A Self-Gravitational Upper Bound on Localized Energy, Including that of Virtual Particles and Quantum Fields, which Yields a Passable "Dark Energy" Density Estimate

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Abstract

The self-gravitational correction to a localized spherically symmetric static energy distribution is obtained from an upgraded Newtonian model which is energetically self-consistent, and is also obtained from the Birkhoff-theorem extension of the unique "Newtonian" form of the free-space Schwarzschild metric into the interior region of its self-gravitationally corrected source. The two approaches yield identical results, which include a strict prohibition on the gravitational redshift factor ever being other than finite, real and positive. Consequently, the self-gravitationally corrected energy within a sphere of radius r is bounded by r times the "Planck force", namely the fourth power of c divided by G. That energy bound rules out any physical singularity at the Schwarzschild radius, and it also cuts off the mass deviation of any interacting quantum virtual particle at the Planck mass. Because quantum uncertainty makes the minimum energy of a quantum field infinite, such a field's self-gravitationally corrected energy essentially attains the Planck force times that field's boundary radius r. Roughly estimating r as c times the age of the universe yields a "dark energy" density of 1.7 joules per cubic kilometer. But if r is put to the Planck length appropriate to the birth of the universe, that energy density changes to the enormous Planck unit value, which could quite conceivably drive primordial "inflation". The density of "dark energy" decreases as the universe expands, but more slowly than the density of ordinary matter decreases.

Self-gravitational correction of spherically symmetric localized static energy

A common tacit idealization made in non-gravitational theoretical physics is to ignore self-gravitational corrections as being vastly too small to matter. The ratio of the electrostatic to the gravitational attraction of electron to proton is a great many orders of magnitude, for example. But the uncertainty principle of the quantum theory can manifest a disconcerting predilection to throw up *infinite* energies, and if we understandably quail at abandoning so firmly established a principle, it behooves us to at least *try* to ponder its self-gravitational implications. Even in gravity theory *itself*, well-known solutions for source-free regions, such as that of Schwarzschild, exhibit troubling behavior near an idealized gravitational source which is sufficiently strong and compact. That raises the question of whether we have adequately modeled how self-gravitational corrections might *modify* so strong and compact an idealized gravitational source.

Here we essay a simplified inroad into self-gravitationally correcting localized energies that are obtained or postulated under the tacit assumption that $G = 0$. To make exact calculation readily feasible, we restrict the localized energies actually treated to be spherically symmetric and static. Such models turn out to be readily formally solvable in a Newtonian gravitational framework which has been upgraded through incorporation of the concept that energy (not mass) is to be self-consistently regarded as gravity's source, *notwithstanding* the complication that gravity unavoidably *alters* this selfsame energy. The consequences of this relatively straightforward *energetically self-consistent* Newtonian approach are presented in the remainder of this section. In the next section we show that a somewhat less straightforward metric-based general relativistic approach yields *identical* results, albeit that latter approach manifests especially transparently, due to its inherent metric nature, the fact that the Schwarzschild singularity isn't physically realizable.

Starting from a *given* spherically symmetric static *cumulative energy distribution* $E_{G=0}(r)$ that is a feature of a *gravitationally bereft* (i.e., $G = 0$) "world", our goal is to calculate its self-gravitational *modification* $E_G(r)$ that is intended to apply to a more "realistic" world where $G > 0$, but we *don't* go so far as to permit $E_G(r)$ to become *nonstatic*. In view of its *cumulative nature*, we assume that $E_{G=0}(r)$ *vanishes at* $r = 0$, i.e.,

$$E_{G=0}(r = 0) = 0, \tag{1a}$$

and that it has *nonnegative derivative*,

$$E'_{G=0}(r) \geq 0. \tag{1b}$$

We shall also implicitly assume that $E_{G=0}(r)$ is a physically realistic *localized* cumulative energy distribution, namely that it satisfies $E'_{G=0}(r) = 0$ for all r which are greater than some positive bounding radius R. We

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also note here that $E'_{G=0}(r)$ is, of course, closely related to the nonnegative spherically symmetric *energy density* $T_{G=0}(r)$ via,

$$E'_{G=0}(r) = 4\pi r^2 T_{G=0}(r). \tag{1c}$$

Now if it were the case *that we actually had* $E_G(r)$ *in hand*, we could calculate $[E_G(r + dr) - E_G(r)]$, namely the gravitationally-modified energy of the immediately enveloping infinitesimally-thick spherical shell, by proceeding to *discount* from that shell's *original energy* $E'_{G=0}(r)dr$ the *negative Newtonian gravitational work* $-[GE_G(r)/(c^4 r)]E'_{G=0}(r)dr$ which is required to *assemble* it at radius r in the presence of the *attractive gravitational field produced by* $E_G(r)$. Assembly of that infinitesimally-thick shell at radius r *also* entails negative Newtonian gravitational work done by the shell's constituents *on each other*, but that work is of *second order in* dr, and therefore can safely be ignored here. Thus we obtain,

$$E_G(r + dr) - E_G(r) = (1 - [GE_G(r)/(c^4 r)])E'_{G=0}(r)dr. \tag{2a}$$

In addition, the fact that $E_{G=0}(r = 0)$ vanishes eliminates any energy carry-through in the degenerate $r = 0$ special case, and leaves $E_G(r = 0)$ with no option but to vanish as well, i.e.,

$$E_G(r = 0) = 0. \tag{2b}$$

From Eq. (2a) we see that the *physical core* of the calculation is the dimensionless Newtonian *gravitational energy-reduction factor* $(1 - [GE_G(r)/(c^4 r)])$ at radius r, which upon multiplication into the radius-r infinitesimally-thick enveloping shell's *original energy* $E'_{G=0}(r)dr$, yields that shell's *gravitationally-modified energy* $[E_G(r + dr) - E_G(r)] = E'_G(r)dr$.

In the limit $dr \to 0$, Eq. (2a) becomes the linear inhomogeneous first-order differential equation,

$$E'_G(r) + (G/c^4)(E'_{G=0}(r)/r)E_G(r) = E'_{G=0}(r), \tag{2c}$$

for $E_G(r)$, with Eq. (2b) as its single boundary condition. Eq. (2c) is readily reduced to quadrature after multiplying it through by the integrating factor $e^{(G/c^4)\int_{r_0}^{r}(E'_{G=0}(r')/r')dr'}$, which, together with Eq. (2b), yields,

$$E_G(r)e^{(G/c^4)\int_{r_0}^{r}(E'_{G=0}(r')/r')dr'} = \int_0^r dr'\, E'_{G=0}(r')e^{(G/c^4)\int_{r_0}^{r'}(E'_{G=0}(r'')/r'')dr''}, \tag{2d}$$

a result which is more simply written as,

$$E_G(r) = \int_0^r dr'\, E'_{G=0}(r')e^{-(G/c^4)\int_{r'}^{r}(E'_{G=0}(r'')/r'')dr''}. \tag{2e}$$

In light of Eq. (1a) we obtain from Eq. (2e) that,

$$\lim_{G\to 0} E_G(r) = \int_0^r dr'\, E'_{G=0}(r') = E_{G=0}(r), \tag{2f}$$

and in light of both Eqs. (1b) and (1a), we obtain from Eq. (2e) that,

$$0 \le E_G(r) \le \int_0^r dr'\, E'_{G=0}(r') = E_{G=0}(r). \tag{2g}$$

The results given by Eqs. (2f) and (2g) are entirely expected, but in *addition* to these there lies concealed in the deceptively humdrum form for $E_G(r)$ which is given by Eq. (2e) a startling physical feature. To expose that property of $E_G(r)$ to the light of day, we note that the integrand on the right-hand side of Eq. (2e) comes remarkably close to being a perfect differential. Therefore $E_G(r)$ can be rewritten,

$$E_G(r) = (c^4/G)\int_0^r dr'\, r'd\left(e^{-(G/c^4)\int_{r'}^{r}(E'_{G=0}(r'')/r'')dr''}\right)/dr', \tag{3a}$$

which neatly lends itself to integration by parts, with the result,

$$E_G(r) = (c^4 r/G)\left[1 - (1/r)\int_0^r dr'\, e^{-(G/c^4)\int_{r'}^{r}(E'_{G=0}(r'')/r'')dr''}\right]. \tag{3b}$$

In conjunction with Eq. (1b), Eq. (3b) implies that,

$$0 \le E_G(r) < (c^4 r/G), \tag{3c}$$

which can be combined with Eq. (2g) to yield,

$$0 \le E_G(r) \le \min\left[(c^4 r/G), E_{G=0}(r)\right]. \tag{3d}$$

We therefore have the fascinating result that *regardless of how large* the cumulative energy $E_{G=0}(r)$ may be *before* it is self-gravitationally corrected, its *self-gravitational modification* $E_G(r)$ *cannot attain* the "Planck force" (c^4/G) times the radius r of the sphere which encloses it. With this we are at long last in possession of a potent instrument with which to confront the pervasive energy infinities that have for so long been the quantum theory's dispiriting affront to physical understanding. Nor does the "Schwarzschild singularity" seem so troubling in the soothing light of Eq. (3c), which makes it very clear that arbitrarily strong and compact gravitational sources simply do not exist.

Although the "Planck force times radius" localized-energy upper bound of Eq. (3c) may seem unfamiliar, it is straightforward to recast it into an *equivalent* form to which which we can readily relate, namely,

$$1 \ge (1 - [GE_G(r)/(c^4 r)]) > 0, \tag{3e}$$

i.e., the dimensionless Newtonian *gravitational energy-reduction factor* that appears in Eq. (2a) is *always real and positive*. *Also* equivalently, the *inverse* of the Newtonian gravitational energy-reduction factor, namely the dimensionless Newtonian *gravitational redshift factor*, is *always finite, real and positive*. Those statements in fact *transcend* this particular calculation and are endowed with *universal validity*, having been repeatedly robustly verified for the general relativistic *gravitational redshift factor* under the most extreme conditions of gravitational collapse [1]. Therefore we will in the sections below be applying Eqs. (3c) and (3b) with confidence in a broad-brush way to some of the self-gravitationally related issues that naturally arise in gravity theory itself, quantum field theory and even cosmology.

Before doing so, however, we close this section with the technical development of the asymptotic expansion of $E_G(r)$ when $E_{G=0}(r)$ *greatly exceeds* the "Planck force times radius" upper bound $(c^4 r/G)$ for $E_G(r)$; that asymptotic expansion will serve as a useful reminder of the basic logic which needs to be applied in dealing with the energy infinities that can result from quantum theory considerations.

To develop this asymptotic expansion of $E_G(r)$ from Eq. (3b), it is clearly useful to define,

$$\kappa(r) \stackrel{\text{def}}{=} (G/c^4)E'_{G=0}(r)/r = 4\pi(G/c^4)rT_{G=0}(r), \tag{4a}$$

where the second equality follows from Eq. (1c). Using $\kappa(r)$, Eq. (3b) reads,

$$E_G(r) = (c^4 r/G)\left[1 - (1/r)\textstyle\int_0^r dr'\, e^{-\int_{r'}^{r}\kappa(r'')dr''}\right]. \tag{4b}$$

Since $\kappa(r'') \ge 0$, the exponential integrand of the integral over the variable r' obviously has its maximum value of unity at $r' = r$. Our basic strategy will therefore be to expand the argument of that exponential (which itself is an integral over the variable r'') around that point $r' = r$. To facilitate this, we change variable from r' to the dimensionless variable $u \stackrel{\text{def}}{=} (r - r')\kappa(r)$, after which we can expand the argument of that exponential in powers of u. The change of integration variable from r' to u causes Eq. (4b) to read,

$$E_G(r) = (c^4 r/G)\left[1 - (1/(r\kappa(r)))\textstyle\int_0^{r\kappa(r)} du\, e^{\int_r^{r-(u/\kappa(r))}\kappa(r'')dr''}\right]. \tag{4c}$$

Expanding the argument of the exponential in powers of u formally produces,

$$\textstyle\int_r^{r-(u/\kappa(r))}\kappa(r'')dr'' = -u + \displaystyle\sum_{j=2}^{\infty}(-1)^j(u/\kappa(r))^j\kappa^{(j-1)}(r)/j!\,. \tag{4d}$$

Putting this formal expansion result into the exponential produces,

$$e^{\int_r^{r-(u/\kappa(r))}\kappa(r'')dr''} = e^{-u}\, e^{\sum_{j=2}^{\infty}(-1)^j(u/\kappa(r))^j\kappa^{(j-1)}(r)/j!}, \tag{4e}$$

and we now proceed to expand the *second* exponential on the right-hand side of Eq. (4e),

$$e^{\sum_{j=2}^{\infty}(-1)^j(u/\kappa(r))^j\kappa^{(j-1)}(r)/j!} = 1 + \sum_{n=1}^{\infty}\left(\textstyle\sum_{j=2}^{\infty}(-1)^j(u/\kappa(r))^j\kappa^{(j-1)}(r)/j!\right)^n/n!\,. \tag{4f}$$

We now intend to substitute the right-hand side of Eq. (4f) into Eq. (4e), and then to substitute the right-hand side of Eq. (4e) into Eq. (4c). At this point we recall that we are interested in the asymptotic form of Eq. (4c) when $E_{G=0}(r) \gg (c^4 r/G)$. That state of affairs can be expected to be coincident with the upper integration limit $r\kappa(r)$ of the dimensionless u-integration in Eq. (4c) being very much greater than unity. Therefore, together with the substitutions mentioned in the first sentence of this paragraph, we shall as well change the upper limit of the dimensionless u-integration in Eq. (4c) to ∞. That change, together with the expansions given by Eqs. (4f) and (4e), enables us to analytically evaluate Eq. (4c) on a term-by-term basis by simply applying the elementary formula,

$$\int_0^\infty du\, e^{-u}\, u^k = k!\,. \tag{4g}$$

Therefore, the first few terms of the desired asymptotic expansion of $E_G(r)$ when $r\kappa(r) \gg 1$ are given by,

$$E_G(r) \sim (c^4 r/G)\left[1 - (1/(r\kappa(r)))\left(1 + \kappa'(r)/(\kappa(r))^2 - \kappa''(r)/(\kappa(r))^3 + 3(\kappa'(r))^2/(\kappa(r))^4 + \cdots\right)\right]. \tag{4h}$$

In the case that the uncorrected initial *energy density* $T_{G=0}(r)$ happens to be *constant in* r, we have that,

$$\kappa'(r)/(\kappa(r))^2 = 1/(r\kappa(r)) = 1/(4\pi(G/c^4)r^2 T_{G=0}), \tag{4i}$$

and also that $\kappa''(r) = 0$. Therefore in the case of *constant* uncorrected initial energy density $T_{G=0}$ we have that when $r\kappa(r) \gg 1$,

$$E_G(r) \sim (c^4 r/G)\left[1 - (1/(r\kappa(r))) - (1/(r\kappa(r)))^2 - 3(1/(r\kappa(r)))^3 + \cdots\right], \tag{4j}$$

where,

$$r\kappa(r) = 4\pi(G/c^4)r^2 T_{G=0} = 3(G/(c^4 r))E_{G=0}(r). \tag{4k}$$

Self-gravitational correction from a general relativistic metric

We have strongly emphasized that the physical core of the energetically self-consistent Newtonian self-gravitational correction of spherically-symmetric static cumulative energy distributions lies with the dimensionless *Newtonian gravitational energy-reduction factor* $(1 - [GE_G(r)/(c^4 r)])$ of Eqs. (2a) and (3e). Now a *metric*, which is in effect a system's general relativistic gravitational potential, *also* provides a dimensionless gravitational energy-reduction factor. For a spherically-symmetric static system that *metric* gravitational energy-reduction factor is $(g_{00}(r))^{\frac{1}{2}}$, which is the *inverse* of the metric's gravitational *time-dilation* or *red-shift* factor $(g_{00}(r))^{-\frac{1}{2}}$ [2]. Such a spherically-symmetric static system is of course *itself* described by a self-gravitationally corrected cumulative energy distribution,

$$E_G(r) = 4\pi \int_0^r (r')^2 T_G(r')dr', \tag{5a}$$

where $T_G(r)$ is its corresponding self-gravitationally corrected spherically-symmetric static *energy density*. Now $T_G(r)$ would be expected to be the energy density of a *localized system*, i.e., $T_G(r) = 0$ for all $r > R$, where R is that localized system's *bounding radius*. Thus if we denote $E_G(R)$ as simply E_G, we see that $E_G(r) = E_G$ for all $r \geq R$, and, in the *free-space region*, namely $r > R$, this system's *metric* will be a *free-space Schwarzschild* metric that has the self-gravitationally corrected *energy constant* E_G. One well-known representation of such a free-space Schwarzschild metric with self-gravitationally corrected energy constant E_G is its "isotropic" form [3],

$$(cd\tau)^2 = \left(\frac{1-\frac{1}{2}[GE_G/(c^4\rho)]}{1+\frac{1}{2}[GE_G/(c^4\rho)]}\right)^2 (cdt)^2 - \left(1+\tfrac{1}{2}[GE_G/(c^4\rho)]\right)^4\left((d\rho)^2 + (\rho d\theta)^2 + (\rho\sin\theta d\phi)^2\right). \tag{5b}$$

Unfortunately, the gravitational energy-reduction factor $(g_{00}(\rho))^{\frac{1}{2}}$ of this Eq. (5b) "isotropic" form of the Schwarzschild metric with energy constant E_G obviously *fails to match* the corresponding *Newtonian* gravitational energy-reduction factor $(1 - [GE_G/(c^4\rho)])$. It turns out, however, that the simple change of radial coordinate variable $r \stackrel{\text{def}}{=} \rho + \frac{1}{2}[GE_G/c^4]$ in the Eq. (5b) "isotropic" form of the free-space Schwarzschild metric produces the unique *desired* form of the free-space Schwarzschild metric, namely,

$$(cd\tau)^2 = \left(1-[GE_G/(c^4r)]\right)^2 (cdt)^2 - \left(1-\tfrac{1}{2}[GE_G/(c^4r)]\right)^{-4} (dr)^2 - \left(1-\tfrac{1}{2}[GE_G/(c^4r)]\right)^{-2} \left((rd\theta)^2 + (r\sin\theta d\phi)^2\right), \quad (5c)$$

that actually *satisfies* $(g_{00}(r))^{\frac{1}{2}} = (1 - [GE_G/(c^4r)])$. It is therefore appropriate to refer to the Eq. (5c) form of the free-space Schwarzschild metric as its "Newtonian" form.

Next we wish to *extend* this "Newtonian" form of the free-space Schwarzschild metric into the *interior* region $r < R$ of its *source*, where *it is no longer necessarily the case that* $E_G(r)$ *is equal to the constant* E_G. The general relativistic instrument of choice for accomplishing this extension is the combination of the *Birkhoff theorem and its corollary* for spherically-symmetric static systems [4].

If we focus on a *particular* radius value r_0, the Birkhoff theorem's *corollary* tells us that the part of the spherically-symmetric static cumulative energy distribution $E_G(r)$ for which $r > r_0$ *makes no contribution* to the *local* r_0 value of the metric, while the Birkhoff theorem *itself* tells us that the part of the spherically-symmetric static self-gravitationally corrected cumulative energy distribution $E_G(r)$ for which $r \le r_0$ produces at r_0 *a local Schwarzschild metric* whose *energy constant* E_G has the value $E_G(r_0)$.

The upshot of the Birkhoff theorem and its corollary is thus that the "Newtonian" *free-space* Schwarzschild metric form of Eq. (5c) can be *extended* to the *interior* region of its self-gravitationally corrected spherically-symmetric static *source*, which is described by the cumulative energy distribution $E_G(r)$, by the very simple expedient of *replacing* the *energy constant* E_G which appears in Eq. (5c) by that cumulative energy distribution $E_G(r)$. This yields the *complete* "Newtonian" metric for the self-gravitationally corrected spherically-symmetric static cumulative energy distribution $E_G(r)$,

$$(cd\tau)^2 = \left(1-[GE_G(r)/(c^4r)]\right)^2 (cdt)^2 - \left(1-\tfrac{1}{2}[GE_G(r)/(c^4r)]\right)^{-4} (dr)^2 - \left(1-\tfrac{1}{2}[GE_G(r)/(c^4r)]\right)^{-2} \left((rd\theta)^2 + (r\sin\theta d\phi)^2\right), \quad (5d)$$

which *exactly* corresponds to the dimensionless Newtonian gravitational energy-reduction factor in Eq. (2a) because,

$$(g_{00}(r))^{\frac{1}{2}} = (1 - [GE_G(r)/(c^4r)]). \quad (5e)$$

Therefore if we multiply the gravitationally-modified energy $E'_G(r)dr$ of an infinitesimally-thick spherical shell at radius r by the *inverse* $1/(1 - [GE_G(r)/(c^4r)])$ of the corresponding radius-r dimensionless gravitational energy-reduction factor given by Eq. (5e), we *undo* the gravitational modification of the energy of that infinitesimally-thick shell and arrive at what its energy *would be* if gravity were "switched off", i.e., at what its energy would be if G *were put to zero*,

$$E'_{G=0}(r)dr = E'_G(r)dr/(1 - [GE_G(r)/(c^4r)]). \quad (5f)$$

Eq. (5f) implies the linear inhomogeneous first-order differential equation,

$$E'_G(r) + (G/c^4)(E'_{G=0}(r)/r)E_G(r) = E'_{G=0}(r), \quad (5g)$$

which is *exactly the same* as the key Eq. (2c) result of the energetically self-consistent Newtonian approach set out in the previous section. This differential equation is, of course, completely solved and exhaustively discussed in that section, and is shown there to imply the *crucial upper bound* on the self-gravitationally corrected cumulative energy $E_G(r)$ that is given by Eq. (3c), namely that,

$$0 \le E_G(r) < (c^4r/G). \quad (5h)$$

The Eq. (5h) upper bound on the self-gravitationally corrected cumulative energy $E_G(r)$ is easily verified to imply that, *despite* one's cursory initial impression, the *complete* "Newtonian" metric given by Eq. (5d) *has no singularities whatsoever*.

Of course the *complete* "Newtonian" metric of Eq. (5d) is *entirely consistent* with the Eq. (5c) "Newtonian" form of the free-space *Schwarzschild* metric that *only applies* in the $r \ge R$ free-space region where the cumulative energy distribution $E_G(r)$ is equal to the energy constant E_G. Therefore once again *despite* one's cursory initial impression, the Eq. (5c) "Newtonian" form of the free-space *Schwarzschild* metric *has no singularities whatsoever in its free-space region of validity*. Therefore the ostensibly "threatening" singularity that occurs in the Eq. (5c) free-space Schwarzschild metric when r equals the *Schwarzschild radius* $r_S \stackrel{\text{def}}{=} GE_G/c^4$ [5] *is never physically realized*, and this is *also* the case for the ostensibly pernicious singularity that occurs in the Eq. (5c) free-space Schwarzschild metric when $r = \frac{1}{2}r_S$.

Interacting quantum particles that have forbidden energy or mass

Because of its wave character, a quantum particle which interacts with a potential can penetrate a short distance into a region where the potential's value exceeds the particle's energy—such penetration is forbidden to classical particles. In such a region the quantum particle's kinetic energy and momentum squared effectively assume *negative* values, and its penetration length λ into that energetically forbidden region can be roughly described as,

$$\lambda \approx \hbar/(-p_{\text{eff}}^2)^{\frac{1}{2}}. \tag{6a}$$

Since for a nonrelativistic interacting particle,

$$E = p^2/(2m) + V, \tag{6b}$$

we can rewrite Eq. (6a) as,

$$\lambda \approx \hbar/(2m(V - E))^{\frac{1}{2}}, \tag{6c}$$

and the energetically forbidden region corresponds to $V > E$. If V should vary significantly from the edge of the forbidden region to the depth λ, Eq. (6c) will need to be regarded as an approximate *implicit relationship* which actually needs to be *solved* for λ. That implicit relationship is set up by reexpressing V *as a function of the distance from the edge of the forbidden region*, and the resulting new independent variable is then identified as λ.

A highly energetic interacting quantum particle can similarly deviate from the natural rest mass which it has when it is free, and thus enter a region of forbidden rest mass. Just as there is an effective length limit for quantum particle penetration into a region of forbidden energy, so there is an effective proper time limit, i.e., lifetime, for quantum particle penetration into a region of forbidden rest mass, namely,

$$\tau \approx \hbar/(\Delta_m c^2), \tag{7a}$$

where Δ_m is the rest-mass deviation experienced by an energetic interacting quantum particle and τ is the lifetime of that deviant-mass state. Given its limited lifetime, such a deviant-mass *virtual particle* is as well *limited in space* to a spherical region of radius $R \approx c\tau$,

$$R \approx c\tau \approx \hbar/(\Delta_m c). \tag{7b}$$

Given that radius R of Eq. (7b), Eq. (3c) then yields a self-gravitational approximate upper bound on the total energy E of this deviant-mass particle,

$$E \mathrel{\widetilde{<}} (c^4 R/G) \approx c^3\hbar/(G\Delta_m). \tag{7c}$$

Since $\Delta_m c^2$ cannot exceed the deviant-mass particle's total energy E, it follows from Eq. (7c) that,

$$\Delta_m c^2 \mathrel{\widetilde{<}} c^3\hbar/(G\Delta_m). \tag{7d}$$

Therefore,

$$(\Delta_m)^2 \mathrel{\widetilde{<}} \hbar c/G, \tag{7e}$$

which implies that,

$$\Delta_m \mathrel{\widetilde{<}} (\hbar c/G)^{\frac{1}{2}}, \tag{7f}$$

namely that Δ_m is approximately bounded by the *Planck mass* $(\hbar c/G)^{\frac{1}{2}}$.

This approximate Planck-mass upper bound on the mass deviation of any interacting quantum virtual particle universally cuts off the *ultraviolet divergences* which bedevil quantum particle scattering amplitude calculations.

Self-gravitational correction of the infinite energies of quantum fields

A noninteracting field (e.g., the source-free electromagnetic field) always decomposes into an infinite number of independent simple harmonic oscillators whose frequency spectrum has no upper bound. Upon quantization, each such oscillator has a *minimum positive energy* (i.e., quantum ground state energy) which is equal to its frequency times $\hbar/2$: that minimum energy is quantum theoretically *inviolable*, being *completely mandated by the quantum uncertainty principle*. The fact that there are an *infinite number* of such simple harmonic oscillators with an *unbounded frequency spectrum* implies that the corresponding quantum field *always has infinite energy*. If the field is set up in a bounded region, then its quantum counterpart has *not only* infinite energy, but necessarily *infinite average energy density* as well.

Therefore for quantum fields the uncertainty principle baldly confronts us with an unphysical nightmare, yet *without* this *selfsame* uncertainty principle *likewise* mandating a definite minimum energy, Rutherford's nuclear atom can't be sustained.

The last apparent hope for the beleaguered theorist in this harrowing circumstance resides with the self-gravitational modification of an *initially infinite* energy density, e.g., Eqs. (4j) and (4k) in the limit that the initial energy density $T_{G=0}$ is *infinite*. A saving grace of this vertiginous maneuver is that its result is particularly unambiguous and simple, namely,

$$E_G(r) = (c^4 r/G), \tag{8a}$$

where r is the bounding radius of the quantum field. Of course it makes no difference whatsoever *how many* or *what types* of quantum fields reside within that bounding radius: *any infinite* initial energy density produces the result of Eq. (8a).

The *total* self-gravitationally modified energy of quantum fields, which is what Eq. (8a) is supposed to describe, isn't *directly* measurable. However, by making use of the fields' bounding radius r, we can obtain from Eq. (8a) their self-gravitationally modified *averaged energy density* $\bar{\rho}$,

$$\bar{\rho} = (3c^4/(4\pi G r^2)). \tag{8b}$$

It now remains to puzzle out what conceivable "real world" physics could relate to the mathematical abstraction of the quantum fields' *bounding radius* r. It is, of course, apparent that *no material substance* can serve to bound *the arbitrarily high frequencies* which such fields are able to muster. The universe' *cosmological redshift*, however, in principle ought to defang *any* frequency, and indeed appears to serve as the "containment" for all that we can hope to survey. Therefore a not altogether implausible crude estimate of the quantum fields' "bounding radius" r ought to be given by the age of the universe [6] times the speed of light, which comes to about 1.3×10^{26} meters. Putting that value of r into Eq. (8b) yields about 1.7×10^{-9} joules per cubic meter (i.e., 1.7×10^{-8} ergs per cubic centimeter or 1.7 joules per cubic kilometer) as a crude estimate of the universe' average "dark energy" density. This is in fact of the same order of magnitude as what is yielded by observations [7, 8].

In addition to the ability of Eq. (8b) to yield a passable crude estimate of the current universe' average "dark energy" density, its systematics also seem fascinating. If we project it all the way back to the universe' birth, when r was presumably of the order of magnitude of the Planck length $(G\hbar/c^3)^{\frac{1}{2}}$, then $\bar{\rho}$ approaches of order unity in Planck units of energy density, which is roughly 120 orders of magnitude greater than its value for the current universe.

Theorists *who did not attempt to actually model the physics which produces self-gravitational energy correction* have favored this particular *enormous* value of "dark energy" density because of their adoption of a physically-blinkered "universal fix" for infinite results, namely the replacement of any such infinity by one Planck unit of the appropriate dimensions [8]. Neither physical modeling of self-gravitational energy correction nor observations have much overlap with such undiscriminating replacement of the quantum energy-density infinity by its Planck-unit value, but it is still fascinating to consider that enormous Planck unit of "dark energy" density as being relevant to the *early* universe, as that would apparently provide an *automatic* mechanism for the heretofore puzzling "inflation" of that early universe.

Finally, Eq. (8b) suggests that the average "dark energy" density ought to decrease toward zero as the universe continues its expansion. This brings to mind the not infrequently expressed theorist preference for *exactly vanishing* "dark energy" density over its *observed* value [8], which while *immensely smaller* than the Planck unit of energy density, nonetheless *absolutely fails to vanish*. In fact, completely to the *contrary*, it *dominates* the *net* average energy density of our universe [7]. We see that Eq. (8b) apparently caters for *all* tastes in average "dark energy" density, whether those tastes gravitate toward the enormous Planck unit of energy density, zero energy density, or anything in between, including a passable rendition of the observed average "dark energy" density which actually obtains at the current stage of evolution of our universe. It is to be cautioned, however, that while Eq. (8b) indeed has average "dark energy" density decreasing toward zero as the universe continues to expand, the average density of *normal* matter would be expected to decrease *at a faster rate*, so that "dark energy" relative *dominance* would continue to grow.

Must self-gravitation be quantized to correct quantum energy infinities?

Gravity, like electromagnetism, is a *gauge theory*, and the issues surrounding its quantization formally parallel those issues in electromagnetism. In both cases there are dynamical, nondynamical and redundant fields present, and the *dynamical* fields in *both* cases are *two* in number and describe *transverse radiation*. *Only these two dynamical radiation fields are subject to quantization.*

What remains after the two transverse dynamical radiation fields are accounted for splits evenly into nondynamical and redundant fields, *neither* of which, of course, are subject to quantization. The four-potential of electromagnetism yields one redundant field and one nondynamical field of Coulombic character. The symmetric metric tensor of gravity yeilds four redundant fields and four nondynamical ones. One of the nondynamical fields very roughly corresponds to Newtonian gravity with roughly an energy-density source, while the other three merely round out a relativistic four-vector representation, and therefore have roughly a momentum-flux source.

If we look back at the previous parts of this article, it is clear that the self-gravitational corrections which are of overarching importance can all be profitably pondered in a quasi-static or outright static framework. The basic ingredients for self-gravitational corrections tend to be *Newtonian*, albeit *an energetically self-consistent form* of gravitational Newtonianism.

Gravitational radiation *doesn't* physically enter into self-gravitational correction, so gravity quantization *cannot* be an issue in such correction, any more than electromagnetic quantization can be an issue in electrostatics. Gravitostatics is merely more subtle than electrostatics because of its energetic self-consistency.

Conclusion

We have constructed a simple spherically-symmetric and energetically self-consistent Newtonian gravitostatic model which yields a simple and apparently very useful upper bound on the amount of self-gravitationally corrected energy which can be contained in a spherical region. That bound is just the "Planck force" (c^4/G) times the radius r of the sphere, a relationship which, inter alia, implies that the Schwarzschild radius *never* lies in free space, making the Schwarzschild singularity physically unrealizable. This same model can *also* be obtained by using the Birkhoff theorem and its corollary to *extend* the "Newtonian" form of the free-space Schwarzschild metric into the *interior* region of its self-gravitationally corrected spherically-symmetric static *source*.

The "Planck force" times radius bound on a sphere's contained energy also cuts off the mass deviation of an interacting quantum virtual particle at approximately the Planck mass, which in principle does away with the ultraviolet divergences that bedevil quantum particle scattering amplitude calculations.

The "Planck force" times radius bound on a sphere's contained energy ought to be *attained* for contained quantum fields, which have *infinite energy* before self-gravitational correction, due to the combination of their unbounded frequency spectra and the quantum uncertainty principle. But *only* the universe *itself*, with its cosmological redshift, is actually capable of "containing" the arbitrarily high frequencies of a quantum field. Roughly estimating the radius of the universe as its age times the speed of light, and then dividing the "Planck force" times this radius by the corresponding spherical volume that has this radius, yields a rough averaged "dark energy" density estimate of about 1.7 joules per cubic kilometer, which is of the same order of magnitude as observational data. The *same* formula suggests that the *early* universe might have had an immensely greater "dark energy" density, perhaps as much as a Planck unit of energy density, which would be roughly 120 orders of magnitude times its present value. It is interesting that this seems to provide an *automatic* mechanism for the inflation of the early universe. The formula also suggests that the "dark energy" density will be decreasing toward zero as the universe expands, but that it won't decrease as rapidly as the density of ordinary matter will, which will increase the relative dominance of dark energy.

Finally, there seems to be no particular reason why the "dark energy" density should not share the small inhomogeneities which are so typical of the rest of the universe, such as the small peaks in the cosmic microwave background, and the galaxies, groups, filaments and voids in the distribution of luminous matter. If "dark energy" indeed has inhomogeneities, then might not those inhomogeneities *themselves* be the thing we call "dark matter"? In spite of all the gravitational evidence for "dark matter", there is apparently no observationally-known non-gravitational signal whatsoever for it. It would be a relief if something so elusive ultimately turned out to *not* have an independent existence.

References

[1] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972), pp. 345–348.

[2] S. Weinberg, op. cit., p. 79, Eq. (3.5.2).

[3] S. Weinberg, op. cit., p. 181, Eq. (8.2.14).

[4] S. Weinberg, op. cit., pp. 337–338.

[5] S. Weinberg, op. cit., p. 207.

[6] Wikipedia, "Age of the universe", http://en.wikipedia.org/wiki/Age_of_the_universe.

[7] Wikipedia, "Dark energy", http://en.wikipedia.org/wiki/Dark_energy.

[8] S. M. Carroll, arXiv:astro-ph/0310342 (2003); AIP Conf. Proc. **743**, 16–32 (2005).