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Algebraic Long-Time Decay in Fractional Classical Mechanics

Hosein Nasrolahpour*

Abstract

Time fractional formalism is a very useful tool in describing systems with memory and delay. In this article, we reformulate the equations presented in [F. Mainardi, *Chaos Sol. Frac*. **7(9)** (1996) 1461-1477] and show that solutions for the relaxation and oscillation equations represent algebraic time-decaying behavior at asymptotic long time which depends on the order of time derivative.

Keywords**:** Fractional classical mechanics; Fractional relaxation; Fractional oscillation

The fractional calculus is a powerful tool to describe physical systems that have long-time memory [1, 2]. Fractional differentiation with respect to time is characterized by long-term memory effects that correspond to intrinsic dissipative processes in the physical systems**.** The commonest way to obtain a fractional differential equation for describing the evolution of a typical system is to generalize the ordinary derivative in the standard differential equation into the fractional derivative. For example in the realm of classical mechanics, fractional form of two important phenomena i.e. relaxation and oscillation can be written as [2-5]:

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}}+\frac{\eta^{1-\alpha}}{\tau}x(t)=0 \tag{1}$$

$$\frac{d^{2\alpha'}x}{dt^{2\alpha'}}+\eta^{2(1-\alpha')}\frac{k}{m}x=0 \tag{2}$$

In above equations the fractional derivative of order α , $n-1<\alpha<n$, $n\in N$ is defined in the Caputo sense:

$$\frac{\partial^{\alpha}f(t)}{\partial t^{\alpha}}=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}\frac{\partial^{n}f(\tau)}{\partial\tau^{n}}d\tau \tag{3}$$

Where Γ denotes the Gamma function. For $\alpha=n$, $n\in N$ the Caputo fractional derivative is defined as the standard derivative of order n . Also, note that we have introduced an arbitrary quantity η with dimension of [second] to ensure that all quantities have correct dimensions. As we can see from (3) Caputo derivative describes a memory effect by means of a convolution between the integer order derivative and a power of time that corresponds to intrinsic dissipation in the system. Exact solutions of (1, 2) in term of Mittag-Leffler function are in the following forms:

$$x(t)=x(0)E_{\alpha}(-\eta^{1-\alpha}(\frac{t^{\alpha}}{\tau})) \tag{4}$$

$$x(t)=x(0)E_{2\alpha'}(-\omega_f^{\,2}t^{2\alpha'})+\dot{x}(0)tE_{2\alpha',2}(-\omega_f^{\,2}t^{2\alpha'}) \tag{5}$$

where $\omega_f^{\,2}=\eta^{2(1-\alpha')}\dfrac{k}{m}$. Now if we choose $x(0)=x_0$ and $\dot{x}(0)=0$ as the initial conditions, the solutions become:

*Correspondence: Hosein Nasrolahpour, Babolsar City, Mazandaran, Iran. E-Mail: hnasrolahpour@gmail.com

$$x(t) = x_0 E_\alpha(-\eta^{1-\alpha}(\frac{t^\alpha}{\tau})) \tag{6}$$

$$x(t) = x_0 E_{2\alpha'}(-\omega_f{}^2 t^{2\alpha'}) \tag{7}$$

the solution for relaxation equation (6) for the special case of $\alpha = \frac{1}{2}$ becomes

$$x(t) = x_0 E_{\frac{1}{2}}(-\frac{\sqrt{\eta t}}{\tau}) = x_0 e^{\frac{\eta t}{\tau^2}} erfc(\frac{\sqrt{\eta t}}{\tau}) \tag{8}$$

where $erfc$ denotes the complimentary error function and the error function is defined as

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt, \quad erfc(z) = 1 - erf(z), \qquad z \in C \tag{9}$$

The algebraic decay of the solutions of the fractional equations is the most important effect of the fractional derivative in the typical fractional equations contrary to the exponential decay of the usual standard form of the equations. For this purpose we use the asymptotic behavior of the complimentary error function and Mittag-Leffler functions that play a very important role in the interpretation and understanding of the solutions of various problems of physics connected with fractional phenomena that occur in complex systems. So, using the asymptotic behavior of complimentary error function and Mittag-Leffler functions for large values of arguments [6]:

$$erfc(z) \simeq \frac{1}{\sqrt{\pi z}} \exp(-z^2) \tag{10}$$

$$E_\alpha(-at^\alpha) \simeq \frac{1}{\Gamma(1-\alpha)} \frac{1}{at^\alpha} \tag{11}$$

in the case of fractional relaxation the asymptotic behavior exhibits an algebraic decay for $t \to \infty$, for the special case of $\alpha = \frac{1}{2}$ becomes

$$x_{\text{relaxation}}(t) \simeq [\frac{\tau x_0}{\sqrt{\pi \eta}}] \frac{1}{\sqrt{t}} \tag{12}$$

and for arbitrary values of α we have

$$x_{\text{relaxation}}(t) \simeq [\frac{\tau x_0}{\eta^{1-\alpha} \Gamma(1-\alpha)}](\frac{1}{t^\alpha}) \tag{13}$$

Also for arbitrary values of α' for the fractional oscillation we have:

$$x_{\text{oscillation}}(t) \simeq [\frac{m x_0}{\eta^{2(1-\alpha')} k \Gamma(1-2\alpha')}](\frac{1}{t^{2\alpha'}}) \tag{14}$$

As we can see in the above results, we arrive to the asymptotic solution for the relaxation and oscillation equation which represents algebraic time-decaying displacements which depend on the parameter α and α' as it was showed by numerical calculations in [4] and by perturbation method in [5]. This is a direct consequence of the time fractional derivative in these systems. In the other word fractional differentiation with respect to time can be interpreted as an existence of memory effects which correspond to intrinsic dissipation in our systems.

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