Einstein's rank-2 tensor compression of Maxwell's equations does not turn them into rank-2 spacetime curvature

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Maxwell's equations of electromagnetism describe three dimensional electric and magnetic field line divergence and curl (rank 1 tensors, or vector calculus), but were compressed by Einstein by including those *rank-1 equations as components of rank 2 tensors*. However, Einstein did not express the electromagnetic force in terms of a rank-2 spacetime curvature. In order to unify or even compare the equations for two forces (gravity and electromagnetism), *you need first to have them expressed in terms of similarly physical descriptions: either rank-1 field lines for both, or spacetime curvature for both.* Fixing the gauge potentials by $B = \nabla \times A$, and $E = (-dA/dt) - \nabla \cdot \phi$, where vector potential A and scalar potential ϕ represent the gauge field, the invariance of the magnetic field is ensured because the curl of A automatically eliminates *any* divergence term added to A in a transformation: $B = \nabla \times (A + \nabla \cdot Y) = \nabla \times A$. Since the curl of a divergence is zero, $\nabla \times (\nabla \cdot Y) = 0$, gauge fixing instantly produces two Maxwell equations:

$$\nabla \times E = \nabla \times (-dA/dt - \nabla \cdot \phi) = \nabla \times (-dA/dt) = -d(\nabla \times A)/dt = -dB/dt.$$ (Faraday's law of induction.)

$$\nabla \cdot B = \nabla \cdot (\nabla \times A) = 0.$$ (Gauss's law for magnetism, or Heaviside's "no magnetic monopoles".)

These are Maxwell's homogeneous equations. The divergence of a curl is zero because the curling field line loop is closed like a circle, with zero net outward divergence. (The partial divergences of an elliptically shaped curling field line are exactly compensated by its partial convergences, which are required to form a closed loop. If the field line does not form a closed loop, for instance a spiral shape, then it does have a net divergence. This is not to be sold as abstract pure mathematics; merely drawing diverging and curling field lines.) The other Maxwell equations are selective (inhomogeneous) components of $d_\mu F^{\mu\nu}$, the covariant partial derivative of the contravariant field strength, $F^{\mu\nu} = d^\mu A^\nu - d^\nu A^\mu$.

By setting $\nu = 1$ we get Ampere's electric current with Maxwell's displacement current law, and by setting $\nu = 0$ we get Gauss's law:

$$d_\mu F^{\mu 1} = d_\mu (d^\mu A^1 - d^1 A^\mu) =$$
$$= (-dE/dt) + [(dB_z/dy) - (dB_y/dz)] + [(dB_x/dz) - (dB_z/dx)] + [(dB_y/dx) - (dB_x/dy)]$$
$$= (-dE/dt) + c^2(\nabla \times B) = j/\varepsilon$$

which is Ampere's law and Maxwell's displacement current: $j/\varepsilon = (-dE/dt) + c^2(\nabla \times B)$. Setting $\nu = 0$:

$$d_\mu F^{\mu 0} = d_\mu (d^\mu A^0 - d^0 A^\mu)$$
$$= (dE_x/dx) + (dE_y/dy) + (dE_z/dz) = \nabla \cdot E = \rho/\varepsilon$$

which is Gauss's law: $\rho/\varepsilon = \nabla \cdot E$. Faraday's electric and magnetic field strengths ("field line" concentrations in space), E and B, are components of the field strength tensor $F^{\mu\nu} = d^\mu A^\nu - d^\nu A^\mu$. For example:

$$E^\nu = -F^{0\nu} = -(d^0 A^\nu - d^\nu A^0) = (-dA/dt) - \nabla \cdot \phi,$$ where in static fields $dA/dt = 0$ by definition.

$$E^\mu = F^{\mu 0}$$
$$B_x = B^1 = F^{32}$$
$$B_y = B^2 = F^{13}$$
$$B_z = B^3 = F^{21}$$

Altogether there are $4^2 = 16$ components to field strength tensor $F^{\mu\nu} = d^\mu A^\nu - d^\nu A^\mu$ with μ and ν each taking spacetime values of 0, 1, 2, 3, and the cancellation symmetry of $d^{\mu}A^{\nu} - d^{\nu}A^{\mu}$ gives an overall value of $F^{\mu\nu} = 0$:

$$\begin{array}{llll} F^{00} = 0 & F^{01} = -E_x & F^{02} = -E_y & F^{03} = -E_z \\ F^{10} = E_x & F^{11} = 0 & F^{12} = -B_z & F^{13} = B_y \\ F^{20} = E_y & F^{21} = B_z & F^{22} = 0 & F^{23} = -B_x \\ F^{30} = E_z & F^{31} = -B_y & F^{32} = B_x & F^{33} = 0 \end{array}$$

Therefore, it is only when components of $F^{\mu\nu}$ are picked out that a net field strength results. The field strength tensor is therefore somewhat contrived or arbitrary and "unphysical", contrasted to the physically-based electric and magnetic fields, E and B. Furthermore, $F^{\mu\nu}$ is a rank-2 tensor since it has two indices, although it summarizes rank-1 components, obtained by selection of the indices, e.g. $E^{\mu} = F^{\mu 0}$. Although, as emphasised above, this can be perceived to be a "elegant" mathematical model of electromagnetism, physically it does not move away from Faraday's representation of fields as magnetic and electric field "lines in space". If we want compatibility of electromagnetism and general relativity, we must move away from lines in space and towards spacetime curvature, which is truly a rank-2 formulation over fully spacetime indices, without the mathematical "trick" of picking out rank-1 equations from a rank-2 tensor by artificially setting one indice to zero, as in $E^{\mu} = F^{\mu 0}$. From this point of view, Maxwell's equations are obfuscated by Einstein's tensor framework because they remain diverging or curling lines in space, as contrasted to spacetime curvature described by the rank-2 Ricci tensor in general relativity, yet this field tensor gauge fixing is extended in the Yang-Mills equation.

The arbitrary way that Maxwell's inhomogeneous equations are "picked out" of $d_{\mu} F^{\mu\nu}$ by setting $\nu = 1$ for Ampere's electric current and Maxwell's displacement current laws, and $\nu = 0$ for Gauss's law looks conveniently contrived in appearance; the student simply asks "what about $\nu = 2$, and $\nu = 3$?" However, the Aharonov-Bohm effect proves that the "physical" E and B components do not include all of the information necessary to describe the electromagnetic field, so on this experimental basis, some larger structure including for example the extra components of potentials A and ϕ are defensible. On the other hand, the Aharonov-Bohm effect is physically the existence of the energy density of "cancelled" fields. It proves that the superposition of two fields which cancel doesn't destroy the energy density of the field in the vacuum, which remains there, unseen apart from subtle quantum effects. It proves that Maxwell's simplistic E, B field line formulation is inadequate, but it does not directly prove the details of the Einstein's tensor formulation based on a neat fixing of field potentials. As shown above, Einstein's gauge fixing elegantly yields the two key homogeneous Maxwell equations.

In addition, the gauge fixing of Maxwell's equations allows very neat calculations for electromagnetic radiation. Thus, Mandl and Shaw's *Quantum Field Theory* (2nd ed., 2010, equation 1.6) points out that $\nabla \cdot A = 0$ "defines the Coulomb or radiation gauge. A vector with vanishing divergence [satisfying $\nabla \cdot A = 0$] is called a transverse field, since for a wave $A_{x,t} = A_0 e^{i(k.x - \omega t)}$, [the equation $\nabla \cdot A = 0$] gives $k \cdot A = 0$, i.e., A is perpendicular to the direction of propagation k of the wave."

A key geometric development from electromagnetic gauge fixing was the Yang-Mills equation for non-Abelian fields where the Lie product, $[A_{\mu}, A_{\nu}] = A_{\mu} A_{\nu} - A_{\nu} A_{\mu}$, is added to the covariant electromagnetic field strength tensor $F_{\mu\nu} = d_{\mu}A_{\nu} - d_{\nu}A_{\mu}$, to allow charge transfer by massive charged field quanta, which occurs in the SU(N) non-Abelian interactions. Steenrod wrote *The Topology of Fibre Bundles* in 1951, leading to work on the connection of gauge theory to fibre bundles by Simons, which was taken up by Wu and Yang in 1975 (*Phys. Rev.* D12, pp. 3845-57). Zeidler's 2011 *Quantum Field Theory III: Gauge Theory* (p. 5) explains how Yang became interested in the geometry of fibre bundles after discovering in the 1960s that his Yang-Mills non-Abelian field equation is transformable into the Riemann tensor (the rank-4 curvature tensor whose contraction is the Ricci tensor of general relativity). The Riemann tensor measures curvature by the holonomy of the mainfold, i.e. the failure of a vector transported around a loop to return to its original position in curved space. But the Yang-Mills equation, an extension of the electromagnetic field strength tensor, just describes "field lines".

This "transformation" between the Yang-Mills and Riemann tensors by Yang is based on two fundamental assumptions (Zeidler, *Quantum Field Theory III: Gauge Theory,* Springer, 2011, pp. 35-6); first, the 16 component Christoffel matrix $\Gamma^{a}{}_{\mu b}$ is assumed to be equal to the covariant vector potential, $A_{\mu} = \Gamma^{a}{}_{\mu b}$ where a and b each take values of 0, 1, 2 and 3 to form the matrix, and second, that the covariant field strength tensor $F_{\mu\nu}$ equals the Riemann tensor, $R^a{}_{\mu\nu b}$ again where *a* and *b* each take values of 0, 1, 2 and 3. The problem here is that the field strength tensor and the Riemann tensor describe fields differently, so this is like a "transformation" between chalk and cheese, based solely upon substitution and the superficial similarity of their colour. These are efforts to paper over the *physical* distinctions between the field line description of gauge theory and the curved spacetime description of general relativity. By "transforming" one into another, through the postulate that $F_{\mu\nu} = R^a{}_{\mu\nu b}$, this obfuscation is complete. This assumption can then used to try to justify itself in a circular argument: *because a piece of chalk can be substituted for a piece of cheese and looks similar, they are equivalent, the propaganda claims, and there is no real physical difference whatsoever.* (If you want a *physically impressive* connection between the Yang-Mills equation and general relativity, you might try to transform the field equation of general relativity into the Yang-Mills equation, so see how the metric is related to the Lie product. You would *not* physically postulate that the *Riemann* curvature is the same thing as the field line strength tensor!)

This Orwellian "doublethink" can harden into an orthodox dogma, where it is then used to try to suppress alternative ideas, such as either rebuilding gauge theory in terms of space time curvature, or else rebuilding general relativity in terms of the field lines of gauge theory. In this way, real progress gets stifled, perhaps indefinitely delaying. On the other hand, a look at published alternative ideas shows that the extremely radical alternatives which might have a chance of being different enough from orthodoxy to really be new, are usually wrong, not even wrong, or in a very infantile stage which ensures their unpopularity. The nature of revolutionary genius, required to overthrow the existing dogma, is therefore unlikely to emerge in piecemeal fashion from a group of outsiders, since any revolutionary ideas will - by definition - appear insulting and objectionable to the mainstream dogmas they seek to overthrow. We see that the fashionability at the heart of modern physics ensures that ideas off the beaten track must be developed long and hard in the bush before they have a chance of invading and overthrowing the mainstream's status quo which has thousands of reseachers and millions of funding.

These dogmas sometimes arise from famous personalities throwing out "not even wrong" pieces of mathematical trivial disguised as profundity, after having become famous for something else which is far more impressive. It is the label of the fame of the author which then helps to market them, a phenomenon which is objectionable to those physicists who entered science precisely to avoid cult fashion politic.

Faraday invented the "field line" or "lines of force" description of electric and magnetic forces. Maxwell's equations of "unified electromagnetism" are based on this description, where the weakening of electric fields with increasing distances from charges are described by the divergence of the field lines. However, one of Maxwell's equations is Gauss's equation, $\nabla \cdot E = \rho/\varepsilon$, which states that the total divergence of electric field lines, $\nabla \cdot E$, is equal to the charge density, ρ Coulombs per cubic metre, divided by permittivity, ε (converting the units of charge into those of electric field divergence). This states that the total divergence of the electric field is directly proportional to the electric charge density. The divergence operator, $\nabla \cdot$, is just Heaviside's mathematical shorthand for the sum of gradients in the three perpendicular directions of space: $\nabla \cdot E = (dE_x/dx) + (dE_y/dy) + (dE_z/dz)$. If we have a point charge, the electric field will be a symmetric spherical shape if it is static relative to an observer, so we can replace *x, y,* and *z* by radial distance from the centre of the charge, *r*, giving $\nabla \cdot E = \rho/\varepsilon = (dE_x/dx) + (dE_y/dy) + (dE_z/dz) = 3\, dE_r/dr$, the integral of which is $E = q/(4\pi\varepsilon\, r^2)$, thereby yielding Coulomb's inverse-square law, $F = EQ = Qq/(4\pi\varepsilon\, r^2)$.

The asymmetry in electromagnetism is the magnetic analogy $\nabla \cdot B = 0$ to Gauss's law $\nabla \cdot E = \rho/\varepsilon$ which shows that magnetic fields always have no total divergence. This is because the divergence of a magnetic field is the sum of the divergence of field lines radiating in all directions from a source. *Because a magnet has two poles of exactly equal and opposite field strength, the divergence of those magnetic field lines which radiate outwards from one pole is precisely cancelled out by the convergence of the magnetic field lines which flow into the opposite pole.* You cannot isolate one magnetic pole from the other by breaking the magnet into two pieces (you get four poles). Maxwell did not promote his equations as being symmetric, and did not predict magnetic monopoles to overcome the asymmetry of $\nabla \cdot B = 0$. It was Heaviside who after 1875 wrote the Maxwell equations in their common compact vector calculus form, leading to questions of symmetry, and a re-writing of history by mathematicians who preferred mathematics to understanding. Webber in 1856 empircally discovered that light velocity is yielded by $c = 1/(\varepsilon\mu)^{1/2}$, using electric and magnetic force equation constants ε and μ. Specifically in order to derive Webber's result from an electromagnetic wave equation, Maxwell from 1861-5 predicted the vacuum "displacement current" formula, which states any time-varying electric field, dE/dt, produces a magnetic field identical to that caused by a real electric current of magnitude $j = \varepsilon\, dE/dt$. Maxwell contrived a mechanical theory to yield Webber's equation. This history of physics shows deep fundamental fiddling. We point out a replacement for Maxwell's equations in http://vixra.org/abs/1111.0111