A New Look at the Position Operator in Quantum Theory

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Abstract:

In standard quantum theory the momentum and position operators are on equal footing and wave functions in momentum and coordinate representations are related to each other by the Fourier transform. However, while the momentum operator is unambiguously defined as one of the operators of the symmetry algebra, the position operator has a physical meaning only in semiclassical approximation and should be defined from additional considerations. We show that, as a consequence of the inconsistent definition of standard position operator, an inevitable effect in standard theory is the wave packet spreading (WPS) of the photon coordinate wave function in directions perpendicular to the photon momentum. This leads to the fundamental paradox that we should see not separate stars but only an almost continuous background from all stars. We propose a new consistent definition of the position operator. In this approach WPS in directions perpendicular to the particle momentum is absent regardless of whether the particle is nonrelativistic or relativistic. Hence the above paradox is resolved. Moreover, for an ultrarelativistic particle the effect of WPS is absent at all. Different components of the new position operator do not commute with each other and, as a consequence, there is no wave function in coordinate representation.

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1 Why do we need position operator in quantum theory?

In standard quantum theory the momentum and position operators are on equal footing and wave functions in the momentum and coordinate representations are related to each other by the Fourier transform. However, those operators should not be on equal footing for the following reason. In quantum theory each elementary particle is described by an irreducible representation (IR) of the symmetry algebra. For example, in Poincare invariant theory the set of momentum operators represents three of ten representation operators of the Poincare algebra and hence those operators are consistently defined. On the other hand, among the representation operators there is no position operator. This operator has a physical meaning only in semiclassical approximation and should be *defined* from additional considerations.

As an example, consider the following question. The Schrödinger equation for the electron in the hydrogen atom correctly describes the atom energy levels. Is this an argument in favor of the statement that the standard position operator has a correct physical meaning? Historically this equation has been first written in coordinate space and in standard textbooks on quantum mechanics it is still discussed in this form. However, from the point of view of the present knowledge this equation should be treated as follows.

A fundamental theory describing electromagnetic interactions is quantum electrodynamics (QED). This theory proceeds from quantizing classical Lagrangian which is only an auxiliary tool for constructing S-matrix. When this construction is accomplished, the results of QED are formulated exclusively in momentum space and the theory does not contain space-time at all. In particular, as follows from the Feynman diagram for the one-photon exchange, in the nonrelativistic approximation the electron can be described in the potential formalism where the potential acts on the wave function in momentum space as

$$V\chi(\mathbf{p}) = \frac{e^2}{2\pi^2}\int \frac{\chi(\mathbf{p}')}{(\mathbf{p}-\mathbf{p}')^2} d^3\mathbf{p}'$$

where e is the electron charge. So for calculating energy levels one should solve the eigenvalue problem for the Hamiltonian with this potential. This is an integral equation which can be solved by different methods. One of the convenient methods is to apply the Fourier transform and to get the standard Schrödinger equation in coordinate representation with the Coulomb potential. Then one can find energy levels, coordinate wave functions etc. Hence the fact that the results for energy levels are in good agreement with experiment shows only that QED defines the potential interaction correctly and the standard coordinate Schrödinger equation is only a convenient *mathematical* way of solving the eigenvalue problem. For this problem the physical meaning of the position operator is not important at all. One can consider other transformations of the original integral equation and define other position operators. The fact that for non-standard choices one might obtain something different from the Coulomb potential is not important on quantum level. We know that on classical level the interaction between two charges can be described by the Coulomb potential but this does not imply that on quantum level the potential in coordinate representation should be necessarily Coulomb. We conclude that the fact that the standard coordinate Schrödinger equation correctly reproduces the hydrogen energy levels cannot be treated as an argument in favor of the standard choice of the position operator.

Another example is as follows. It is said that the spatial distribution of the electric charge inside a system can be extracted from measurements of form-factors in the electron scattering on this system. However, the information about the experiment is again given only in terms of momenta and conclusions about the spatial distribution can be drawn only if we assume additionally how the position operator is expressed in terms of momentum variables. On quantum level the physical meaning of such a spatial distribution is not fundamental.

Our general conclusion is as follows. Since the *results* of existing fundamental quantum theories describing interactions on quantum level (QED, electroweak theory and QCD) are formulated exclusively in terms of the S-matrix in momentum space without any mentioning of space-time, for investigating such *stationary quantum* problems as calculating energy levels, form-factors etc., the notion of the position operator is not needed.

However, the choice of the position operator is important in nonstationary problems when the evolution is described by the time dependent Schrödinger equation (with the nonrelativistic or relativistic Hamiltonian). For any new theory there should exist a correspondence principle that at some conditions the new theory should reproduce results of the old well tested theory with a good accuracy. In particular, quantum theory should reproduce the motion of a classical particle along the classical trajectory defined by classical equations of motion. Hence the position operator is needed only in semiclassical approximation.

In standard approaches to quantum theory the existence of space-time background is assumed from the beginning. Then the position operator for a particle in this background is the operator of multiplication by the particle radius-vector $\mathbf{r}$. As explained in standard textbooks on quantum mechanics, the result $-i\hbar\partial/\partial\mathbf{r}$ for the momentum operator can be justified from the requirement that quantum theory should correctly reproduce classical result in semiclassical approximation. However, this requirement does not define the operator uniquely.

A standard approach to Poincare symmetry on quantum level is as follows. Since Poincare group is the group of motions of Minkowski space, quantum states should be described by representations of the Poincare group. In turn, this implies that the representation generators should commute according to the commutation relations of the Poincare group Lie algebra:

$$\begin{aligned}&[P^\mu,P^\nu]=0 \quad [P^\mu,M^{\nu\rho}]=-i(\eta^{\mu\rho}P^\nu-\eta^{\mu\nu}P^\rho)\\&[M^{\mu\nu},M^{\rho\sigma}]=-i(\eta^{\mu\rho}M^{\nu\sigma}+\eta^{\nu\sigma}M^{\mu\rho}-\eta^{\mu\sigma}M^{\nu\rho}-\eta^{\nu\rho}M^{\mu\sigma})\end{aligned} \tag{1}$$

where P^μ are the operators of the four-momentum, $M^{\mu\nu}$ are the operators of Lorentz angular momenta, the diagonal metric tensor $\eta^{\mu\nu}$ has the nonzero components $\eta^{00}=-\eta^{11}=-\eta^{22}=-\eta^{33}=1$ and $\mu,\nu=0,1,2,3$. It is usually said that the above relations are written in the system of units $c=\hbar=1$. However, as we argue in Ref. [1], quantum theory should not contain c and $\hbar$ at all; those quantities arise only because we wish to measure velocities in m/s and angular momenta in $kg\times m^2/s$.

The above approach is in the spirit of the well-known Klein's Erlangen program in mathematics. However, as we argue in Refs. [1, 2], quantum theory should not be based on classical space-time background. The notion of space-time background contradicts the basic principle of physics that a definition of a physical quantity is a description of how this quantity should be measured. Indeed one cannot measure the coordinates of a manifold which exists only in our imagination.

As we argue in Refs. [1, 2] and other publications, the approach should be the opposite. Each system is described by a set of independent operators. By definition, the rules how these operators commute with each other define the symmetry algebra. In particular, *by definition*, Poincare symmetry on quantum level means that the operators commute according to Eq. (1). This definition does not involve Minkowski space at all. Such a definition of symmetry on quantum level is in the spirit of Dirac's paper [3].

The fact that an elementary particle in quantum theory is described by an IR of the symmetry algebra can be treated as a definition of the elementary particle. In Poincare invariant theory the IRs can be implemented in a space of functions $\chi(\mathbf{p})$ such that $\int |\chi(\mathbf{p}|^2 d^3\mathbf{p} < \infty$ (see Sec. 4). In this representation the momentum operator $\mathbf{P}$ is defined *unambiguously* and is simply the operator of multiplication by $\mathbf{p}$. A standard *assumption* is that the position operator in this representation is $i\hbar\partial/\partial\mathbf{p}$.

As noted above, there is no position operator among the representation operators of the Poincare algebra and the position operator is needed only in semiclassical approximation. As explained in standard textbooks on quantum mechanics (see e.g. Ref. [4]), semiclassical approximation cannot be valid in situations when the momentum is rather small. Consider first a one-dimensional case. If the value of the x component of the momentum p_x is rather large, the definition of the coordinate operator $x = i\hbar\partial/\partial p_x$ can be justified but this definition does not have a physical meaning in situations when p_x is small.

Consider now the three-dimensional case. If all the components p_j ($j = 1, 2, 3$) are rather large then there are situations when all the operators $i\hbar\partial/\partial p_j$ are semiclassical. A semiclassical wave function $\chi(\mathbf{p})$ in momentum space should describe a narrow distribution around the mean value $\mathbf{p}_0$. Suppose now that the coordinate axes are chosen such $\mathbf{p}_0$ is directed along the z axis. Then in view of the above remarks the operators $i\hbar\partial/\partial p_j$ cannot be physical for $j = 1, 2$, i.e. in directions perpendicular to the particle momentum. Hence the standard definition of all the components of the position operator can be physical only for special choices of the coordinate axes and there exist choices when the definition is not physical. The situation when a definition of an operator is physical or not depending on the choice of the coordinate axes is not acceptable and hence the standard definition of the position operator is not physical.

It is believed that the standard definition of the position operator is reasonable since in semiclassical approximation the nonstationary Schrödinger equation correctly describes the motion of a quantum mechanical wave packet along the classical trajectory. As explained in standard textbooks on quantum mechanics, if the coordinate wave function $\psi(\mathbf{r}, t)$ contains a rapidly oscillating factor $exp[iS(\mathbf{r}, t)/\hbar]$, where $S(\mathbf{r}, t)$ is the classical action as a function of coordinates and time, then in the formal limit $\hbar \to 0$ the Schrödinger equation becomes the Hamilton-Jacoby equation.

At the same time, it has been known since the discovery of quantum mechanics that its inevitable consequence is the effect of wave packet spreading (WPS) described in standard textbooks and many papers (see e.g. Ref. [5] and references therein). In particular, this effect has been investigated by Schrödinger. The fact that WPS is inevitable has not been treated as a drawback of the theory for the following reasons. For macroscopic bodies this effect is extremely small (see Sec. 3) while in experiments on the Earth with atoms and elementary particles spreading does not have enough time to manifest itself.

However, it seems rather strange that no one has posed a problem of what happens to photons from distant stars which can travel to the Earth even for billions of years. As shown in our Ref. [6] (see also Sec. 7), the effect of WPS for photons emitted even by close stars is so strong that if this effect is calculated in the framework of standard theory then we have a fundamental glaring paradox that we should see not separate stars but rather an almost continuous background from all stars. The consideration given in the present paper shows that the reason of the paradox is that the standard position operator is not consistently defined. Hence the inconsistent definition of the position operator is not an academic problem but leads to the above paradox.

In the present paper we propose a consistent definition of the position operator. As a consequence, in our approach WPS in directions perpendicular to the particle momentum is absent regardless of whether the particle is nonrelativistic or relativistic. Hence the above paradox is resolved. Moreover, for an ultrarelativistic particle the effect of WPS is absent at all. In our approach different components of the position operator do not commute with each other and, as a consequence, there is no wave function in coordinate representation.

Our presentation is selfcontained and for reproducing the results of the calculations no special knowledge is needed. Hence we belive that the paper can be understood by a wide audience.

The paper is organized as follows. In Secs. 2 and 4 we discuss the approach to the position operator in the standard nonrelativistic and relativistic quantum theory, respectively. An inevitable consequence of this approach is the effect of WPS of the coordinate wave function which is discussed in Secs. 3 and 5 for the nonrelativistic and relativistic cases, respectively. As shown in Sec. 7, this leads to a fundamental glaring paradox that we should see not separate stars but almost continuous background from all stars. Out approach to a consistent definition of the position operator and its application to WPS are discussed in Secs. 8-10. Finally, Chap. 11 is discussion.

2 Position operator in nonrelativistic quantum mechanics

In quantum theory, states of a system are represented by elements of a projective Hilbert space. The fact that a Hilbert space H is projective means that if $\psi \in H$ is a state then $const\,\psi$ is the same state. The matter is that not the probability itself but only relative probabilities of different measurement outcomes have a physical meaning. In this paper we will work with states ψ normalized to one, i.e. such that $||\psi|| = 1$ where $||...||$ is a norm. It is defined such that if $(...,...)$ is a scalar product in H then $||\psi|| = (\psi,\psi)^{1/2}$.

In quantum theory every physical quantity is described by a selfadjoint operator. Each selfadjoint operator is Hermitian i.e. satisfies the property $(\psi_2, A\psi_1) = (A\psi_2, \psi_1)$ for any states belonging to the domain of A. If A is an operator of some quantity then the mean value of the quantity and its uncertainty in state ψ are given by $\bar{A} = (\psi, A\psi)$ and $\Delta A = ||(A-\bar{A})\psi||$, respectively. The condition that a quantity corresponding to the operator A is semiclassical in state ψ can be defined such that $|\Delta A| \ll |\bar{A}|$. This implies that the quantity can be semiclassical only if $|\bar{A}|$ is rather large. In particular, if $\bar{A} = 0$ then the quantity cannot be semiclassical.

Let B be an operator corresponding to another physical quantity and $\bar{B}$ and ΔB be the mean value and the uncertainty of this quantity, respectively. We can write $AB = \{A,B\}/2 + [A,B]/2$ where the commutator $[A,B] = AB - BA$ is anti-Hermitian and the anticommutator $\{A,B\} = AB + BA$ is Hermitian. Let $[A,B] = -iC$ and $\bar{C}$ be the mean value of the operator C.

A question arises whether two physical quantities corresponding to the operators A and B can be simultaneously semiclassical in state ψ. Since $||\psi_1||||\psi_2|| \geq |(\psi_1,\psi_2)|$, we have that

$$\Delta A \Delta B \geq \frac{1}{2}|(\psi, (\{A-\bar{A}, B-\bar{B}\} + [A,B])\psi)| \tag{2}$$

Since $(\psi, \{A-\bar{A}, B-\bar{B}\}\psi)$ is real and $(\psi, [A,B]\psi)$ is imaginary, we get

$$\Delta A \Delta B \geq \frac{1}{2}|\bar{C}| \tag{3}$$

This condition is known as a general uncertainty relation between two quantities. A well-known special case is that if P is the x component of the momentum operator and X is the operator of multiplication by x then $[P,X] = -i\hbar$ and $\Delta p \Delta x \geq \hbar/2$. The states where $\Delta p \Delta x = \hbar/2$ are called coherent ones. They are treated such that the momentum and the coordinate are simultaneously semiclassical in a maximal possible extent. A well-known example is that if

$$\psi(x) = \frac{1}{a^{1/2}\pi^{1/4}} exp[\frac{i}{\hbar}p_0 x - \frac{1}{2a^2}(x-x_0)^2]$$

then $\bar{X} = x_0$, $\bar{P} = p_0$, $\Delta x = a/\sqrt{2}$ and $\Delta p = \hbar/(a\sqrt{2})$.

Consider first a one dimensional motion. In standard textbooks on quantum mechanics, the presentation starts with a wave function $\psi(x)$ in coordinate space since it is implicitly assumed that the meaning of space coordinates is known. Then a question arises why $P = -i\hbar d/dx$ should be treated as the momentum operator. The explanation is as follows.

Consider wave functions having the form $\psi(x) = exp(ip_0x/\hbar)a(x)$ where the amplitude $a(x)$ has a sharp maximum near $x = x_0 \in [x_1, x_2]$ such that $a(x)$ is not small only when $x \in [x_1, x_2]$. Then Δx is of the order $x_2 - x_1$ and the condition that the coordinate is semiclassical is $\Delta x \ll |x_0|$. Since $-i\hbar d\psi(x)/dx = p_0\psi(x) - i\hbar exp(ip_0x/\hbar)da(x)/dx$, we see that $\psi(x)$ will be approximately the eigenfunction of $-i\hbar d/dx$ with the eigenvalue p_0 if $|p_0a(x)| \gg \hbar|da(x)/dx|$. Since $|da(x)/dx|$ is of the order of $|a(x)/\Delta x|$, we have a condition $|p_0\Delta x| \gg \hbar$. Therefore if the momentum operator is $-i\hbar d/dx$, the uncertainty of momentum Δp is of the order of $\hbar/\Delta x$, $|p_0| \gg \Delta p$ and this implies that the momentum is also semiclassical. At the same time, $|p_0\Delta x|/2\pi\hbar$ is approximately the number of oscillations which the exponent makes on the segment $[x_1, x_2]$. Therefore the number of oscillations should be much greater than unity. In particular, semiclassical approximation cannot be valid if Δx is very small, but on the other hand, Δx cannot be very large since it should be much less than x_0. Another justification of the fact that $-i\hbar d/dx$ is the momentum operator is that in the formal limit $\hbar \to 0$ the Schrödinger equation becomes the Hamilton-Jacobi equation.

We conclude that the choice of $-i\hbar d/dx$ as the momentum operator is justified from the requirement that in semiclassical approximation this operator becomes the classical momentum. However, it is obvious that this requirement does not define the operator uniquely: any operator $\tilde{P}$ such that $\tilde{P} - P$ disappears in semiclassical limit, also can be called the momentum operator.

One might say that the choice $P = -i\hbar d/dx$ can also be justified from the following considerations. In nonrelativistic quantum mechanics we assume that the theory should be invariant under the action of the Galilei group, which is a group of transformations of Galilei space-time. The x component of the momentum operator should be the generator corresponding to spatial translations along the x axis and $-i\hbar d/dx$ is precisely the required operator. In this consideration one assumes that the space-time background has a physical meaning while, as discussed in Refs. [1, 2] and references therein, this is not the case.

As noted in Refs. [1, 2] and references therein, one should start not from space-time but from a symmetry algebra. Therefore in nonrelativistic quantum mechanics we should start from the Galilei algebra and consider its IRs. For simplicity we again consider a one dimensional case. Let $P_x = P$ be one of representation operators in an IR of the Galilei algebra. We can implement this IR in a Hilbert space of functions $\chi(p)$ such that $\int_{-\infty}^{\infty} |\chi(p)|^2 dp < \infty$ and P is the operator of multiplication by p, i.e. $P\chi(p) = p\chi(p)$. Then a question arises how the operator of the x coordinate should be defined. In contrast to the momentum operator, the coordinate one is not defined by the representation and so it should be defined from additional assumptions. Probably a future quantum theory of measurements will make it possible to construct operators of physical quantities from the rules how these quantities should be measured. However, at present we can construct necessary operators only from rather intuitive considerations.

By analogy with the above discussion, one can say that semiclassical wave functions should be of the form $\chi(p) = exp(-ix_0p/\hbar)a(p)$ where the amplitude $a(p)$ has a sharp maximum near $p = p_0 \in [p_1, p_2]$ such that $a(p)$ is not small only when $p \in [p_1, p_2]$. Then Δp is of the order of $p_2 - p_1$ and the condition that the momentum is semiclassical is $\Delta p \ll |p_0|$. Since $i\hbar d\chi(p)/dp = x_0\chi(p) + i\hbar exp(-ix_0p/\hbar)da(p)/dp$, we see that $\chi(p)$ will be approximately the eigenfunction of $i\hbar d/dp$ with the eigenvalue x_0 if $|x_0a(p)| \gg \hbar|da(p)/dp|$. Since $|da(p)/dp|$ is of the order of $|a(p)/\Delta p|$, we have a condition $|x_0\Delta p| \gg \hbar$. Therefore if the coordinate operator is $X = i\hbar d/dp$, the uncertainty of coordinate Δx is of the order of $\hbar/\Delta p$, $|x_0| \gg \Delta x$ and this implies that the coordinate defined in such a way is also semiclassical. We can also note that $|x_0\Delta p|/2\pi\hbar$ is approximately the number of oscillations which the exponent makes on the segment $[p_1, p_2]$ and therefore the number of oscillations should be much greater than unity. It is also clear that semiclassical approximation cannot be valid if Δp is very small, but on the other hand, Δp cannot be very large since it should be much less than p_0. By analogy with the above discussion, the requirement that the operator $i\hbar d/dp$ becomes the coordinate in classical limit does not define the operator uniquely. In nonrelativistic quantum mechanics it is assumed that the coordinate is a well defined physical quantity even on quantum level and that $i\hbar d/dp$ is the most pertinent choice.

The above results can be directly generalized to the three-dimensional case. For example, if the coordinate wave function is chosen in the form

$$\psi(\mathbf{r}) = \frac{1}{\pi^{3/4}a^{3/2}}exp[-\frac{(\mathbf{r}-\mathbf{r}_0)^2}{2a^2} + \frac{i}{\hbar}\mathbf{p}_0\mathbf{r}] \tag{4}$$

then the momentum wave function is

$$\chi(\mathbf{p}) = \int exp(-\frac{i}{\hbar}\mathbf{p}\mathbf{r})\psi(\mathbf{r})\frac{d^3\mathbf{r}}{(2\pi\hbar)^{3/2}} = \frac{a^{3/2}}{\pi^{3/4}\hbar^{3/2}}exp[-\frac{(\mathbf{p}-\mathbf{p}_0)^2a^2}{2\hbar^2} - \frac{i}{\hbar}(\mathbf{p}-\mathbf{p}_0)\mathbf{r}_0] \tag{5}$$

It is easy to verify that

$$||\psi||^2 = \int |\psi(\mathbf{r})|^2d^3\mathbf{r} = 1, \quad ||\chi||^2 = \int |\chi(\mathbf{p})|^2d^3\mathbf{p} = 1, \tag{6}$$

the uncertainty of each component of the coordinate operator is $a/\sqrt{2}$ and the uncertainty of each component of the momentum operator is $\hbar/(a\sqrt{2})$. Hence one might think that Eqs. (4) and (5) describe a state which is semiclassical in a maximal possible extent.

Let us make the following remark about semiclassical vector quantities. We defined a quantity as semiclassical if its uncertainty is much less than its mean value. In particular, as noted above, a quantity cannot be semiclassical if its mean value is small. In the case of vector quantities we have sets of three physical quantities. Some of them can be small and for them it is meaningless to discuss whether they are semiclassical or not. We say that a vector quantity is semiclassical if all its components which are not small are semiclassical and there should be at least one semiclassical component.

For example, if the mean value of the momentum $\mathbf{p}_0$ is directed along the z axes then the xy components of the momentum are not semiclassical but the three-dimensional vector quantity $\mathbf{p}$ can be semiclassical if $\mathbf{p}_0$ is rather large. However, in that case the definitions of the x and y components of the position operator as $x = i\hbar\partial/\partial p_x$ and $y = i\hbar\partial/\partial p_y$ become inconsistent. The situation when the validity of an operator depends on the choice of directions of the coordinate axes is not acceptable and hence the above definition of the position operator is at least problematic. Moreover, as already mentioned, the standard choice of the position operator leads to a paradox with WPS of the coordinate photon wave function.

Let us note that semiclassical states can be constructed not only in momentum or coordinate representations. For example, instead of momentum wave functions $\chi(\mathbf{p})$ one can work in the representation where the quantum numbers (p, l, μ) in wave functions $\chi(p, l, \mu)$ mean the magnitude of the momentum p, the orbital quantum number l (such that a state is the eigenstate of the orbital momentum squared $\mathbf{L}^2$ with the eigenvalue $l(l+1)$) and the magnetic quantum number μ (such that a state is the eigenvector or L_z with the eigenvalue μ). A state described by a $\chi(p, l, \mu)$ will be semiclassical with respect to those quantum numbers if $\chi(p, l, \mu)$ has a sharp maximum at $p = p_0$, $l = l_0$, $\mu = \mu_0$ and the widths of the maxima in p, l and μ are much less than p_0, l_0 and μ_0, respectively. However, by analogy with the above discussion, those widths cannot be arbitrarily small if one wishes to have other semiclassical variables (e.g. the coordinates). Examples of such situations will be discussed in Sec. 9.

3 Wave packet spreading in nonrelativistic quantum mechanics

A well-known fact of quantum theory is that there is no operator having the meaning of the time operator. Hence a problem arises how time should be understood in quantum theory and this problem is discussed in a wide literature (see e.g. Ref. [7]). It is usually assumed that time is a classical parameter such that the dependence of the wave function on time is defined by the Hamiltonian according to the Schrödinger equation. In nonrelativistic quantum mechanics the Hamiltonian of the particle with the mass m is $H = \mathbf{p}^2/2m$ and hence, as follows from Eq. (5), in the model discussed above the dependence of the momentum wave function on t is given by

$$\chi(\mathbf{p},t) = \frac{a^{3/2}}{\pi^{3/4}\hbar^{3/2}} exp[-\frac{(\mathbf{p}-\mathbf{p}_0)^2 a^2}{2\hbar^2} - \frac{i}{\hbar}(\mathbf{p}-\mathbf{p}_0)\mathbf{r}_0 - \frac{i\mathbf{p}^2 t}{2m\hbar}] \tag{7}$$

It is easy to verify that for this state the mean value of the operator $\mathbf{p}$ and the uncertainty of each momentum component are the same as for the state $\chi(\mathbf{p})$, i.e. those quantities do not change with time.

Consider now the dependence of the coordinate wave function on t. This dependence can be calculated by using Eq. (7) and the fact that

$$\psi(\mathbf{r},t) = \int exp(\frac{i}{\hbar}\mathbf{pr})\chi(\mathbf{p},t)\frac{d^3\mathbf{p}}{(2\pi\hbar)^{3/2}} \tag{8}$$

The result of a direct calculation is

$$\psi(\mathbf{r},t) = \frac{1}{\pi^{3/4}a^{3/2}}(1+\frac{i\hbar t}{ma^2})^{-3/2} exp[-\frac{(\mathbf{r}-\mathbf{r}_0-\mathbf{v}_0 t)^2}{2a^2(1+\frac{\hbar^2 t^2}{m^2 a^4})}(1-\frac{i\hbar t}{ma^2}) + \frac{i}{\hbar}\mathbf{p}_0\mathbf{r} - \frac{i\mathbf{p}_0^2 t}{2m\hbar}] \tag{9}$$

where $\mathbf{v}_0 = \mathbf{p}_0/m$ is the classical velocity. This result shows that the semiclassical wave packet is moving along the classical trajectory $\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t$. At the same time, it is now obvious that the uncertainty of each coordinate depends on time as

$$\Delta x_j(t) = \Delta x_j(0)(1+\hbar^2 t^2/m^2 a^4)^{1/2}, \quad (j=1,2,3) \tag{10}$$

where $\Delta x_j(0) = a/\sqrt{2}$, i.e. the width of the wave packet in coordinate representation is increasing. This fact, known as the wave-packet spreading (WPS), is described in many textbooks and papers (see e.g. the textbooks [5] and references therein). It shows that if a state was semiclassical in the maximal extent at $t=0$, it will not have this property at $t>0$ and the accuracy of semiclassical approximation will decrease with the increase of t. The characteristic time of spreading can be defined as $t_* = ma^2/\hbar$. For macroscopic bodies this is an extremely large quantity and hence in macroscopic physics the effect of the wave-packet spreading can be neglected. In the formal limit $\hbar \to \infty$, t_* becomes infinite, i.e. spreading does not take place. This shows that WPS is a pure quantum phenomenon. For the first time the result (9) has been probably obtained by Schrödinger.

One might pose a problem whether the WPS effect is specific only for Gaussian wave functions. One might expect that this effect will take place in general situations since each component of the standard position operator $\mathbf{r} = i\hbar\partial/\partial\mathbf{p}$ does not commute with the Hamiltonian and so for each component of $\mathbf{r}$ the distribution of the corresponding physical quantity will be time dependent.

As shown in Ref. [8] titled "Nonspreading wave packets", for a one-dimensional wave function in the form of an Airy function, spreading does not take place and the maximum of the quantity $|\psi(x)|^2$ propagates with constant acceleration even in the absence of external forces. Those properties of Airy packets have been observed in optical experiments [9]. However, since such a wave function is not normalizable, we believe that the term "wave packet" in the given situation might be misleading since the mean values and uncertainties of the coordinate and momentum cannot be calculated in a standard way. Such a wave function can be constructed only in a limited region of space. As explained in Ref. [8], this wave function describes not a particle but rather families of particle orbits. As shown in Ref. [8], one can construct a normalized state which is a superposition of Airy functions with Gaussian coefficients and "eventually the spreading due to the Gaussian cutoff takes over". This is an additional argument that the effect of WPS is an inevitable consequence of standard quantum theory.

Since quantum theory is invariant under time reversal, one might ask the following question: is it possible that the width of the wave packet in coordinate representation is decreasing with time? From the formal point of view, the answer is "yes". Indeed, the solution given by Eq. (9) is valid not only when $t \geq 0$ but when $t < 0$ as well. Then, as follows from Eq. (10), the uncertainty of each coordinate is decreasing when t changes from some negative value to zero. However, eventually the value of t will become positive and the quantities $\Delta x_j(t)$ will grow to infinity. In the present paper we consider situations when a photon is created on atomic level and hence one might expect that its initial coordinate uncertainties are not large. However, when the photon travels a long distance to the Earth, those uncertainties become much greater, i.e. the term WPS reflects the physics adequately.

In Sec. 5 we consider how the WPS effect manifests itself in relativistic quantum theory.

4 Position operator in relativistic quantum mechanics

The next step in our construction is the definition of elementary particle. Although theory of elementary particles exists for a rather long period of time, there is no commonly accepted definition of elementary particle in this theory. In Refs. [1, 2] and references cited therein we argue that, in the spirit of Wigner's approach to Poincare symmetry [10], a general definition, not depending on the choice of the classical background and on whether we consider a local or nonlocal theory, is that a particle is elementary if the set of its wave functions is the space of an IR of the symmetry algebra in the given theory.

There exists a wide literature describing how IRs of the Poincare algebra can be constructed. In particular, an IR can be implemented in a space of functions $\xi(\mathbf{p})$ satisfying the condition

$$\int |\xi(\mathbf{p})|^2 d\rho(\mathbf{p}) < \infty, \quad d\rho(\mathbf{p}) = \frac{d^3\mathbf{p}}{\epsilon(\mathbf{p})} \tag{11}$$

where $\epsilon(\mathbf{p}) = (m^2 + \mathbf{p}^2)^{1/2}$ is the energy of the particle with the mass m. The convenience of the above requirement is that the volume element $d\rho(\mathbf{p})$ is Lorentz invariant.

It can be easily shown by direct calculations (see e.g. Ref. [11]) that in the space of functions satisfying Eq. (11) the representation operators in the spinless case have the form

$$\mathbf{L} = -i\mathbf{p} \times \frac{\partial}{\partial \mathbf{p}}, \quad \mathbf{N} = -i\epsilon(\mathbf{p})\frac{\partial}{\partial \mathbf{p}}, \quad \mathbf{P} = \mathbf{p}, \quad E = \epsilon(\mathbf{p}) \tag{12}$$

where $\mathbf{L} = (M^{23}, M^{31}, M^{12})$ is the orbital angular momentum operator, $\mathbf{N} = (M^{10}, M^{20}, M^{30})$ is the Lorentz boost operator, $\mathbf{P} = (P^1, P^2, P^3)$ is the momentum operator and $E = P^0$ is the energy operator. For particles with spin, the functions $\xi(\mathbf{p})$ also depend on the spin projections but we will not write this dependence explicitly. Also for particles with spin the Lorentz algebra operators $M^{\mu\nu}$ contain additional spin terms but the momentum and energy operators are the same as in the spinless case. We assume as usual that in semiclassical approximation orbital parts of the Lorentz algebra operators are much greater than the corresponding spin parts and hence spin effects can be neglected.

As follows from Eqs. (1), the operator $I_2 = E^2 - \mathbf{P}^2$ is the Casimir operator of the second order, i.e. it is a bilinear combination of representation operators commuting with all the operators of the algebra. As follows from the well-known Schur lemma, all states belonging to an IR are the eigenvectors of I_2 with the same eigenvalue m^2. Note that Eqs. (12) contain only m^2 but not m. The choice of the energy sign is only a matter of convention but not a matter of principle. Indeed, the energy can be measured only if the momentum $\mathbf{p}$ is measured and then it is only a matter of convention what sign of the square root should be chosen. However, it is important that the sign should be the same for all particles. For example, if we consider a system of two particles with the same values of m^2 and the opposite momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ such that $\mathbf{p}_1 + \mathbf{p}_2 = 0$, we cannot define the energies of the particles as $\epsilon(\mathbf{p}_1)$ and $-\epsilon(\mathbf{p}_2)$, respectively, since in that case the total four-momentum of the two-particle system will be zero what contradicts experiment.

The notation $I_2 = m^2$ is justified by the fact that for all known particles I_2 is greater or equal than zero. Then the mass m is *defined* as the square root of m^2 and the sign of m is only a matter of convention. The usual convention is that $m \geq 0$. However, from mathematical point of view, IRs with $I_2 < 0$ are not prohibited. If the velocity operator $\mathbf{v}$ is *defined* as $\mathbf{v} = \mathbf{P}/E$ then for known particles $|\mathbf{v}| \leq 1$, i.e. $|\mathbf{v}| \leq c$ in standard units. However, for IRs with $I_2 < 0$ we will have that $|\mathbf{v}| > c$ and, at least from the point of view of mathematical construction of IRs, this case is not prohibited. The hypothetical particles with such properties are called tachyons and their possible existence is widely discussed in the literature. If the tachyon mass m is also defined as the square root of m^2 then this quantity will be imaginary. However, this does not mean than the corresponding IRs are unphysical since all the operators of the Poincare group Lie algebra depend only on m^2.

As follows from Eqs. (11) and (12), in the nonrelativistic approximation $d\rho(\mathbf{p}) = d^3\mathbf{p}/m$ and $\mathbf{N} = -im\partial/\partial\mathbf{p}$. Therefore in this approximation $\mathbf{N}$ is proportional to the *standard* position operator and one can say that the position operator is in fact present in the description of the IR.

In relativistic case the operator $i\partial/\partial\mathbf{p}$ is not selfadjoint since $d\rho(\mathbf{p})$ is not proportional to $d^3\mathbf{p}$. However, one can perform a unitary transformation $\xi(\mathbf{p}) \to \chi(\mathbf{p}) = \xi(\mathbf{p})/\epsilon(\mathbf{p})^{1/2}$ such that the Hilbert space becomes the space of functions $\chi(\mathbf{p})$ satisfying the condition $\int |\chi(\mathbf{p})|^2 d^3\mathbf{p} < \infty$. It is easy to verify that in this implementation of the IR the operators $(\mathbf{L}, \mathbf{P}, E)$ will have the same form as in Eq. (12) but the expression for $\mathbf{N}$ will be

$$\mathbf{N} = -i\epsilon(\mathbf{p})^{1/2}\frac{\partial}{\partial\mathbf{p}}\epsilon(\mathbf{p})^{1/2} \tag{13}$$

In this case one can *define* $\mathbf{r} = i\hbar\partial/\partial\mathbf{p}$ as a position operator but now we do not have a situation when the position operator is present among the other representation operators.

A problem of the definition of the position operator in relativistic quantum theory has been discussed since the beginning of the 1930s and it has been noted that when quantum theory is combined with relativity the existence of the position operator with correct physical properties becomes a problem. The above definition has been proposed by Newton and Wigner in Ref. [12]. With this definition the coordinate wave function $\psi(\mathbf{r})$ can be again defined by Eq. (4) and a question arises whether this operator has all the required properties of the physical coordinate operator.

For example, in the introductory section of the well-known textbook [13] the following arguments are given in favor of the statement that in relativistic quantum theory it is not possible to define a physical position operator. Suppose that we measure coordinates of an electron with the mass m. When the uncertainty of the coordinates is of the order of $\hbar/mc$, the uncertainty of momenta is of the order of mc, the uncertainty of energy is of the order of mc^2 and hence creation of electron-positron pairs is allowed. As a consequence, it is not possible to localize the electron with the accuracy better than its Compton wave length $\hbar/mc$. Hence, for a particle with a nonzero mass exact measurement is possible only either in the non-relativistic limit (when $c \to \infty$) or classical limit (when $\hbar \to 0$). If $m = 0$ is possible, the problem becomes even more complicated since the photon can create other photons with lesser energies. However, those arguments do not exclude a possibility that the Newton-Wigner position operator can be meaningful in semiclassical approximation.

Another argument that the Newton-Wigner position operator does not have all the required properties follows. If at $t = 0$ the function $\psi(\mathbf{r})$ has a finite carrier (i.e. $\psi(\mathbf{r}) \neq 0$ only if $\mathbf{r}$ belongs to a vicinity of some vector $\mathbf{r}_0$) and the evolution of $\psi(\mathbf{r}, t)$ is governed by the Schrödinger equation with the relativistic energy operator then it is easy to show that at any $t > 0$ the carrier of $\psi(\mathbf{r}, t)$ will belong to the whole three-dimensional space. Then at any $t > 0$ the particle can be detected at any point of the space and this contradicts the requirement that no information should be transmitted with the speed greater than c. However, one might say that the requirement that no signal can be transmitted with the speed greater than c has been obtained in Special Relativity which is a classical (i.e. nonquantum) theory operating only with classical space-time coordinates. As noted above, from the point of view of quantum theory the existence of tachyons is not prohibited. Note also that when two electrically charged particles exchange by a virtual photon, a typical situation is that the four-momentum of the photon is spacelike, i.e. the photon is the tachyon.

In view of the abovementioned paradox with WPS of the photon wave function, we consider the photon case in greater details. Let us first make a few remarks about the terminology of quantum theory. The terms "wave function" and "particle-wave duality" have arisen at the beginning of quantum era in efforts to explain quantum behavior in terms of classical waves but now it is clear that no such explanation exists. The notion of wave is purely classical; it has a physical meaning only as a way of describing systems of many particles by their average characteristics. In particular, such notions as frequency and wave length can be applied only to classical waves, i.e. to systems consisting of many particles such that space-time characteristics of those systems are measured on classical level. If a particle state vector contains $exp[i(px - Et)/\hbar]$ then by analogy with the theory of classical waves one might say that the particle is a wave with the frequency $\omega = E/\hbar$ and the (de Broglie) wave length $\lambda = 2\pi\hbar/p$. However, such defined quantities ω and λ are not real frequencies and wave lengths measured e.g. in spectroscopic experiments where only characteristics of many-particle systems are measured. In quantum theory the photon and other particles can be characterized by their energies, momenta and other quantities for which there exist well defined operators. Those quantities might be measured in collisions of those particles with other particles. The term "wave function" might be misleading since in quantum theory it defines not amplitudes of waves but only amplitudes of probabilities. So, although in our opinion the term "state vector" is more pertinent than "wave function" we will use the latter in accordance with the usual terminology.

In classical theory the notion of field, as well as that of wave, is used for describing systems of many particles by their average characteristics. For example, the electromagnetic field consists of many photons. In classical theory each photon is not described individually but the field as a whole is described by the field strengths $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ which can be measured (in principle) by using macroscopic test bodies. In particular, the notions of electric and magnetic fields of a single photon have no physical meaning.

In standard textbooks on quantum electrodynamics (see e.g. Ref. [14]) it is stated that in this theory there is no way to define a photon wave function in coordinate representation and the arguments are as follows. The electric and magnetic fields of the photon in coordinate representation are proportional to Fourier transforms of $|\mathbf{p}|^{1/2}\chi(\mathbf{p})$, rather than $\chi(\mathbf{p})$. As a consequence, the quantities $\mathbf{E}(\mathbf{r})$

and $\mathbf{B}(\mathbf{r})$ are defined not by $\psi(\mathbf{r})$ but by integrals of $\psi(\mathbf{r})$ over a region of the order of the wave length. However, this argument also does not exclude the possibility that $\psi(\mathbf{r})$ can have a physical meaning in semiclassical approximation since, as noted above, the notions of the electric and magnetic fields of the single photon do not have a physical meaning. Note also that the wave lengths of photons belonging to visible light are so small that if a wave function gives a good description with the accuracy of wave lengths then on semiclassical level this description is quite satisfactory.

The above discussion shows that on quantum level the physical meaning of the coordinate is not clear but at least there are reasons to think that the transverse component of the Newton-Wigner position operator has a physical meaning in semiclassical approximation (see also Sec. 6). Indeed, one might expect that the relativistic nature of the photon will be somehow manifested in the longitudinal direction while in transverse directions the behavior of the wave function should be similar to that in standard nonrelativistic quantum mechanics.

In any case for describing the motion of photons in semiclassical approximation we need a position operator and it seems that the existing theory has not managed to propose anything better than the Newton-Wigner position operator. For this reason in the remaining part of this chapter we consider what happens if the space-time evolution of relativistic wave packets is described by using this operator.

5 Wave packet spreading in relativistic quantum mechanics

Consider first a construction of the wave packet for a particle with nonzero mass. A possible way of the construction follows. We first consider the particle in its rest system, i.e. in the reference frame where the mean value of the particle momentum is zero. The wave function $\chi_0(\mathbf{p})$ in this case can be taken as in Eq. (5) with $\mathbf{p}_0 = 0$. As noted in Sec. 2, such a state cannot be semiclassical. However, it is possible to obtain a semiclassical state by applying a Lorentz transformation to $\chi_0(\mathbf{p})$. As shown in a wide literature, in standard quantum theory any IR representation of the algebra (1) by Hermitiam operators can be extended to an unitary IR of the Poincare group. One can show (see e.g. Eq. (2.4) in Ref. [11]) that for a spinless particle the unitary representation operator $U(g)$ corresponding to a Lorentz transformation g can be defined as

$$U(g)\chi_0(\mathbf{p}) = [\frac{\epsilon(\mathbf{p}')}{\epsilon(\mathbf{p})}]^{1/2}\chi_0(\mathbf{p}') \tag{14}$$

where $\mathbf{p}'$ is the momentum obtained from $\mathbf{p}$ by the Lorentz transformation g^{-1}. If g is the Lorentz boost along the z axis with the velocity v then

$$\mathbf{p}'_\perp = \mathbf{p}_\perp, \quad p'_z = \frac{p_z - v\epsilon(\mathbf{p})}{(1-v^2)^{1/2}} \tag{15}$$

where we use the subscript $\perp$ to denote projections of vectors onto the xy plane.

As follows from this expression, $exp(-\mathbf{p}'^2a^2/2\hbar^2)$ as a function of $\mathbf{p}$ has the maximum at $\mathbf{p}_\perp = 0$, $p_z = p_{z0} = v[(m^2+\mathbf{p}_\perp^2)/(1-v^2)]^{1/2}$ and near the maximum

$$exp(-\frac{a^2\mathbf{p}'^2}{2\hbar^2}) \approx exp\{-\frac{1}{2\hbar^2}[a^2\mathbf{p}_\perp^2 + b^2(p_z - p_{z0})^2]\}$$

where $b = a(1-v^2)^{1/2}$ what represents the effect of the Lorentz contraction. If $m \gg \hbar/a$ (in units where $c = 1$) then $m \gg |\mathbf{p}_\perp|$ and $p_{z0} \approx mv/(1-v^2)^{1/2}$. In this case the transformed state is semiclassical and the mean value of the momentum is exactly the classical (i.e. nonquantum) value of the momentum of a particle with mass m moving along the z axis with the velocity v. However, in the opposite case when $m \ll \hbar/a$ the transformed state is not semiclassical since the uncertainty of p_z is of the same order as the mean value of p_z.

If the photon mass is exactly zero then the photon cannot have the rest state. However, even if the photon mass is not exactly zero, it is so small that the relation $m \ll \hbar/a$ is certainly satisfied for any realistic value of a. Hence a semiclassical state for the photon or a particle with a very small mass cannot be obtained by applying the Lorentz transformation to $\chi_0(\mathbf{p})$ and considering the case when v is very close to unity. In this case we will describe a semiclassical state by a wave function which is a generalization of the function (5):

$$\chi(\mathbf{p},0) = \frac{ab^{1/2}}{\pi^{3/4}\hbar^{3/2}} exp[-\frac{\mathbf{p}_\perp^2 a^2}{2\hbar^2} - \frac{(p_z-p_0)^2b^2}{2\hbar^2} - \frac{i}{\hbar}\mathbf{p}_\perp\mathbf{r}_{0\perp} - \frac{i}{\hbar}(p_z-p_0)z_0] \qquad (16)$$

Here we assume that the vector $\mathbf{p}_0$ is directed along the z axis and its z component is p_0. In the general case the parameters a and b defining the momentum distributions in the transverse and longitudinal directions, respectively, can be different. In that case the uncertainty of each transverse component of momentum is $\hbar/(a\sqrt{2})$ while the uncertainty of the z component of momentum is $\hbar/(b\sqrt{2})$. In view of the above discussion one might think that, as a consequence of the Lorentz contraction, the parameter b should be very small. However, the above discussion shows that the notion of the Lorentz contraction has a physical meaning only if $m \gg \hbar/a$ while for the photon the opposite relation takes place. We will see below that in typical situations the quantity b is large and much greater than a.

In relativistic quantum theory the situation with time is analogous to that in the nonrelativistic case (see Sec. 3) and time can be treated only as a good approximate parameter describing the evolution according to the Schrödinger equation with the relativistic Hamiltonian. Then the dependence of the momentum wave function (16) on t is given by

$$\chi(\mathbf{p},t) = exp(-\frac{i}{\hbar}pct)\chi(\mathbf{p},0) \qquad (17)$$

where $p = |\mathbf{p}|$ and we assume that the particle is ultrarelativistic, i.e. $p \gg m$. Since at different moments of time the wave functions in momentum space differ each other only by a phase factor, the mean value and uncertainty of each momentum component do not depend on time. In other words, there is no WPS for the wave function in momentum space. As noted in Sec. 2, the same is true in the nonrelativistic case.

In view of the above discussion, the function $\psi(\mathbf{r},t)$ can be again defined by Eq. (8) where now $\chi(\mathbf{p},t)$ is defined by Eq. (17). If the variable p_z in the integrand is replaced by p_0+p_z then as follows from Eqs. (8,16,17)

$$\psi(\mathbf{r},t) = \frac{ab^{1/2}exp(i\mathbf{p}_0\mathbf{r}/\hbar)}{\pi^{3/4}\hbar^{3/2}(2\pi\hbar)^{3/2}}\int exp\{-\frac{\mathbf{p}_\perp^2 a^2}{2\hbar^2} - \frac{p_z^2 b^2}{2\hbar^2} + \frac{i}{\hbar}\mathbf{p}(\mathbf{r}-\mathbf{r}_0)$$
$$-\frac{ict}{\hbar}[(p_z+p_0)^2+\mathbf{p}_\perp^2]^{1/2}\}d^3\mathbf{p} \quad (18)$$

We now take into account the fact that in semiclassical approximation the quantity p_0 should be much greater than the uncertainties of the momentum in the longitudinal and transversal directions, i.e. $p_0 \gg p_z$ and $p_0 \gg |\mathbf{p}_\perp|$. Hence with a good accuracy we can expand the square root in the integrand in powers of $|\mathbf{p}|/p_0$. Taking into account the linear and quadratic terms in the square root we get

$$[(p_z+p_0)^2+\mathbf{p}_\perp^2]^{1/2} \approx p_0 + p_z + \mathbf{p}_\perp^2/2p_0 \quad (19)$$

Then the integral over $d^3\mathbf{p}$ can be calculated as the product of integrals over $d^2\mathbf{p}_\perp$ and dp_z and the calculation is analogous to that in Eq. (9). The result of the calculation is

$$\psi(\mathbf{r},t) = [\pi^{3/4}ab^{1/2}(1+\frac{i\hbar ct}{p_0a^2})]^{-1}exp[\frac{i}{\hbar}(\mathbf{p}_0\mathbf{r}-p_0ct)]$$
$$exp[-\frac{(\mathbf{r}_\perp-\mathbf{r}_{0\perp})^2(1-\frac{i\hbar ct}{p_0a^2})}{2a^2(1+\frac{\hbar^2c^2t^2}{p_0^2a^4})} - \frac{(z-z_0-ct)^2}{2b^2}] \quad (20)$$

This result shows that the wave packet describing an ultrarelativistic particle (including a photon) is moving along the classical trajectory $z(t) = z_0 + ct$, in the longitudinal direction there is no spreading while in the transversal direction spreading is characterized by the function

$$a(t) = a(1+\frac{\hbar^2c^2t^2}{p_0^2a^4})^{1/2} \quad (21)$$

The characteristic time of spreading can be defined as $t_* = p_0a^2/\hbar c$. The fact that $t_* \to \infty$ in the formal limit $\hbar \to 0$ shows that in relativistic case WPS also is a pure quantum phenomenon (see the end of Sec. 2). From the formal point of view the result for t_* is the same as in nonrelativistic theory but m should be replaced by E/c^2 where E is the energy of the ultrarelativistic particle. This fact could be expected since, as noted above, it is reasonable to think that spreading in the direction perpendicular to the particle momentum is similar to that in standard nonrelativistic quantum mechanics. However, in the ultrarelativistic case spreading takes place only in this direction. If $t \gg t_*$ the transversal width of the packet is $a(t) = \hbar ct/p_0a$. Hence the speed of spreading in the perpendicular direction is $v_* = \hbar c/p_0a$.

6 Geometrical optics

The relation between quantum and classical electrodynamics is well-known and is described in textbooks (see e.g. Ref. [14]). As already noted, classical electromagnetic field consists of many photons and in classical electrodynamics the photons are not described individually. Instead, classical electromagnetic field is described by field strengths which represent average characteristics of a large set of photons. For constructing the field strengths one can use the photon wave functions $\chi(\mathbf{p},t)$ or $\psi(\mathbf{r},t)$ where E is replaced by $\hbar\omega$ and $\mathbf{p}$ is replaced by $\hbar\mathbf{k}$. In this connection it is interesting to note that since ω is a classical quantity used for describing a classical electromagnetic field, the photon is a pure quantum particle since its energy disappears in the formal limit $\hbar \to 0$. Even this fact shows that the photon cannot be treated as a classical particle and the effect of WPS for the photon cannot be neglected.

With the above replacements the functions χ and ψ will not contain any dependence on $\hbar$ (note that the normalization factor $\hbar^{-3/2}$ in $\chi(\mathbf{k},t)$ will disappear since the normalization integral for $\chi(\mathbf{k},t)$ is now over $d^3\mathbf{k}$, not $d^3\mathbf{p}$). The quantities ω and $\mathbf{k}$ are now treated, respectively, as the frequency and the wave vector of the classical electromagnetic field and the functions $\chi(\mathbf{k},t)$ and $\psi(\mathbf{r},t)$ are interpreted not such that they describe probabilities for a single photon but such that they describe classical electromagnetic field and $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ can be constructed from these functions as described in textbooks on quantum electrodynamics (see e.g. Ref. [14]).

As noted in the preceding section, some authors (see e.g. Ref. [14]) state that the function $\psi(\mathbf{r},t)$ cannot be interpreted as the photon wave function in coordinate representation since for each value of $\mathbf{r}$, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ depend not only on $\psi(\mathbf{r},t)$ but on the values of the function ψ in some vicinity of $\mathbf{r}$. This vicinity has dimensions of the order of the wave length. Hence, for example for visible light, where the wave length is of the order of hundreds of nanometers, the quantity $\mathbf{r}$ can be treated as a good coordinate in semiclassical approximation. Another argument in favor of this statement is that in classical electrodynamics the quantities $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ for the free field should satisfy the wave equation $\partial^2\mathbf{E}/c^2\partial t^2 = \Delta\mathbf{E}$ and analogously for $\mathbf{B}(\mathbf{r},t)$. Hence if $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ are constructed from $\psi(\mathbf{r},t)$ as described in textbooks (see e.g. Ref. [14]), they will satisfy the wave equation since, as follows from Eqs. (8,16,17), $\psi(\mathbf{r},t)$ also satisfies this equation.

The approximation of geometrical optics can be formulated in full analogy with semiclassical approximation in quantum theory. This approximation implies that if $\mathbf{k}_0$ and $\mathbf{r}_0$ are the average values of the wave vector and the spatial radius vector for a wave packet describing the electromagnetic wave then the uncertainties Δk and Δr, which are the average values of $|\mathbf{k}-\mathbf{k}_0|$ and $|\mathbf{r}-\mathbf{r}_0|$, respectively, should satisfy the requirements $\Delta k \ll |\mathbf{k}_0|$ and $\Delta r \ll |\mathbf{r}_0|$. Analogously, in full analogy with the derivation of Eq. (3), one can show that for each $j = 1,2,3$ the uncertainties of the corresponding projections of the vectors $\mathbf{k}$ and $\mathbf{r}$ satisfy the requirement $\Delta k_j \Delta r_j \geq 1/2$ (see e.g. Ref. [15]). In particular, an electromagnetic wave satisfies the approximation of geometrical optics in the greatest possible extent if $\Delta k \Delta r$ is of the order of unity.

We conclude that in the language of classical waves the parameters of spreading can be characterized by the function $a(t)$ (see Eq. (21)) and the quantities t_* and v_* (see the end of the preceding section) such that in terms of the wave length $\lambda = 2\pi c/\omega_0$

$$a(t) = a(1 + \frac{\lambda^2 c^2 t^2}{4\pi^2 a^4})^{1/2}, \quad t_* = \frac{2\pi a^2}{\lambda c}, \quad v_* = \frac{\lambda c}{2\pi a} \tag{22}$$

In Ref. [16] the problem of WPS for classical electromagnetic waves has been discussed in the Fresnel approximation (i.e. in the approximation of geometrical optics) for a two-dimensional wave packet. Equation (25) of Ref. [16] is a special case of Eq. (19) and the author of Ref. [16] shows that, in his model the wave packet spreads in the direction perpendicular to the group velocity of the packet. As noted at the end of the preceding section, in the ultrarelativistic case the function $a(t)$ is given by the same expression as in the nonrelativistic case but m is replaced by E/c^2. Hence if the results of the preceding section are reformulated in terms of classical waves then m should be replaced by $\hbar\omega_0/c^2$ and this fact has been pointed out in Ref. [16].

The quantity $N_{||} = b/\lambda$ shows how many oscillations the oscillating exponent in Eq. (20) makes in the region where the wave function or the amplitude of the classical wave is significantly different from zero. As noted in Sec. 2, for the validity of semiclassical approximation this quantity should be very large. In nonrelativistic quantum mechanics a and b are of the same order and hence the same can be said about the quantity $N = a/\lambda$. As noted above, in the case of the photon we don't know the relation between a and b. In terms of the quantity N we can rewrite the expressions for t_* and v_* in Eq. (22) as

$$t_* = 2\pi N^2 T, \quad v_* = \frac{c}{2\pi N} \tag{23}$$

where T is the period of the classical wave. Hence the accuracy of semiclassical approximation (or the geometrical optics approximation in classical electrodynamics) increases with the increase of N.

7 Experimental consequences of WPS in standard theory

The problem of explaining the redshift phenomenon has a long history. Different competing approaches can be divided into two big sets which we call Theory A and Theory B. In Theory A the redshift has been originally explained as a manifestation of the Doppler effect but in recent years the cosmological and gravitational redshifts have been added to the consideration. In this theory the interaction of photons with the interstellar medium is treated as practically not important, i.e. it is assumed that with a good accuracy we can treat photons as propagating in the empty space. On the contrary, in Theory B, which is often called the tired-light theory, the interaction of photons with the interstellar medium is treated as a main reason for the redshift. At present the majority of physicists believe that Theory A explains the astronomical data better than Theory B. Even some physicists working on Theory B acknowledged that any sort of scattering of light would predict more blurring that is seen (see e.g. the article "Tired Light" in Wikipedia).

A problem arises whether or not the effect of wave packet spreading (WPS) of the photon wave function is important for explaining the redshift. One might think that this effect is not important since a considerable WPS would also blur the images more than what is seen. Moreover, the very fact that we can see stars is an indication that for some reasons WPS is not explicitly manifested in observational astronomy. However, WPS is an inevitable consequence of standard quantum theory and moreover this effect is also known in classical electrodynamics. Hence it is not sufficient to just say that a considerable WPS is excluded by observations. One should try to estimate the importance of WPS and to understand whether our intuition is correct or not.

As follows from these remarks, in Theory A it is assumed that with a good accuracy we can treat photons as propagating in the empty space. It is also reasonable to expect (see the discussion in Ref. [6]) that the effect of WPS can be considered for each photon independently. Hence the results of the preceding sections make it possible to understand what experimental consequences of WPS are.

A question arises what can be said about the characteristics of photons coming to the Earth from distance objects. Typical conclusions based on numerous experiments with light coming to the Earth from the Sun are as follows. We know that with a good accuracy this light can be described in the framework of geometrical optics i.e. one can approximately treat the light as a collection of particles moving along classical trajectories. Since we know that the photons came from the Sun then with a good accuracy we know the direction of their momenta and since we know the distribution of wave lengths then (in the approximation described in Sec. 6) we know the distribution of photon energies.

The next question is what we know about the width of the coordinate photon wave functions in the direction perpendicular to the photon momentum in the approximation when the coordinate operators are defined as in Sec. 4. Suppose that a wide beam of light falls on a screen which is perpendicular to the direction of light. Suppose that the total area of the screen is S but the surface contains slits with the total area S_1. We are interested in the question of what part of the light will pass the screen. One might think that the obvious answer is that the part equals S_1/S. This answer follows from the picture that the light consists of many photons moving along geometrical trajectories and hence only the S_1/S part of the photons will pass the surface. Numerous experiments show that deviations from the above answer begin to manifest in interference experiments where dimensions of slits and distances between them have the order of tens or hundreds of microns or even less. Hence one can conclude that the width of the photon wave functions cannot be of the order of say centimeters or meters since in that case deviations from the S_1/S law would be visible if the slits and the distances between them would have the corresponding dimensions but this does not happen.

Consider, for example, the Lyman transition $2P \to 1S$ in the hydrogen atom on the Sun. In this case the energy of the photon is $E = 10.2eV$, its wave length is $\lambda = 121.6nm$, the lifetime is $\tau = 1.6 \cdot 10^{-9}s$ and the period of the wave is $T \approx 4 \cdot 10^{-16}s$. Hence the phrase that the lifetime is τ can be interpreted such that the uncertainty of the energy is $\hbar/\tau$, the uncertainty of the longitudinal momentum is $\hbar/c\tau$ and b is of the order of $c\tau \approx 0.48m$ or greater. In view of the above discussion, the estimation $a \approx b$ seems to be very favorable since one might expect that the value of a is much less than $0.48m$. With this estimation $N = a/\lambda \approx 4 \cdot 10^6$. So the value of N is rather large and in view of Eq. (23) one might think that the effect of spreading is not important. However, this is not the case since, as follows from Eq. (23), $t_* \approx 0.04s$. Since the distance between the Sun and the Earth is approximately $t = 8$ light minutes and this time is much greater than t_*, the width of the wave packet when it arrives to the Earth is $v_* t \approx 5760m$. It is obvious from the above discussion that such a value of the width is unrealistically large. On the other hand, if we assume that the initial value of a is of the order of several wave lengths then the value of N is much less and the width of the wave packet coming to the Earth is much greater. We conclude that a standard treatment of WPS for the photon wave function contradicts the well known data on interference of solar light.

Consider now a photon which was created in the same reaction but on Sirius which is the brightest star on our sky. Since the distance to Sirius is 8.6 light years, an analogous estimation shows that even in the favorable scenario the width of the wave packet coming to the Earth from Sirius will be approximately equal to $3 \cdot 10^6 km$ but in less favorable situations the width will be much greater.

A standard understanding of light coming to the Earth from the Sun and other stars is such that the major part of the light comes not from transitions between atomic levels but from processes which can be approximately described as a black body radiation. In that case the spectrum of the radiation is approximately continuous and we cannot estimate the quantity a as above. However, even if we take for a a very favorable value of the same order as above we will come to the same conclusion that the width of the wave packet will be unreasonably high. For illustration of this point we consider the following example.

Let the Earth be at point A and the center of Sirius be at point B. Suppose for simplicity that the Earth is a pointlike particle. Suppose that Sirius emitted a photon such that its wave function in momentum space has a narrow distribution around the mean value directed not along BA but along BC such that the angle between BA and BC is α. As noted in Sec. 5, there is no WPS in momentum space but, as follows from Eq. (22), the function $a(t)$ defining the mean value of the radius of the coordinate photon wave function in the perpendicular direction is a rapidly growing function of t. Let us assume for simplicity that $\alpha \ll 1$. Then if L is the length of AB, the distance from A to BC is approximately $d = L\alpha$. So if this photon is treated as a point moving along the classical trajectory then the observer on the Earth will not see the photon. Let us now take into account the effect of WPS in the direction perpendicular to the photon momentum. The front of the photon wave function passes the Earth when t approximately equals $t_* = L/c$. If $a(t_*)$ is of the order of d or greater and we look in the direction AD such that AD is antiparallel to BC then there is a nonzero probability that we will detect this photon. So we can see photons coming from Sirius in the angular range which is of the order of $a(t_*)/L$. If R is the radius of Sirius and $a(t_*)$ is of the order of R or greater, the image of Sirius will be blurred. As noted above, a very optimistic estimation of $a(t_*)$ is $3 \cdot 10^6 km$ but a more realistic estimation gives a much greater value. Since $R = 1.1 \cdot 10^6 km$ this means that the image of Sirius will be extremely blurred (to say nothing about the image of Sirius B which has the radius $6000 km$). Moreover, in the above angular range we can detect photons emitted not only by Sirius but also by other objects.

Since the distance to Sirius is "only" 8.6 light years, for the majority of stars the effect of WPS will be pronounced even in a much greater extent. So if WPS is considerable then we will see not separate stars but an almost continuous background from many objects. On the other hand, it is obvious that the effect of WPS is important only if light travels a rather long distance while in experiments on the Earth this effect is negligible. Indeed, in experiments on the Earth the quantity t_* is extremely small and so $a(t_*)$ is much less than the size of any macroscopic source of light.

We conclude that WPS of photons coming to the Earth from astronomical objects is the fundamental unsolved problem of physics. In the remaining part of the paper we propose a solution of the problem proceeding from a consistent definition of the position operator.

8 Consistent construction of position operator

The above results give grounds to think that the reason of the paradox with the photon wave function in the transverse direction is that the standard definition of the position operator in this direction is not consistent. One might try to define the position operator from the following considerations. Since the operators $\mathbf{P}$ and $\mathbf{L}$ are consistently defined as representation operators of the Poincare algebra one might seek the position operator such that on classical level the relation $\mathbf{r} \times \mathbf{p} = \mathbf{L}$ will take place. Note that on quantum level this relation is not necessary. Indeed, the very fact that some elementary particles has a half-integer spin (in usual units) shows that the total angular momentum for those particles does not have the orbital nature but on classical level the angular momentum can be always represented as a cross product of the radius-vector and standard momentum. However, if the values of $\mathbf{p}$ and $\mathbf{L}$ are known and $\mathbf{p} \neq 0$ then the requirement that $\mathbf{r} \times \mathbf{p} = \mathbf{L}$ does not define $\mathbf{r}$ uniquely. One can define parallel and perpendicular components of $\mathbf{r}$ as $\mathbf{r} = r_{||}\mathbf{p}/p + \mathbf{r}_\perp$ where $p = |\mathbf{p}|$. Then the relation $\mathbf{r} \times \mathbf{p} = \mathbf{L}$ defines uniquely only $\mathbf{r}_\perp$. Namely, taking the cross product of the both parts of this relation with $\mathbf{p}$ we get $\mathbf{r}_\perp = (\mathbf{p} \times \mathbf{L})/p^2$. On quantum level $\mathbf{r}_\perp$ should be replaced by a Hermitian operator $\mathcal{R}_\perp$ defined as

$$\begin{aligned} \mathcal{R}_{\perp j} &= \frac{\hbar}{2p^2} e_{jkl}(p_k L_l + L_l p_k) = \frac{\hbar}{p^2} e_{jkl} p_k L_l - \frac{i\hbar}{p^2} p_j \\ &= i\hbar \frac{\partial}{\partial p_j} - i\frac{\hbar}{p^2} p_j p_k \frac{\partial}{\partial p_k} - \frac{i\hbar}{p^2} p_j \end{aligned} \tag{24}$$

where a sum over repeated indices is assumed and we now assume that all the quantities are taken in standard units such that if $\mathbf{L}$ is given by Eq. (12) then the orbital momentum is $\hbar\mathbf{L}$.

We define the operators $\mathbf{F}$ and $\mathbf{G}$ such that $\mathcal{R}_\perp = \hbar\mathbf{F}/p$ and $\mathbf{G}$ is the operator of multiplication by the unit vector $\mathbf{n} = \mathbf{p}/p$. A direct calculation shows that these operators satisfy the following relations:

$$\begin{aligned} &[L_j, F_k] = ie_{jkl}F_l, \quad [L_j, G_k] = ie_{jkl}F_l, \quad \mathbf{G}^2 = 1, \quad \mathbf{F}^2 = l^2 + l + 1 \\ &[G_j, G_k] = 0, \quad [F_j, F_k] = -ie_{jkl}L_l \quad e_{jkl}\{F_k, G_l\} = -2L_j \\ &\mathbf{LG} = \mathbf{GL} = \mathbf{LF} = \mathbf{FL} = \mathbf{FG} + \mathbf{GF} = 0 \end{aligned} \tag{25}$$

The first two relations show that $\mathbf{F}$ and $\mathbf{G}$ are the vector operators as expected. The result for the anticommutator shows that on classical level $\mathbf{F} \times \mathbf{G} = -\mathbf{L}$ and the last relation shows that on classical level the operators in the triplet $(\mathbf{F}, \mathbf{G}, \mathbf{L})$ are mutually orthogonal.

In contrast to the standard definition of the position operator, the operator $\mathcal{R}_\perp$ defined by Eq. (24) depends only on the momentum and orbital angular momentum and does not depend on the choice of the coordinate axes. In particular, if the momentum distribution is narrow and such that the mean value of the momentum is directed along the z axis then it does not mean that on the operator level the z component of the operator $\mathcal{R}_\perp$ should be zero. The matter is that the direction of the momentum does not have a definite value. One might expect that only the mean value of the operator $\mathcal{R}_\perp$ will be zero or very small.

In addition, an immediate consequence of the definition (24) is as follows: *Since the momentum and angular momentum operators commute with the Hamiltonian, the distribution of all the components of* $\mathbf{r}_\perp$ *does not depend on time. In particular, there is no WPS in the direction defined by* $\mathcal{R}_\perp$. This is also clear from the fact that $\mathcal{R}_\perp = \hbar\mathbf{F}/p$ where the operator $\mathbf{F}$ acts only over angular variables and the Hamiltonian depends only on p. On classical level the conservation of $\mathcal{R}_\perp$ is obvious since it is defined by the conserving quantities $\mathbf{p}$ and $\mathbf{L}$. It is also obvious that since a free particle is moving along a straight line, a vector from the origin perpendicular to this line does not change with time.

The next question is how to implement the relation $\mathbf{r} = r_{||}\mathbf{p}/|\mathbf{p}| + \mathbf{r}_\perp$ on quantum level. A direct calculation shows that if $\partial/\partial\mathbf{p}$ is written in terms of p and angular variables then

$$i\hbar\frac{\partial}{\partial\mathbf{p}} = \mathbf{G}\mathcal{R}_{||} + \mathcal{R}_\perp \tag{26}$$

where the operator $\mathcal{R}_{||}$ acts only over the variable p:

$$\mathcal{R}_{||} = i\hbar(\frac{\partial}{\partial p} + \frac{1}{p}) \tag{27}$$

The correction $1/p$ is related to the fact that the operator $\mathcal{R}_{||}$ is Hermitian since in variables $(p, \mathbf{n})$ the scalar product is given by

$$(\chi_2, \chi_1) = \int \chi_2(p, \mathbf{n})^* \chi_1(p, \mathbf{n}) p^2 dp do \tag{28}$$

where do is the element of the solid angle.

Hence Eq. (26) gives a decomposition of the standard position operator which does not depend on the choice of the coordinate axes. So a consistent definition of the position operator tells us that physical coordinates are described by the operators $\mathcal{R}_{||}$ and $\mathcal{R}_\perp$ but not by the set $(i\hbar\partial/\partial p_x, i\hbar\partial/\partial p_y, i\hbar\partial/\partial p_z)$. In classical approximation the both sets are equivalent but on quantum level they are different (as noted in Secs. 3 and 5, the WPS effect is pure quantum). This is also clear from the fact that while the components of the standard position operator commute with each other, the operators $\mathcal{R}_{||}$ and $\mathcal{R}_\perp$ satisfy the following commutation relation:

$$[\mathcal{R}_{||}, \mathcal{R}_\perp] = -\frac{i\hbar}{p}\mathcal{R}_\perp \quad [\mathcal{R}_{\perp j}, \mathcal{R}_{\perp k}] = -\frac{i\hbar^2}{p^2} e_{jkl} L_l \tag{29}$$

An immediate consequence of these relation is as follows: *Since the operator* $\mathcal{R}_{||}$ *and different components of* $\mathcal{R}_\perp$ *do not commute with each other, the corresponding quantities cannot be simultaneously measured and hence there is no wave function* $\psi(r_{||}, \mathbf{r}_\perp)$ *in coordinate representation.*

As follows from Eq. (29), $[\mathcal{R}_{||}, p] = -i\hbar$, i.e. in the longitudinal direction the commutation relation between the coordinate and momentum is the same as in standard theory. One can also calculate the commutators between the different components of $\mathcal{R}_\perp$ and $\mathbf{p}$. Those commutators are not given by such simple expressions as in standard theory but it is easy to see that all of them are of the order of $\hbar$ as it should be.

9 New position operator and semiclassical states

As noted in Sec. 2, in standard theory states are treated as semiclassical in greatest possible extent if $\Delta r_j \Delta p_j = \hbar/2$ for each j and such states are called coherent.

The existence of coherent states in standard theory is a consequence of commutation relations $[p_j, r_k] = -i\hbar\delta_{jk}$. Since in our approach there are no such relations, a problem arises how to construct states in which all physical quantities p, $r_{||}$, $\mathbf{n}$ and $\mathbf{r}_\perp$ are semiclassical.

One of the ways to prove this is to calculate the mean values and uncertainties of the opeator $\mathcal{R}_{||}$ and all the components of the operator $\mathcal{R}_\perp$ in the state defined by Eq. (16). The calculation is not simple since it involves three-dimensional integrals with Gaussian functions devided by p^2. The result is that these operators are semiclassical in the state (16) if $p_0 \gg \hbar/b$, $p_0 \gg \hbar/a$ and r_{0z} has the same order of magnitude as r_{0x} and r_{0y}.

However, a more natural approach is as follows. Since $\mathcal{R}_\perp = \hbar\mathbf{F}/p$, the operator $\mathbf{F}$ acts only over the angular variable $\mathbf{n}$ and $\mathcal{R}_{||}$ acts only over the variable p, it is convenient to work in the representation where the Hilbert space is the space of functions $\chi(p, l, \mu)$ such that the scalar product is

$$(\chi_2, \chi_1) = \sum_{l\mu} \int_0^\infty \chi_2(p, l, \mu)^* \chi_1(p, l, \mu) dp \tag{30}$$

and l and μ are the orbital and magnetic quantum numbers, respectively, i.e.

$$\mathbf{L}^2\chi(p, l, \mu) = l(l+1)\chi(p, l, \mu), \quad L_z\chi(p, l, \mu) = \mu\chi(p, l, \mu) \tag{31}$$

The operator $\mathbf{L}$ in this space does not act over the variable p and the action of the remaining components is given by

$$L_+\chi(l, \mu) = [(l+\mu)(l+1-\mu)]^{1/2}\chi(l, \mu-1), \quad L_-\chi(l, \mu) = [(l-\mu)(l+1+\mu)]^{1/2}\chi(l, \mu+1) \tag{32}$$

where the $\pm$ components of vectors are defined such that $L_x = L_+ + L_-$, $L_y = -i(L_+ - L_-)$.

A direct calculation shows that, as a consequence of Eq. (24)

$$\begin{aligned}
F_+\chi(l, \mu) &= \frac{i}{2}[\frac{(l+\mu)(l+\mu-1)}{(2l-1)(2l+1)}]^{1/2} l\chi(l-1, \mu-1) \\
&+ \frac{i}{2}[\frac{(l+2-\mu)(l+1-\mu)}{(2l+1)(2l+3)}]^{1/2}(l+1)\chi(l+1, \mu-1) \\
F_-\chi(l, \mu) &= -\frac{i}{2}[\frac{(l-\mu)(l-\mu-1)}{(2l-1)(2l+1)}]^{1/2} l\chi(l-1, \mu+1) \\
&- \frac{i}{2}[\frac{(l+2+\mu)(l+1+\mu)}{(2l+1)(2l+3)}]^{1/2}(l+1)\chi(l+1, \mu+1) \\
F_z\chi(l, \mu) &= -i[\frac{(l-\mu)(l+\mu)}{(2l-1)(2l+1)}]^{1/2} l\chi(l-1, \mu) \\
&+ i[\frac{(l+1-\mu)(l+1+\mu)}{(2l+1)(2l+3)}]^{1/2}(l+1)\chi(l+1, \mu)
\end{aligned} \tag{33}$$

The operator **G** acts on such states as follows

$$G_+\chi(l,\mu) = \frac{1}{2}[\frac{(l+\mu)(l+\mu-1)}{(2l-1)(2l+1)}]^{1/2}\chi(l-1,\mu-1)$$
$$-\frac{1}{2}[\frac{(l+2-\mu)(l+1-\mu)}{(2l+1)(2l+3)}]^{1/2}\chi(l+1,\mu-1)$$
$$G_-\chi(l,\mu) = -\frac{1}{2}[\frac{(l-\mu)(l-\mu-1)}{(2l-1)(2l+1)}]^{1/2}\chi(l-1,\mu+1)$$
$$+\frac{1}{2}[\frac{(l+2+\mu)(l+1+\mu)}{(2l+1)(2l+3)}]^{1/2}\chi(l+1,\mu+1)$$
$$G_z\chi(l,\mu) = -[\frac{(l-\mu)(l+\mu)}{(2l-1)(2l+1)}]^{1/2}\chi(l-1,\mu)$$
$$-[\frac{(l+1-\mu)(l+1+\mu)}{(2l+1)(2l+3)}]^{1/2}\chi(l+1,\mu) \qquad (34)$$

and now the operator $\mathcal{R}_{||}$ has a familiar form $\mathcal{R}_{||} = i\hbar\partial/\partial p$.

Therefore by analogy with Secs. 2 and 3 one can construct states which are coherent with respect to $(r_{||}, p)$, i.e. such that $\Delta r_{||}\Delta p = \hbar/2$. Indeed (see Eq. (5)), the wave function

$$\chi(p) = \frac{b^{1/2}}{\pi^{1/4}\hbar^{1/2}}exp[-\frac{(p-p_0)^2b^2}{2\hbar^2} - \frac{i}{\hbar}(p-p_0)r_0] \qquad (35)$$

describes a state where the mean values of p and $r_{||}$ are p_0 and r_0, respectively and their uncertainties are $\hbar/(b\sqrt{2})$ and $b/\sqrt{2}$, respectively. Strictly speaking, the analogy between the given case and that discussed in Secs. 2 and 3 is not full since in the given case the quantity p can be in the range $[0.\infty)$, not in $(-\infty,\infty)$ as momentum variables used in those sections. However, if $p_0b/\hbar \gg 1$ then the formal expression for $\chi(p)$ at $p<0$ is extremely small and so the normalization integral for $\chi(p)$ can be formally taken from $-\infty$ to ∞.

In such an approximation one can define wave functions $\psi(r)$ in the $r_{||}$ representation. By analogy with the consideration in Secs. 2 and 3 we define

$$\psi(r) = \int exp(\frac{i}{\hbar}pr)\chi(p)\frac{dp}{(2\pi\hbar)^{1/2}} \qquad (36)$$

where the integral is formally taken from $-\infty$ to ∞. Then

$$\psi(r) = \frac{1}{\pi^{1/4}b^{1/2}}exp[-\frac{(r-r_0)^2}{2b^2} + \frac{i}{\hbar}p_0r] \qquad (37)$$

Note that here the quantities r and r_0 have the meaning of coordinates in the direction parallel to the particle momentum, i.e. they can be positive or negative.

Consider now states where the quantities **F** and **G** are semiclassical. One might expect that in semiclassical states the quantities l and μ are very large. In this approximation, as follows from Eqs. (33) and (34), the action of the operators **F** and **G** can be written as

$$
\begin{aligned}
F_+\chi(l,\mu) &= \frac{i}{4}(l+\mu)\chi(l-1,\mu-1) + \frac{i}{4}(l-\mu)\chi(l+1,\mu-1)\\
F_-\chi(l,\mu) &= -\frac{i}{4}(l-\mu)\chi(l-1,\mu+1) - \frac{i}{4}(l+\mu)\chi(l+1,\mu+1)\\
F_z\chi(l,\mu) &= \frac{i}{2l}(l^2-\mu^2)^{1/2}[\chi(l+1,\mu) - \chi(l-1,\mu)]\\
G_+\chi(l,\mu) &= \frac{l+\mu}{4l}\chi(l-1,\mu-1) - \frac{l-\mu}{4l}\chi(l+1,\mu-1)\\
G_-\chi(l,\mu) &= -\frac{l-\mu}{4l}\chi(l-1,\mu+1) + \frac{l+\mu}{4l}\chi(l+1,\mu+1)\\
G_z\chi(l,\mu) &= -\frac{1}{2l}(l^2-\mu^2)^{1/2}[\chi(l+1,\mu) + \chi(l-1,\mu)]
\end{aligned}
\tag{38}
$$

In view of the remark in Sec. 2 about semiclassical vector quantities, consider a state $\chi(l,\mu)$ such that $\chi(l,\mu) \neq 0$ only if $l \in [l_1, l_2]$, $\mu \in [\mu_1, \mu_2]$ where $l_1, \mu_1 > 0$, $\delta_1 = l_2+1-l_1$, $\delta_2 = \mu_2+1-\mu_1$, $\delta_1 \ll l_1$, $\delta_2 \ll \mu_1$ $\mu_2 < l_1$ and $\mu_1 \gg (l_1-\mu_1)$. This is the state where the quantity μ is close to its maximum value l. As follows from Eqs. (31) and (32), in this state the quantity L_z is much greater than L_x and L_y and, as follows from Eq. (38), the quantities F_z and G_z are small. So on classical level this state describes a motion of the particle in the xy plane. The quantity L_z in this state is obviously semiclassical since $\chi(l,\mu)$ is the eigenvector of the operator L_z with the eigenvalue μ. As follows from Eq. (38), the action of the operators (F_+, F_-, G_+, G_-) on this state can be described by the following approximate formulas:

$$
\begin{aligned}
F_+\chi(l,\mu) &= \frac{il_0}{2}\chi(l-1,\mu-1), \quad F_-\chi(l,\mu) = -\frac{il_0}{2}\chi(l+1,\mu+1)\\
G_+\chi(l,\mu) &= \frac{1}{2}\chi(l-1,\mu-1), \quad G_-\chi(l,\mu) = \frac{1}{2}\chi(l+1,\mu+1)
\end{aligned}
\tag{39}
$$

where l_0 is a value from the interval $[l_1, l_2]$.

Consider a simple model when $\chi(l,\mu) = exp[i(l\alpha - \mu\beta)]/(\delta_1\delta_2)^{1/2}$ when $l \in [l_1, l_2]$ and $\mu \in [\mu_1, \mu_2]$. Then a simple direct calculation using Eq. (39) gives

$$
\begin{aligned}
&\bar{G}_x = cos\gamma, \quad \bar{G}_y = -sin\gamma \quad \bar{F}_x = l_0 sin\gamma \quad \bar{F}_y = l_0 cos\gamma\\
&\Delta G_x = \Delta G_y = (\frac{1}{\delta_1} + \frac{1}{\delta_2})^{1/2}, \quad \Delta F_x = \Delta F_y = l_0(\frac{1}{\delta_1} + \frac{1}{\delta_2})^{1/2}
\end{aligned}
\tag{40}
$$

where $\gamma = \alpha - \beta$. Hence the vector quantities **F** and **G** are semiclassical since either $|cos\gamma|$ or $|sin\gamma|$ or both are much greater than $(\delta_1 + \delta_2)/(\delta_1\delta_2)$.

10 New position operator and wave packet spreading

If the space of states is implemented according to the scalar product (30) then the dependence of the wave function on t is

$$\chi(p,k,\mu,t) = exp[-\frac{i}{\hbar}(m^2c^2+p^2)^{1/2}ct]\chi(p,k,\mu,t=0) \tag{41}$$

As noted in Secs. 3 and 5, there is no WPS in momentum space and this is natural in view of momentum conservation. Then, as already noted, the distribution of the quantity $\mathbf{r}_\perp$ does not depend on time and this is natural from the considerations described in Sec. 8.

At the same time, the dependence of the $r_{||}$ distribution on time can be calculated in full analogy with Sec. 3. Indeed, consider, for example a function $\chi(p,l,\mu,t=0)$ having the form

$$\chi(p,l,\mu,t=0) = \chi(p,t=0)\chi(l,\mu) \tag{42}$$

Then, as follows from Eqs. (36) and (41),

$$\psi(r,t) = \int exp[-\frac{i}{\hbar}(m^2c^2+p^2)^{1/2}ct + \frac{i}{\hbar}pr]\chi(p,t=0)\frac{dp}{(2\pi\hbar)^{1/2}} \tag{43}$$

Suppose that the function $\chi(p,t=0)$ is given by Eq. (35). Then in full analogy with the calculations in Sec. 3 we get that in the nonrelativistic case the $r_{||}$ distribution is defined by the wave function

$$\psi(r,t) = \frac{1}{\pi^{1/4}b^{1/2}}(1+\frac{i\hbar t}{mb^2})^{-1/2}exp[-\frac{(r-r_0-v_0t)^2}{2b^2(1+\frac{\hbar^2t^2}{m^2b^4})}(1-\frac{i\hbar t}{mb^2})+\frac{i}{\hbar}p_0r-\frac{ip_0^2t}{2m\hbar}] \tag{44}$$

where $v_0 = p_0/m$ is the classical speed of the particle in the direction of the particle momentum. Hence the WPS effect in this direction is similar to that given by Eq. (9) in standard theory.

In the opposite case when the particle is ultrarelativistic, Eq. (43) can be written as

$$\psi(r,t) = \int exp[\frac{i}{\hbar}p(r-ct)]\chi(p,t=0)\frac{dp}{(2\pi\hbar)^{1/2}} \tag{45}$$

Hence, as follows from Eq. (37):

$$\psi(r,t) = \frac{1}{\pi^{1/4}b^{1/2}}exp[-\frac{(r-r_0-ct)^2}{2b^2}+\frac{i}{\hbar}p_0(r-ct)] \tag{46}$$

In particular, for an ultrarelativistic particle there is no WPS in the direction of particle momentum and this is in agreement with the results of Sec. 5.

We conclude that in our approach an ultrarelativistic particle (e.g. the photon) experiences WPS neither in the direction of its momentum nor in the transverse direction, i.e. the WPS effect for an ultrarelativistic particle is absent at all.

Let us note that the absence of WPS in transverse directions is simply a consequence of the fact that a consistently defined operator $\mathcal{R}_\perp$ commutes with the Hamiltonian, i.e. $\mathbf{r}_\perp$ is a conserving physical quantity. On the other hand, the longitudinal coordinate cannot be conserving since a particle is moving along the direction of its momentum. However, in a special case of ultrarelativistic particle the absence of WPS is simply a consequence of the fact that the wave function given by Eq. (45) depends on r and t only via a combination of $r - ct$.

11 Discussion

In the present paper we consider a problem of constructing position operator in quantum theory. As noted in Sec. 1, this operator is needed only in situations where semiclassical approximation works with a high accuracy and where quantum theory should reproduce the results of classical one.

A standard choice of the position operator in momentum space is $i\hbar\partial/\partial\mathbf{p}$. A motivation for this choice is discussed in Sec. 2. We note that the standard definition is not consistent since $i\hbar\partial/\partial p_j$ cannot be a physical position operator in directions where the momentum is small. Physicists did not pay attention to the inconsistency probably for the following reason: as explained in standard textbooks on quantum mechanics, the transition from quantum to classical theory can be performed such that if the coordinate wave function contains a rapidly oscillating exponent $exp(iS/\hbar)$ where S is the classical action then in the formal limit $\hbar \to 0$ the Schrödinger equation becomes the Hamilton-Jacobi equation.

However, an inevitable consequence of standard quantum theory is the effect of wave packet spreading (WPS). This fact has not been considered as a drawback of the theory since for macroscopic bodies this effect is extremely small while in experiments on the Earth with atoms and elementary particles spreading does not have enough time to manifest itself. However, for photons travelling to the Earth from distant stars this effect is considerable, and it seems that this fact has been overlooked by physicists.

As shown in Sec. 7, if the WPS effect for photons travelling to the Earth from distant stars is as given by standard theory then we should see not stars but only an almost continuous background from all stars. Hence we have a fundamental glaring paradox which should be resolved. The calculations in Sec. 5 show that the reason of the paradox is that in directions perpendicular to the particle momentum the standard position operator is defined inconsistently.

We propose a new definition of the position operator which we treat as consistent for the following reasons. Our position operator does not depend on the choice of coordinate axes and depends only on the direction of the particle momentum.

So the operator is defined by two components - in the direction along the momentum and in the perpendicular directions. The first part has a familiar form $i\hbar\partial/\partial p$ and is treated as the operator of the longitudinal coordinate if the magnitude of p is rather large. At the same condition the position operator in the perpendicular directions is defined as a quantum generalization of the relation $\mathbf{r}_\perp \times \mathbf{p} = \mathbf{L}$. So in contrast to the standard definition of the position operator, the new operator is expected to be physical only if the *magnitude* of the momentum is rather large.

As a consequence of our construction, WPS in directions perpendicular to the particle momentum is absent regardless of whether the particle is nonrelativistic or relativistic. Hence the above paradox is resolved. Moreover, for an ultrarelativistic particle the effect of WPS is absent at all.

Different components of the new position operator do not commute with each other and, as a consequence, there is no wave function in coordinate representation. This shows that the problem of transition from quantum theory to classical one should be reformulated. This is not an academic but extremely important problem of modern physics. Indeed, if we believe that quantum theory is fundamental then it should describe not only atoms and elementary particles but even the motion of bodies in the Solar System and in the Universe. So we need to know how the evolution of macroscopic bodies should be described in quantum theory and what is the correct choice of position operator.

Our result for ultrarelativistic particles shows that in that case the result can be treated as ideal: quantum theory reproduces the motion along a classical trajectory without any spreading. However, this is only a special case of one free elementary particle and a problem arises whether this result can be generalized to cases when interactions are present and even when particles are macroscopic.

As noted above, in the direction perpendicular to the particle momentum the choice of the position operation is based only on the requirement that semiclassical approximation should reproduce the standard relation $\mathbf{r}_\perp \times \mathbf{p} = \mathbf{L}$. This requirement seems to be beyond any doubts since *on classical level* this relation is confirmed in numerous experiments. At the same time, the choice $i\hbar\partial/\partial p$ of the coordinate operator in the longitudinal direction is analogous to that in standard theory and hence one might expect that this operator is physical if the magnitude of p is rather large.

It will be shown in a separate publication that the construction of the position operator described in this paper for the case of Poincare invariant theory can be generalized to the case of de Sitter (dS) invariant theory. In this case the interpretation of the position operator is even more important than in Poincare invariant theory. The reason is that even the free two-body mass operator in the dS theory depends not only on the relative two-body momentum but also on the distance between the particles.

As argued in Ref. [17], in dS theory over a Galois field the assumption that the dS analog of the operator $i\hbar\partial/\partial p$ is the operator of the longitudinal coordinate is not valid *for macroscopic bodies* (even if p is large) since in that case semiclassical approximation is not valid. We have proposed a modification of the position operator such that quantum theory reproduces for the two-body mass operator the mean value compatible with the Newton law of gravity and precession of Mercury's perihelion. Then a problem arises how quantum theory can reproduce classical evolution for macroscopic bodies.

We believe that the assumption that the evolution of macroscopic bodies can be described by the Schrödinger equation is unphysical. For example, if the motion of the Earth is described by the evolution operator $exp[-iH(t_2-t_1)/\hbar]$ where H is the Hamiltonian of the Earth then the quantity $H(t_2-t_1)/\hbar$ becomes of the order of unity when t_2-t_1 is a quantity of the order of $10^{-68}s$ if the Hamiltonian is written in nonrelativistic form and $10^{-76}s$ if it is written in relativistic form. Such time intervals seem to be unphysical and so in the given case the approximation when t is a continuous parameter seems to be unphysical too.

The above examples show that at macroscopic level a consistent definition of the transition from quantum to classical theory is the fundamental open problem.

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