $y = x_{\Psi}$. We write $y[\mathbf{Th}_{\omega,1}^{\#}]$ iff y is a $\mathbf{Th}_{\omega,1}^{\#}$-set.

Remark 2.18. Note that

$$\begin{gathered} y[\mathbf{Th}_{\omega,1}^{\#}] \Leftrightarrow \exists \Psi \Big\{ (y = x_{\Psi}) \wedge \mathbf{Pr}_{\mathbf{Th}_{\omega,1}^{\#}}([\exists! x_{\Psi}[\Psi(x_{\Psi})]]^c) \\ \Big\{ \mathbf{Pr}_{\mathbf{Th}_{\omega,1}^{\#}}([\exists! x_{\Psi}[\Psi(x_{\Psi})]]^c) \Rightarrow \exists! x_{\Psi}[\Psi(x_{\Psi})] \Big\} \Big\}. \end{gathered} \tag{2.79}$$

Definition 2.19.Let $\Im_{\omega,1}$ be a collection such that :

$$\forall x\left[x \in \Im_{\omega,1} \leftrightarrow x \text{ is a } \mathbf{Th}^{\#}_{\omega,1}\text{-set}\right]. \tag{2.80}$$

Proposition 2.15. Collection $\Im_{\omega,1}$ is a $\mathbf{Th}^{\#}_{\omega,1}$-set.

Proof. Let us consider an one-place open wff $\Psi(x)$ such that conditions (2.37) is satisfied, i.e. $\mathbf{Th}^{\#}_{\omega,1} \vdash \exists! x_\Psi[\Psi(x_\Psi)]$. We note that there exists countable collection $\mathcal{F}_\Psi$ of the one-place open wff's $\mathcal{F}_\Psi = \{\Psi_n(x)\}_{n\in\mathbb{N}}$ such that: (i) $\Psi(x) \in \mathcal{F}_\Psi$ and (ii)

$$\begin{cases} \mathbf{Th}_\omega \triangleq \mathbf{Th}^{\#}_{\omega,1} \vdash \exists! x_\Psi[[\Psi(x_\Psi)] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]\}] \\ \text{or in the equivalent form} \\ \mathbf{Th}_\omega \triangleq \mathbf{Th}^{\#}_{\omega,1} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,1}}([\exists! x_\Psi[\Psi(x_\Psi)]]^c) \wedge \\ \left[\mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,1}}([\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]^c)\right], \end{cases} \tag{2.81}$$

or in the following equivalent form

$$\begin{cases} \mathbf{Th}^{\#}_{\omega,1} \vdash \exists! x_1[[\Psi_1(x_1)] \wedge \{\forall n(n \in \mathbb{N})[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]\}] \\ \text{or} \\ \mathbf{Th}^{\#}_{\omega,1} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,1}}([\exists! x_1\Psi(x_1)]^c) \wedge \\ \left[\mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,1}}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^c)\right], \end{cases} \tag{2.82}$$

where we have set $\Psi(x) = \Psi_1(x_1), \Psi_n(x_1) = \Psi_{n,1}(x_1)$ and $x_\Psi = x_1$. We note that any collection $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n\in\mathbb{N}}, k = 1,2,\ldots$ such mentioned above, defines an unique set x_{Ψ_k}, i.e. $\mathcal{F}_{\Psi_{k_1}} \cap \mathcal{F}_{\Psi_{k_2}} = \varnothing$ iff $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$. We note that collections $\mathcal{F}_{\Psi_k}, k = 1,2,\ldots$ is not a part of the ZFC_2, i.e. collection $\mathcal{F}_{\Psi_k}$ there is no set in sense of ZFC_2. However that is no problem, because by using Gödel numbering one can to replace any collection $\mathcal{F}_{\Psi_k}, k = 1,2,\ldots$ by collection $\Theta_k = g(\mathcal{F}_{\Psi_k})$ of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n\in\mathbb{N}}, k = 1,2,\ldots\ . \tag{2.83}$$

It is easy to prove that any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,\ldots$ is a $\mathbf{Th}^{\#}_{\omega,1}$-set.This is done by Gödel encoding [8],[10] (2.83), by the statament (2.81) and by axiom schema of separation [9]. Let $g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1,2,\ldots$ be a Gödel number of the wff $\Psi_{n,k}(x_k)$. Therefore

$g(\mathcal{F}_k) = \{g_{n,k}\}_{n\in\mathbb{N}}$, where we have set $\mathcal{F}_k = \mathcal{F}_{\Psi_k},\ k = 1,2,\ldots$ and

$$\forall k_1 \forall k_2[\{g_{n,k_1}\}_{n\in\mathbb{N}} \cap \{g_{n,k_2}\}_{n\in\mathbb{N}} = \varnothing \leftrightarrow x_{k_1} \neq x_{k_2}]. \tag{2.84}$$

Let $\{\{g_{n,k}\}_{n\in\mathbb{N}}\}_{k\in\mathbb{N}}$ be a family of the all sets $\{g_{n,k}\}_{n\in\mathbb{N}}$. By axiom of choice [9] one obtains unique set $\Im'_1 = \{g_k\}_{k\in\mathbb{N}}$ such that $\forall k[g_k \in \{g_{n,k}\}_{n\in\mathbb{N}}]$. Finally one obtains a set $\Im_{\omega,1}$ from the set $\Im'_{\omega,1}$ by axiom schema of replacement [9].

Proposition 2.16. Any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,\ldots$ is a $\mathbf{Th}^{\#}_{\omega,1}$-set.

Proof. We define $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$. Therefore $g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$ (see [10]). Let us define now predicate $\Pi(g_{n,k}, v_k)$

$$\begin{gathered}\Pi(g_{n,k}, v_k) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,1}}([\exists! x_k[\Psi_{1,k}(x_1)]]^c) \wedge \\ \wedge \exists! x_k(v_k = [x_k]^c)\Big[\forall n(n \in \mathbb{N})\Big[\mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,1}}([[\Psi_{1,k}(x_k)]]^c) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,1}}(\mathbf{Fr}(g_{n,k}, v_k))\Big]\Big].\end{gathered} \tag{2.85}$$

We define now a set Θ_k such that

$$\begin{cases} \Theta_k = \Theta'_k \cup \{g_k\}, \\ \forall n(n \in \mathbb{N})[g_{n,k} \in \Theta'_k \leftrightarrow \Pi(g_{n,k}, v_k)] \end{cases} \tag{2.86}$$

Obviously definitions (2.81) and (2.86) is equivalent.

Definition 2.20.We define now the following $\mathbf{Th}^{\#}_{\omega,1}$-set $\Re_{\omega,1} \subsetneq \Im_{\omega,1}$:

$$\forall x\Big[x \in \Re_{\omega,1} \iff (x \in \Im_{\omega,1}) \wedge \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,1}}([x \notin x]^c)\Big]. \tag{2.87}$$

Proposition 2.17. (i) $\mathbf{Th}^{\#}_{\omega,1} \vdash \exists \Re_{\omega,1}$, (ii) $\Re_1$ is a countable $\mathbf{Th}^{\#}_{\omega,1}$-set.

Proof.(i) Statement $\mathbf{Th}^{\#}_{\omega,1} \vdash \exists \Re_{\omega,1}$ follows immediately from the statement $\exists \Im_{\omega,1}$ and axiom schema of separation [4], (ii) follows immediately from countability of a set $\Im_{\omega,1}$.

Proposition 2.18. A set $\Re_{\omega,1}$ is inconsistent.

Proof.From formla (2.87) we obtain

$$\mathbf{Th}^{\#}_{\omega,1} \vdash \Re_{\omega,1} \in \Re_{\omega,1} \iff \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,1}}([\Re_{\omega,1} \notin \Re_{\omega,1}]^c). \tag{2.88}$$

From (2.88) we obtain

$$\mathbf{Th}^{\#}_{\omega,1} \vdash \Re_{\omega,1} \in \Re_{\omega,1} \iff \Re_{\omega,1} \notin \Re_{\omega,1} \tag{2.89}$$

and therefore

$$\mathbf{Th}^{\#}_{\omega,1} \vdash (\Re_{\omega,1} \in \Re_{\omega,1}) \wedge (\Re_{\omega,1} \notin \Re_{\omega,1}). \tag{2.90}$$

But this is a contradiction.

Definition 2.21. Let $\Psi = \Psi(x)$ be one-place open $\mathbf{Th}$-wff such that the following condition:

$$\mathbf{Th}^{\#}_{\omega,i} \vdash \exists! x_\Psi[\Psi(x_\Psi)] \tag{2.91}$$

is satisfied.

Remark 2.19.We rewrite now the condition (2.91) using only lenguage of the theory $\mathbf{Th}^{\#}_{\omega,i}$:

$$\{\mathbf{Th}^{\#}_{\omega,i} \vdash \exists! x_\Psi[\Psi(x_\Psi)]\} \iff \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,i}}([\exists! x_\Psi[\Psi(x_\Psi)]]^c). \tag{2.92}$$

Definition 2.22. We will say that, a set y is a $\mathbf{Th}^{\#}_{\omega,i}$-set if there exist one-place open wff $\Psi(x)$ such that $y = x_\Psi$. We write $y[\mathbf{Th}^{\#}_{\omega,i}]$ iff y is a $\mathbf{Th}^{\#}_{\omega,i}$-set.

Remark 2.20. Note that

$$y[\mathbf{Th}^{\#}_{\omega,i}] \Leftrightarrow \exists\Psi\Big[(y = x_{\Psi}) \wedge \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,i}}([\exists!x_{\Psi}[\Psi(x_{\Psi})]]^{c})\Big]. \quad (2.93)$$

Definition 2.23.Let $\Im_{\omega,i}$ be a collection such that :

$$\forall x\Big[x \in \Im_{\omega,i} \leftrightarrow x \text{ is a } \mathbf{Th}^{\#}_{\omega,i}\text{-set}\Big]. \quad (2.94)$$

Proposition 2.19. Collection $\Im_{\omega,i}$ is a $\mathbf{Th}^{\#}_{\omega,i}$-set.

Proof. Let us consider an one-place open wff $\Psi(x)$ such that conditions (2.91) is satisfied, i.e. $\mathbf{Th}^{\#}_{\omega,i} \vdash \exists!x_{\Psi}[\Psi(x_{\Psi})]$.We note that there exists countable collection $\mathcal{F}_{\Psi}$ of the one-place open wff's $\mathcal{F}_{\Psi} = \{\Psi_n(x)\}_{n\in\mathbb{N}}$ such that: (i) $\Psi(x) \in \mathcal{F}_{\Psi}$ and (ii)

$$\mathbf{Th}^{\#}_{\omega,i} \vdash \exists!x_{\Psi}[[\Psi(x_{\Psi})] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi})]\}]$$

or in the equivalent form

$$\mathbf{Th}^{\#}_{\omega,i} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,i}}([\exists!x_{\Psi}[\Psi(x_{\Psi})]]^{c}) \wedge \quad (2.95)$$

$$\Big[\mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,i}}([\forall n(n \in \mathbb{N})[\Psi(x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi})]]^{c})\Big],$$

or in the following equivalent form

$$\mathbf{Th}^{\#}_{\omega,i} \vdash \exists!x_1[[\Psi_1(x_1)] \wedge \{\forall n(n \in \mathbb{N})[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]\}]$$

or

$$\mathbf{Th}^{\#}_{\omega,i} \vdash \quad (2.96)$$

$$\mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,i}}([\exists!x_1\Psi(x_1)]^{c}) \wedge$$

$$\Big[\mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,i}}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^{c})\Big].$$

where we have set $\Psi(x) = \Psi_1(x_1), \Psi_n(x_1) = \Psi_{n,1}(x_1)$ and $x_{\Psi} = x_1$. We note that any collection $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n\in\mathbb{N}}, k = 1,2,\ldots$ such mentioned above, defines an unique set x_{Ψ_k},i.e. $\mathcal{F}_{\Psi_{k_1}} \cap \mathcal{F}_{\Psi_{k_2}} = \varnothing$ iff $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$.We note that collections $\mathcal{F}_{\Psi_k}, k = 1,2,..$ is not a part of the ZFC_2,i.e. collection $\mathcal{F}_{\Psi_k}$ there is no set in sense of ZFC_2. However that is no problem, because by using Gödel numbering one can to replace any collection $\mathcal{F}_{\Psi_k}, k = 1,2,..$ by collection $\Theta_k = g(\mathcal{F}_{\Psi_k})$ of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n\in\mathbb{N}}, k = 1,2,\ldots\ . \quad (2.97)$$

It is easy to prove that any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}^{\#}_{\omega,i}$-set.This is done by Gödel encoding [8],[10] (2.97), by the statament (2.91) and by axiom schema of separation [9]. Let $g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1,2,..$ be a Gödel number of the wff $\Psi_{n,k}(x_k)$. Therefore $g(\mathcal{F}_k) = \{g_{n,k}\}_{n\in\mathbb{N}}$, where we have set $\mathcal{F}_k = \mathcal{F}_{\Psi_k},\ k = 1,2,..$ and

$$\forall k_1\forall k_2[\{g_{n,k_1}\}_{n\in\mathbb{N}} \cap \{g_{n,k_2}\}_{n\in\mathbb{N}} = \varnothing \leftrightarrow x_{k_1} \neq x_{k_2}]. \quad (2.98)$$

Let $\{\{g_{n,k}\}_{n\in\mathbb{N}}\}_{k\in\mathbb{N}}$ be a family of the all sets $\{g_{n,k}\}_{n\in\mathbb{N}}$. By axiom of choice [9] one obtains unique set $\Im'_i = \{g_k\}_{k\in\mathbb{N}}$ such that $\forall k[g_k \in \{g_{n,k}\}_{n\in\mathbb{N}}]$.Finally one obtains a set

$\mathfrak{I}_{\omega,i}$ from the set $\mathfrak{I}'_i$ by axiom schema of replacement [9].

Proposition 2.20. Any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}^{\#}_{\omega,i}$-set.

Proof. We define $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$. Therefore $g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$ (see [10]). Let us define now predicate $\Pi_{\omega,i}(g_{n,k}, v_k)$

$$\Pi_{\omega,i}(g_{n,k}, v_k) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,i}}([\exists! x_k[\Psi_{1,k}(x_1)]]^c) \wedge$$
$$\wedge \exists! x_k(v_k = [x_k]^c)\Big[\forall n(n \in \mathbb{N})\Big[\mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,i}}([[\Psi_{1,k}(x_k)]]^c) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,i}}(\mathbf{Fr}(g_{n,k}, v_k))\Big]\Big]. \tag{2.99}$$

We define now a set Θ_k such that

$$\Theta_k = \Theta'_k \cup \{g_k\},$$
$$\forall n(n \in \mathbb{N})[g_{n,k} \in \Theta'_k \leftrightarrow \Pi_{\omega,i}(g_{n,k}, v_k)]. \tag{2.100}$$

Obviously definitions (2.91 and (2.100) is equivalent.

Definition 2.24.We define now the following $\mathbf{Th}^{\#}_{\omega,i}$-set $\mathfrak{R}_{\omega,i} \subsetneqq \mathfrak{I}_{\omega,i}$:

$$\forall x\Big[x \in \mathfrak{R}_{\omega,i} \iff (x \in \mathfrak{I}_{\omega,i}) \wedge \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,i}}([x \notin x]^c)\Big]. \tag{2.101}$$

Proposition 2.21. (i) $\mathbf{Th}^{\#}_{\omega,i} \vdash \exists \mathfrak{R}_{\omega,i}$,(ii) $\mathfrak{R}_{\omega,i}$ is a countable $\mathbf{Th}^{\#}_{\omega,i}$-set,$i \in \mathbb{N}$.

Proof.(i) Statement $\mathbf{Th}^{\#}_{\omega,i} \vdash \exists \mathfrak{R}_{\omega,i}$ follows immediately by using statement $\exists \mathfrak{I}_{\omega,i}$ and axiom

schema of separation [9]. (ii) follows immediately from countability of a set $\mathfrak{I}_{\omega,i}$.

Proposition 2.22. Any set $\mathfrak{R}_{\omega,i}, i \in \mathbb{N}$ is inconsistent.

Proof.From formla (2.101) we obtain

$$\mathbf{Th}^{\#}_{\omega,i} \vdash \mathfrak{R}_{\omega,i} \in \mathfrak{R}_{\omega,i} \iff \mathbf{Pr}_{\mathbf{Th}^{\#}_{\omega,i}}([\mathfrak{R}_{\omega,i} \notin \mathfrak{R}_{\omega,i}]^c). \tag{2.102}$$

From (2.102) we obtain

$$\mathbf{Th}^{\#}_{\omega,i} \vdash \mathfrak{R}_{\omega,i} \in \mathfrak{R}_{\omega,i} \iff \mathfrak{R}_{\omega,i} \notin \mathfrak{R}_{\omega,i} \tag{2.103}$$

and therefore

$$\mathbf{Th}^{\#}_{\omega,i} \vdash (\mathfrak{R}_{\omega,i} \in \mathfrak{R}_{\omega,}) \wedge (\mathfrak{R}_{\omega,i} \notin \mathfrak{R}_{\omega,i}). \tag{2.104}$$

But this is a contradiction.

Definition 2.25.Let $\Psi = \Psi(x)$ be one-place open $\mathbf{Th}^{\#}_{\infty;\omega}$-wff such that the following condition:

$$\mathbf{Th}^{\#}_{\infty;\omega} \vdash \exists! x_\Psi[\Psi(x_\Psi)] \tag{2.105}$$

is satisfied.

Remark 2.20.We rewrite now the condition (2.65) using only lenguage of the theory $\mathbf{Th}^{\#}_{\infty}$:

$$\{\mathbf{Th}^{\#}_{\infty;\omega} \vdash \exists! x_\Psi[\Psi(x_\Psi)]\} \iff \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}([\exists! x_\Psi[\Psi(x_\Psi)]]^c) \tag{2.106}$$

Definition 2.26.We will say that, a set y is a $\mathbf{Th}^{\#}_{\infty;\omega}$-set if there exist one-place open wff

$\Psi(x)$ such that $y = x_\Psi$. We write $y[\mathbf{Th}^{\#}_{\infty;\omega}]$ iff y is a $\mathbf{Th}^{\#}_{\infty;\omega}$-set.

Definition 2.27. Let $\Im_{\infty;\omega}$ be a collection such that

: $\forall x\big[x \in \Im_{\infty;\omega} \leftrightarrow x$ is a $\mathbf{Th}^{\#}_{\infty;\omega}$-set$\big]$.

Proposition 2.23. Collection $\Im_{\infty;\omega}$ is a $\mathbf{Th}^{\#}_{\infty;\omega}$-set.

Proof. Let us consider an one-place open wff $\Psi(x)$ such that condition (2.65) is satisfied, i.e. $\mathbf{Th}^{\#}_{\infty;\omega} \vdash \exists! x_\Psi[\Psi(x_\Psi)]$. We note that there exists countable collection $\mathcal{F}_\Psi$ of the one-place open wff's $\mathcal{F}_\Psi = \{\Psi_n(x)\}_{n\in\mathbb{N}}$ such that: (i) $\Psi(x) \in \mathcal{F}_\Psi$ and (ii)

$$\begin{gathered}\mathbf{Th}^{\#}_{\infty;\omega} \vdash \exists! x_\Psi[[\Psi(x_\Psi)] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]\}]\\ \text{or in the equivalent form}\\ \mathbf{Th}^{\#}_{\infty;\omega} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}([\exists! x_\Psi[\Psi(x_\Psi)]]^c) \wedge\\ \Big[\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}([\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]^c)\Big],\end{gathered} \tag{2.107}$$

or in the following equivalent form

$$\begin{gathered}\mathbf{Th}^{\#}_{\infty;\omega} \vdash \exists! x_1[[\Psi_1(x_1)] \wedge \{\forall n(n \in \mathbb{N})[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]\}]\\ \text{or}\\ \mathbf{Th}^{\#}_{\infty;\omega} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}([\exists! x_1 \Psi(x_1)]^c) \wedge\\ \Big[\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^c)\Big],\end{gathered} \tag{2.108}$$

where we set $\Psi(x) = \Psi_1(x_1), \Psi_n(x_1) = \Psi_{n,1}(x_1)$ and $x_\Psi = x_1$. We note that any collection $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n\in\mathbb{N}}, k = 1,2,\ldots$ such above defines an unique set x_{Ψ_k}, i.e. $\mathcal{F}_{\Psi_{k_1}} \bigcap \mathcal{F}_{\Psi_{k_2}} = \varnothing$ iff $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$. We note that collections $\mathcal{F}_{\Psi_k}, k = 1,2,..$ is no part of the ZFC_2, i.e. collection $\mathcal{F}_{\Psi_k}$ there is no set in sense of ZFC_2. However that is no problem, because by using Gödel numbering one can to replace any collection $\mathcal{F}_{\Psi_k}, k = 1,2,..$ by collection $\Theta_k = g(\mathcal{F}_{\Psi_k})$ of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n\in\mathbb{N}}, k = 1,2,\ldots . \tag{2.109}$$

It is easy to prove that any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}^{\#}_{\infty;\omega}$-set.This is done by Gödel encoding [8],[10] by the statament (2.109) and by axiom schema of separation [9]. Let $g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1,2,..$ be a Gödel number of the wff $\Psi_{n,k}(x_k)$. Therefore $g(\mathcal{F}_k) = \{g_{n,k}\}_{n\in\mathbb{N}}$, where we have set $\mathcal{F}_k = \mathcal{F}_{\Psi_k},\ k = 1,2,..$ and

$$\forall k_1 \forall k_2[\{g_{n,k_1}\}_{n\in\mathbb{N}} \bigcap \{g_{n,k_2}\}_{n\in\mathbb{N}} = \varnothing \leftrightarrow x_{k_1} \neq x_{k_2}]. \tag{2.110}$$

Let $\{\{g_{n,k}\}_{n\in\mathbb{N}}\}_{k\in\mathbb{N}}$ be a family of the all sets $\{g_{n,k}\}_{n\in\mathbb{N}}$. By axiom of choice [9] one obtains unique set $\Im' = \{g_k\}_{k\in\mathbb{N}}$ such that $\forall k[g_k \in \{g_{n,k}\}_{n\in\mathbb{N}}]$. Finally one obtains a set $\Im_{\infty;\omega}$ from the set $\Im'_{\infty;\omega}$ by axiom schema of replacement [9].Thus one can define

$\mathbf{Th}^{\#}_{\infty;\omega}$-set $\Re_{\infty;\omega} \subsetneqq \Im_{\infty;\omega}$:

$$\forall x\Big[x \in \Re_{\infty;\omega} \Leftrightarrow (x \in \Im_{\infty;\omega}) \wedge \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}([x \notin x]^c)\Big]. \tag{2.111}$$

Proposition 2.24. Any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}^{\#}_{\infty;\omega}$-set.

Proof. We define $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$. Therefore $g_{n,k} = g(\Psi_{n,k}(x_k)) \Leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$ (see [10]). Let us define now predicate $\Pi_{\infty;\omega}(g_{n,k}, v_k)$

$$\begin{gathered}\Pi_{\infty;\omega}(g_{n,k}, v_k) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}([\exists! x_k[\Psi_{1,k}(x_1)]]^c) \wedge \\ \exists! x_k(v_k = [x_k]^c)\Big[\forall n(n \in \mathbb{N})\Big[\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}([[\Psi_{1,k}(x_k)]]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}(\mathbf{Fr}(g_{n,k}, v_k))\Big]\Big].\end{gathered} \tag{2.112}$$

We define now a set Θ_k such that

$$\begin{cases} \Theta_k = \Theta'_k \cup \{g_k\}, \\ \forall n(n \in \mathbb{N})[g_{n,k} \in \Theta'_k \leftrightarrow \Pi_{\infty;\omega}(g_{n,k}, v_k)] \end{cases} \tag{2.113}$$

Obviously definitions (2.106) and (2.113) is equivalent by Proposition 2.1.

Proposition 2.25. (i) $\mathbf{Th}^{\#}_{\infty;\omega} \vdash \exists \Re_{\infty;\omega}$, (ii) $\Re_{\infty;\omega}$ is a countable $\mathbf{Th}^{\#}_{\infty;\omega}$-set.

Proof.(i) Statement $\mathbf{Th}^{\#}_{\infty;\omega} \vdash \exists \Re_{\infty;\omega}$ follows immediately from the statement $\exists \Im$ and axiom

schema of separation [9], (ii) follows immediately from countability of the set $\Im_\infty$.

Proposition 2.26. Set $\Re_{\infty;\omega}$ is inconsistent.

Proof.From the formula (2.71) we obtain

$$\mathbf{Th}^{\#}_{\infty;\omega} \vdash \Re_{\infty;\omega} \in \Re_{\infty;\omega} \leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;\omega}}([\Re_{\infty;\omega} \notin \Re_{\infty;\omega}]^c). \tag{2.114}$$

From the formula (2.114) and Proposition 2.1 we obtain

$$\mathbf{Th}^{\#}_{\infty;\omega} \vdash \Re_{\infty;\omega} \in \Re_{\infty;\omega} \leftrightarrow \Re_{\infty;\omega} \notin \Re_{\infty;\omega} \tag{2.115}$$

and therefore

$$\mathbf{Th}^{\#}_{\infty;\omega} \vdash (\Re_{\infty;\omega} \in \Re_{\infty;\omega}) \wedge (\Re_{\infty;\omega} \notin \Re_{\infty;\omega}). \tag{2.116}$$

But this is a contradiction.

Proposition 2.26.Assume that (i) $Con(\mathbf{Th})$ and (ii) $\mathbf{Th}$ has an nonstandard model $M^{\mathbf{Th}}_{Nst}$ and $M^{Z_2}_{\omega} \subset M^{\mathbf{Th}}_{Nst}$. Then theory $\mathbf{Th}$ can be extended to a maximally consistent nice theory $\mathbf{Th}^{\#}_{\infty} \triangleq \mathbf{Th}^{\#}_{\infty}[M^{\mathbf{Th}}_{Nst}]$.

Proof. Let $\Phi_1 \ldots \Phi_i \ldots$ be an enumeration of all wff's of the theory $\mathbf{Th}$ (this can be achieved if the set of propositional variables can be enumerated). Define a chain $\wp = \{\mathbf{Th}^{\#}_{Nst,i} | i \in \mathbb{N}\}, \mathbf{Th}^{\#}_{Nst,1} = \mathbf{Th}$ of consistent theories inductively as follows: assume that theory $\mathbf{Th}_i$ is defined. (i) Suppose that a statement (2.117) is satisfied

$$\mathbf{Th}^{\#}_{Nst,i} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\Phi_i]^c) \text{ and } [\mathbf{Th}^{\#}_{Nst,i} \nvdash \Phi_i] \wedge [M^{\mathbf{Th}}_{Nst} \vDash \Phi_i]. \tag{2.117}$$

Then we define a theory $\mathbf{Th}_{Nst,i+1}$ as follows $\mathbf{Th}_{Nst,i+1} \triangleq \mathbf{Th}_{Nst,i} \cup \{\Phi_i\}$. Using Lemma 2.1 we will rewrite the condition (2.117) symbolically as follows

$$\begin{cases} \mathbf{Th}^{\#}_{Nst,i} \vdash \mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{Nst,i}}([\Phi_i]^c), \\ \mathbf{Pr}^{\#}_{\mathbf{Th}_i}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\Phi_i]^c) \wedge [M^{\mathbf{Th}}_{Nst} \models \Phi_i]. \end{cases} \tag{2.118}$$

(ii) Suppose that the statement (2.119) is satisfied

$$\mathbf{Th}^{\#}_{Nst,i} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\neg\Phi_i]^c) \text{ and } [\mathbf{Th}^{\#}_{Nst,i} \nvdash \neg\Phi_i] \wedge [M^{\mathbf{Th}}_{Nst} \models \neg\Phi_i]. \tag{2.119}$$

Then we define theory $\mathbf{Th}_{i+1}$ as follows: $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\neg\Phi_i\}$. Using Lemma 2.2 we will rewrite the condition (2.119) symbolically as follows

$$\begin{cases} \mathbf{Th}^{\#}_{Nst,i} \vdash \mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{Nst,i}}([\neg\Phi_i]^c) \ , \\ \mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{Nst,i}}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\neg\Phi_i]^c) \wedge [M^{\mathbf{Th}}_{\omega} \models \neg\Phi_i]. \end{cases} \tag{2.120}$$

(iii) Suppose that a statement (2.121) is satisfied

$$\mathbf{Th}^{\#}_{Nst,i} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\Phi_i]^c) \text{ and } \mathbf{Th}^{\#}_{Nst,i} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\Phi_i]^c) \Rightarrow \Phi_i. \tag{2.121}$$

We will rewrite the condition (2.121) symbolically as follows

$$\begin{cases} \mathbf{Th}^{\#}_{Nst,i} \vdash \mathbf{Pr}^{*}_{\mathbf{Th}^{\#}_{Nst,i}}([\Phi_i]^c), \\ \mathbf{Pr}^{*}_{\mathbf{Th}^{\#}_{Nst,i}}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \Rightarrow \Phi_i] \end{cases} \tag{2.122}$$

Then we define a theory $\mathbf{Th}^{\#}_{Nst,i+1}$ as follows: $\mathbf{Th}^{\#}_{Nst,i+1} \triangleq \mathbf{Th}^{\#}_{Nst,i}$.

(iv) Suppose that the statement (2.123) is satisfied

$$\mathbf{Th}^{\#}_{Nst,i+1} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\neg\Phi_i]^c) \text{ and } \mathbf{Th}^{\#}_{Nst,i} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\neg\Phi_i]^c) \Rightarrow \neg\Phi_i. \tag{2.123}$$

We will rewrite the condition (2.123) symbolically as follows

$$\begin{gathered} \mathbf{Th}^{\#}_{Nst,i} \vdash \mathbf{Pr}^{*}_{\mathbf{Th}^{\#}_{Nst,i}}([\neg\Phi_i]^c), \\ \mathbf{Pr}^{*}_{\mathbf{Th}^{\#}_{Nst,i}}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\neg\Phi_i]^c) \wedge \left[\mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\neg\Phi_i]^c) \Rightarrow \neg\Phi_i\right] \end{gathered} \tag{2.124}$$

Then we define a theory $\mathbf{Th}^{\#}_{Nst,i+1}$ as follows: $\mathbf{Th}^{\#}_{Nst,i+1} \triangleq \mathbf{Th}^{\#}_{Nst,i}$. We define now a theory $\mathbf{Th}^{\#}_{\infty;Nst}$ as follows:

$$\mathbf{Th}^{\#}_{\infty;Nst} \triangleq \bigcup_{i\in\mathbb{N}} \mathbf{Th}^{\#}_{Nst,i}. \tag{2.125}$$

First, notice that each $\mathbf{Th}^{\#}_{Nst,i}$ is consistent. This is done by induction on i and by Lemmas 2.1-2.2. By assumption, the case is true when $i = 1$. Now, suppose $\mathbf{Th}^{\#}_{Nst,i}$ is consistent. Then its deductive closure $\mathbf{Ded}(\mathbf{Th}^{\#}_{Nst,i}) \triangleq \{A | \mathbf{Th}^{\#}_{Nst,i} \vdash A\}$ is also consistent. If a statement (2.121) is satisfied,i.e. $\mathbf{Th}^{\#}_{Nst,i} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\Phi_i]^c)$ and $\mathbf{Th}^{\#}_{Nst,i} \vdash \Phi_i$, then clearly $\mathbf{Th}^{\#}_{Nst,i+1} \triangleq \mathbf{Th}^{\#}_{Nst,i} \cup \{\Phi_i\}$ is consistent since it is a subset of closure $\mathbf{Ded}(\mathbf{Th}^{\#}_{Nst,i})$. If a statement (2.123) is satisfied,i.e. $\mathbf{Th}^{\#}_{Nst,i} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\neg\Phi_i]^c)$ and

$\mathbf{Th}^{\#}_{Nst,i} \vdash \neg\Phi_i$, then clearly $\mathbf{Th}^{\#}_{Nst,i+1} \triangleq \mathbf{Th}^{\#}_{Nst,i} \cup \{\neg\Phi_i\}$ is consistent since it is a subset of closure $\mathbf{Ded}(\mathbf{Th}^{\#}_{Nst,i})$. If a statement (2.117) is satisfied,i.e. $\mathbf{Th}^{\#}_{Nst,i} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\Phi_i]^c)$ and $[\mathbf{Th}^{\#}_{Nst,i} \nvdash \Phi_i] \wedge [M^{\mathbf{Th}}_{Nst} \models \Phi_i]$ then clearly $\mathbf{Th}^{\#}_{Nst,i+1} \triangleq \mathbf{Th}^{\#}_{Nst,i} \cup \{\Phi_i\}$ is consistent by Lemma 2.1 and by one of the standard properties of consistency: $\Delta \cup \{A\}$ is consistent iff $\Delta \nvdash \neg A$. If a statement (2.119) is satisfied,i.e. $\mathbf{Th}^{\#}_{Nst,i} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\neg\Phi_i]^c)$ and $[\mathbf{Th}^{\#}_{Nst,i} \nvdash \neg\Phi_i] \wedge [M^{\mathbf{Th}}_{Nst} \models \neg\Phi_i]$ then clearly $\mathbf{Th}^{\#}_{Nst,i+1} \triangleq \mathbf{Th}^{\#}_{Nst,i} \cup \{\neg\Phi_i\}$ is consistent by Lemma 2.2 and by one of the standard properties of consistency: $\Delta \cup \{\neg A\}$ is consistent iff $\Delta \nvdash A$. Next, notice $\mathbf{Ded}(\mathbf{Th}^{\#}_{\infty;Nst})$ is maximally consistent nice extension of the $\mathbf{Ded}(\mathbf{Th})$. $\mathbf{Ded}(\mathbf{Th}^{\#}_{\infty;Nst})$ is consistent because, by the standard Lemma 2.3 above, it is the union of a chain of consistent sets. To see that $\mathbf{Ded}(\mathbf{Th}^{\#}_{\infty;Nst})$ is maximal, pick any wff Φ. Then Φ is some Φ_i in the enumerated list of all wff's. Therefore for any Φ such that $\mathbf{Th}^{\#}_{Nst,i} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\Phi]^c)$ or $\mathbf{Th}^{\#}_{Nst,i} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{Nst,i}}([\neg\Phi]^c)$, either $\Phi \in \mathbf{Th}^{\#}_{\infty;Nst}$ or $\neg\Phi \in \mathbf{Th}^{\#}_{\infty;Nst}$. Since $\mathbf{Ded}(\mathbf{Th}^{\#}_{Nst,i+1}) \subsetneqq \mathbf{Ded}(\mathbf{Th}^{\#}_{\infty;Nst})$, we have $\Phi \in \mathbf{Ded}(\mathbf{Th}^{\#}_{\infty;Nst})$ or $\neg\Phi \in \mathbf{Ded}(\mathbf{Th}^{\#}_{\infty;Nst})$, which implies that $\mathbf{Ded}(\mathbf{Th}^{\#}_{\infty;Nst})$ is maximally consistent nice extension of the $\mathbf{Ded}(\mathbf{Th})$.

Definition 2.28. We define now predicate $\mathbf{Pr}_{\mathbf{Th}^{\#}}([\Phi_i]^c)$ asserting provability in $\mathbf{Th}^{\#}_{\infty;Nst}$:

$$\begin{cases} \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;Nst}}([\Phi_i]^c) \Leftrightarrow \left[\mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{\infty;Nst}}([\Phi_i]^c)\right] \vee \left[\mathbf{Pr}^{*}_{\mathbf{Th}^{\#}_{\infty;Nst}}([\Phi_i]^c)\right], \\ \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;Nst}}([\neg\Phi_i]^c) \Leftrightarrow \left[\mathbf{Pr}^{\#}_{\mathbf{Th}^{\#}_{\infty;Nst}}([\neg\Phi_i]^c)\right] \vee \left[\mathbf{Pr}^{*}_{\mathbf{Th}^{\#}_{\infty;Nst}}([\neg\Phi_i]^c)\right]. \end{cases} \tag{2.126}$$

Definition 2.29. Let $\Psi = \Psi(x)$ be one-place open wff such that the conditions:

$(*)$ $\mathbf{Th}^{\#}_{\infty;Nst} \vdash \exists! x_\Psi[\Psi(x_\Psi)]$ or

$(**)$ $\mathbf{Th}^{\#}_{\infty;Nst} \vdash \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;Nst}}([\exists! x_\Psi[\Psi(x_\Psi)]]^c)$ and $M^{\mathbf{Th}}_{Nst} \models \exists! x_\Psi[\Psi(x_\Psi)]$ is satisfied.

Then we said that, a set y is a $\mathbf{Th}^{\#}$-set iff there is exist one-place open wff $\Psi(x)$ such that

$y = x_\Psi$. We write $y[\mathbf{Th}^{\#}_{\infty;Nst}]$ iff y is a $\mathbf{Th}^{\#}_{\infty;Nst}$-set.

Remark 2.21. Note that $[(*) \vee (**)] \Rightarrow \mathbf{Th}^{\#}_{\infty;Nst} \vdash \exists! x_\Psi[\Psi(x_\Psi)]$.

Remark 2.22. Note that $y[\mathbf{Th}^{\#}_{\infty;Nst}] \Leftrightarrow \exists\Psi\left[(y = x_\Psi) \wedge \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;Nst}}([\exists! x_\Psi[\Psi(x_\Psi)]]^c)\right]$

Definition 2.30.Let $\Im^{\#}_{\infty;Nst}$ be a collection such that

: $\forall x\left[x \in \Im^{\#}_{\infty;Nst} \leftrightarrow x \text{ is a } \mathbf{Th}^{\#}\text{-set}\right]$.

Proposition 2.27.Collection $\Im^{\#}_{\infty;Nst}$ is a $\mathbf{Th}^{\#}_{\infty;Nst}$-set.

Proof. Let us consider an one-place open wff $\Psi(x)$ such that conditions $(*)$ or $(**)$ is satisfied, i.e. $\mathbf{Th}^{\#} \vdash \exists! x_\Psi[\Psi(x_\Psi)]$. We note that there exists countable collection $\mathcal{F}_\Psi$ of the one-place open wff's $\mathcal{F}_\Psi = \{\Psi_n(x)\}_{n\in\mathbb{N}}$ such that: (i) $\Psi(x) \in \mathcal{F}_\Psi$ and (ii)

$$\mathbf{Th}^{\#}_{\infty;Nst} \vdash \exists! x_{\Psi}\Big[[\Psi(x_{\Psi})] \wedge \Big\{\forall n\Big(n \in M^{Z_2^{Hs}}_{\boldsymbol{\omega}}\Big)[\Psi(x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi})]\Big\}\Big]$$

or

$$\mathbf{Th}^{\#}_{\infty;Nst} \vdash \exists! x_{\Psi}\Big[\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;Nst}}([\Psi(x_{\Psi})]^c) \wedge \Big\{\forall n\Big(n \in M^{Z_2^{Hs}}_{\boldsymbol{\omega}}\Big)\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;Nst}}([\Psi(x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi})]^c)\Big\}\Big] \quad (2.127)$$

and

$$M^{\mathbf{Th}}_{Nst} \models \exists! x_{\Psi}\Big[[\Psi(x_{\Psi})] \wedge \Big\{\forall n\Big(n \in M^{Z_2^{Hs}}_{\boldsymbol{\omega}}\Big)[\Psi(x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi})]\Big\}\Big]$$

or of the equivalent form

$$\mathbf{Th}^{\#}_{\infty;Nst} \vdash \exists! x_1\Big[[\Psi_1(x_1)] \wedge \Big\{\forall n\Big(n \in M^{Z_2^{Hs}}_{\boldsymbol{\omega}}\Big)[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]\Big\}\Big]$$

or

$$\mathbf{Th}^{\#}_{\infty;Nst} \vdash \exists! x_{\Psi}\Big[\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;Nst}}([\Psi(x_1)]^c) \wedge \Big\{\forall n\Big(n \in M^{Z_2^{Hs}}_{\boldsymbol{\omega}}\Big)\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;Nst}}([\Psi(x_1) \leftrightarrow \Psi_n(x_1)]^c)\Big\}\Big] \quad (2.128)$$

and

$$M^{\mathbf{Th}}_{Nst} \models \exists! x_{\Psi}\Big[[\Psi(x_1)] \wedge \Big\{\forall n\Big(n \in M^{Z_2^{Hs}}_{\boldsymbol{\omega}}\Big)[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]\Big\}\Big]$$

where we set $\Psi(x) = \Psi_1(x_1), \Psi_n(x_1) = \Psi_{n,1}(x_1)$ and $x_{\Psi} = x_1$. We note that any collection $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n\in\mathbb{N}}, k = 1,2,\ldots$ such above defines an unique set x_{Ψ_k}, i.e. $\mathcal{F}_{\Psi_{k_1}} \bigcap \mathcal{F}_{\Psi_{k_2}} = \varnothing$ iff $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$. We note that collections $\mathcal{F}_{\Psi_k}, k = 1,2,..$ is no part of the ZFC_2^{Hs}, i.e. collection $\mathcal{F}_{\Psi_k}$ there is no set in sense of ZFC_2^{Hs}. However that is no problem, because by using Gödel numbering one can to replace any collection $\mathcal{F}_{\Psi_k}, k = 1,2,..$ by collection $\Theta_k = g(\mathcal{F}_{\Psi_k})$ of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n\in\mathbb{N}}, k = 1,2,\ldots\ . \quad (2.129)$$

It is easy to prove that any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}^{\#}_{\infty;Nst}$-set. This is done by Gödel encoding [8],[10] (2.129) and by axiom schema of separation [9]. Let $g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1,2,..$ be a Gödel number of the wff $\Psi_{n,k}(x_k)$. Therefore $g(\mathcal{F}_k) = \{g_{n,k}\}_{n\in\mathbb{N}}$, where we set $\mathcal{F}_k = \mathcal{F}_{\Psi_k},\ k = 1,2,..$ and

$$\forall k_1 \forall k_2[\{g_{n,k_1}\}_{n\in\mathbb{N}} \bigcap \{g_{n,k_2}\}_{n\in\mathbb{N}} = \varnothing \leftrightarrow x_{k_1} \neq x_{k_2}]. \quad (2.130)$$

Let $\{\{g_{n,k}\}_{n\in\mathbb{N}}\}_{k\in\mathbb{N}}$ be a family of the all sets $\{g_{n,k}\}_{n\in\mathbb{N}}$. By axiom of choice [9] one obtain unique set $\mathfrak{I}^{\#\prime}_{\infty;Nst} = \{g_k\}_{k\in\mathbb{N}}$ such that $\forall k[g_k \in \{g_{n,k}\}_{n\in\mathbb{N}}]$. Finally one obtain a set $\mathfrak{I}^{\#}_{\infty;Nst}$ from a set $\mathfrak{I}^{\#\prime}_{\infty;Nst}$ by axiom schema of replacement [9]. Thus we can define a $\mathbf{Th}^{\#}_{\infty;Nst}$-set

$\mathfrak{R}^{\#}_{\infty;Nst} \subsetneqq \mathfrak{I}^{\#}_{\infty;Nst}$:

$$\forall x\Big[x \in \mathfrak{R}^{\#}_{\infty;Nst} \leftrightarrow (x \in \mathfrak{I}^{\#}_{\infty;Nst}) \wedge \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;Nst}}([x \notin x]^c) \wedge \Big(\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;Nst}}([x \notin x]^c \Rightarrow x \notin x)\Big)\Big]. \quad (2.131)$$

Proposition 2.28. Any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}^{\#}_{\infty;Nst}$-set.

Proof. We define $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$. Therefore $g_{n,k} = g(\Psi_{n,k}(x_k)) \iff \mathbf{Fr}(g_{n,k}, v_k)$ (see [10]). Let us define now predicate $\Pi_\infty(g_{n,k}, v_k)$

$$\Pi_\infty(g_{n,k}, v_k) \iff \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;Nst}}([\exists! x_k[\Psi_{1,k}(x_1)]]^c) \wedge$$
$$\wedge \exists! x_k(v_k = [x_k]^c) \quad (2.132)$$
$$\left[\forall n\left(n \in M_{\mathbf{st}}^{Z_2^{Hs}}\right)\left[\mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;Nst}}([[\Psi_{1,k}(x_k)]]^c) \iff \mathbf{Pr}_{\mathbf{Th}^{\#}_{\infty;Nst}}(\mathbf{Fr}(g_{n,k}, v_k))\right]\right].$$

We define now a set Θ_k such that

$$\begin{cases} \Theta_k = \Theta'_k \cup \{g_k\}, \\ \forall n(n \in \mathbb{N})[g_{n,k} \in \Theta'_k \leftrightarrow \Pi_\infty(g_{n,k}, v_k)] \end{cases} \quad (2.133)$$

But obviously definitions (2.29) and (2.133) is equivalent by Proposition 2.26.

Proposition 2.28. (i) $\mathbf{Th}^{\#}_{\infty;Nst} \vdash \exists \mathfrak{R}^{\#}_{\infty;Nst}$, (ii) $\mathfrak{R}^{\#}_{\infty;Nst}$ is a countable $\mathbf{Th}^{\#}_{\infty;Nst}$-set.

Proof.(i) Statement $\mathbf{Th}^{\#} \vdash \exists \mathfrak{R}_c$ follows immediately from the statement $\exists \mathfrak{I}^{\#}_{\infty;Nst}$ and axiom

schema of separation [9]. (ii) follows immediately from countability of the set $\mathfrak{I}^{\#}_{\infty;Nst}$.

Proposition 2.29. A set $\mathfrak{R}^{\#}_{\infty;Nst}$ is inconsistent.

Proof.From formla (2.131) we obtain

$$\mathbf{Th}^{\#}_{\infty;Nst} \vdash \mathfrak{R}^{\#}_{\infty;Nst} \in \mathfrak{R}^{\#}_{\infty;Nst} \iff \mathfrak{R}^{\#}_{\infty;Nst} \notin \mathfrak{R}^{\#c}_{\infty;Nst}. \quad (2.134)$$

From formula (2.41) and Proposition 2.6 one obtains

$$\mathbf{Th}^{\#}_{\infty;Nst} \vdash \mathfrak{R}^{\#}_{\infty;Nst} \in \mathfrak{R}^{\#}_{\infty;Nst} \iff \mathfrak{R}^{\#}_{\infty;Nst} \notin \mathfrak{R}^{\#}_{\infty;Nst} \quad (2.135)$$

and therefore

$$\mathbf{Th}^{\#}_{\infty;Nst} \vdash (\mathfrak{R}^{\#}_{\infty;Nst} \in \mathfrak{R}^{\#}_{\infty;Nst}) \wedge (\mathfrak{R}^{\#}_{\infty;Nst} \notin \mathfrak{R}^{\#}_{\infty;Nst}). \quad (2.136)$$

But this is a contradiction.

Derivation inconsistensy of the set theory $ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$ using Generalized Tarski's undefinability theorem.

Now we will prove that a set theory $ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$ is inconsistent, without any refference

to the set $\mathfrak{I}_\infty$ and inconsistent set $\mathfrak{R}_\infty$.

Proposition 2.30.(Generalized Tarski's undefinability theorem).Let $\mathbf{Th}^{Hs}_{\mathcal{L}}$ be second order

theory with Henkin semantics and with formal language $\mathcal{L}$, which includes negation and has a Gödel encoding $g(\cdot)$ such that for every $\mathcal{L}$-formula $A(x)$ there is a formula B such

that $B \Leftrightarrow A(g(B)) \wedge [A(g(B) \Rightarrow B]$ holds. Assume that $\mathbf{Th}_{\mathcal{L}}^{Hs}$ has an standard Model M.

Then there is no $\mathcal{L}$-formula $\mathbf{True}(n)$ such that for every $\mathcal{L}$-formula A such that $M \models A$, the

following equivalence

$$A \Leftrightarrow \mathbf{True}(g(A)) \wedge [\mathbf{True}(g(A)) \Rightarrow A] \tag{1.137}$$

holds.

Proof.The diagonal lemma yields a counterexample to this equivalence, by giving a "Liar"

sentence S such that $S \Leftrightarrow \neg\mathbf{True}(g(S))$ holds.

Remark 2.23. Above we defined the set $\Im_\infty$ (see Definition 2.10) in fact using generalized

"truth predicate" $\mathbf{True}_\infty^\#([\Phi]^c, \Phi)$ such that

$$\mathbf{True}_\infty^\#([\Phi]^c, \Phi) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\Phi]^c) \wedge \{\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\Phi]^c) \Rightarrow \Phi\}. \tag{2.138}$$

In order to prove that set theory $ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$ is inconsistent without any refference to

the set $\Im_\infty$,notice that by the properties of the nice extension $\mathbf{Th}_\infty^\#$ follows that definition

given by (2.138) is correct, i.e.,for every ZFC_2^{Hs}-formula Φ such that $M^{ZFC_2^{Hs}} \models \Phi$ the following equivalence

$$\Phi \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\Phi]^c) \wedge \{\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\Phi]^c) \Rightarrow \Phi\}. \tag{2.139}$$

holds.

Proposition 2.31.Set theory $\mathbf{Th}_\mathbf{1}^\# = ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$ is inconsistent.

Proof.Notice that by the properties of the nice extension $\mathbf{Th}_\infty^\#$ of the$\mathbf{Th}_\mathbf{1}^\#$ follows that

$$M^{ZFC_2^{Hs}} \models \Phi \Rightarrow \mathbf{Th}_\infty^\# \vdash \Phi. \tag{2.140}$$

Therefore (2.138) gives generalized "truth predicate" for set theory $\mathbf{Th}_\mathbf{1}^\#$.By Proposition

2.30 one obtains a contradiction.

Remark 2.24.A cardinal κ is inaccessible if and only if κ has the following reflection property: for all subsets $U \subset V_\kappa$, there exists $\alpha < \kappa$ such that $(V_\alpha, \in, U \cap V_\alpha)$ is an elementary substructure of $(V_\kappa, \in, U)$. (In fact, the set of such α is closed unbounded in κ.) Equivalently, κ is Π_n^0 -indescribable for all $n \geq 0$.

Remark 2.25.Under ZFC it can be shown that κ is inaccessible if and only if $(V_\kappa, \in)$ is a model of second order ZFC,[5].

Remark 2.26. By the reflection property, there exists $\alpha < \kappa$ such that $(V_\alpha, \in)$ is a standard model of (first order) ZFC. Hence, the existence of an inaccessible cardinal is a stronger hypothesis than the existence of the standard model of ZFC_2^{Hs}.

3.Derivation inconsistent countable set in set theory ZFC_2 with the full semantics.

Let $\mathbf{Th} = \mathbf{Th}^{fss}$ be an second order theory with the full second order semantics.We assume now that $\mathbf{Th}$ contains ZFC_2^{fss}.We will write for short $\mathbf{Th}$, instead $\mathbf{Th}^{fss}$.

Remark 3.1.Notice that M is a model of ZFC_2^{fss} if and only if it is isomorphic to a model of

the form $V_\kappa, \in \cap(V_\kappa \times V_\kappa)$, for κ a strongly inaccessible ordinal.

Remark 3.2.Notice that a standard model for the language of first-order set theory is an ordered pair $\{D,I\}$ .Its domain, D, is a nonempty set and its interpretation function, I, assigns a set of ordered pairs to the two-place predicate "$\in$".A sentence is true in $\{D,I\}$ just in case it is satisfied by all assignments of first-order variables to members of D and second-order variables to subsets of D; a sentence is satisfiable just in case it is true in some standard model; finally, a sentence is valid just in case it is true in all standard models.

Remark 3.3.Notice that:

(**I**)The assumption that D and I be sets is not without consequence. An immediate effect of this stipulation is that no standard model provides the language of set theory with its intended interpretation. In other words, there is no standard model $\{D,I\}$ in which D consists of all sets and I assigns the standard element-set relation to "$\in$". For it is a theorem of ZFC that there is no set of all sets and that there is no set of ordered-pairs $\{x,y\}$ for x an element of y.

(**II**)Thus, on the standard definition of model:

(1) it is not at all obvious that the validity of a sentence is a guarantee of its truth;

(2) similarly, it is far from evident that the truth of a sentence is a guarantee of its satisfiability in some standard model.

(3)If there is a connection between satisfiability, truth, and validity, it is not one that can be

"read off" standard model theory.

(**III**) Nevertheless this is not a problem in the first-order case since set theory provides us

with two reassuring results for the language of first-order set theory. One result is the first

order completeness theorem according to which first-order sentences are provable, if

true in all models. Granted the truth of the axioms of the first-order predicate calculus

and the truth preserving character of its rules of inference, we know that a sentence of the first-order language of set theory is true, if it is provable. Thus, since valid sentences are provable and provable sentences are true, we know that valid sentences are true. The connection between truth and satisfiability immediately follows: if ϕ is unsatisfiable, then $\neg\phi$, its negation, is true in all models and hence valid. Therefore, $\neg\phi$ is true and ϕ is false.

Definition 3.1. The language of second order arithmetic Z_2 is a two-sorted language: there are two kinds of terms, numeric terms and set terms.

$\mathbf{0}$ is a numeric term,

1.There are in nitely many numeric variables, $x_0, x_1, \ldots, x_n, \ldots$ each of which is a numeric term;

2.If s is a numeric term then $\mathbf{S}s$ is a numeric term;

3.If s, t are numeric terms then $+st$ and $\cdot st$ are numeric terms (abbreviated $s + t$ and $s \cdot t$);

3.There are infinitely many set variables, $X_0, X_1, \ldots, X_n \ldots$ each of which is a set term;

4.If t is a numeric term and S then $\in tS$ is an atomic formula (abbreviated $t \in S$);

5.If s and t are numeric terms then $= st$ and $< st$ are atomic formulas (abbreviated $s = t$ and $s < t$ correspondingly).

The formulas are built from the atomic formulas in the usual way.

As the examples in the definition suggest, we use upper case letters for set variables and lower case letters for numeric terms. (Note that the only set terms are the variables.) It will be more convenient to work with functions instead of sets, but within arithmetic, these are equivalent: one can use the pairing operation, and say that X represents a function if for each n there is exactly one m such that the pair (n, m) belongs to X.

We have to consider what we intend the semantics of this language to be. One possibility is the semantics of full second order logic: a model consists of a set M, representing the numeric objects, and interpretations of the various functions and relations (probably with the requirement that equality be the genuine equality relation), and a statement $\forall X \Phi(X)$ is satisfied by the model if for every possible subset of M, the corresponding statement holds.

Remark 3.1.Full second order logic has no corresponding proof system. An easy way to see this is to observe that it has no compactness theorem. For example, the only model (up to isomorphism) of Peano arithmetic together with the second order induction axiom: $\forall X(\mathbf{0} \in X \wedge \forall x(x \in X \Rightarrow \mathbf{S}x \in X) \Rightarrow \forall x(x \in X))$ is the standard model $\mathbb{N}$. This is easily seen: any model of Peano arithmetic has an initial segment isomorphic to $\mathbb{N}$;

applying the induction axiom to this set, we see that it must be the whole of the model.

Remark 3.2.There is no completeness theorem for second-order logic. Nor do the axioms of second-order ZFC imply a reflection principle which ensures that if a sentence of second-order set theory is true, then it is true in some standard model. Thus there may be sentences of the language of second-order set theory that are true but unsatisfiable, or sentences that are valid, but false. To make this possibility vivid, let Z be the conjunction of all the axioms of second-order ZFC. Z is surely true. But the existence of a model for Z requires the existence of strongly inaccessible cardinals. The axioms of second-order ZFC don't entail the existence of strongly inaccessible cardinals, and hence the satisfiability of Z is independent of second-order ZFC. Thus, Z is true but its unsatisfiability is consistent with second-order ZFC [5].

Thus with respect to ZFC_2^{fss}, this is a semantically defined system and thus it is not standard to speak about it being contradictory if anything, one might attempt to prove that it has no models, which to be what is being done in section 2 for ZFC_2^{Hs}.

Definition 3.2. Using formula (2.3) one can define predicate $\mathbf{Pr}^{\#}_{\mathbf{Th}}(y)$ really asserting provability in $\mathbf{Th} = ZFC_2^{fss}$

$$\begin{gathered}\mathbf{Pr}^{\#}_{\mathbf{Th}}(y) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}}(y) \wedge [\mathbf{Pr}_{\mathbf{Th}}(y) \Rightarrow \Phi], \\ \mathbf{Pr}_{\mathbf{Th}}(y) \Leftrightarrow \exists x\left(x \in M_{\omega}^{Z_2^{fss}}\right)\mathbf{Prov}_{\mathbf{Th}}(x,y), \\ y = [\Phi]^c. \end{gathered} \tag{3.1}$$

Theorem 3.1.[12].(Löb's Theorem for ZFC_2^{fss}) Let Φ be any closed formula with code $y = [\Phi]^c \in M_{\omega}^{Z_2}$, then $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$ implies $\mathbf{Th} \vdash \Phi$ (see [12] Theorem 5.1).

Proof. Assume that

(#) $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$.

Note that

(1) $\mathbf{Th} \nvdash \neg\Phi$. Otherwise one obtains $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi]^c) \wedge \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$, but this is a contradiction.

(2) Assume now that (2.i) $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$ and (2.ii) $\mathbf{Th} \nvdash \Phi$.

From (1) and (2.ii) follows that

(3) $\mathbf{Th} \nvdash \neg\Phi$ and $\mathbf{Th} \nvdash \Phi$.

Let $\mathbf{Th}_{\neg\Phi}$ be a theory

(4)$\mathbf{Th}_{\neg\Phi} \triangleq \mathbf{Th} \cup \{\neg\Phi\}$. From (3) follows that

(5) $Con(\mathbf{Th}_{\neg\Phi})$.

From (4) and (5) follows that

(6) $\mathbf{Th}_{\neg\Phi} \vdash \mathbf{Pr}_{\mathbf{Th}_{\neg\Phi}}([\neg\Phi]^c)$.
From (4) and (#) follows that
(7) $\mathbf{Th}_{\neg\Phi} \vdash \mathbf{Pr}_{\mathbf{Th}_{\neg\Phi}}([\Phi]^c)$.
From (6) and (7) follows that
(8) $\mathbf{Th}_{\neg\Phi} \vdash \mathbf{Pr}_{\mathbf{Th}_{\neg\Phi}}([\Phi]^c) \wedge \mathbf{Pr}_{\mathbf{Th}_{\neg\Phi}}([\neg\Phi]^c)$, but this is a contradiction.

Definition 3.3. Let $\Psi = \Psi(x)$ be one-place open wff such that:

$$\mathbf{Th} \vdash \exists! x_\Psi[\Psi(x_\Psi)] \tag{3.2}$$

Then we will says that, a set y is a $\mathbf{Th}$-set iff there is exist one-place open wff $\Psi(x)$ such
that $y = x_\Psi$. We write $y[\mathbf{Th}]$ iff y is a $\mathbf{Th}$-set.

Remark 3.2. Note that

$$\begin{gathered} y[\mathbf{Th}] \Leftrightarrow \\ \exists\Psi[(y = x_\Psi) \wedge \mathbf{Pr}_{\mathbf{Th}}([\exists! x_\Psi[\Psi(x_\Psi)]]^c) \wedge [(\mathbf{Pr}_{\mathbf{Th}}([\exists! x_\Psi[\Psi(x_\Psi)]]^c) \Rightarrow \exists! x_\Psi[\Psi(x_\Psi)])]] \end{gathered} \tag{3.3}$$

Definition 3.4. Let $\Im$ be a collection such that : $\forall x\big[x \in \Im \leftrightarrow x$ is a $\mathbf{Th}$-set$\big]$.

Proposition 3.1. Collection $\Im$ is a $\mathbf{Th}$-set.

Definition 3.4. We define now a $\mathbf{Th}$-set $\Re_c \subsetneqq \Im$:

$$\forall x[x \in \Re_c \leftrightarrow (x \in \Im) \wedge \mathbf{Pr}_{\mathbf{Th}}([x \notin x]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}}([x \notin x]^c) \Rightarrow x \notin x]]. \tag{3.4}$$

Proposition 3.2. (i) $\mathbf{Th} \vdash \exists \Re_c$, (ii) $\Re_c$ is a countable $\mathbf{Th}$-set.

Proof.(i) Statement $\mathbf{Th} \vdash \exists \Re_c$ follows immediately by using statement $\exists \Im$ and axiom
schema of separation [4], (ii) follows immediately from countability of a set $\Im$.

Proposition 3.3. A set $\Re_c$ is inconsistent.

Proof.From formla (3.2) one obtains

$$\mathbf{Th} \vdash \Re_c \in \Re_c \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}}([\Re_c \notin \Re_c]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}}([\Re_c \notin \Re_c]^c) \Rightarrow \Re_c \notin \Re_c]. \tag{3.5}$$

From formula (3.4) and definition 3.5 one obtains

$$\mathbf{Th} \vdash \Re_c \in \Re_c \Leftrightarrow \Re_c \notin \Re_c \tag{3.6}$$

and therefore

$$\mathbf{Th} \vdash (\Re_c \in \Re_c) \wedge (\Re_c \notin \Re_c). \tag{3.7}$$

But this is a contradiction.

Thus finally we obtain:

Theorem 3.2.[12].$\neg Con(ZFC_2^{fss})$.

It well known that under ZFC it can be shown that κ is inaccessible if and only if $(V_\kappa, \in)$ is a
model of ZFC_2 [5],[11].Thus finally we obtain.

Theorem 3.3.[12].$\neg Con(ZFC + \exists M_{st}^{ZFC}(M_{st}^{ZFC} = H_k))$.

4.Non consistency Results in Topology.

Definition 4.1.[19].A Lindelöf space is indestructible if it remains Lindelöf after forcing with any countably closed partial order.

Theorem 4.1.[20].If it is consistent with ZFC that there is an inaccessible cardinal, then it is consistent with ZFC that every Lindelöf T_3 indestructible space of weight $\leq \aleph_1$ has size $\leq \aleph_1$.

Corollary 4.1.[20] The existence of an inaccessible cardinal and the statement:
$\mathcal{L}[T_3, \leq \aleph_1, \leq \aleph_1] \triangleq$ "every Lindelöf T_3 indestructible space of weight $\leq \aleph_1$ has size $\leq \aleph_1$"
are equiconsistent.

Theorem 4.2.[12].$\neg Con(ZFC + \mathcal{L}[T_3, \leq \aleph_1, \leq \aleph_1])$.

Proof.Theorem 4.2 immediately follows from Theorem 3.3 and Corollary 4.1.

Definition 4.2.The $\aleph_1$-Borel Conjecture is the statement: $BC[\aleph_1] \triangleq$ "a Lindelöf space is indestructible if and only if all of its continuous images in $[0;1]^{\omega_1}$ have cardinality $\leq \aleph_1$".

Theorem 4.3.[12]. If it is consistent with ZFC that there is an inaccessible cardinal, then it is consistent with ZFC that the $\aleph_1$-Borel Conjecture holds.

Corollary 4.2.The $\aleph_1$-Borel Conjecture and the existence of an inaccessible cardinal are equiconsistent.

Theorem 4.4.[12] $\neg Con(ZFC + BC[\aleph_1])$.

Proof.Theorem 4.4 immediately follows from Theorem 3.3 and Corollary 4.2.

Theorem 4.5.[20]. If ω_2 is not weakly compact in $\mathbf{L}$, then there is a Lindelöf T_3 indestructible space of pseudocharacter $\leq \aleph_1$ and size $\aleph_2$.

Corollary 4.3.The existence of a weakly compact cardinal and the statement:
$\widetilde{\mathcal{L}}[T_3, \leq \aleph_1, \aleph_2] \triangleq$ "there is no Lindelöf T_3 indestructible space of pseudocharacter $\leq \aleph_1$ and size $\aleph_2$ are equiconsistent.

Theorem 4.6.[12].There is a Lindelöf T_3 indestructible space of pseudocharacter $\leq \aleph_1$ and size $\aleph_2$ in $\mathbf{L}$.

Proof.Theorem 4.6 immediately follows from Theorem 3.3 and Theorem 4.5.

Theorem 4.7.[12]. $\neg Con\left(ZFC + \widetilde{\mathcal{L}}[T_3, \leq \aleph_1, \aleph_2]\right)$.

Proof.Theorem 3.7 immediately follows from Theorem 3.3 and Corollary 4.3.

5.Conclusion.

In this paper we have proved that the second order ZFC with the full second-order semantic is inconsistent,i.e. $\neg Con(ZFC_2^{fss})$. Main result is: let k be an inaccessible cardinal and H_k is a set of all sets having hereditary size less then k, then $\neg Con(ZFC + (V = H_k))$. This result also was obtained in [7],[12],[13] by using essentially another approach. For the first time this result has been declared to AMS in [14],[15]. An important applications in topology and homotopy theory are obtained in [16],[17],[18].

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