Inconsistent countable set in second order ZFC

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Abstract In this article we derived an importent example of the inconsistent countable set in second order ZFC (ZFC_2). Main result is:
$\sim con(ZFC_2 + \exists(\omega\text{-model of } ZFC_2))$.

Keywords Gödel encoding • Completion of ZFC_2 • Russell's paradox •

ω-model.

1.Introduction.

Let's remind that accordingly to naive set theory, any definable collection is a set. Let R be the set of all sets that are not members of themselves. If R qualifies as a member of itself, it would contradict its own definition as a set containing all sets that are not members of themselves. On the other hand, if such a set is not a member of itself, it would qualify as a member of itself by the same definition. This contradiction is Russell's paradox. In 1908, two ways of avoiding the paradox were proposed, Russell's type theory and the Zermelo set theory, the first constructed axiomatic set theory. Zermelo's axioms went well beyond Frege's axioms of extensionality and unlimited set abstraction, and evolved into the now-canonical Zermelo–Fraenkel set theory *ZFC.*

"But how do we know that ZFC is a consistent theory, free of contradictions? The short answer is that we don't; it is a matter of faith (or of skepticism)"— E.Nelson wrote in paper [1]. However, it is deemed unlikely that ZFC_2 harbors an unsuspected contradiction; it is widely believed that if ZFC_2 were inconsistent, that fact would have been uncovered by now. This much is certain — ZFC_2 is immune to the classic paradoxes of naive set theory: Russell's paradox, the Burali-Forti paradox, and Cantor's paradox.

Nevertheless it is easy to see that ZFC_2 is inconsistent and it guards. Let $\Im$ be the countable collection of all sets X such that $ZFC_2 \vdash \exists! X \Psi(X)$, where $\Psi(X)$ is any 1-place open wff i.e.,

$$\forall Y\{Y \in \mathfrak{I} \leftrightarrow \exists \Psi(\bullet)\exists! X[\Psi(X) \wedge Y = X]\}. \tag{1.1}$$

Let $X \notin_{\vdash_{ZFC}} Y$ be a predicate such that $X \notin_{\vdash_{ZFC}} Y \leftrightarrow ZFC_2 \vdash X \notin Y$. Let $\mathfrak{R}$ be the countable collection of all sets such that

$$\forall X\left[X \in \mathfrak{R} \leftrightarrow X \notin_{\vdash_{ZFC}} X\right]. \tag{1.2}$$

From (1.1) one obtain

$$\mathfrak{R} \in \mathfrak{R} \leftrightarrow \mathfrak{R} \notin_{\vdash_{ZFC}} \mathfrak{R}. \tag{1.3}$$

But obviously this is a contradiction. However contradiction (1.3) it is not a contradiction inside ZFC for the reason that predicates $\exists \Psi(\bullet)\exists! X[\Psi(X) \wedge Y = X]$ and $X \notin_{\vdash_{ZFC}} Y$ not is a predicates in ZFC and therefore countable collections $\mathfrak{I}$ and $\mathfrak{R}$ not is a sets. Nevertheless by using Gödel encoding the above stated contradiction can be shipped in special consistent completion of ZFC.

Remark 1.1. We note that in order to deduce $\sim con(ZFC)$ from $con(ZFC)$ by using Gödel encoding, one needs something more than the consistency of ZFC_2, e.g., that ZFC_2 has an omega-model i.e., a model in which the *integers are the standard integers.* To put it another way, why should we believe a statement just because there's a ZFC_2-proof of it? It's clear that if ZFC_2 is inconsistent, then we won't believe ZFC_2-proofs. What's slightly more subtle is that the mere consistency of ZFC_2 isn't quite enough to get us to believe arithmetical theorems of ZFC_2; we must also believe that these arithmetical theorems are asserting something about the standard naturals. It is "conceivable" that ZFC_2 might be consistent but that the only models it has are those in which the integers are nonstandard, in which case we might not "believe" an arithmetical statement such as "ZFC_2 is inconsistent" even if there is a ZFC_2-proof of it.

We assume that: (i) $con(ZFC_2)$, (ii) $con(ZFC_2 + \exists(\omega\text{-model of } ZFC_2))$.

Main result is: $\sim con(ZFC_2 + \exists(\omega\text{-model of } ZFC_2))$.

2.Inconsistent countable set derivation.

Let **Th** be some fixed, but unspecified, consistent formal theory. For later convenience, we assume that the encoding is done in some fixed formal theory **S** and that **Th** contains **S**.We do not specify **S** — it is usually taken to be a formal system of arithmetic, although a weak set theory is often more convenient. The sense in which **S** is contained in **Th** is better exemplified than explained: If **S** is a formal system of arithmetic and **Th** is, say, ZFC_2, then **Th** contains **S** in the sense that there is a well-known embedding, or interpretation, of **S** in **Th**.Since encoding is to take place in **S**, it will have to have a large supply of constants and closed terms to be used as codes. (E.g. in formal arithmetic, one has $\overline{0},\overline{1},\ldots$.) **S** will also have certain function symbols to be described shortly.To each formula, Φ, of the language of **Th** is assigned a closed term, $[\Phi]^c$, called the code of Φ. [N.B. If $\Phi(x)$ is a formula with free variable x, then $[\Phi(x)]^c$ is a closed term encoding the formula $\Phi(x)$ with x viewed as a syntactic object and not as a parameter.] Corresponding to the logical connectives and quantifiers are function symbols, $neg(\cdot)$, $imp(\cdot)$, etc., such that, for all formulae Φ, Ψ : $\mathbf{S} \vdash neg([\Phi]^c) = [\neg\Phi]^c$, $\mathbf{S} \vdash imp([\Phi]^c,[\Psi]^c) = [\Phi \to \Psi]^c$ etc. Of particular importance is the substitution operator, represented by the function symbol $sub(\cdot,\cdot)$. For formulae $\Phi(x)$, terms t with codes $[t]^c$:

$$\mathbf{S} \vdash sub([\Phi(x)]^c,[t]^c) = [\Phi(t)]^c. \tag{2.1}$$

It well known [3] that one can also encode derivations and have a binary relation $\mathbf{Prov_{Th}}(x,y)$ (read "x proves y " or "x is a proof of y") such that for closed t_1,t_2 :

$\mathbf{S} \vdash \mathbf{Prov_{Th}}(t_1,t_2)$ iff t_1 is the code of a derivation in **Th** of the formula with code t_2 . It follows that

$$\mathbf{Th} \vdash \Phi \text{ iff } \mathbf{S} \vdash \mathbf{Prov_{Th}}(t,[\Phi]^c) \tag{2.2}$$

for some closed term t.Thus one can define

$$\mathbf{Pr_{Th}}(y) \leftrightarrow \exists x \mathbf{Prov_{Th}}(x,y), \tag{2.3}$$

and therefore one obtain a predicate asserting provability. We note that is not always the case that [3]:

$$\mathbf{Th} \vdash \Phi \text{ iff } \mathbf{S} \vdash \mathbf{Pr_{Th}}([\Phi]^c). \tag{2.4}$$

It well known [3] that the above encoding can be carried out in such a way that the following important conditions $\mathbf{D}1, \mathbf{D}2$ and $\mathbf{D}3$ are met for all sentences [3]:

$$\mathbf{D}1.\,\mathbf{Th} \vdash \Phi \text{ implies } \mathbf{S} \vdash \mathbf{Pr_{Th}}([\Phi]^c),$$

$$\mathbf{D}2.\,\mathbf{S} \vdash \mathbf{Pr_{Th}}([\Phi]^c) \to \mathbf{Pr_{Th}}([\mathbf{Pr_{Th}}([\Phi]^c)]^c), \tag{2.5}$$

$$\mathbf{D}3.\,\mathbf{S} \vdash \mathbf{Pr_{Th}}([\Phi]^c) \wedge \mathbf{Pr_{Th}}([\Phi \to \Psi]^c) \to \mathbf{Pr_{Th}}([\Psi]^c).$$

Conditions $\mathbf{D}1, \mathbf{D}2$ and $\mathbf{D}3$ are called the Derivability Conditions.

Lemma 2.1. Assume that: (i) $Con(\mathbf{Th})$ and (ii) $\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\Phi]^c)$, where Φ is a closed

formula.Then $\mathbf{Th} \nvdash \mathbf{Pr_{Th}}([\neg\Phi]^c)$.

Proof. Let $Con_{\mathbf{Th}}(\Phi)$ be a formula

$$Con_{\mathbf{Th}}(\Phi) \triangleq \forall t_1 \forall t_2 \neg[\mathbf{Prov_{Th}}(t_1,[\Phi]^c) \wedge \mathbf{Prov_{Th}}(t_2, neg([\Phi]^c))] \leftrightarrow$$
$$\neg\exists t_1 \neg\exists t_2[\mathbf{Prov_{Th}}(t_1,[\Phi]^c) \wedge \mathbf{Prov_{Th}}(t_2, neg([\Phi]^c))]. \tag{2.6}$$

where t_1, t_2 is a closed term. We note that $\mathbf{Th} + Con(\mathbf{Th}) \vdash Con_{\mathbf{Th}}(\Phi)$ for any closed Φ. Suppose that $\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\neg\Phi]^c)$,then (ii) gives

$$\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\Phi]^c) \wedge \mathbf{Pr_{Th}}([\neg\Phi]^c). \tag{2.7}$$

From (2.3) and (2.7) we obtain

$$\exists t_1 \exists t_2[\mathbf{Prov_{Th}}(t_1,[\Phi]^c) \wedge \mathbf{Prov_{Th}}(t_2, neg([\Phi]^c))]. \tag{2.8}$$

But the formula (2.6) contradicts the formula (2.8). Therefore $\mathbf{Th} \nvdash \mathbf{Pr_{Th}}([\neg\Phi]^c)$.

Lemma 2.2. Assume that: (i) $Con(\mathbf{Th})$ and (ii) $\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\neg\Phi]^c)$, where Φ is a closed

formula.Then $\mathbf{Th} \nvdash \mathbf{Pr_{Th}}([\Phi]^c)$.

Assumption 2.1. We assume now that:

(i) the language of $\mathbf{Th}$ consists of:

numerals $\overline{0}, \overline{1}$,...

countable set of the numerical variables: $\{v_0, v_1, \ldots\}$

countable set $\mathcal{F}$ of the set variables: $\mathcal{F} = \{x, y, z, X, Y, Z, \mathfrak{R}, \ldots\}$

countable set of the n-ary function symbols: $f_0^n, f_1^n, \ldots$

countable set of the n-ary relation symbols: $R_0^n, R_1^n, \ldots$

connectives: $\neg, \rightarrow$
quantifier: $\forall$.
(ii) $\mathbf{Th}$ contains $\mathbf{Th}^*$=ZFC_2;
(iii) Let Φ be any closed formula, then $[\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\Phi]^c)]\&[M_\omega^{\mathbf{Th}} \models \Phi]$ implies $\mathbf{Th} \vdash \Phi$.

Definition 2.1. An $\mathbf{Th}$-wff Φ (well-formed formula Φ) is closed - i.e. Φ is a sentence - if
it has no free variables; a wff is open if it has free variables.We'll use
the slang 'k-place open wff ' to mean a wff with k distinct free variables.

Definition 2.2.We said that,$\mathbf{Th}^\#$ is a nice theory or a nice extension of the $\mathbf{Th}$ iff
(i) $\mathbf{Th}^\#$ contains $\mathbf{Th}$;
(ii) Let Φ be any closed formula, then $\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\Phi]^c)$ implies $\mathbf{Th}^\# \vdash \Phi$.

Definition 2.3.We said that,$\mathbf{Th}^\#$ is a maximally nice theory or a maximally nice
extension of the $\mathbf{Th}$ iff $\mathbf{Th}^\#$ is consistent and for any consistent nice extension $\mathbf{Th}'$ of
the $\mathbf{Th}$: $\mathbf{Ded}(\mathbf{Th}^\#) \subseteqq \mathbf{Ded}(\mathbf{Th}')$ implies $\mathbf{Ded}(\mathbf{Th}^\#) = \mathbf{Ded}(\mathbf{Th}')$.

Proposition 2.1. Assume that (i) $Con(\mathbf{Th})$ and (ii) $\mathbf{Th}$ has an ω-model $M_\omega^{\mathbf{Th}}$. Then
theory $\mathbf{Th}$ can be extended to a maximally consistent nice theory $\mathbf{Th}^\#$.

Proof. Let $\Phi_1 \ldots \Phi_i \ldots$ be an enumeration of all wff's of the theory $\mathbf{Th}$ (this can be achieved if the set of propositional variables can be enumerated). Define a chain $\wp = \{\mathbf{Th}_i | i \in \mathbb{N}\}, \mathbf{Th}_1 = \mathbf{Th}$ of consistent theories inductively as follows: assume that theory $\mathbf{Th}_i$ is defined. (i) Suppose that a statement (2.9) is satisfied

$$\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\Phi_i]^c) \text{ and } [\mathbf{Th}_i \nvdash \Phi_i]\&[M_\omega^{\mathbf{Th}} \models \Phi]. \quad (2.9)$$

Then we define theory $\mathbf{Th}_{i+1}$ as follows $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\Phi_i\}$.
(ii) Suppose that a statement (2.10) is satisfied

$$\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\neg\Phi_i]^c) \text{ and } [\mathbf{Th}_i \nvdash \neg\Phi_i]\&[M_\omega^{\mathbf{Th}} \models \neg\Phi]. \quad (2.10)$$

Then we define theory $\mathbf{Th}_{i+1}$ as follows: $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\neg\Phi_i\}$.
(iii) Suppose that a statement (2.11) is satisfied

$$\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\Phi_i]^c) \text{ and } \mathbf{Th} \vdash \Phi_i. \quad (2.11)$$

Then we define theory $\mathbf{Th}_{i+1}$ as follows: $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\Phi_i\}$.
(iv) Suppose that a statement (2.12) is satisfied

$$\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi_i]^c) \text{ and } \mathbf{Th} \vdash \neg\Phi_i. \tag{2.12}$$

Then we define theory $\mathbf{Th}_{i+1}$ as follows: $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i$.

We define theory $\mathbf{Th}^{\#}$ as follows:

$$\mathbf{Th}^{\#} \triangleq \bigcup_{i \in \mathbb{N}} \mathbf{Th}_i \tag{2.13}$$

First, notice that each $\mathbf{Th}_i$ is consistent. This is done by induction on i and by Lemmas 2.1-2.2. By assumption, the case is true when $i = 1$. Now, suppose $\mathbf{Th}_i$ is consistent. Then its deductive closure $\mathbf{Ded}(\mathbf{Th}_i)$ is also consistent. If a statement (2.11) is satisfied,i.e. $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi_i]^c)$ and $\mathbf{Th} \vdash \Phi_i$, then clearly $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\Phi_i\}$ is consistent since it is a subset of closure $\mathbf{Ded}(\mathbf{Th}_i)$. If a statement (2.12) is satisfied,i.e. $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi_i]^c)$ and $\mathbf{Th} \vdash \neg\Phi_i$, then clearly $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\neg\Phi_i\}$ is consistent since it is a subset of closure $\mathbf{Ded}(\mathbf{Th}_i)$.

Otherwise:

(i) if a statement (2.9) is satisfied,i.e. $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi_i]^c)$ and $\mathbf{Th}_i \nvdash \Phi_i$ then clearly $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\Phi_i\}$ is consistent by Lemma1 and by one of the standard properties of consistency: $\Delta \cup \{A\}$ is consistent iff $\Delta \nvdash \neg A$;

(ii) if a statement (2.10) is satisfied,i.e. $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi_i]^c)$ and $\mathbf{Th}_i \nvdash \neg\Phi_i$ then clearly $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\neg\Phi_i\}$ is consistent by Lemma 2 and by one of the standard properties of consistency: $\Delta \cup \{\neg A\}$ is consistent iff $\Delta \nvdash A$.

Next, notice $\mathbf{Ded}(\mathbf{Th}^{\#})$ is maximally consistent nice extension of the $\mathbf{Ded}(\mathbf{Th})$. $\mathbf{Ded}(\mathbf{Th}^{\#})$ is consistent because, by the standard Lemma 2.3 belov, it is the union of a chain of consistent sets. To see that $\mathbf{Ded}(\mathbf{Th}^{\#})$ is maximal, pick any wff Φ. Then Φ is some Φ_i in the enumerated list of all wff's. Therefore for any Φ such that $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$ or $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi]^c)$, either $\Phi \in \mathbf{Th}^{\#}$ or $\neg\Phi \in \mathbf{Th}^{\#}$. Since $\mathbf{Ded}(\mathbf{Th}_{i+1}) \subseteq \mathbf{Ded}(\mathbf{Th}^{\#})$, we have $\Phi \in \mathbf{Ded}(\mathbf{Th}^{\#})$ or $\neg\Phi \in \mathbf{Ded}(\mathbf{Th}^{\#})$, which implies that $\mathbf{Ded}(\mathbf{Th}^{\#})$ is maximally consistent nice extension of the $\mathbf{Ded}(\mathbf{Th})$.

Lemma 2.3. The union of a chain $\wp = \{\Gamma_i | i \in \mathbb{N}\}$ of consistent sets Γ_i, ordered by $\subseteq$, is consistent.

Definition 2.4. Let $\Psi(x)$ be a one-place open wff such that conditions:

$(*)$ $\mathbf{Th} \vdash \exists! x_\Psi[\Psi(x_\Psi)]$ or

$(**)$ $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\exists! x_\Psi[\Psi(x_\Psi)]]^c)$ and $M_\omega^{\mathbf{Th}} \models \exists! x_\Psi[\Psi(x_\Psi)]$ is satisfied.

Then we said that, an set y is a $\mathbf{Th}^{\#}$-set iff there is exist a one-place open $\mathbf{Th}$-wff $\Psi(x)$

such that $y = x_\Psi$.

Definition 2.5. Let $\Im$ be a collection such that : $\forall x\big[x \in \Im \leftrightarrow x \text{ is a } \mathbf{Th}^{\#}\text{-set}\big]$.

Proposition 2.2. Collection $\Im$ is a $\mathbf{Th}^{\#}$-set.

Proof. Let us consider an one-place open wff $\Psi(x)$ such that conditions $(*)$ or

$(**)$ is satisfied, i.e. $\mathbf{Th} \vdash \exists! x_\Psi[\Psi(x_\Psi)]$. We note that there is exists countable collection $\mathcal{F}_\Psi$ of the one-place open wff's $\mathcal{F}_\Psi = \{\Psi_n(x)\}_{n\in\mathbb{N}}$ such that: (i) $\Psi(x) \in \mathcal{F}_\Psi$ and (ii)

$$\mathbf{Th} \vdash \exists! x_\Psi[[\Psi(x_\Psi)] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]\}]$$

or

$$\mathbf{Th} \vdash \exists! x_\Psi[\mathbf{Pr_{Th}}([\Psi(x_\Psi)]^c) \wedge \{\forall n(n \in \mathbb{N})\mathbf{Pr_{Th}}([\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]^c)\}] \qquad (2.14)$$

and

$$M_\omega^{\mathbf{Th}} \vDash \exists! x_\Psi[[\Psi(x_\Psi)] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]\}]$$

or in the equivalent form

$$\mathbf{Th} \vdash \exists! x_1[[\Psi_1(x_1)] \wedge \{\forall n(n \in \mathbb{N})[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]\}]$$

or

$$\mathbf{Th} \vdash \exists! x_\Psi[\mathbf{Pr_{Th}}([\Psi(x_1)]^c) \wedge \{\forall n(n \in \mathbb{N})\mathbf{Pr_{Th}}([\Psi(x_1) \leftrightarrow \Psi_n(x_1)]^c)\}] \qquad (2.15)$$

and

$$M_\omega^{\mathbf{Th}} \vDash \exists! x_\Psi[[\Psi(x_1)] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]\}]$$

where we set $\Psi(x) = \Psi_1(x_1), \Psi_n(x_1) = \Psi_{n,1}(x_1)$ and $x_\Psi = x_1$. We note that everyone collection $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n\in\mathbb{N}}, k = 1,2,\ldots$ such above defines an unique set x_{Ψ_k}, i.e. $\mathcal{F}_{\Psi_{k_1}} \bigcap \mathcal{F}_{\Psi_{k_2}} = \varnothing$ iff $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$.

We note that collections $\mathcal{F}_{\Psi_k}, k = 1,2,..$ is no part of the ZFC, i.e. collection $\mathcal{F}_{\Psi_k}$ there is no set in sense of ZFC. However that is no problem, because by using Gödel numbering one can to replace any collection $\mathcal{F}_{\Psi_k}, k = 1,2,..$ by collection $\Theta_k = g(\mathcal{F}_{\Psi_k})$ of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n\in\mathbb{N}}, \qquad (2.16)$$
$$k = 1,2,\ldots .$$

It is easy to prove that any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}$-set. This is done by Gödel encoding [3],[5] of the statament (2.15) by Proposition 2.1 and by axiom schema of separation [4], (**see Proposition 2.3**). Let

$g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1,2,..$ be a Gödel number of the wff $\Psi_{n,k}(x_k)$. Therefore $g(\mathcal{F}_k) = \{g_{n,k}\}_{n\in\mathbb{N}}$, where we set $\mathcal{F}_k = \mathcal{F}_{\Psi_k},\ k = 1,2,..$ and

$$\forall k_1 \forall k_2[\{g_{n,k_1}\}_{n\in\mathbb{N}} \bigcap \{g_{n,k_2}\}_{n\in\mathbb{N}} = \varnothing \leftrightarrow x_{k_1} \neq x_{k_2}]. \tag{2.17}$$

Let $\{\{g_{n,k}\}_{n\in\mathbb{N}}\}_{k\in\mathbb{N}}$ be a family of the all sets $\{g_{n,k}\}_{n\in\mathbb{N}}$. By axiom of choice [4] one obtain unique set $\Im' = \{g_k\}_{k\in\mathbb{N}}$ such that $\forall k[g_k \in \{g_{n,k}\}_{n\in\mathbb{N}}]$. Finally one obtain a set $\Im$ from a set $\Im'$ by axiom schema of replacement [4].Thus one can define a $\mathbf{Th}^{\#}$-set

$\Re_c \subsetneqq \Im$:

$$\forall x[x \in \Re_c \leftrightarrow (x \in \Im) \wedge \mathbf{Pr}_{\mathbf{Th}}([x \notin x]^c)]. \tag{2.18}$$

Proposition 2.3. Any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}$-set.

Proof. We define $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$. Therefore $g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$ (see [7] Appendix,def.10). Let us define predicate $\Pi(g_{n,k}, v_k)$

$$\begin{gathered}\Pi(g_{n,k}, v_k) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}}([\exists! x_k[\Psi_{1,k}(x_1)]]^c) \wedge \\ \wedge \exists! x_k(v_k = [x_k]^c)[\forall n(n \in \mathbb{N})[\mathbf{Pr}_{\mathbf{Th}}([[\Psi_{1,k}(x_k)]]^c) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}}(\mathbf{Fr}(g_{n,k}, v_k))]].\end{gathered} \tag{2.19}$$

We define now a set Θ_k such that

$$\begin{gathered}\Theta_k = \Theta'_k \cup \{g_k\}, \\ \forall n(n \in \mathbb{N})[g_{n,k} \in \Theta'_k \leftrightarrow \Pi(g_{n,k}, v_k)]\end{gathered} \tag{2.20}$$

But obviously definitions (2.16) and (2.20) is equivalent by Proposition 2.1.

Proposition 2.4. (i) $\mathbf{Th}^{\#} \vdash \exists \Re_c$,(ii) $\Re_c$ is a countable $\mathbf{Th}^{\#}$-set.

Proof.(i) Statement $\mathbf{Th}^{\#} \vdash \exists \Re_c$ follows immediately by using statement $\exists \Im$ and axiom schema of separation [4]. (ii) follows immediately from countability of a set $\Im$.

Proposition 2.5. A set $\Re_c$ is inconsistent.

Proof.From formla (2.18) one obtain

$$\mathbf{Th}^{\#} \vdash \Re_c \in \Re_c \leftrightarrow \mathbf{Pr}_{\mathbf{Th}}([\Re_c \notin \Re_c]^c). \tag{2.21}$$

From formla (2.21) and Proposition 2.1 one obtain

$$\mathbf{Th}^{\#} \vdash \Re_c \in \Re_c \leftrightarrow \Re_c \notin \Re_c \qquad (2.22)$$

and

$$\mathbf{Th}^{\#} \vdash (\Re_c \notin \Re_c) \leftrightarrow \mathbf{Th}^{\#} \nvdash (\Re_c \notin \Re_c). \qquad (2.23)$$

But this is a contradictions.

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