Prespacetime Journal| November 2012 | Volume 3| Issue 13 | pp. 1247-1250

Unified Integro-Differential Equation for Relaxation and Oscillation

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Abstract

In this paper we discuss some important consequences of application of fractional operators in physics. Also we present a unified integro-differential equation for relaxation and oscillation. We focus on time fractional formalism whose derivative is in Caputo sense.

Keywords: Fractional Calculus; Fractional Operators; Memory Effects; Algebraic Decay.

1. Introduction

Fractional calculus is a very useful tool in describing the evolution of systems with memory, which typically are dissipative and to complex systems. In recent decades the fractional calculus and in particular the fractional differential equations has attracted interest of researches in several areas including mathematics, physics, chemistry, biology, engineering and economics. Applications of fractional calculus in the field of physics have gained considerable popularity and many important results were obtained during the last years [1-4]. Despite these various applications, there are some important challenges. For example physical interpretation for the fractional derivative is not completely clarified yet. In this paper we focus on time fractional formalism whose derivative is in Caputo sense. We emphasize that fractional differentiation with respect to time can be interpreted as an existence of memory effects which correspond to intrinsic dissipation in our system. Fractional relaxation and oscillation has been subject of many researches in recent years [8-15]. Equations for standard and fractional relaxation and oscillation [8- 10] are given in Tab. 1 below.

	Relaxation	Oscillation
Standard form	$\frac{dx(t)}{dt}+\frac{1}{\tau}x(t)=0$	$\frac{d^2x}{dt^2}+\frac{k}{m}x=0$
Fractional form	$\frac{d^\alpha x(t)}{dt^\alpha}+\frac{\eta^{1-\alpha}}{\tau}x(t)=0$	$\frac{d^{2\alpha'}x}{dt^{2\alpha'}}+\eta^{2(1-\alpha')}\frac{k}{m}x=0$

Table 1: standard and fractional relaxation and oscillation equations

In the above equations the fractional derivative of order α $, n-1<\alpha<n$, $n\in N$ is defined in the Caputo sense:

$$ {}_0^cD_t^\alpha f(t)=\frac{\partial^\alpha f(t)}{\partial t^\alpha}=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}\frac{\partial^n f(\tau)}{\partial\tau^n}d\tau \tag{1}$$

Where Γ denotes the Gamma function. For $\alpha=n$, $n\in N$ the Caputo fractional derivative is defined as the standard derivative of order n . Also, note that we have introduced an arbitrary quantity η with dimension of [second] to ensure that all quantities have correct dimensions. As we can see from Eq.(1) Caputo derivative describes a memory effect by means of a convolution between the integer order derivative and a power of time that corresponds to intrinsic dissipation in the system. Now, using the asymptotic behavior of the Mittag-Leffler functions for large values of arguments:

$$E_\alpha(-at^\alpha)\simeq\frac{1}{\Gamma(1-\alpha)}\frac{1}{at^\alpha} \tag{2}$$

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It is showed [9] that in the case of fractional relaxation the asymptotic behavior of the solution exhibit an algebraic decay for $t \to \infty$ i.e. for arbitrary values of α we have:

$$x_{\text{relaxation}}(t) \sim (\frac{1}{t^{\alpha}}) \tag{3}$$

Also for arbitrary values of α' for the fractional oscillation we have:

$$x_{\text{oscillation}}(t) \sim (\frac{1}{t^{2\alpha'}}) \tag{4}$$

These results prove that the algebraic decay of the solutions of the fractional equations contrary to the exponential decay of the usual standard form of the equations is the important intrinsic effect of the fractional derivative in the typical fractional equations.

2. Effects of power-law memory

As we can see from Eq. (1) Caputo derivative describes a memory effect by means of a convolution between the integer order derivative and a power of time that corresponds to intrinsic dissipation in the system. It becomes apparent that fractional equations naturally represent systems with memory [5-7]. To see this consider the following integro-differential equation

$$\frac{d^n x(t)}{dt^n} + \lambda_0 x(t) = -\lambda_1 \int_0^t k(t-\tau)x(\tau)d\tau \tag{5}$$

where x is the quantity of interest, K the memory kernel, and λ_0, λ_1 the parameters. This integro-differential equation can be used to describe both Markovian and non-Markovian evolutions in the realm of classical physics. Let us consider two cases for Eq. (5):

1- For a system without memory, we have the Markov processes, and the time dependence of the memory function is $k(t-\tau) = \delta(t-\tau)$ where $\delta(t-\tau)$ is the Dirac delta-function and one gets:

$$\frac{d^n x(t)}{dt^n} + (\lambda_0 + \lambda_1)x(t) = 0 \tag{6}$$

For $n = 1$ it becomes:

$$\frac{dx(t)}{dt} + (\lambda_0 + \lambda_1)x(t) = 0 \tag{7}$$

Now choosing $\lambda_0 + \lambda_1 = \frac{1}{\tau}$ above equation becomes standard relaxation equation and for $n = 2$ we have:

$$\frac{d^2 x(t)}{dt^2} + (\lambda_0 + \lambda_1)x(t) = 0 \tag{8}$$

If we set $\lambda_0 + \lambda_1 = \frac{k}{m}$ we will have the standard oscillation equation.

2- For a system with power-law memory we have the non-Markovian processes, and the time dependence of the memory function is $k(t-\tau) = A(t-\tau)^{-\alpha}$ where A , is a new parameter. Now substituting it into Eq. (5) we will have

$$\frac{d^n x(t)}{dt^n} + (\lambda_2)\,{}_0^cD_t^{\alpha} x(t) + (\lambda_0 + \lambda_1)x(t) = 0 \tag{9}$$

where $\lambda_2 = A\Gamma(1-\alpha)\lambda_1$. Then for $n = 1$ we have:

$$\frac{dx(t)}{dt} + (\lambda_2)\,{}_0^cD_t^{\alpha} x(t) + (\lambda_0 + \lambda_1)x(t) = 0 \tag{10}$$

and for $n = 2$

$$\frac{d^2x(t)}{dt^2} + (\lambda_2)_0^c D_t^\alpha x(t) + (\lambda_0 + \lambda_1)x(t) = 0 \tag{11}$$

where above equations are linear fractionally damped relaxation and oscillation equations respectively and can describe many fundamental processes in physics.

3. Conclusion

Fractional calculus is a very useful tool in describing the evolution of systems with memory, which typically are dissipative**.** In this work, by use of the concept of relaxation and oscillation we discuss tow important consequences of application of fractional operators in physics. First we show that displacements of relaxation and oscillation show an algebraic decay in the asymptotic long time $t \to \infty$ and then we introduce an integro-differential equation that by suitable choosing of the parameters and memory kernel it produces standard and linear fractionally damped equations for relaxation and oscillation.

Acknowledgement

We thank Richard Herrmann for helpful comment.

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