Symmetry-Nondependent Self-Gravitational Upper Bound on Static Local Energy from Use of a Nonperturbative Iteration Method for Lippmann-Schwinger Equations

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Abstract

It has recently been shown that self-gravitation reduces static spherically-symmetric cumulative energy distributions below the value of their radii times the "Planck force", which is the inverse of G times the fourth power of c. In this article quantitative treatment of self-gravitation is extended to any static energy density that is nonnegative, smooth and globally integrable. The resulting dimensionless local gravitational energy-reduction factor (namely the inverse of the local gravitational time-dilation factor) is shown to satisfy the zero-momentum nonrelativistic Lippmann-Schwinger quantum scattering equation for a repulsive potential which is proportional (with a known coefficient) to that static energy density. Standard perturbative Born-type iteration of Lippmann-Schwinger equations can diverge for sufficiently strong potentials, which in the gravitational case correspond to sufficiently large static energy densities. We have been able, however, to devise an alternate, completely nonperturbative iteration method for Lippmann-Schwinger equations in coordinate representation. Every one of this nonperturbative method's successive approximations to the local gravitational energy-reduction factor turns out to be positive and less than or equal to unity. In consequence, the self-gravitationally corrected static energy contained in any sphere is bounded by that sphere's diameter times the "Planck force".

Self-gravitational correction of nonnegative static energy densities

It has recently been shown that any spherically-symmetric static *cumulative energy distribution* $E_{G=0}(r)$ that satisfies $E_{G=0}(r=0)=0$ and $d\left(E_{G=0}(r)\right)/dr \geq 0$ has a corresponding *self-gravitationally corrected* spherically-symmetric static cumulative energy distribution $E_G(r)$ which satisfies the inequalities $0 \leq E_G(r) \leq E_{G=0}(r)$ and which also, *irrespective of how large* $E_{G=0}(r)$ *may be*, is bounded above by the product of its radius r and the "Planck force" (c^4/G), i.e., $E_G(r) < r(c^4/G)$ [1].

Here we *extend* the self-gravitational energy-correction process introduced in Ref. [1] to any specified *static energy density* $T_{G=0}(\mathbf{r})$ which is nonnegative, smooth and globally integrable, i.e.,

$$T_{G=0}(\mathbf{r}) \geq 0, \tag{1a}$$

$$\nabla_{\mathbf{r}}\left(T_{G=0}(\mathbf{r})\right) \text{ is continuous,} \tag{1b}$$

and,

$$\int T_{G=0}(\mathbf{r}')d^3\mathbf{r}' < \infty. \tag{1c}$$

We *specifically refrain* here, however, from making the Ref. [1] assumption that $T_{G=0}(\mathbf{r})$ possesses *spherical symmetry*, nor do we assume that it possesses *any other particular symmetry*.

Now if it were the case that we *actually had in hand the self-gravitational correction* $T_G(\mathbf{r})$ of the specified static energy density $T_{G=0}(\mathbf{r})$, we could calculate the negative Newtonian gravitational work done to bring the *infinitesimal original static energy* $T_{G=0}(\mathbf{r})d^3\mathbf{r}$ from infinity to its position at $\mathbf{r}$ while subject to the static gravitational field that is provided by $T_G(\mathbf{r})$, which yields the result $-(G/c^4)\int d^3\mathbf{r}' T_G(\mathbf{r}')|\mathbf{r}-\mathbf{r}'|^{-1}T_{G=0}(\mathbf{r})d^3\mathbf{r}$. However, because the static gravitational interaction inherently occurs between *pairs* of infinitesimal energies, we must take care to *avoid double-counting*, so we assign only *half* of this *negative gravitational work correction* to the infinitesimal static energy located at $\mathbf{r}$, and thus obtain,

$$T_G(\mathbf{r})d^3\mathbf{r} = \left[1 - \tfrac{1}{2}(G/c^4)\int d^3\mathbf{r}' T_G(\mathbf{r}')|\mathbf{r}-\mathbf{r}'|^{-1}\right]T_{G=0}(\mathbf{r})d^3\mathbf{r}. \tag{2a}$$

From Eq. (2a) we see that the dimensionless *local gravitational energy-reduction factor* $\mathcal{F}_G(\mathbf{r})$ that satisfies,

$$T_G(\mathbf{r})d^3\mathbf{r} = \mathcal{F}_G(\mathbf{r})T_{G=0}(\mathbf{r})d^3\mathbf{r}, \tag{2b}$$

is given by,

$$\mathcal{F}_G(\mathbf{r}) \stackrel{\text{def}}{=} 1 - \tfrac{1}{2}(G/c^4)\int |\mathbf{r}-\mathbf{r}'|^{-1}T_G(\mathbf{r}')d^3\mathbf{r}'. \tag{2c}$$

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This static local gravitational energy-reduction factor $\mathcal{F}_G(\mathbf{r})$ is obviously the *inverse* of the corresponding gravitational time-dilation factor, and thus is equal to $(g_{00}(\mathbf{r}))^{\frac{1}{2}}$ [2]. If we insert the instruction implicit in Eq. (2b) into the right-hand side of Eq. (2c), we obtain the following inhomogeneous linear integral equation for $\mathcal{F}_G(\mathbf{r})$,

$$\mathcal{F}_G(\mathbf{r}) = 1 - \tfrac{1}{2}(G/c^4) \int |\mathbf{r} - \mathbf{r}'|^{-1} T_{G=0}(\mathbf{r}')\mathcal{F}_G(\mathbf{r}')d^3\mathbf{r}'. \tag{2d}$$

In light of Eq. (1c), we can deduce from Eq. (2d) that,

$$\lim_{|\mathbf{r}|\to\infty} \mathcal{F}_G(\mathbf{r}) = 1. \tag{3a}$$

Furthermore, since the integral transform *kernel* $-1/(4\pi|\mathbf{r} - \mathbf{r}'|)$ is the Green's function of the Laplacian operator $\nabla^2_{\mathbf{r}}$, we in addition deduce from Eq. (2d) that,

$$\nabla^2_{\mathbf{r}}\mathcal{F}_G(\mathbf{r}) = (2\pi G/c^4)T_{G=0}(\mathbf{r})\mathcal{F}_G(\mathbf{r}),$$

which is readily reexpressed as a zero-energy stationary-state nonrelativistic *Schrödinger equation* [3] for the dimensionless wave function $\mathcal{F}_G(\mathbf{r})$, namely,

$$\left(-\hbar^2\nabla^2_{\mathbf{r}}/(2m) + V(\mathbf{r})\right)\mathcal{F}_G(\mathbf{r}) = 0, \tag{3b}$$

whose repulsive potential $V(\mathbf{r})$ is defined as,

$$V(\mathbf{r}) \stackrel{\text{def}}{=} \left[\pi\hbar^2 G/(mc^4)\right] T_{G=0}(\mathbf{r}). \tag{3c}$$

Because of Eq. (1c) it is clear from Eq. (3c) that,

$$\lim_{|\mathbf{r}|\to\infty} V(\mathbf{r}) = 0, \tag{3d}$$

which, in turn, implies that the large-$|\mathbf{r}|$ limit of $\mathcal{F}_G(\mathbf{r})$ that is given by Eq. (3a) is *consistent* with the Eq. (3b) zero-energy Schrödinger equation.

Having established the connection of the Eq. (2d) *gravitational* integral equation to the Eq. (3b) *zero-energy* stationary-state Schrödinger equation, we now furthermore note that for stationary states of *positive energy* $E > 0$ this Schrödinger equation becomes,

$$\left(-\hbar^2\nabla^2_{\mathbf{r}}/(2m) + V(\mathbf{r})\right)\langle\mathbf{r}|\psi_E\rangle = E\langle\mathbf{r}|\psi_E\rangle. \tag{4a}$$

From Eq. (3d) we see that as $|\mathbf{r}| \to \infty$, Eq. (4a) becomes simply,

$$\left(-\hbar^2\nabla^2_{\mathbf{r}}/(2m)\right)\langle\mathbf{r}|\psi_E\rangle = E\langle\mathbf{r}|\psi_E\rangle, \tag{4b}$$

whose solutions include *all* the dimensionless *plane waves* $e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}$ for which $\mathbf{p}$ satisfies $|\mathbf{p}|^2 = 2mE$, as well, of course, as the *linear combinations* of these which comprise the the full set of angularly-modulated outgoing and ingoing free *spherical waves* which have *this same scalar wave number* $k = (2mE)^{\frac{1}{2}}/\hbar$ [3]. Thus we see that Eq. (4a) has a massive *inherent solution degeneracy*. A useful *resolution* of this solution degeneracy can be achieved by *reexpressing* Eq. (4a) in an *inhomogeneous* linear form that *can only be satisfied* by the *particular* permissible $|\mathbf{r}| \to \infty$ asymptotic behavior which *properly accords* with the *design* of a specified experiment.

That idea underlies the Lippmann-Schwinger *inhomogeneous modification of* Eq. (4a), which *forces* its wave function to behave as a specifically chosen *single* permitted plane wave $e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}$ plus *only outgoing* angularly-modulated spherical waves in the asymptotic region $|\mathbf{r}| \to \infty$ where Eq. (4a) is adequately described by Eq. (4b). If we denote as $\langle\mathbf{r}|\psi^+_{\mathbf{p}}\rangle$ the solution of Eq. (4a) which *satisfies* this *particular* permitted $|\mathbf{r}| \to \infty$ asymptotic behavior, then the *inhomogeneous* Lippmann-Schwinger equation that *uniquely* describes $\langle\mathbf{r}|\psi^+_{\mathbf{p}}\rangle$ is [4],

$$\langle\mathbf{r}|\psi^+_{\mathbf{p}}\rangle = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} - \langle\mathbf{r}|(-\hbar^2\widehat{\nabla^2}/(2m) - |\mathbf{p}|^2/(2m) - i\epsilon)^{-1}\widehat{V}|\psi^+_{\mathbf{p}}\rangle. \tag{5a}$$

From a *static gravitational* standpoint the *relevant feature* of the Eq. (5a) inhomogeneous Lippmann-Schwinger equation and its wave function $\langle\mathbf{r}|\psi^+_{\mathbf{p}}\rangle$ is that,

$$\langle\mathbf{r}|\psi^+_{\mathbf{p}}\rangle\big|_{\mathbf{p}=\mathbf{0}} = \mathcal{F}_G(\mathbf{r}), \tag{5b}$$

as is seen from comparison of the $\mathbf{p} = \mathbf{0}$ case of Eq. (5a) with Eq. (2d)—to make this comparison one must use Eq. (3c) to express $V(\mathbf{r})$ as the appropriate constants times $T_{G=0}(\mathbf{r})$, and one must also use the fact that the integral transform kernel $-1/(4\pi|\mathbf{r}-\mathbf{r}'|)$ is the coordinate-representation inverse (i.e., Green's function) of the Hilbert-space "Laplacian" operator $\widehat{\nabla^2} = -|\widehat{\mathbf{p}}|^2/\hbar^2$, namely that,

$$\langle\mathbf{r}|(\widehat{\nabla^2})^{-1}|\mathbf{r}'\rangle = -1/(4\pi|\mathbf{r}-\mathbf{r}'|)\,.$$

Note that the *negative* imaginary infinitesimal $-i\epsilon$ which appears in Eq. (5a) is *unnecessary* when $\mathbf{p} = \mathbf{0}$, which represents a *purely static* state of affairs that has *no* distinguishable outgoing versus ingoing spherical waves. Indeed the massive solution degeneracy of the stationary-state Schrödinger equation given by Eq. (4a) *collapses* when $E = 0$.

Given the Eq. (5b) close relationship of the local gravitational energy-reduction factor $\mathcal{F}_G(\mathbf{r})$ to the Lippmann-Schwinger wave function $\langle\mathbf{r}|\psi^+_{\mathbf{p}}\rangle$, it would seem logical to apply well-known general solution methods for Lippmann-Schwinger equations to our gravitational Eq. (2d). Unfortunately, however, the only widely-applied fully general solution method for Lippmann-Schwinger equations is *perturbative* in character, and therefore is *inherently* subject to *failure*.

The struggle to transcend the perturbative Born trap

If we bring the second term on the right-hand side of the Eq. (5a) Lippmann-Schwinger equation to its left-hand side, we obtain,

$$\langle\mathbf{r}|\psi^+_{\mathbf{p}}\rangle + \langle\mathbf{r}|(\widehat{K} - E_{\mathbf{p}} - i\epsilon)^{-1}\widehat{V}|\psi^+_{\mathbf{p}}\rangle = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}, \tag{6a}$$

where $\widehat{K} \stackrel{\text{def}}{=} (-\hbar^2\widehat{\nabla^2}/(2m))$ is the kinetic energy *operator* and $E_{\mathbf{p}} \stackrel{\text{def}}{=} (|\mathbf{p}|^2/(2m))$ is the kinetic energy *c-number scalar* that corresponds to the c-number momentum vector $\mathbf{p}$. Taking $\langle\mathbf{r}|\mathbf{p}\rangle \stackrel{\text{def}}{=} e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}$, the formal solution of Eq. (6a) is,

$$\langle\mathbf{r}|\psi^+_{\mathbf{p}}\rangle = \langle\mathbf{r}|[1 + (\widehat{K} - E_{\mathbf{p}} - i\epsilon)^{-1}\widehat{V}]^{-1}|\mathbf{p}\rangle. \tag{6b}$$

The only way forward at this point would seem to be expansion of the inverse of the operator in square brackets in the well-known Born geometric perturbative series, which involves successive *powers* with alternating signs of the particular operator,

$$\widehat{X} \stackrel{\text{def}}{=} (\widehat{K} - E_{\mathbf{p}} - i\epsilon)^{-1}\widehat{V},$$

acting on the momentum eigenstate $|\mathbf{p}\rangle$ [5]. If the operator $\widehat{X}$ dominates the identity on the momentum eigenstate $|\mathbf{p}\rangle$, it is not unlikely that the Born geometric series *diverges*. One might suppose that in such instances one could simply *recast* the Born geometric expansion to be in powers of the *inverse* of $\widehat{X}$, since one is formally free to choose *either* the expansion $[1+\widehat{X}]^{-1} = 1-\widehat{X}+\widehat{X}^2-\cdots$ *or* the expansion $[1+\widehat{X}]^{-1} = \widehat{X}^{-1}[1+\widehat{X}^{-1}]^{-1} = \widehat{X}^{-1}-(\widehat{X}^{-1})^2+(\widehat{X}^{-1})^3-\cdots$. Most unfortunately, however, since $\widehat{X}^{-1} = \widehat{V}^{-1}\left(\widehat{K} - E_{\mathbf{p}}\right)$, the operator $\widehat{X}^{-1}$ *vanishes altogether when applied to the momentum eigenstate* $|\mathbf{p}\rangle$. This unanticipated abrupt setback is a stark warning that hidden snares beset Born-style geometric perturbative expansions for the Lippmann-Schwinger equation.

For the $\mathbf{p} = \mathbf{0}$ case of the Lippmann-Schwinger equation that applies to *static gravitation*, we are, of course, *particularly* interested in *arbitrarily large* energy densities $T_{G=0}(\mathbf{r})$, and therefore in *arbitrarily strong* operators $\widehat{V}$ and $\widehat{X}$. Thus it is clear that we need to *entirely abjure* Born-style geometric perturbative expansion, but how could that *conceivably* be accomplished *in practice*? The *only* possibility is through exploration of *nonstandard manipulations* of the Lippmann-Schwinger equation.

Returning to Eq. (6a) we now *deliberately shun* the *elegant and natural* factorization of the *operator* $[1+\widehat{X}]$ on its left-hand side, which can only lead us down the primrose path to the Born geometric perturbative series, and *instead* opt to forcibly factor that side into two *mere functions of the vector coordinate* $\mathbf{r}$, the *first* of which is *still* $\langle\mathbf{r}|\psi^+_{\mathbf{p}}\rangle$ because we aim for a result which is at least *akin* to Eq. (6b), but the *second* of which is *repulsively inelegant*, being *merely the product of* $(\langle\mathbf{r}|\psi^+_{\mathbf{p}}\rangle)^{-1}$ with the *original left-hand side of* Eq. (6a). There *is* in fact "method" in that gross ugliness, however, because we can now *actually arithmetically divide* the plane wave $e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}$ on the *right-hand side* of Eq. (6a) by that gauche *second* factor *without resorting to any kind of perturbative expansion*. A very heavy price has been paid in the coin of gross inelegance, but the goal of *no perturbative expansion whatsoever* has been achieved. To be sure, the almost childish manipulations just described *haven't* extracted any final *result* from Eq. (6a), what they have produced is *only* a basis for *refinement through iteration*. It is readily seen, however, that the iteration process is *devoid of perturbative characteristics*; it *instead* resembles a *continued fraction*.

The iteration formula we have just extracted from Eq. (6a) is explicitly,

$$\langle \mathbf{r}|\psi_{\mathbf{p}}^{(n+1)+}\rangle = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}/[1 + (\langle \mathbf{r}|\psi_{\mathbf{p}}^{(n)+}\rangle)^{-1}\langle \mathbf{r}|(\widehat{K} - E_{\mathbf{p}} - i\epsilon)^{-1}\widehat{V}|\psi_{\mathbf{p}}^{(n)+}\rangle], \tag{6c}$$

for $n = 0, 1, 2, \ldots$, where, of course, $\langle \mathbf{r}|\psi_{\mathbf{p}}^{(0)+}\rangle = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}$. Therefore $(\langle \mathbf{r}|\psi_{\mathbf{p}}^{(0)+}\rangle)^{-1}$ is obviously well-defined, and the form of Eq. (6c) makes it apparent that for $n = 1, 2, \ldots$, $(\langle \mathbf{r}|\psi_{\mathbf{p}}^{(n)+}\rangle)^{-1}$ is well-defined as well. That the iteration formula of Eq. (6c) does not have perturbative characteristics, but rather those of a continued fraction is also manifest.

Finally, on inserting $\mathbf{p} = \mathbf{0}$ into Eq. (6c) we obtain the iteration formula for the gravitational energy-reduction factor $\mathcal{F}_G(\mathbf{r})$, which is,

$$\mathcal{F}_G^{(n+1)}(\mathbf{r}) = 1/[1 + \tfrac{1}{2}(G/c^4)(\mathcal{F}_G^{(n)}(\mathbf{r}))^{-1} \textstyle\int |\mathbf{r} - \mathbf{r}'|^{-1} T_{G=0}(\mathbf{r}')\mathcal{F}_G^{(n)}(\mathbf{r}')d^3\mathbf{r}'], \tag{7a}$$

for $n = 0, 1, 2, \ldots$, where, of course, $\mathcal{F}_G^{(0)}(\mathbf{r}) = 1$.

Since $T_{G=0}(\mathbf{r}) \geq 0$ from Eq. (1a), $T_{G=0}(\mathbf{r})$ is smooth from Eq. (1b), and $\int T_{G=0}(\mathbf{r}')d^3\mathbf{r}' < \infty$ from Eq. (1c), it is clear from Eq. (7a) that,

$$\text{if } 1 \geq \mathcal{F}_G^{(n)}(\mathbf{r}) > 0, \text{ then } 1 \geq \mathcal{F}_G^{(n+1)}(\mathbf{r}) > 0. \tag{7b}$$

Therefore we can conclude that,

$$1 \geq \mathcal{F}_G(\mathbf{r}) > 0. \tag{7c}$$

From Eqs. (1a), (2b) and (7c) we can deduce that,

$$0 \leq T_G(\mathbf{r}) \leq T_{G=0}(\mathbf{r}). \tag{7d}$$

From Eq. (2c) and the fact that the gravitational energy-reduction factor $\mathcal{F}_G(\mathbf{r})$ satisfies $\mathcal{F}_G(\mathbf{r}) > 0$ we can deduce that,

$$2(c^4/G) > \int |\mathbf{r} - \mathbf{r}'|^{-1} T_G(\mathbf{r}')d^3\mathbf{r}' = \int |\mathbf{r}''|^{-1} T_G(\mathbf{r} + \mathbf{r}'')d^3\mathbf{r}'',$$

where $\mathbf{r}'' \stackrel{\text{def}}{=} (\mathbf{r}' - \mathbf{r})$. Furthermore, we have that,

$$\int |\mathbf{r}''|^{-1} T_G(\mathbf{r} + \mathbf{r}'')d^3\mathbf{r}'' \geq \int_{|\mathbf{r}''| \leq R} |\mathbf{r}''|^{-1} T_G(\mathbf{r} + \mathbf{r}'')d^3\mathbf{r}'' \geq R^{-1} \int_{|\mathbf{r}''| \leq R} T_G(\mathbf{r} + \mathbf{r}'')d^3\mathbf{r}''.$$

Therefore from the two foregoing lines of displayed integral inequalities we can conclude that,

$$(2R)(c^4/G) > \int_{|\mathbf{r}''| \leq R} T_G(\mathbf{r} + \mathbf{r}'')d^3\mathbf{r}'', \tag{7e}$$

namely that the self-gravitationally corrected static energy contained in any sphere cannot exceed the diameter of that sphere times the "Planck force" (c^4/G). Note that this result is *completely independent of any assumption concerning symmetry properties of the energy distribution*. Therefore it is likely to be overly conservative in practice. A more practical energy upper-bound estimate can be obtained by simply averaging the maximum possible and minimum possible values of $|\mathbf{r}''|$ that occur when integrating over the sphere of radius R, which yields $R/2$, and therefore the "rough" bound,

$$R(c^4/G) \mathrel{\widetilde{>}} \int_{|\mathbf{r}''| \leq R} T_G(\mathbf{r} + \mathbf{r}'')d^3\mathbf{r}'',$$

which is in line with the result obtained in Ref. [1], where spherical symmetry was assumed.

Conclusion

No doubt the most interesting aspect of what has been presented here is the unfastening of the shackles of the perturbative Born-expansion paradigm for a class of equation systems that incorporate linear operators. A superior iteration method has been developed by deliberately shunning an attractive natural relationship that involves those linear operators in favor of concocting a clumsy artificial relationship that involves *only function values*. The point of proceeding in this way is that purely *arithmetic* operations with *function values* require *no approximations*, whereas even quite elementary-looking operations involving *operators* may *not* be practically feasible without making use of potentially disastrous perturbation expansions. Indeed function manipulations can, on the contrary, be *directed* toward the goal of achieving iteration schemes that have *continued fraction* rather than perturbative character.

References

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