Obtaining the Probability Vector Current Density in Canonical Quantum Mechanics by Linear Superposition

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Abstract

The quantum mechanics status of the probability vector current density has long seemed to be marginal. On one hand no systematic prescription for its construction is provided, and the special examples of it that are obtained for particular types of Hamiltonian operator could conceivably be attributed to happenstance. On the other hand this concept's key physical interpretation as local average particle flux, which flows from the equation of continuity that it is supposed to satisfy in conjunction with the probability scalar density, has been claimed to breach the uncertainty principle. Given the dispiriting impact of that claim, we straightaway point out that the subtle directional nature of the uncertainty principle makes it consistent with the measurement of local average particle flux. We next focus on the fact that the unique closed-form linear-superposition quantization of any classical Hamiltonian function yields in tandem the corresponding unique linear-superposition closed-form divergence of the probability vector current density. Because the probability vector current density is linked to the quantum physics only through the occurrence of its divergence in the equation of continuity, it is theoretically most appropriate to construct this vector field exclusively from its divergence—analysis of the best-known "textbook" special example of a probability vector current density shows that it is thus constructed. That special example in fact leads to the physically interesting "Ehrenfest subclass" of probability vector current densities, which are closely related to their classical peers.

Introduction

The quantum mechanical probability vector current density concept has long been at best hazily understood. Although special examples of probability vector current density have been obtained for particular types of Hamiltonian operator [1], there is no systematic prescription for constructing it, so its special examples could conceivably be attributed to happenstance. Its essential feature is supposed to be that, in consequence of the quantum mechanical conservation of probability, it jointly with the probability scalar density satisfies the equation of continuity, which compellingly suggests that it physically represents local average particle flux. It has, however, been claimed that the measurement of local average particle flux breaches the uncertainty principle because such a measurement involves arbitrary particle localization while it simultaneously yields information concerning particle velocity, and for that reason probability vector current density *cannot have its only truly natural physical interpretation*, namely that of local average particle flux [1]. This contention, if valid, would raise two linked vexing issues: (1) if probability vector current density cannot in principle be physically interpreted as local average particle flux, then *what* is its correct physical interpretation, and (2) is that correct physical interpretation still something that is of actual interest to physicists?

Very forturnately, however, while it is indeed the case that measurement of local average particle flux restricts particle position while delivering information regarding particle velocity, these things do not in fact occur in such a manner as to challenge the uncertainty principle. To measure one of the vector components of local average particle flux at a given point, one passes a plane which is perpendicular to that vector component through that point. One then selects an arbitrarily small region of that plane centered on that point and measures the average rate that particles pass through that selected planar region. This average rate, divided by the area of the selected planar region, is an approximation to that particular vector component of average particle flux at that particular point, an approximation which is, in principle, refined by additionally shrinking the planar region on which that point is centered. The flux one thus measures does indeed reflect the component of average particle velocity in the direction *perpendicular* to the plane, provided, of course, that one knows the average particle density. The uncertainty principle, however, *does not restrict the accuracy* with which a particular component of particle momentum can be determined due to restrictions that are imposed on the particle's position in directions which are *perpendicular* to that momentum component. To see this one need look no further than the Dirac canonical commutation rule,

$$[\widehat{x}_i, \widehat{x}_j] = 0, \;\; [\widehat{p}_i, \widehat{p}_j] = 0, \;\; [\widehat{x}_i, \widehat{p}_j] = i\hbar\delta_{ij}, \;\; i, j = 1, 2, 3,$$

and recall that the uncertainty principle *only applies* to quantized dynamical variable pairs *that fail to commute*, which is clearly *not the case* for a position component and a momentum component *that are mutually*

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perpendicular. Therefore *the measurement of local average particle flux*, notwithstanding that it restricts particle position while delivering averaged particle velocity information—namely in the direction *perpendicular* to the plane in which particle position is restricted—*does not challenge the uncertainty principle*.

Thus there is in fact no known valid reason to refrain from physically interpreting probability vector current density as local average particle flux, even as the equation of continuity ostensibly satisfied by probability vector current density in conjunction with probability scalar density so cogently suggests is sound. The bona fide issue here is a *different* one, namely the lack of a *general* prescription for *constructing* that desired probability vector current density which *actually satisfies* the equation of continuity in conjunction with the probability scalar density.

We therefore in the rest of this article concentrate on systematically working out such a probability vector current density for any canonical Hamiltonian operator whatsoever (at least in principle—as one can well imagine, anything worked out to such a degree of generality yields formidable expressions that cannot be expected to have transparency as their strong suit). We begin with the *divergence* of the probability vector current density, which, from the equation of continuity, must be equal to the negative of the time derivative of *the well-defined probability scalar density.* That time derivative, in turn, is the state-vector expectation value of $(i/\hbar)$ times the commutator of the set of position projection operators *with the Hamiltonian operator*. Because there exists a linear superposition technique for the unique closed-form quantization of *any classical Hamiltonian function* [2], it turns out that for Hamiltonian operators which have classical Hamiltonian function antecedents we can *also* uniquely reduce the just-mentioned expression for the *divergence* of the probability vector current density, which involves the state-vector expectation value of a *Hamiltonian-operator* commutator, to a linear-superposition closed form. Thus the *divergence* of the probability vector current density can always be fully and uniquely worked out when the Hamiltonian operator has a classical Hamiltonian function antecedent, i.e., when it is a *canonical* Hamiltonian operator.

The probability vector current density *itself* of course *is naturally mathematically ambiguous*, e.g., it tolerates *the addition of an arbitrary vector field that is a curl.* However, since its *divergence* is a state-vector *expectation value* (of a set of operators), it is physically entirely reasonable to *restrict* the probability vector current density to being a *homogeneously linear functional* of that *divergence* so that *it likewise will be a state-vector expectation value*. Indeed, the larger idea of *restricting* the probability vector current density to the very barest *minimum* that is compatible with its divergence makes impeccable sense because it is *only that divergence* which makes *actual contact with the system's quantum physics*, doing so, of course, through the *equation of continuity* that it satisfies in conjunction with the probability scalar density. Therefore the probability vector current density is devoid of quantum physical information *that is not already implicit in its divergence*. We shall formally *implement* this crucial point by stipulating not only that the probability current density is *homogeneously linear* in its divergence, but that it furthermore *must not* depend on any *constants* which are *additional* to those that are *intrinsic to its divergence*, and also that among the forms which are mathematical candidates for this *vector* field, *only the most symmetric are to be considered* because these *add no further information* to that available from its *scalar* divergence. These three stipulations result in a *unique closed-form expression* for the probability vector current density in terms of its unique divergence, and we shall see that the "textbook" best-known special example of a probability vector current density [1] is indeed consistent with these three stipulations.

We now turn to the presentation of *the ingredients that are needed* to work out the probability vector current density in the manner that has just been outlined. As is well-known, the *conservation of probability* in quantum mechanics [1] follows from the Hermitian property of the Hamiltonian operator, namely $\widehat{H}^\dagger(t) = \widehat{H}(t)$, in conjunction with the Schrödinger equation,

$$i\hbar d|\psi(t)\rangle/dt = \widehat{H}(t)|\psi(t)\rangle, \tag{1a}$$

which together imply that,

$$-i\hbar d\langle\psi(t)|/dt = \langle\psi(t)|\widehat{H}(t), \tag{1b}$$

and therefore that probability is conserved,

$$d\langle\psi(t)|\psi(t)\rangle/dt = \langle\psi(t)|\left[(i/\hbar)\widehat{H}(t) + (-i/\hbar)\widehat{H}(t)\right]|\psi(t)\rangle = 0. \tag{1c}$$

In the case of *canonical* quantum mechanics, $\widehat{H}(t)$ is *uniquely obtained* from a *classical Hamiltonian function* $H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$ by the imposition of the following *self-consistent* slight extension of Dirac's canonical commutation rule [2],

$$[f_1(\widehat{\mathbf{x}}) + g_1(\widehat{\mathbf{p}}), f_2(\widehat{\mathbf{x}}) + g_2(\widehat{\mathbf{p}})] = i\hbar\ \widehat{\{f_1(\mathbf{x}) + g_1(\mathbf{p}),\ f_2(\mathbf{x}) + g_2(\mathbf{p})\}}\ , \tag{2}$$

where the { , } are the *classical Poisson brackets*, and the overbrace is used to denote *quantization*—the overbrace is *only* used here for that purpose because the traditional hat accent for denoting quantization *is not sufficiently extensible*. The Eq. (2) slight extension of Dirac's canonical commutation rule turns out to self-consistently *completely resolve* the Dirac *operator-ordering ambiguity* in favor of Born-Jordan operator ordering [2], which is *exactly the same operator ordering that is implicit in the Hamiltonian path integral* [3]. The *unique* Hamiltonian operator which follows from the classical Hamiltonian function $H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$ via the application of Eq. (2) is, of course, denoted as $\overbrace{H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)}$, or, when *explicit reference* to the underlying classical Hamiltonian function $H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$ *isn't necessary*, as simply the *canonical Hamiltonian operator* $\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)$.

When such a *canonical* $\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)$ describes the quantum mechanics, probability conservation, given by Eq. (1c), can be expressed in terms of the *probability scalar density* $(\langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\psi(t)\rangle)$ because of the following expansion in the complete set of position states $|\mathbf{x}\rangle$,

$$d\langle\psi(t)|\psi(t)\rangle/dt = (d/dt)\int\langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\psi(t)\rangle\, d^N\mathbf{x} = 0, \tag{3a}$$

which relation might plausibly be expected to imply that the time derivative $d(\langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\psi(t)\rangle)/dt$ of the probability scalar density $(\langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\psi(t)\rangle)$ *is a perfect differential in the vector variable* $\mathbf{x}$, namely that there exists *a probability vector current density* $\mathbf{j}(\mathbf{x},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])$ which satisfies,

$$d(\langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\psi(t)\rangle)/dt + \nabla_{\mathbf{x}}\cdot\mathbf{j}(\mathbf{x},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)]) = 0. \tag{3b}$$

Eq. (3b) is the *equation of continuity* that the probability vector current density $\mathbf{j}(\mathbf{x},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])$ *is supposed to satisfy* in conjunction with the probability scalar density $(\langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\psi(t)\rangle)$ as a plausible consequence of the conservation of probability that is given by Eq. (3a). In light of the Schrödinger equations given by Eqs. (1a) and (1b), as they apply to the particular case that $\widehat{H}(t) = \widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)$, we can see that *actually fulfilling the* Eq. (3b) *equation of continuity requires that*,

$$\begin{gathered}\nabla_{\mathbf{x}}\cdot\mathbf{j}(\mathbf{x},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)]) = -d(\langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\psi(t)\rangle)/dt = \\ (i/\hbar)\Big[\langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)|\psi(t)\rangle - \langle\psi(t)|\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)|\mathbf{x}\rangle\langle\mathbf{x}|\psi(t)\rangle\Big] = \\ \Re\Big[(2i/\hbar)\langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)|\psi(t)\rangle\Big],\end{gathered} \tag{3c}$$

where the symbol $\Re$ that occurs after the last equal sign of Eq. (3c) denotes the real part of the bracketed expression which follows it. We thus see that to obtain the *divergence* of $\mathbf{j}(\mathbf{x},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])$ we need to evaluate the *core part* $\langle\mathbf{x}|\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)|\psi(t)\rangle$ of that bracketed expression for an *arbitrary* canonical Hamiltonian operator $\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)$. To *achieve* this goal it will be extremely useful to emulate the procedure of Ref. [2] wherein such an *arbitrary* $\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)$ is *linearly decomposed* into "Fourier component" *operators* of the form $e^{\mp i\widehat{\mathbf{x}}\cdot\mathbf{k}}e^{\pm i\widehat{\mathbf{p}}\cdot\mathbf{l}/\hbar}$ because, of course,

$$\langle\mathbf{x}|e^{\mp i\widehat{\mathbf{x}}\cdot\mathbf{k}}e^{\pm i\widehat{\mathbf{p}}\cdot\mathbf{l}/\hbar}|\psi(t)\rangle = e^{\mp i\mathbf{x}\cdot\mathbf{k}}\langle\mathbf{x}|e^{\pm i\widehat{\mathbf{p}}\cdot\mathbf{l}/\hbar}|\psi(t)\rangle = e^{\mp i\mathbf{x}\cdot\mathbf{k}}\langle\mathbf{x}\pm\mathbf{l}|\psi(t)\rangle. \tag{3d}$$

The probability vector current density divergence by linear superposition

Following Ref. [2], we note that the key step for decomposing $\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)$ into "Fourier component" *operators* is the *orthodox* corresponding Fourier decomposition of its *underlying classical Hamiltontonian function* $H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$. This, of course, follows from the identity,

$$H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t) = \int d^N\mathbf{x}'\int d^N\mathbf{p}'\,\delta^{(N)}(\mathbf{x}-\mathbf{x}')\,\delta^{(N)}(\mathbf{p}-\mathbf{p}')\,H_{\mathrm{cl}}([\mathbf{x}',\mathbf{p}'],t), \tag{4a}$$

after we insert into it the Fourier delta-function representations,

$$\delta^{(N)}(\mathbf{x}-\mathbf{x}') = (2\pi)^{-N}\int d^N\mathbf{k}\,e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')},\ \ \delta^{(N)}(\mathbf{p}-\mathbf{p}') = (2\pi\hbar)^{-N}\int d^N\mathbf{l}\,e^{i\mathbf{l}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}. \tag{4b}$$

Since $H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$ is a real-valued function, after inserting Eq. (4b) into Eq. (4a) it is convenient to furthermore explicitly discard the vanishing imaginary part of the result, which yields,

$$\begin{gathered}H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t) = (4\pi^2\hbar)^{-N}\int d^N\mathbf{x}'\int d^N\mathbf{p}'\,H_{\mathrm{cl}}([\mathbf{x}',\mathbf{p}'],t)\int d^N\mathbf{k}\int d^N\mathbf{l}\,\times \\ \big(\cos(-\mathbf{x}'\cdot\mathbf{k}+\mathbf{p}'\cdot\mathbf{l}/\hbar)\cos(-\mathbf{x}\cdot\mathbf{k}+\mathbf{p}\cdot\mathbf{l}/\hbar) + \sin(-\mathbf{x}'\cdot\mathbf{k}+\mathbf{p}'\cdot\mathbf{l}/\hbar)\sin(-\mathbf{x}\cdot\mathbf{k}+\mathbf{p}\cdot\mathbf{l}/\hbar)\big).\end{gathered} \tag{4c}$$

The fact that quantization is a *linear process* [2] permits us to conclude from Eq. (4c) that,

$$\begin{gathered}\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t) = \overbrace{H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)} = (4\pi^2\hbar)^{-N}\int d^N\mathbf{x}'\int d^N\mathbf{p}'\, H_{\mathrm{cl}}([\mathbf{x}',\mathbf{p}'],t)\int d^N\mathbf{k}\int d^N\mathbf{l}\,\times \\ \Big(\cos(-\mathbf{x}'\cdot\mathbf{k}+\mathbf{p}'\cdot\mathbf{l}/\hbar)\,\overbrace{\cos(-\mathbf{x}\cdot\mathbf{k}+\mathbf{p}\cdot\mathbf{l}/\hbar)} + \sin(-\mathbf{x}'\cdot\mathbf{k}+\mathbf{p}'\cdot\mathbf{l}/\hbar)\,\overbrace{\sin(-\mathbf{x}\cdot\mathbf{k}+\mathbf{p}\cdot\mathbf{l}/\hbar)}\Big).\end{gathered} \tag{4d}$$

As in Ref. [2], we obtain $\overbrace{\exp(-i\mathbf{x}\cdot\mathbf{k}+i\mathbf{p}\cdot\mathbf{l}/\hbar)}$ from Eq. (2). From this we further immediately obtain $\overbrace{\exp(i\mathbf{x}\cdot\mathbf{k}-i\mathbf{p}\cdot\mathbf{l}/\hbar)}$, and those two results together yield $\overbrace{\cos(-\mathbf{x}\cdot\mathbf{k}+\mathbf{p}\cdot\mathbf{l}/\hbar)}$ and $\overbrace{\sin(-\mathbf{x}\cdot\mathbf{k}+\mathbf{p}\cdot\mathbf{l}/\hbar)}$, which is what we require for insertion into Eq. (4d). We shall, however, *insure* that these cosine and sine quantizations are expressed as linear combinations of products of exponential operators of the "Fourier component" type that were introduced in Eq. (3d), because it is *those* "Fourier component" type operators which are *transparently useful* for the evaluation of the *core part* of the probability vector current density's *divergence* $\nabla_{\mathbf{x}}\cdot\mathbf{j}(\mathbf{x},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])$, as is apparent from Eqs. (3c) and (3d).

We now carry out the Eq. (2) quantization of $\exp(-i\mathbf{x}\cdot\mathbf{k}+i\mathbf{p}\cdot\mathbf{l}/\hbar)$ with the goal of expressing its result in terms of Eq. (3d) "Fourier component" type operators. With the needed exponential ingredients inserted, Eq. (2) reads,

$$[e^{-i\widehat{\mathbf{x}}\cdot\mathbf{k}}, e^{i\widehat{\mathbf{p}}\cdot\mathbf{l}/\hbar}] = i\hbar\ \overbrace{\{e^{-i\mathbf{x}\cdot\mathbf{k}},\, e^{i\mathbf{p}\cdot\mathbf{l}/\hbar}\}}\,, \tag{4e}$$

which, written out, is,

$$e^{-i\widehat{\mathbf{x}}\cdot\mathbf{k}}e^{i\widehat{\mathbf{p}}\cdot\mathbf{l}/\hbar} - e^{i\widehat{\mathbf{p}}\cdot\mathbf{l}/\hbar}e^{-i\widehat{\mathbf{x}}\cdot\mathbf{k}} = i(\mathbf{k}\cdot\mathbf{l})\ \overbrace{e^{-i\mathbf{x}\cdot\mathbf{k}+i\mathbf{p}\cdot\mathbf{l}/\hbar}}\,. \tag{4f}$$

Eq. (4f) can then be reexpressed as,

$$\left(e^{i(\mathbf{k}\cdot\mathbf{l})/2} - e^{-i(\mathbf{k}\cdot\mathbf{l})/2}\right)e^{-i\widehat{\mathbf{x}}\cdot\mathbf{k}+i\widehat{\mathbf{p}}\cdot\mathbf{l}/\hbar} = i(\mathbf{k}\cdot\mathbf{l})\ \overbrace{e^{-i\mathbf{x}\cdot\mathbf{k}+i\mathbf{p}\cdot\mathbf{l}/\hbar}}\,. \tag{4g}$$

which yields the unique exponential quantization,

$$\overbrace{e^{-i\mathbf{x}\cdot\mathbf{k}+i\mathbf{p}\cdot\mathbf{l}/\hbar}} = e^{-i\widehat{\mathbf{x}}\cdot\mathbf{k}+i\widehat{\mathbf{p}}\cdot\mathbf{l}/\hbar}\sin(\tfrac12\mathbf{k}\cdot\mathbf{l})/(\tfrac12\mathbf{k}\cdot\mathbf{l}) = e^{-i\widehat{\mathbf{x}}\cdot\mathbf{k}}e^{i\widehat{\mathbf{p}}\cdot\mathbf{l}/\hbar}e^{-i\frac12\mathbf{k}\cdot\mathbf{l}}\sin(\tfrac12\mathbf{k}\cdot\mathbf{l})/(\tfrac12\mathbf{k}\cdot\mathbf{l}), \tag{4h}$$

where the form farthest to the right in Eq. (4h) is expressed in terms of the desired "Fourier component" type operator. We now make the simple substitutions $\mathbf{k}\to-\mathbf{k}$ and $\mathbf{l}\to-\mathbf{l}$ in Eq. (4h), which produce the additional useful result that,

$$\overbrace{e^{i\mathbf{x}\cdot\mathbf{k}-i\mathbf{p}\cdot\mathbf{l}/\hbar}} = e^{i\widehat{\mathbf{x}}\cdot\mathbf{k}-i\widehat{\mathbf{p}}\cdot\mathbf{l}/\hbar}\sin(\tfrac12\mathbf{k}\cdot\mathbf{l})/(\tfrac12\mathbf{k}\cdot\mathbf{l}) = e^{i\widehat{\mathbf{x}}\cdot\mathbf{k}}e^{-i\widehat{\mathbf{p}}\cdot\mathbf{l}/\hbar}e^{-i\frac12\mathbf{k}\cdot\mathbf{l}}\sin(\tfrac12\mathbf{k}\cdot\mathbf{l})/(\tfrac12\mathbf{k}\cdot\mathbf{l}). \tag{4i}$$

Combining Eqs. (4h) and (4i) then yields,

$$\overbrace{\cos(-\mathbf{x}\cdot\mathbf{k}+\mathbf{p}\cdot\mathbf{l}/\hbar)} = (1/2)\left(e^{-i\widehat{\mathbf{x}}\cdot\mathbf{k}}e^{i\widehat{\mathbf{p}}\cdot\mathbf{l}/\hbar} + e^{i\widehat{\mathbf{x}}\cdot\mathbf{k}}e^{-i\widehat{\mathbf{p}}\cdot\mathbf{l}/\hbar}\right)e^{-i\frac12\mathbf{k}\cdot\mathbf{l}}\sin(\tfrac12\mathbf{k}\cdot\mathbf{l})/(\tfrac12\mathbf{k}\cdot\mathbf{l}), \tag{4j}$$

and,

$$\overbrace{\sin(-\mathbf{x}\cdot\mathbf{k}+\mathbf{p}\cdot\mathbf{l}/\hbar)} = (1/(2i))\left(e^{-i\widehat{\mathbf{x}}\cdot\mathbf{k}}e^{i\widehat{\mathbf{p}}\cdot\mathbf{l}/\hbar} - e^{i\widehat{\mathbf{x}}\cdot\mathbf{k}}e^{-i\widehat{\mathbf{p}}\cdot\mathbf{l}/\hbar}\right)e^{-i\frac12\mathbf{k}\cdot\mathbf{l}}\sin(\tfrac12\mathbf{k}\cdot\mathbf{l})/(\tfrac12\mathbf{k}\cdot\mathbf{l}), \tag{4k}$$

which are expressed in terms of the desired "Fourier component" type operators. We can now insert Eqs. (4j) and (4k) into Eq. (4d), which in turn is inserted into Eq. (3c), after which Eq. (3d) is straightforwardly applied. The upshot is the general *linear superposition expression* for the *divergence* of the probability vector current density that results from an *arbitrary canonical Hamiltonian operator* $\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)$ *whose classical antecedent Hamiltonian function is* $H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$,

$$\begin{gathered}\nabla_{\mathbf{x}}\cdot\mathbf{j}(\mathbf{x},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)]) = \\ (4\pi^2\hbar)^{-N}\int d^N\mathbf{x}'\int d^N\mathbf{p}'\,(H_{\mathrm{cl}}([\mathbf{x}',\mathbf{p}'],t)/\hbar)\int d^N\mathbf{k}\int d^N\mathbf{l}\ (\sin(\tfrac12\mathbf{k}\cdot\mathbf{l})/(\tfrac12\mathbf{k}\cdot\mathbf{l}))\ \times \\ \Big(\cos(-\mathbf{x}'\cdot\mathbf{k}+\mathbf{p}'\cdot\mathbf{l}/\hbar)\,\Re\big[ie^{-i\frac12\mathbf{k}\cdot\mathbf{l}}\langle\psi(t)|\mathbf{x}\rangle\left(e^{-i\mathbf{x}\cdot\mathbf{k}}\langle\mathbf{x}+\mathbf{l}|\psi(t)\rangle + e^{i\mathbf{x}\cdot\mathbf{k}}\langle\mathbf{x}-\mathbf{l}|\psi(t)\rangle\right)\big] + \\ \sin(-\mathbf{x}'\cdot\mathbf{k}+\mathbf{p}'\cdot\mathbf{l}/\hbar)\,\Re\big[e^{-i\frac12\mathbf{k}\cdot\mathbf{l}}\langle\psi(t)|\mathbf{x}\rangle\left(e^{-i\mathbf{x}\cdot\mathbf{k}}\langle\mathbf{x}+\mathbf{l}|\psi(t)\rangle - e^{i\mathbf{x}\cdot\mathbf{k}}\langle\mathbf{x}-\mathbf{l}|\psi(t)\rangle\right)\big]\Big).\end{gathered} \tag{5a}$$

One interesting *special case* of Eq. (5a) occurs when $H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$ has no dependence on the configuration variables $\mathbf{x}$. In that case the integrations over the variables $\mathbf{x}'$ and $\mathbf{k}$ that occur in Eq. (4d) are obviously superfluous, and, if actually carried out in Eq. (5a), simply eliminate $\mathbf{x}'$ along with a factor of $(2\pi)^{-N}$, while setting $\mathbf{k}$ to $\mathbf{0}$. Thus in the case that $H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$ has no dependence on $\mathbf{x}$, Eq. (5a) simplifies to,

$$\begin{aligned}\nabla_{\mathbf{x}}\cdot\mathbf{j}(\mathbf{x},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{p}}],t)]) = (2\pi\hbar)^{-N}\textstyle\int d^N\mathbf{p}'\,(H_{\mathrm{cl}}([\mathbf{p}'],t)/\hbar)\int d^N\mathbf{l}\,\times \\ \Big(\cos(\mathbf{p}'\cdot\mathbf{l}/\hbar)\,\Re[i\langle\psi(t)|\mathbf{x}\rangle\,(\langle\mathbf{x}+\mathbf{l}|\psi(t)\rangle+\langle\mathbf{x}-\mathbf{l}|\psi(t)\rangle)]\,+ \\ \sin(\mathbf{p}'\cdot\mathbf{l}/\hbar)\,\Re[\langle\psi(t)|\mathbf{x}\rangle\,(\langle\mathbf{x}+\mathbf{l}|\psi(t)\rangle-\langle\mathbf{x}-\mathbf{l}|\psi(t)\rangle)]\Big).\end{aligned} \tag{5b}$$

Similarly, if $H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$ has no dependence on the momentum variables $\mathbf{p}$, then the integrations over $\mathbf{p}'$ and $\mathbf{l}$ in Eq. (5a) eliminate $\mathbf{p}'$ along with a factor of $(2\pi\hbar)^{-N}$, while setting $\mathbf{l}$ to $\mathbf{0}$. Thus in the case that $H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$ has no dependence on $\mathbf{p}$, Eq. (5a) yields,

$$\begin{aligned}\nabla_{\mathbf{x}}\cdot\mathbf{j}(\mathbf{x},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}}],t)]) = (2\pi)^{-N}\textstyle\int d^N\mathbf{x}'\,(H_{\mathrm{cl}}([\mathbf{x}'],t)/\hbar)\int d^N\mathbf{k}\,\times \\ \Big(\cos(-\mathbf{x}'\cdot\mathbf{k})\,\Re\big[i|\langle\psi(t)|\mathbf{x}\rangle|^2\,\big(e^{-i\mathbf{x}\cdot\mathbf{k}}+e^{i\mathbf{x}\cdot\mathbf{k}}\big)\big]\,+ \\ \sin(-\mathbf{x}'\cdot\mathbf{k})\,\Re\big[|\langle\psi(t)|\mathbf{x}\rangle|^2\,\big(e^{-i\mathbf{x}\cdot\mathbf{k}}-e^{i\mathbf{x}\cdot\mathbf{k}}\big)\big]\Big) = 0,\end{aligned} \tag{5c}$$

namely that $\nabla_{\mathbf{x}}\cdot\mathbf{j}(\mathbf{x},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}}],t)])$ *vanishes identically*. This can *also* be verified *directly* from Eq. (3c) for such a canonical Hamiltonian operator $\widehat{H}([\widehat{\mathbf{x}}],t)$, because $\langle\mathbf{x}|$ is one of its *eigenvectors*, with the corresponding *real* eigenvalue $H_{\mathrm{cl}}([\mathbf{x}],t)$, and $\langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\psi(t)\rangle$ is *real-valued* as well. We therefore see that *any terms of* $H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$ *which have no dependence on* $\mathbf{p}$ *can simply be truncated from* $H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$ *before* $H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$ *is inserted* into the general linear superposition expression of Eq. (5a) for the divergence of the probability vector current density $\nabla_{\mathbf{x}}\cdot\mathbf{j}(\mathbf{x},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])$.

Our *ultimate* goal, of course, is the probability vector current density $\mathbf{j}(\mathbf{x},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])$ *itself*, rather than its divergence. We now turn our focus to obtaining it, recalling that in the Introduction we set out three stipulations to be imposed on it to make it formally consistent with the fact that it is devoid of physical information that is not already implicit in its divergence.

Obtaining the probability vector current density from its divergence

A homogeneously linear form in terms of its divergence for the nth component of the probability vector current density is given by,

$$\begin{aligned}\big(\mathbf{j}(\mathbf{x},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])\big)_n = \\ w_n\int_{x_n^{(0)}}^{x_n}\nabla_{\mathbf{x}}\cdot\mathbf{j}(x_1,\ldots,x_n',\ldots,x_N,t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])dx_n',\ \ n=1,\ldots,N,\end{aligned} \tag{6a}$$

where the *weights* w_n satisfy $w_1+\cdots+w_N=1$. We of course stipulated *not only* that the probability vector current density is homogeneously linear in its divergence, but *also* that it has *no dependence* on constants which are *additional* to those that are intrinsic to its divergence. The *dependence* of the expression on the right-hand side of Eq. (6a) on the set of constants $\{x_1^{(0)},\ldots,x_N^{(0)}\}$ that are its *lower limits of integration* is readily *removed* by setting *all* those lower limits of integration to $-\infty$. That the result of thus introducing an infinite integration interval is well-defined (i.e., does not diverge) is directly tied in with the conservation of probability, as one can see from Eqs. (3a) and (3b) (with the latter repeated in Eq. (3c)). Finally, we have *also* stipulated that among the forms of the mathematical candidates for the probability vector current density, *only the most symmetric are to be considered*. That stipulation, and to a certain extent also the injunction against additional constants not intrinsic to its divergence, requires us to set *every weight value* w_n of Eq. (6a) to $1/N$, $n=1,\ldots,N$. With these specifications, Eq. (6a) becomes,

$$\begin{aligned}\big(\mathbf{j}(\mathbf{x},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])\big)_n = \\ (1/N)\int_{-\infty}^{x_n}\nabla_{\mathbf{x}}\cdot\mathbf{j}(x_1,\ldots,x_n',\ldots,x_N,t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])dx_n',\ \ n=1,\ldots,N.\end{aligned} \tag{6b}$$

Eqs. (6b) and (5a) *together* are the linear superposition construction of the probability vector current density $\mathbf{j}(\mathbf{x},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])$ for any canonical Hamiltonian operator $\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)$ whose classical Hamiltonian function antecedent is $H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$—see Eq. (5a) for the way that $H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$ is utilized.

We now turn to the best-known Hamiltonian operator, namely $|\widehat{\mathbf{p}}-\mathbf{p}_0(\widehat{\mathbf{x}},t)|^2/(2m)$, for which *direct application of* Eq. (3c) suffices to extract the corresponding probability vector current density divergence

$\nabla_{\mathbf{x}} \cdot \mathbf{j}(\mathbf{x}, t; [|\psi(t)\rangle, |\widehat{\mathbf{p}} - \mathbf{p}_0(\widehat{\mathbf{x}}, t)|^2/(2m)])$ [1], which we abbreviate as $\nabla_{\mathbf{x}} \cdot \mathbf{j}$ for convenience. It turns out that this divergence can simply be *algebraically manipulated* into the explicit form of a *divergence operator* $\nabla_{\mathbf{x}}\cdot$ *acting on a certain vector field* [1], *without actually making use of* Eq. (6b). We can then inquire whether the resulting "textbook" probability vector current density $\mathbf{j}(\mathbf{x}, t; [|\psi(t)\rangle, |\widehat{\mathbf{p}} - \mathbf{p}_0(\widehat{\mathbf{x}}, t)|^2/(2m)])$ [1] that is specifically thus obtained by *algebraic manipulation* of its divergence $\nabla_{\mathbf{x}} \cdot \mathbf{j}$ *in fact adheres to the three stipulations used to derive* Eq. (6b), namely homogeneous linearity in $\nabla_{\mathbf{x}} \cdot \mathbf{j}$, no additional constants beyond the ones intrinsic to $\nabla_{\mathbf{x}} \cdot \mathbf{j}$, and the maximum possible symmetry.

Do the three postulated stipulations hold for the best-known special example?

The key to applying Eq. (3c) to the Hamiltonian operator $|\widehat{\mathbf{p}} - \mathbf{p}_0(\widehat{\mathbf{x}}, t)|^2/(2m)$ is obviously to work out its Eq. (3c) *core part*, $\langle\mathbf{x}||\widehat{\mathbf{p}} - \mathbf{p}_0(\widehat{\mathbf{x}}, t)|^2/(2m)|\psi(t)\rangle$,

$$\begin{gathered}\langle\mathbf{x}||\widehat{\mathbf{p}} - \mathbf{p}_0(\widehat{\mathbf{x}}, t)|^2/(2m)|\psi(t)\rangle = \langle\mathbf{x}|(|\widehat{\mathbf{p}}|^2 - \widehat{\mathbf{p}}\cdot\mathbf{p}_0(\widehat{\mathbf{x}}, t) - \mathbf{p}_0(\widehat{\mathbf{x}}, t)\cdot\widehat{\mathbf{p}} + |\mathbf{p}_0(\widehat{\mathbf{x}}, t)|^2)|\psi(t)\rangle/(2m) = \\ \langle\mathbf{x}|(|\widehat{\mathbf{p}}|^2 + i\hbar\nabla_{\widehat{\mathbf{x}}}\cdot\mathbf{p}_0(\widehat{\mathbf{x}}, t) - 2\mathbf{p}_0(\widehat{\mathbf{x}}, t)\cdot\widehat{\mathbf{p}} + |\mathbf{p}_0(\widehat{\mathbf{x}}, t)|^2)|\psi(t)\rangle/(2m) = \\ \big[-\hbar^2\nabla^2_{\mathbf{x}}\psi(\mathbf{x}, t) + 2i\hbar\mathbf{p}_0(\mathbf{x}, t)\cdot\nabla_{\mathbf{x}}\psi(\mathbf{x}, t) + (i\hbar\nabla_{\mathbf{x}}\cdot\mathbf{p}_0(\mathbf{x}, t) + |\mathbf{p}_0(\mathbf{x}, t)|^2)\psi(\mathbf{x}, t)\big]/(2m),\end{gathered} \tag{7a}$$

where $\psi(\mathbf{x}, t) \stackrel{\text{def}}{=} \langle\mathbf{x}|\psi(t)\rangle$, and of course there is also its complex conjugate $\bar{\psi}(\mathbf{x}, t) \stackrel{\text{def}}{=} \langle\psi(t)|\mathbf{x}\rangle$. We now obtain the divergence $\nabla_{\mathbf{x}}\cdot\mathbf{j}(\mathbf{x}, t; [|\psi(t)\rangle, |\widehat{\mathbf{p}} - \mathbf{p}_0(\widehat{\mathbf{x}}, t)|^2/(2m)])$, which we abbreviate as $\nabla_{\mathbf{x}}\cdot\mathbf{j}$ for convenience, by merely putting Eq. (7a) into Eq. (3c). But we *also* find that with *considerable additional algebraic manipulation* that result can be explicitly presented as a *divergence operator* $\nabla_{\mathbf{x}}\cdot$ acting on a certain vector field,

$$\begin{gathered}\nabla_{\mathbf{x}}\cdot\mathbf{j} = \Re\big[(2i/\hbar)\langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}||\widehat{\mathbf{p}} - \mathbf{p}_0(\widehat{\mathbf{x}}, t)|^2/(2m)|\psi(t)\rangle\big] = \\ m^{-1}\Re\big[-i\hbar\bar{\psi}(\mathbf{x}, t)(\nabla^2_{\mathbf{x}}\psi(\mathbf{x}, t)) - 2\bar{\psi}(\mathbf{x}, t)(\mathbf{p}_0(\mathbf{x}, t)\cdot\nabla_{\mathbf{x}}\psi(\mathbf{x}, t)) - |\psi(\mathbf{x}, t)|^2(\nabla_{\mathbf{x}}\cdot\mathbf{p}_0(\mathbf{x}, t))\big] = \\ (2m)^{-1}[\bar{\psi}(\mathbf{x}, t)((-i\hbar)\nabla^2_{\mathbf{x}}\psi(\mathbf{x}, t)) + ((i\hbar)\nabla^2_{\mathbf{x}}\bar{\psi}(\mathbf{x}, t))\psi(\mathbf{x}, t)] - \\ m^{-1}[\bar{\psi}(\mathbf{x}, t)(\mathbf{p}_0(\mathbf{x}, t)\cdot\nabla_{\mathbf{x}}\psi(\mathbf{x}, t)) + (\mathbf{p}_0(\mathbf{x}, t)\cdot\nabla_{\mathbf{x}}\bar{\psi}(\mathbf{x}, t))\psi(\mathbf{x}, t) + |\psi(\mathbf{x}, t)|^2(\nabla_{\mathbf{x}}\cdot\mathbf{p}_0(\mathbf{x}, t)] = \\ (2m)^{-1}\nabla_{\mathbf{x}}\cdot[\bar{\psi}(\mathbf{x}, t)((-i\hbar)\nabla_{\mathbf{x}}\psi(\mathbf{x}, t)) + ((i\hbar)\nabla_{\mathbf{x}}\bar{\psi}(\mathbf{x}, t))\psi(\mathbf{x}, t)] - \\ m^{-1}[(\mathbf{p}_0(\mathbf{x}, t)\cdot\nabla_{\mathbf{x}}|\psi(\mathbf{x}, t)|^2) + |\psi(\mathbf{x}, t)|^2(\nabla_{\mathbf{x}}\cdot\mathbf{p}_0(\mathbf{x}, t))] = \\ \nabla_{\mathbf{x}}\cdot[\bar{\psi}(\mathbf{x}, t)((-i\hbar)\nabla_{\mathbf{x}}\psi(\mathbf{x}, t)) + ((i\hbar)\nabla_{\mathbf{x}}\bar{\psi}(\mathbf{x}, t))\psi(\mathbf{x}, t) - 2\mathbf{p}_0(\mathbf{x}, t)|\psi(\mathbf{x}, t)|^2]/(2m).\end{gathered} \tag{7b}$$

Here the divergence *operator* $\nabla_{\mathbf{x}}\cdot$ *has been simply factored out of the divergence expression* $\nabla_{\mathbf{x}}\cdot\mathbf{j}$ which follows from Eq. (3c). Therefore it is abundantly clear that the *resulting* "textbook" probability vector current density [1],

$$\begin{gathered}\mathbf{j}(\mathbf{x}, t; [|\psi(t)\rangle, |\widehat{\mathbf{p}} - \mathbf{p}_0(\widehat{\mathbf{x}}, t)|^2/(2m)]) \stackrel{\text{def}}{=} \\ [\bar{\psi}(\mathbf{x}, t)((-i\hbar)\nabla_{\mathbf{x}}\psi(\mathbf{x}, t)) + ((i\hbar)\nabla_{\mathbf{x}}\bar{\psi}(\mathbf{x}, t))\psi(\mathbf{x}, t) - 2\mathbf{p}_0(\mathbf{x}, t)|\psi(\mathbf{x}, t)|^2]/(2m),\end{gathered} \tag{7c}$$

is *homogeneously linear* in its Eq. (3c) divergence $\nabla_{\mathbf{x}}\cdot\mathbf{j}$ and also that it has *no additional constants* beyond those that are *intrinsic* to $\nabla_{\mathbf{x}}\cdot\mathbf{j}$. Inspection of its Eq. (7c) expression also reveals this particular probability vector current density to be highly symmetric; indeed its N-dimensional form is the *completely symmetric generalization of its one-dimensional case*. Thus this "textbook" best-known special example of a probability vector current density [1] *definitely adheres to the three stipulations which underlie* Eq. (6b).

One naturally wonders which subclass of the class of canonical Hamiltonian operators $\widehat{H}([\widehat{\mathbf{x}}, \widehat{\mathbf{p}}], t)$ has members which all emulate $|\widehat{\mathbf{p}} - \mathbf{p}_0(\widehat{\mathbf{x}}, t)|^2/(2m)$ insofar as having the property that the *divergence operator* $\nabla_{\mathbf{x}}\cdot$ *can simply be explicitly algebraically factored out of* $\nabla_{\mathbf{x}}\cdot\mathbf{j}(\mathbf{x}, t; [|\psi(t)\rangle, \widehat{H}([\widehat{\mathbf{x}}, \widehat{\mathbf{p}}], t)])$. An important clue for working out that subclass is the fact that this property is manifest in the classical limit, where the single-particle vector current density can readily be shown through *precisely* such explicit algebraic factorization to be equal to the singular classical single-particle scalar density times the classical particle velocity. Interestingly, the straightforward quantization of the singular classical single-particle scalar density turns out to be equal to the quantum particle-position projection operator in the Heisenberg picture, and, of course, the *quantum* probability scalar density *is simply an expectation value of that particle-position projection operator*. Thus it isn't greatly surprising that there exists *a subclass of the canonical Hamiltonian operators* $\widehat{H}([\widehat{\mathbf{x}}, \widehat{\mathbf{p}}], t)$ for which, *in Ehrenfest-theorem style*, the probability vector current density turns out to be equal to the *expectation value* of the *quantization* of the just-mentioned result for the *classical* single-particle vector current density, namely the singular classical single-particle scalar density times the classical particle velocity, a result that, *exactly as in the classical case*, is arrived at for this "Ehrenfest subclass" of the canonical Hamiltonian operators *by the explicit algebraic factorization* of the *divergence operator* $\nabla_{\mathbf{x}}\cdot$ out of $\nabla_{\mathbf{x}} \cdot \mathbf{j}(\mathbf{x}, t; [|\psi(t)\rangle, \widehat{H}([\widehat{\mathbf{x}}, \widehat{\mathbf{p}}], t)])$.

The classical vector current density and an Ehrenfest-like theorem

As is familiar from its electromagnetic application [4], the *classical* single-particle scalar density is given by the singular expression,

$$\rho_{\rm cl}(\mathbf{y}, t) = \delta^{(N)}(\mathbf{x}_{\rm cl}(t) - \mathbf{y}), \tag{8a}$$

which, *irrespective* of the value of t, satisfies,

$$\int \rho_{\rm cl}(\mathbf{y}, t)\, d^N\mathbf{y} = 1, \tag{8b}$$

in precise analogy to the probability-conservation property of the *quantum* single-particle probability scalar density $\langle\psi(t)|\mathbf{y}\rangle\langle\mathbf{y}|\psi(t)\rangle$, which satisfies,

$$\int \langle\psi(t)|\mathbf{y}\rangle\langle\mathbf{y}|\psi(t)\rangle\, d^N\mathbf{y} = \langle\psi(t)|\psi(t)\rangle = 1. \tag{8c}$$

Therefore it is plausible that there exists a classical single-particle vector current density $\mathbf{j}_{\rm cl}(\mathbf{y}, t)$ which satisfies the equation of continuity in conjunction with $\rho_{\rm cl}(\mathbf{y}, t)$,

$$d\rho_{\rm cl}(\mathbf{y}, t)/dt + \nabla_{\mathbf{y}} \cdot \mathbf{j}_{\rm cl}(\mathbf{y}, t) = 0. \tag{8d}$$

For Eq. (8d) to hold it must be the case that,

$$\begin{gathered}
\nabla_{\mathbf{y}} \cdot \mathbf{j}_{\rm cl}(\mathbf{y}, t) = -d\rho_{\rm cl}(\mathbf{y}, t)/dt = -\{\delta^{(N)}(\mathbf{x}_{\rm cl}(t) - \mathbf{y}),\, H_{\rm cl}([\mathbf{x}_{\rm cl}(t),\, \mathbf{p}_{\rm cl}(t)], t)\} = \\
-\big(\nabla_{\mathbf{x}_{\rm cl}(t)}\delta^{(N)}(\mathbf{x}_{\rm cl}(t) - \mathbf{y})\big) \cdot \nabla_{\mathbf{p}_{\rm cl}(t)} H_{\rm cl}([\mathbf{x}_{\rm cl}(t),\, \mathbf{p}_{\rm cl}(t)], t) = \\
\big(\nabla_{\mathbf{y}}\delta^{(N)}(\mathbf{x}_{\rm cl}(t) - \mathbf{y})\big) \cdot \nabla_{\mathbf{p}_{\rm cl}(t)} H_{\rm cl}([\mathbf{x}_{\rm cl}(t),\, \mathbf{p}_{\rm cl}(t)], t) = \\
\nabla_{\mathbf{y}} \cdot \big[\delta^{(N)}(\mathbf{x}_{\rm cl}(t) - \mathbf{y})\, \nabla_{\mathbf{p}_{\rm cl}(t)} H_{\rm cl}([\mathbf{x}_{\rm cl}(t),\, \mathbf{p}_{\rm cl}(t)], t)\big] = \nabla_{\mathbf{y}} \cdot \big[\delta^{(N)}(\mathbf{x}_{\rm cl}(t) - \mathbf{y})\, d\mathbf{x}_{\rm cl}(t)/dt\big],
\end{gathered} \tag{8e}$$

where $\{\ ,\ \}$ denotes the classical Poisson bracket, and we have used Hamilton's first classical equation of motion, $d\mathbf{x}_{\rm cl}(t)/dt = \nabla_{\mathbf{p}_{\rm cl}(t)} H_{\rm cl}([\mathbf{x}_{\rm cl}(t),\, \mathbf{p}_{\rm cl}(t)], t)$. Eq. (8e) clearly shows that the *divergence operator* $\nabla_{\mathbf{y}}\cdot$ *explicitly algebraically factors out* of the divergence of the classical single-particle vector current density $\nabla_{\mathbf{y}} \cdot \mathbf{j}_{\rm cl}(\mathbf{y}, t)$, thus yielding for the classical single-particle vector current density $\mathbf{j}_{\rm cl}(\mathbf{y}, t)$ *itself*,

$$\mathbf{j}_{\rm cl}(\mathbf{y}, t) = \delta^{(N)}(\mathbf{x}_{\rm cl}(t) - \mathbf{y})\, \nabla_{\mathbf{p}_{\rm cl}(t)} H_{\rm cl}([\mathbf{x}_{\rm cl}(t),\, \mathbf{p}_{\rm cl}(t)], t) = \rho_{\rm cl}(\mathbf{y}, t)\, d\mathbf{x}_{\rm cl}(t)/dt. \tag{8f}$$

From Eq. (8f) we see that the classical single-particle vector current density is equal to *the classical single-particle scalar density times the classical particle velocity*, which is a simple, physically graphic result that turns out to have significant resonance in canonical quantum mechanics as well.

The quantum/classical linkage arises *both* from the Eq. (2) relation between commutator and Poisson brackets *and* from the fact that the *quantization* of the classical single-particle scalar density $\rho_{\rm cl}(\mathbf{y}, t) = \delta^{(N)}(\mathbf{x}_{\rm cl}(t) - \mathbf{y})$ is the Heisenberg-picture version of the quantum position projection operator $|\mathbf{y}\rangle\langle\mathbf{y}|$, *whose expectation value in the state* $|\psi(t)\rangle$ *is the quantum probability scalar density* $\langle\psi(t)|\mathbf{y}\rangle\langle\mathbf{y}|\psi(t)\rangle$. This aspect of the quantum position projection operator $|\mathbf{y}\rangle\langle\mathbf{y}|$ can be obtained from the fact that its application to any position eigenstate vector $|\mathbf{x}\rangle$ yields,

$$|\mathbf{y}\rangle\langle\mathbf{y}|\mathbf{x}\rangle = \delta^{(N)}(\mathbf{y} - \mathbf{x})|\mathbf{y}\rangle = \delta^{(N)}(\mathbf{x} - \mathbf{y})|\mathbf{x}\rangle, \tag{9a}$$

which is *identical* to what the application of $\delta^{(N)}(\widehat{\mathbf{x}} - \mathbf{y})$ to that position eigenstate vector $|\mathbf{x}\rangle$ yields, namely,

$$\delta^{(N)}(\widehat{\mathbf{x}} - \mathbf{y})|\mathbf{x}\rangle = \delta^{(N)}(\mathbf{x} - \mathbf{y})|\mathbf{x}\rangle, \tag{9b}$$

and the fact that, in the *canonical* quantum regime, the set $\{|\mathbf{x}\rangle\}$ of position eigenstate vectors is *complete*. Therefore, the quantum probability scalar density $\langle\psi(t)|\mathbf{y}\rangle\langle\mathbf{y}|\psi(t)\rangle$ has the *alternate* expression,

$$\langle\psi(t)|\mathbf{y}\rangle\langle\mathbf{y}|\psi(t)\rangle = \langle\psi(t)|\delta^{(N)}(\widehat{\mathbf{x}} - \mathbf{y})|\psi(t)\rangle, \tag{9c}$$

which is easily shown to produce the following *alternate form of* Eq. (3c),

$$\nabla_{\mathbf{y}} \cdot \mathbf{j}(\mathbf{y}, t; [|\psi(t)\rangle, \widehat{H}([\widehat{\mathbf{x}}, \widehat{\mathbf{p}}], t)]) = (i/\hbar)\langle\psi(t)|[\delta^{(N)}(\widehat{\mathbf{x}} - \mathbf{y}),\, \widehat{H}([\widehat{\mathbf{x}}, \widehat{\mathbf{p}}], t)]|\psi(t)\rangle. \tag{9d}$$

Eq. (9d), in turn, implies the following alternate form of the combination of Eqs. (6b) and (5a),

$$\begin{gathered}\big(\mathbf{j}(\mathbf{y},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])\big)_n = \\ (1/N)\int_{-\infty}^{y_n}(i/\hbar)\langle\psi(t)|[\delta^{(N)}(\widehat{\mathbf{x}}-(y_1,\ldots,y'_n,\ldots,y_N)),\ \widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)]|\psi(t)\rangle dy'_n,\ \ n=1,\ldots,N.\end{gathered} \tag{9e}$$

Eq. (9e) does not, however, provide information on the consequences of undertaking Fourier decomposition of the canonical Hamiltonian operator $\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)$, as Eq. (5a) does.

Much less general than Eq. (9e), but doubtless of greater physical interest, is the probability vector current density result for the subclass of canonical Hamiltonian operators $\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)$ which satisfy a particular "Ehrenfest" requirement, namely that,

$$[\delta^{(N)}(\widehat{\mathbf{x}}-\mathbf{y}),\ \widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)] = i\hbar\ \overbrace{\{\delta^{(N)}(\mathbf{x}-\mathbf{y}),\ H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)\}}\ , \tag{10a}$$

where, of course,

$$\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t) = \overbrace{H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)}\ . \tag{10b}$$

From Eq. (2) it is immediately apparent that canonical Hamiltonian operators of the form,

$$\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t) = \overbrace{K([\mathbf{p}],t)+V([\mathbf{x}],t)} = K([\widehat{\mathbf{p}}],t)+V([\widehat{\mathbf{x}}],t), \tag{10c}$$

do indeed *satisfy* the "Ehrenfest" requirement of Eq. (10a).

Given a canonical Hamiltonian operator $\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)$ which satisfies this "Ehrenfest" requirement, the substitution of Eq. (10a) into Eq. (9d) permits a series of steps that are *analogous to those of the classical* Eq. (8e), and that *likewise* result in the *explicit algebraic factorization* of the *divergence operator* $\nabla_{\mathbf{y}}\cdot$ out of the Eq. (9d) expression for the divergence $\nabla_{\mathbf{y}}\cdot\mathbf{j}(\mathbf{y},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])$ of the probability vector current density, thereby yielding the probability vector current density $\mathbf{j}(\mathbf{y},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])$ *itself* as the *expectation value of the quantization of the classical single-particle scalar density* $\delta^{(N)}(\mathbf{x}-\mathbf{y})$ *times the classical particle velocity* $\nabla_{\mathbf{p}}H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$. Carrying out this substitution of Eq. (10a) into Eq. (9d), and then proceeding in analogy with Eq. (8e), we obtain,

$$\begin{gathered}\nabla_{\mathbf{y}}\cdot\mathbf{j}(\mathbf{y},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)]) = -\langle\psi(t)|\ \overbrace{\{\delta^{(N)}(\mathbf{x}-\mathbf{y}),\ H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)\}}\ |\psi(t)\rangle = \\ -\langle\psi(t)|\ \overbrace{\big(\nabla_{\mathbf{x}}\delta^{(N)}(\mathbf{x}-\mathbf{y})\big)\cdot\nabla_{\mathbf{p}}H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)}\ |\psi(t)\rangle = \langle\psi(t)|\ \overbrace{\big(\nabla_{\mathbf{y}}\delta^{(N)}(\mathbf{x}-\mathbf{y})\big)\cdot\nabla_{\mathbf{p}}H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)}\ |\psi(t)\rangle = \\ \langle\psi(t)|\ \overbrace{\nabla_{\mathbf{y}}\cdot\big[\delta^{(N)}(\mathbf{x}-\mathbf{y})\,\nabla_{\mathbf{p}}H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)\big]}\ |\psi(t)\rangle = \\ \nabla_{\mathbf{y}}\cdot\big[\langle\psi(t)|\ \overbrace{\delta^{(N)}(\mathbf{x}-\mathbf{y})\,\nabla_{\mathbf{p}}H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)}\ |\psi(t)\rangle\big],\end{gathered} \tag{10d}$$

where, because quantization and taking the divergence with respect to the non-quantized vector variable $\mathbf{y}$ are independent linear processes, we can extract the explicitly factored divergence operator $\nabla_{\mathbf{y}}\cdot$ out of the quantization—and also, of course, out of the expectation value with the state $|\psi(t)\rangle$. Therefore, for the subclass of canonical Hamiltonian operators $\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)$ which satisfy the "Ehrenfest" requirement of Eq. (10a), the probability vector current density is given by,

$$\mathbf{j}(\mathbf{y},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)]) = \langle\psi(t)|\ \overbrace{\delta^{(N)}(\mathbf{x}-\mathbf{y})\,\nabla_{\mathbf{p}}H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)}\ |\psi(t)\rangle, \tag{10e}$$

which is *the expectation value of the quantization of the classical single-particle scalar density* $\delta^{(N)}(\mathbf{x}-\mathbf{y})$ *times the classical particle velocity* $\nabla_{\mathbf{p}}H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$.

Now in *addition* to those *particular* "Ehrenfest-subclass" canonical Hamiltonian operators which have the form given by Eq. (10c), canonical Hamiltonian operators which have the form,

$$\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t) = \overbrace{\mathbf{p}\cdot\mathbf{v}_0(\mathbf{x},t)} = \tfrac{1}{2}(\widehat{\mathbf{p}}\cdot\mathbf{v}_0(\widehat{\mathbf{x}},t)+\mathbf{v}_0(\widehat{\mathbf{x}},t)\cdot\widehat{\mathbf{p}}), \tag{10f}$$

as well satisfy the "Ehrenfest" requirement of Eq. (10a), which we now show in an abbreviated manner,

$$\begin{gathered}[\delta^{(N)}(\widehat{\mathbf{x}}-\mathbf{y}),\ \tfrac{1}{2}(\widehat{\mathbf{p}}\cdot\mathbf{v}_0(\widehat{\mathbf{x}},t)+\mathbf{v}_0(\widehat{\mathbf{x}},t)\cdot\widehat{\mathbf{p}})] = i\hbar\big(\nabla_{\widehat{\mathbf{x}}}\delta^{(N)}(\widehat{\mathbf{x}}-\mathbf{y})\big)\cdot\mathbf{v}_0(\widehat{\mathbf{x}},t) = \\ i\hbar\big(-\nabla_{\mathbf{y}}\delta^{(N)}(\widehat{\mathbf{x}}-\mathbf{y})\big)\cdot\mathbf{v}_0(\widehat{\mathbf{x}},t) = -i\hbar\nabla_{\mathbf{y}}\cdot\big[\delta^{(N)}(\widehat{\mathbf{x}}-\mathbf{y})\,\mathbf{v}_0(\widehat{\mathbf{x}},t)\big],\end{gathered}$$

and,

$$\overbrace{\{\delta^{(N)}(\mathbf{x}-\mathbf{y}),\, \mathbf{p}\cdot\mathbf{v}_0(\mathbf{x},t)\}} = \overbrace{\big(\nabla_{\mathbf{x}}\delta^{(N)}(\mathbf{x}-\mathbf{y})\big)\cdot\mathbf{v}_0(\mathbf{x},t)} = \overbrace{\big(-\nabla_{\mathbf{y}}\delta^{(N)}(\mathbf{x}-\mathbf{y})\big)\cdot\mathbf{v}_0(\mathbf{x},t)} =$$
$$-\nabla_{\mathbf{y}}\cdot\overbrace{\big[\delta^{(N)}(\mathbf{x}-\mathbf{y})\,\mathbf{v}_0(\mathbf{x},t)\big]} = -\nabla_{\mathbf{y}}\cdot\big[\delta^{(N)}(\widehat{\mathbf{x}}-\mathbf{y})\,\mathbf{v}_0(\widehat{\mathbf{x}},t)\big].$$

These results make it apparent that for this case as well, the divergence operator $\nabla_{\mathbf{y}}\cdot$ explicitly algebraically factors out, paving the way for the resulting probability vector current density to be equal to the expectation value of the quantization of the classical single-particle scalar density $\delta^{(N)}(\mathbf{x}-\mathbf{y})$ times the classical particle velocity $\mathbf{v}_0(\mathbf{x},t) = \nabla_{\mathbf{p}}H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$.

Combining the special cases of Eqs. (10c) and (10f), we see that canonical Hamiltonian operators $\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)$ which adhere to the "Ehrenfest" requirement of Eq. (10a) have the form,

$$\begin{aligned}\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t) = \overbrace{K([\mathbf{p}],t)+\mathbf{p}\cdot\mathbf{v}_0(\mathbf{x},t)+V([\mathbf{x}],t)} = \\ K([\widehat{\mathbf{p}}],t)+\tfrac{1}{2}(\widehat{\mathbf{p}}\cdot\mathbf{v}_0(\widehat{\mathbf{x}},t)+\mathbf{v}_0(\widehat{\mathbf{x}},t)\cdot\widehat{\mathbf{p}})+V([\widehat{\mathbf{x}}],t).\end{aligned} \tag{10g}$$

For this "Ehrenfest subclass" of the canonical Hamiltonian operators, the divergence operator $\nabla_{\mathbf{y}}\cdot$ *always explicitly algebraically factors out* of the Eq. (9d) expression $\nabla_{\mathbf{y}}\cdot\mathbf{j}(\mathbf{y},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])$ for the divergence of the probability vector current density, as we see from Eq. (10d), and the consequent probability vector current density $\mathbf{j}(\mathbf{y},t;[|\psi(t)\rangle,\widehat{H}([\widehat{\mathbf{x}},\widehat{\mathbf{p}}],t)])$ *itself* is always given by *the expectation value of the quantization of the classical single-particle scalar density* $\delta^{(N)}(\mathbf{x}-\mathbf{y})$ *times the classical particle velocity* $\nabla_{\mathbf{p}}H_{\mathrm{cl}}([\mathbf{x},\mathbf{p}],t)$, as we see from Eq. (10e).

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