Product of Distributions Applied to Discrete Differential Geometry

V. Nardozza*

Nov. 2012†

Abstract

A method for dealing with the product of step discontinuous and delta functions is proposed. A new space of generalised functions extending the space D', together with a well defined product, is constructed. The new space of generalized functions is used to prove interesting equalities involving products among elements of D'.

A standard method, for applying the above defined product of distributions to polyhedron vertices, is analysed and the method is applied to a special case where the well known defect angle formula, for the discrete curvature of polyhedra, is derived using the tools of tensor calculus.

Key Words: distribution theory, product of distributions, discrete differential geometry.

1 Introduction

Products of distributions are quite common in several fields of both mathematics and physics. Examples arise naturally in quantum field theory, gravitation and, in partial differential equation, such as shock wave solutions, in hydrodynamics, (see [1]). An important issue, related to product of distributions, is the fact that the product, in the general case, is not well defined in D', issue known as the Schwartz impossibility result (see [1] §1.3) and that only the product between a smooth function and a distribution is well defined.

Discrete differential geometry is a rather new field of mathematics which borrows concepts and ideas from both differential geometry and discrete mathematics. Main applications are concerned with the discrete version of several classical concepts of differential geometry such as discrete curvature, minimal surfaces, geodesics coordinates, minimal paths, surfaces of constant curvature, curvature line parametrisation and the discrete version of continuous functionals (see [2]). At the moment, discrete differential geometry uses many tools of discrete mathematics while the classical tools of differential geometry (e.g. tensors and coordinate free exterior calculus) are difficult to be applied. This leads to an ambiguous definition of the various operators (see [3]) which are instead well defined in the continuous counterpart of the theory.

*Electronic Engineer (MSc). Turin, Italy. mailto: vinardo@libero.it

†Posted at: http://www.vixra.org/abs/1211.0099

In this paper, we propose a method for evaluating the product of step discontinuous functions and Dirac delta functions, related each other by an integrable function. Moreover, the method is applied to a special class of non differentiable varieties for which, the classical idea of curvature, together with all tools of differential geometry, needs to be redefined in terms of distribution functions. In particular, the class of varieties analysed is the one composed of a collection of several Riemannian varieties glued in such a way the final surface is not differentiable on the resulting edges and vertices. In this case, it is possible to show that vertices and edges carry a concentrated discrete curvature which gives a contribution to the total curvature of the surface, contribution that has to be taken into account in order for the Gauss-Bonnet theorem to work.

For vertices, an important result was already known since the time of Descartes which proved, in the first half of the 17th century, its defect angle theorem for polyhedra. That idea, using the modern concept of curvature and applied to the class of surfaces defined above, can be stated by saying that the discrete total curvature of a vertex is equal to 2π minus the sum of the angles between edges.

For edges, using the Gauss-Bonnet theorem, it is easy to see that the discrete curvature carried by an edge L_{ij} is given by:

$$k_{L_{ij}} = \int_{L_{ij}} (k_{g_i} + k_{g_j}) ds \tag{1}$$

where k_{g_i} and k_{g_j} are the geodesic curvatures, evaluated on the edge L_{ij}, of the two variety S_i and S_j for which L_{ij} is the boundary. If the surface is differentiable on L_{ij}, then k_{g_i} and k_{g_j} are opposite and the integral vanishes. If the surface is not differentiable, the integral (1) gives in general a finite result witch corresponds to the discrete curvature concentrated on L_{ij} and $(k_{g_i} + k_{g_j})$ is the discrete curvature for unit length of the surface on L_{ij}.

This kind of surfaces, characterised by a step discontinuous metric, are typical of problems ranging from theoretical physics up to computer graphics, where the usual way to proceed is to brake down the problem and to define boundary conditions (with conserved quantities) in order to keep the whole problem definition consistent (see [4]) or to use methods of discrete mathematics to define the relevant operators (see [3]). The approach proposed in this paper is to use a more direct method directly derived from the classical differential geometry.

In Paragraphs 2, 3 and 4, we derive a method to evaluate products of step discontinuous and Dirac delta functions of the type

$$f(g_1(x_1), \ldots, g_n(x_n))\delta(x_1, \ldots, x_n) \tag{2}$$

where $g_i(x_i))$ are step discontinuous functions and f is a locally integrable function.

In Paragraphs from 5 to 8, we construct a new space of generalised functions, extending the space D'. We use the new space of generalised functions to define a product among steps, deltas and delta derivatives and we show in which cases, the result of the product, can be projected in D'. Also, we use the theory developed in these paragraphs to derive interesting equalities involving products among elements of D'.

In Paragraphs 9 and 10, we use the product of step discontinuous and Dirac delta functions, mentioned above, to evaluate the discrete curvature of a polyhedron vertex. In order to do that, we define the step discontinuous metric of polyhedron vertices and we evaluate their Riemann tensors by applying the classical rules of the differential geometry but taking the derivatives in D'. By using this approach, the final result is, as expected, the defect angle formula for the total curvature of a polyhedron vertex.

2 Product of steps and delta functions

Proposition 1. *Let $g(x)$ be a function defined as follows:*

$$g(x) = \begin{cases} a & for\, x < 0 \\ b & for\, x > 0 \end{cases} \tag{3}$$

with $a, b \in \mathbb{R}$, and let $f(x)$ be any function locally integrable in $A \supseteq [a, b]$ (or $[b, a]$ if $b < a$). Also let $(b-a)\delta(x)$ be the derivative of $g(x)$. Then:

$$f(g(x))\delta(x) = \frac{1}{b-a}\left(\int_a^b f(x)dx\right)\delta(x) \tag{4}$$

where the above product has to be intended as:

$$f(g(x))\delta(x) = \frac{1}{b-a}\lim_{n\to\infty} f(g_n(x))g_n'(x) \tag{5}$$

for any sequence g_n such that:

$$\begin{array}{l} 1)\ g_n(x) \in C^1\ \ \forall n \in \mathbb{N} \\ 2)\ \lim_{n\to\infty} g_n(x) = g(x) \\ 3)\ \lim_{x\to-\infty} g_n(x) = a\ \ \forall n \in \mathbb{N} \\ 4)\ \lim_{x\to+\infty} g_n(x) = b\ \ \forall n \in \mathbb{N} \\ 5)\ g_n(x)\ is\ monotonic\ \forall n \in \mathbb{N} \end{array} \tag{6}$$

Moreover, we choose each $g_n(x)$ such that, if $g_n(x)$ is constant in any $]\alpha, \beta[$ and equal to k, then $f(x)$ is continuous in k.

Proof. The proof is given for $a < b$ only, changes to the proof, for the case $b < a$, are trivial. We note immediately that, given the (6), the $g_n(x)$ are bounded and converge to $g(x)$. For the dominated convergence theorem, $g_n(x)$ converges in L^1_{loc} and therefore in D'. Also g_n' converges to $(b-a)\delta(x)$ in D'.

First, we prove two useful equations. For any $f \in L^1_{loc}(A)$, for any $g_n(x)$ having the characteristics (6) and given any $\alpha, \beta \in \mathbb{R}$ we have:

$$\int_\alpha^\beta f(g_n(x))g_n'(x)dx = \int_\alpha^\beta \frac{d}{dx}F(g_n(x))dx = F(g_n(\beta)) - F(g_n(\alpha)) \tag{7}$$

where $F(x)$ is the primitive of $f(x)$.

Now, $\lim_{\alpha\to-\infty} g_n(\alpha) = a$ and $\lim_{\beta\to+\infty} g_n(\beta) = b$ and therefore we have:

$$\int_{-\infty}^{+\infty} f(g_n(x))g_n'(x)dx = \int_a^b f(x)dx \tag{8}$$

The (8) does not depend from the function $g_n(x)$ since it depends only on $f(x)$, a and b.

Also, let $[\alpha,\beta]$ be any interval. Given the (6), $g_n' \geq 0$. We write $f(x) = f_+(x) - f_-(x)$ as the sum of its positive and negative part. Note that $f_+(x)$ and $f_-(x)$ are locally integrable on A. We have:

$$\begin{aligned}
\int_\alpha^\beta |f(g_n(x))g_n'(x)|dx &= \int_\alpha^\beta f_+(g_n(x))g_n'(x)dx + \int_\alpha^\beta f_-(g_n(x))g_n'(x)dx \\
&= \int_{g_n(\alpha)}^{g_n(\beta)} f_+(x)dx + \int_{g_n(\alpha)}^{g_n(\beta)} f_-(x)dx \\
&= \int_{g_n(\alpha)}^{g_n(\beta)} |f(x)|dx \\
&\leq \int_a^b |f(x)|dx = M > 0 \qquad (9)
\end{aligned}$$

Now we can prove the proposition. Let $\phi(x)$ be a test function, taking into account the (8) it is possible to write:

$$\begin{aligned}
&\left| \int_{-\infty}^{+\infty} f(g_n(x))g_n'(x)\phi(x)dx - \left(\int_a^b f(x)dx\right)\phi(0) \right| \\
&= \left| \int_{-\infty}^{+\infty} f(g_n(x))g_n'(x)[\phi(x)-\phi(0)]dx \right| \\
&\leq I_{m1} + I_{m2} + I_{m3} \qquad (10)
\end{aligned}$$

where m is any positive integer and:

$$I_{m1} = \int_{-\infty}^{-1/m} |f(g_n(x))g_n'(x)|\,|\phi(x)-\phi(0)|dx \qquad (11)$$

$$I_{m2} = \int_{-1/m}^{+1/m} |f(g_n(x))g_n'(x)|\,|\phi(x)-\phi(0)|dx \qquad (12)$$

$$I_{m3} = \int_{+1/m}^{+\infty} |f(g_n(x))g_n'(x)|\,|\phi(x)-\phi(0)|dx \qquad (13)$$

Since ϕ is a test function, it is continuous at $x = 0$. Given any $\epsilon > 0$, it is possible to find $\delta > 0$ such that, whenever $|x| < \delta$, $|\phi(x) - \phi(0)| < \epsilon$. So, given any $m > \frac{1}{\delta}$, if we choose any $n > m$, we have:

$$\begin{aligned}
I_{m2} &= \int_{-1/m}^{+1/m} |f(g_n(x))g_n'(x)|\,|\phi(x)-\phi(0)|dx \\
&\leq \epsilon \int_{-1/m}^{+1/m} |f(g_n(x))g_n'(x)|dx \leq M\epsilon \qquad (14)
\end{aligned}$$

Where we have used the (9).

Now, ϕ is a continuous function with compact support S and therefore it is bounded. We can find $L > 0$ such that $|\phi(x) - \phi(0)| < L$. We have:

$$\begin{aligned} I_{m1} &= \int_{-\infty}^{-1/m} |f(g_n(x))g_n'(x)|\,|\phi(x) - \phi(0)|dx \\ &\leq \int_S Ldx \int_{-\infty}^{-1/m} |f(g_n(x))g_n'(x)|dx \\ &= N \int_a^{g_n(-1/m)} |f(x)|dx \end{aligned} \tag{15}$$

where we have used the (9) and $N > 0$ is the integral of the constant L on S. Since $g_n(-1/m)$ converge to a and given the ϵ above, it is possible to find k such that, whenever $n > k$ then $I_{m1} < N\epsilon$. Applying the same argument to I_{m3} we find that, it is also possible to find k such that, whenever $n > k$ then $I_{m3} < N\epsilon$.

To conclude, given the (10) and given any $\epsilon > 0$, it is possible to find first m and then k such that, whenever we choose $n > k > m$ we have:

$$\left| \int_{-\infty}^{+\infty} f(g_n(x))g_n'(x)\phi(x)dx - \left(\int_a^b f(x)dx \right) \phi(0) \right| \leq (M + 2N)\epsilon \tag{16}$$

This proves that:

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} f(g_n(x))g_n'(x)\phi(x)dx = \left(\int_a^b f(x)dx \right) \phi(0) \tag{17}$$

Now, if we call $(b - a)f(g(x))\delta(x)$ the limit of the sequence of distributions $f(g_n(x))g_n'(x)$, the (17) proves the following:

- the limit exists
- the limit is a Dirac delta function
- the amplitude of the delta function is given by the (4) □

Note that, even in the case where $f(a) = f(b)$ and therefore there is no step in the discontinuity, proposition 1 is essential to evaluate the product of the discontinuity with a related delta function. For example, is easy to show that $sign^2(x)\delta(x) = \frac{1}{3}\delta(x)$.

3 The multidimensional case

Proposition 2. *Let $g_1(x)$ and $g_2(y)$ be two functions defined as follows:*

$$g_1(x) = \begin{cases} a & for\, x < 0 \\ b & for\, x > 0 \end{cases} \tag{18}$$

$$g_2(y) = \begin{cases} c & for\, y < 0 \\ d & for\, y > 0 \end{cases} \tag{19}$$

with $a, b, c, d \in \mathbb{R}$ *and let* $f(x, y)$ *be any function locally integrable in* $A \supseteq [a, b] \times [c, d]$ *(if* $b < a$ *and/or* $d < c$ *the definition of* A *has to be changed accordingly). Also let* $(b-a)(d-c)\delta(x, y)$ *be the product of the derivatives of* $g_1(x)$ *and* $g_2(y)$*. Then:*

$$f(g_1(x), g_2(y))\delta(x, y) = \frac{1}{(b-a)(d-c)} \left(\int_c^d dy \int_a^b f(x, y)dx \right) \delta(x, y) \qquad (20)$$

where the above product has to be intended as:

$$f(g_1(x), g_2(y))\delta(x, y) = \frac{1}{(b-a)(d-c)} \lim_{n \to \infty} f(g_{1n}, g_{2n}) g'_{1n} g'_{2n} \qquad (21)$$

for any pair of sequences g_{1n}*,* g_{2n} *such that:*

$$\begin{array}{l} 1)\ g_{1n}(x),\ g_{2n}(y) \in C^1\ \ \forall n \in \mathbb{N} \\ 2)\ \lim_{n\to\infty} g_{1n}(x) = g_1(x),\ \lim_{n\to\infty} g_{2n}(y) = g_2(y) \\ 3)\ \lim_{x\to-\infty} g_{1n}(x) = a,\ \ \lim_{x\to+\infty} g_{1n}(x) = b\ \ \forall n \in \mathbb{N} \\ 4)\ \lim_{y\to-\infty} g_{2n}(y) = c,\ \ \lim_{y\to+\infty} g_{2n}(y) = d\ \ \forall n \in \mathbb{N} \\ 5)\ g_{1n}(x),\ g_{2n}(y)\ \textit{are monotonic}\ \forall n \in \mathbb{N} \end{array} \qquad (22)$$

Moreover, we choose each pair $(g_{1n}(x), g_{2n}(y))$ *such that, if* $(g_{1n}(x), g_{2n}(y))$ *are constant in any* $]\alpha_1, \beta_1[\times]\alpha_2, \beta_2[$ *and equal to* (k_1, k_2)*, then* $f(x, y)$ *is continuous in* (k_1, k_2)*.*

Obviously, we can interchange the roles of x and y since we may integrate first with respect of y and then with respect of x. Note that the discontinuity $f(g_1(x), g_2(y))$ addressed by this proposition is not the most general step discontinuity we may have in two dimensions.

As for proposition 1, in order to prove the above proposition, we first need to prove some useful equations. As an example, we will prove the equivalent of the (8). Let $g_{1n}(x), g_{2n}(y)$ be two functions having the characteristics (22) and let $F(x, y)$ be a function such that $F_{xy} = F_{yx} = f(x, y)$. We have:

$$\begin{aligned} &\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} f(g_{1n}(x), g_{2n}(y)) g'_{1n}(x) g'_{2n}(y) dx \\ &= \int_{-\infty}^{+\infty} dy \frac{\partial}{\partial y} \int_{-\infty}^{+\infty} \frac{\partial}{\partial x} F(g_{1n}(x), g_{2n}(y)) dx \\ &= F(b, d) - F(a, d) - F(b, c) + F(a, c) \end{aligned} \qquad (23)$$

where, to prove the (23), we have taken the symbol $\frac{\partial}{\partial y}$ inside the integral (for the linearity of integrals) and applied the definition of $F(x, y)$. The (23) is independent from g_{1n}, g_{2n} and depends only on $f(x, y), a, b, c, d$. Proposition 2 will not be proven in this paper. However, it is possible to prove it by following similar steps to the ones used for proving proposition 1.

Note that proposition 2 gives a clear path on the possible way to generalise the idea of products of step discontinuities and delta functions to the case with as many dimensions as we like.

4 Further remarks on product of distributions

So far, we have been mainly interested in step discontinuities and Dirac delta functions. Each discontinuity of this kind can be defined by means of the limit of an infinite number of sequences of distributions, all different each other and having, in a distributional sense, the same limit in D'. For the purpose of this paper, we define the structure of such discontinuities to be the specific sequence we use to define them. In general, since we want a distribution to be as generic as possible, we never define its own structure and we leave it indeterminate. However, the concept of structure of a discontinuity is essential to this paper as it will be clear shortly.

Moreover, for products of distributions, every time we define the product in a point x_0, where the distributions are discontinuous, we always want the discontinuities to have each other structure related by a well known law so that, if the structure of one distribution in x_0, which is unknown to us, changes, the structure of all other distributions in the same point will change accordingly.

Since we never want to define the structures of the distributions, we are mostly interested in products of distributions, like the ones of proposition 1 and 2, which work regardless their underlying structures. This is why in proposition 1 and 2 we define the product as the limit of any possible sequence (i.e. structure) and we want this limit to be independent from it.

The idea that a particular distribution may have an infinite number of different structures is very similar to the notion of associated distributions present in the Colombeau theory (see [1] §3.2), where the product make sense if it is independent from the particular representative of the involved generalised functions (see [1] §3.1).

As a final remark, note that the fact proposition 1 and 2 are valid for f locally integrable, is an important feature. An example, where we use this feature is given in paragraph 9 and 10 of this paper.

5 The need for new generalised functions.

So far, we have focused our attention only on the structure of step discontinuities and the way they are modified (by composition with a locally integrable function f). When it comes to Dirac delta functions, it is possible to show that they change their own structure by means of multiplication by step discontinuous functions. Let us consider the function $f(g(x))$ where g is a step discontinuous function defined as in (3) and $f \in L^1_{loc}([a,b])$. Since we may define our function as the limit of a sequence of functions $f(g_n(x))$ with $g_n(x) \in C^1$, and since the Leibniz rule may be applied to each term of the sequence, we will suppose that we can apply the Leibniz rule also to its limit. This point will be justified in further paragraphs. We have:

$$Df(g(x)) = (b-a)f'(g(x))\delta(x) \tag{24}$$

from which we see that by multiplying a delta function having structure $g(x)$ (i.e. derivative of a step discontinuous function $g(x)$) by $f'(g(x))$ we get a delta function with structure $f(g(x))$ (i.e. derivative of a step discontinuous function $f(g(x))$).

We have seen that, in a product of distributions, if we change the structure of a term we get a different result. In order to overcame this limitation, we want now to extend the space of distributions D' by adding to it, as separate generalised functions, additional elements representing any possible discontinuity structure needed for describing products of step and delta functions.

We will assume now that all step discontinuous and delta functions, we are dealing with, are all related to the same Heaviside function and their structure can be described by the way they are related to it. From this new point of view, the function f, which before was used to relate distribution structures, became now the structure itself of the distribution. We will say that a step discontinuity has structure f if it is of the form $f(u(x))$. We will say that a delta function has structure f if it is the derivative of a step discontinuous function of structure f. We will consider steps and delta functions, with different structures, as separate generalised functions.
We will use the following notation:

$$\begin{array}{ll} u_{[f(x)]} = f(u(x)) & step\ function\ having\ structure\ f \\ \delta_{[f'(x)]} = f'(u(x))\delta(x) & delta\ function\ having\ structure\ f \end{array} \tag{25}$$

where $u_{[f(x)]}$ and $\delta_{[f(x)]}$ are not normalised (i.e they may have amplitude different from 1) and $u_{[x]} = u(x) \in D'$, $\delta_{[1]} = \delta(x) \in D'$. We will show, with an example at the end of this paragraph, that the above defined generalised functions have components outside D' and therefore there is a need for defining a larger space of generalised functions including D'. We will do that in the next paragraphs.
Using the (25), we define the multiplication as follows:

$$u_{[f_1]}u_{[f_2]} \cdot \ \ldots \ \cdot u_{[f_n]}\delta_{[f_{n+1}]} = \delta_{[f_1 f_2 \cdot \ldots \cdot f_n f_{n+1}]} \tag{26}$$

Finally we define a projector operator $P_{D'}$, which project any generalised function of the kind (25), on the space D'. For step discontinuous functions the way $P_{D'}$ works is trivial (e.g. $u^2(x)$ goes to $u(x)$). For delta functions, we apply the theory developed in paragraph 2 and, by using proposition 1, we have:

$$P_{D'}\left(\delta_{[f_1 f_2 \cdot \ldots \cdot f_n f_{n+1}]}\right) = \left(\int_0^1 f_1 f_2 \cdot \ \ldots \ \cdot f_n f_{n+1} dx\right)\delta(x) \in D' \tag{27}$$

where the integration is performed between 0 and 1, which is the jump of our reference step discontinuity $u(x)$. Note that the (26) and (27) provide a well defined product of the (25) which is fully coherent with the theory developed in the previous paragraphs. The product is also commutative and associative since commutative and associative is the product of the f_i functions used in the definition of the (26).

Let us make an example. Consider the product of distributions $sign^2(x)\delta(x)$ (compare with [5] §1.1 ex. iii). By using proposition 1 we find easily that:

$$sign^2(x)\delta(x) = \frac{1}{3}\delta(x) \tag{28}$$

Let us check associativity by using, once again, proposition 1:

$$sign^2(x)\delta(x) = sign(x)[sign(x)\delta(x)] = sign(x) \cdot 0 = 0 \tag{29}$$

we conclude that, in D', our product is not associative. Let us see what happen using the (26):

$$sign(x)[sign(x)\delta(x)] = sign(x)\delta_{[(2x-1)\cdot 1]} = sign(x)[\delta_{[2x]} - \delta_{[1]}] \qquad (30)$$

In D', $\delta_{[1]} = \delta$ and $P_{D'}(\delta_{[2x]}) = \delta$. However, as generalised function of the kind (25), they are separate objects and they do not cancel each other. We have eventually:

$$sign^2(x)\delta(x) = P_{D'}\left(\delta_{[(2x-1)^2]}\right) = \frac{1}{3}\delta(x) \qquad (31)$$

6 New generalised functions

Definition 1. *We define the generalised function $\eta^{p,q}$ to be the limit of the following sequence of distributions:*

$$\eta^{p,q}(x) = \lim_{n\to\infty} n^{q-1}\sum_{k=0}^{p}(-1)^k\binom{p}{k}\delta(x-\frac{k}{n}) \;\; with\, p,q \geq 0 \qquad (32)$$

It is easy to see that:

$$\eta^{p,p+1}(x) = \delta^{(p)}(x) \qquad (33)$$

What kind of generalised function is $\eta^{p,q}$? If the sequence of distributions f_n converges to $\eta^{p,q}$, then $\frac{f_n}{n^{q-p-1}}$ converges to $\delta^{(p)}$. So, with an abuse of notation, we may say that:

$$\eta^{p,q} = n^{q-p-1}\delta^{(p)} \qquad (34)$$

The $\eta^{p,q}$ are therefore the limit of sequences of functions that are shaped like $\delta^{(p)}$ and that, when we take the limit, grow at a lower or faster rate (according to the sign of p-q+1).

The $\eta^{p,q}$ can be defined by means of the limit of a sequence of functions $f_n(x)$. In this paper we will deal only with generalised functions defined by means of the limit of a sequence of the form:

$$\lim_{n\to\infty} n^q f(nx) \qquad (35)$$

Where $f \in C^\infty$. Note that the above sequences are not the most general way to define distributions. For example, there is not sequence of the form (35) converging to $\delta + \delta'$. We will call $f(x)$ the generating function, $n^q f(nx)$ the generating sequence and q the growing index of the generalised function defined by the (35).

Next, we define $\mathbb{F}$ to be the set of all the function $f(x)$ having the following characteristics.

$$\begin{array}{l} 1)\ f(x) \in C^\infty \\ 2)\ \lim_{x\to-\infty} f(x)x^k = 0 \;\; for\; any\; k \in \mathbb{N} \\ 3)\ \lim_{x\to+\infty} f(x)x^k = 0 \;\; for\; any\; k \in \mathbb{N} \end{array} \qquad (36)$$

Now, let us see how to determine all the $\eta^{p,q}$ components of a generalised function defined by means of the (35) and having generating function $f(x) \in \mathbb{F}$.

First of all, we note that all the components of the distribution (35) have the same growing index q and therefore are of the form $\eta^{p,q}$. We have:

$$d = \lim_{n\to\infty} n^q f(nx) = \sum_{p=0}^{\infty} a_p \eta^{p,q} \tag{37}$$

which contains one distribution $\eta^{q-1,q}(x) = \delta^{(q-1)}(x) \in D'$. We know, by definition of the $\eta^{p,q}$, that if:

$$\lim_{n\to\infty} n^q f(nx) = a_p \eta^{p,q} \tag{38}$$

then:

$$\lim_{n\to\infty} \frac{n^q f(nx)}{n^{q-p-1}} = a_p \delta^{(p)} \tag{39}$$

So, for the distribution defined by the (37), we can determine the a_p coefficients by applying the Schwartz theory of distribution to our sequence of functions divided by n^{q-p-1}. Let ϕ be a test function and given p, we have:

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} n^{p+1} f(nx)\phi(x)dx = \lim_{n\to\infty} \sum_{k=0}^{\infty} (-1)^k a_k \frac{\phi^{(k)}(0)}{n^{k-p}} \tag{40}$$

In the right side of the above equation, when n goes to infinity, some terms go to infinity (the ones with $k < p$) and some other terms go to 0 (the ones with $k > p$). To better evaluate all a_p we decide to use a test function ϕ that has all derivatives $\phi^{(i)}(0) = 0$ for $i \neq p$. A test function with this characteristic is $\phi(x) = x^p$. Of course a test function should vanish outside a compact interval and x^p does not. However, given the (36), integrability for $|x|$ going to infinity in ensured, and therefore this is not a problem. We have:

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} n^{p+1} f(nx)x^p dx = (-1)^p\, a_p\, p! \tag{41}$$

where $p!$ is the value of the p^{th} derivatives of x^p. From the (41) we can easily evaluate the a_p as follows:

$$\begin{aligned} a_p &= \lim_{n\to\infty} \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} n^{p+1} f(nx) x^p dx \\ &= \lim_{n\to\infty} \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} n\, f(nx)(nx)^p dx \end{aligned} \tag{42}$$

We note that the right part of the (42), for n that goes to infinity, in the (x, y) plane, shrinks (along x) and grows (along y) like n, which leaves the integral unchanged. For the above reason, the limit of the (42) is simply the value of the integrals for any n. We may as well evaluate it for n=1. We have:

$$a_p = \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} f(x) x^p dx \tag{43}$$

We are now ready to define our new space of generalised functions.

Definition 2. *We define $\mathbb{G}^\eta$ to be the following space of generalised functions:*

$$\mathbb{G}^\eta = D^{'} \cup \{d : d = \lim_{n \rightarrowtail \infty} n^q f(nx) \; with \; q \in \mathbb{N}, f \in \mathbb{F}\} \tag{44}$$

We also define $\mathbb{A}$ to be the set of all sequences of coefficients $a_f = (a_0, a_1, \ldots)$ associated by the (43) to the generating function f.

Let us see an example. If we choose $f(x)$ to be a Gaussian distribution as follows:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{45}$$

and we choose q=1, from the (43) we find:

$$\lim_{n \to \infty} n\, f(nx) = d \in \mathbb{G}^\eta = \delta(x) + \frac{1}{2}\eta^{2,1} + \frac{1}{8}\eta^{4,1} + R\left(\eta^{6,1}\right) \tag{46}$$

where $R(\eta^{6,1})$ means that, to have the above equality exact, you need to add components of growing index 1 and order ≥ 6.

7 Main generating functions

In the paragraphs above, we have defined the concept of structure of a discontinuity. We note that, for a generalised function $d \in \mathbb{G}^\eta$, if q is the growing index, $f \in \mathbb{F}$ is the generating functions and $a_f \in \mathbb{A}$ are the coefficients of the $\eta^{p,q}$, we can fully characterize the structure of a discontinuity (i.e. fully define the relevant generalised function) by providing either couples (f, q) or (a_f, q). Moreover, if $a_f = (a_0, a_1, \ldots)$, then $a_{f'} = (0, a_0, a_1, \ldots)$ and therefore, in $\mathbb{G}^\eta$, the derivative of $d = (f, q)$ is $d' = (f', q + 1)$.

Let $f(x) \in \mathbb{F}$ be a generating function for δ in D' (by means of the generating sequence $n\, f(nx)$). If we evaluate the coefficients $a_f \in \mathbb{A}$, we know that $a_1 = 1$. We also know that all the others coefficient can have any value (see example in the previous paragraph). We are interested, among all the $f \in \mathbb{F}$, to the ones for which a_f is of the form $a_0 = 1$ and $a_p = 0$ for $p > 1$.
We give the following definitions:

Definition 3. *Let $\xi(x) \in \mathbb{F}$. If $\xi(x)$ verifies the following equations:*

$$\int_{-\infty}^{+\infty} \xi(x) x^p dx = \begin{cases} 1 & for\, p = 0 \\ 0 & for\, p > 0 \end{cases} \tag{47}$$

then we call ξ a main generating function for $\delta(x)$.

We have:

$$\lim_{n \to \infty} n^{p+1} \xi^{(p)} = \delta^{(p)}(x) \;\; in\; \mathbb{G}^\eta \tag{48}$$

The (48) states that, if we use main generating functions, we can define delta and delta derivatives that have no components outside D'. In a few word, if we accept generalised function $\eta^{p,q}$ to be real things (i.e. we work in $\mathbb{G}^\eta$), we have also to accept that only sequences $n\xi(nx)$ composed of main generating functions converge to δ.

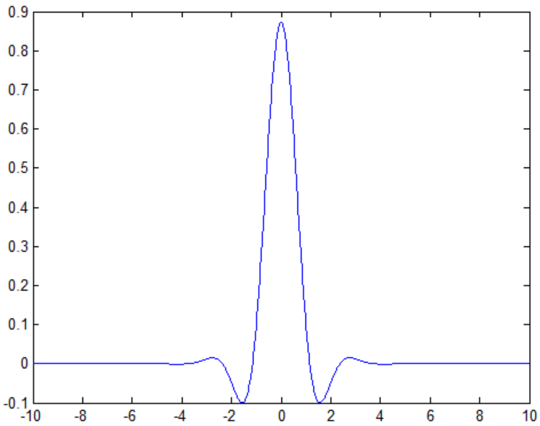

Figure 1: ξ function

The above figure is a plot of a $\xi(x)$.

Given $f \in \mathbb{F}$, there is only one element $a_f \in \mathbb{A}$. On the contrary, given $a_f \in \mathbb{A}$, there exist at least one generating function $g \in \mathbb{F}$, with $f \neq g$, such that $a_f = a_g$. In particular, is easy to prove that, if $\xi(x) \in \mathbb{F}$ is a main generating function for δ, then $\alpha\xi(\alpha x)$ with $\alpha > 0$ is a main generating function for δ and $\alpha^{p+1}\xi^{(p)}(\alpha x)$ is a main generating function for $\delta^{(p)}$. Finally, if $\xi_1^{(p)}(x)$ is a main generating function for $\delta^{(p)}$ and its derivative $\xi_1^{(p+1)}$ is a main generating function for $\delta^{(p+1)}$, then, given the (43), $\xi_2^{(p)}(x) = \frac{-x}{p+1}\xi_1^{(p+1)}(x)$ is a main generating function for $\delta^{(p)}$ with $\xi_2^{(p)} \neq \xi_1^{(p)}$.

We will call the functions $\alpha^{p+1}\xi^{(p)}(\alpha x)$ the generating functions relevant to the $\delta^{(p)}$ induced by the generating function $\alpha\xi(\alpha x)$ chosen for δ.

We show now an important fact about the coefficient a_p of the (43). Given $\xi^{(p)}$, a main generating functions $\delta^{(p)}$, we have:

$$\lim_{n\to\infty} a_p(\xi)n^q\xi^{(p)}(nx) = a_p(\xi)\eta^{p,q} \tag{49}$$

If we choose $\xi_\alpha = \alpha^{p+1}\xi^{(p)}(\alpha x)$, as a different generating function for $\delta^{(p)}$, we have:

$$\lim_{n\to\infty} a_p(\xi_\alpha)n^q\alpha^{p+1}\xi^{(p)}(n\,\alpha x) = a_p(\xi_\alpha)\eta^{p,q} \tag{50}$$

If the (49) and (50) are the same generalised function then:

$$\lim_{n\to\infty} a_p(\xi)n^q\xi^{(p)}(nx) = \lim_{n\to\infty} \frac{a_p(\xi_\alpha)}{\alpha^{q-p-1}}(n\alpha)^q\xi^{(p)}(n\,\alpha x) \tag{51}$$

since

$$\lim_{n\to\infty} n^q\xi^{(p)}(nx) = \lim_{n\to\infty} (n\alpha)^q\xi^{(p)}(n\,\alpha x) = \eta^{p,q} \tag{52}$$

because the left and the right side limit of the (52) are the same function growing and shrinking at the same rate with n, and therefore converge to the same generalised function $\eta^{p,q}$. We conclude that:

$$a_p(\xi_\alpha) = \alpha^{q-p-1}a_p(\xi) \tag{53}$$

Let us see now, how to represent step discontinuous functions in $\mathbb{G}^\eta$. Let $\xi \in \mathbb{F}$ be a main generating function for δ. We define the following function:

$$\chi(x) = \int_{-\infty}^{x} \xi(t)dt \tag{54}$$

to be a generating function for $u(x)$, the Heaviside function, where we use a growing index $q = 0$. Moreover, $\chi(\alpha x)$, with $\alpha > 0$, is also a main generating function for u and we will call it the main generating function induced by the generating function $\alpha\xi(\alpha x)$ chosen for δ. Finally, if $f \in C^\infty$ is a function and χ is a main generating function for u, then $f(\chi(x)))$ is a main generating function for $f(u(x))$.

Let $\chi(\alpha x)$ be a generating function for u, if $f \in C^\infty$ then, for the generating function $f(\chi(x))$, there are always $\beta, \gamma \in \mathbb{R}$ such that:

$$[f(\chi(x)) - \beta - \gamma\chi(x)] \in \mathbb{F} \tag{55}$$

the (55), together with the (53), allow us to evaluate $f(u(x))$ in terms of elements of $\mathbb{G}^\eta$ as follows:

$$f(g(x)) = \beta + \gamma u(x) + \sum_{p=0}^{\infty} \frac{a_p(\xi)}{\alpha^{p+1}} \eta^{p,0} \tag{56}$$

For example:

$$u^2(x) = u(x) + \sum_{p=0}^{\infty} \frac{a_p}{\alpha^{p+1}} \eta^{p,0} \tag{57}$$

$$sign^2(x) = (2u(x) - 1)^2 = 1 + \sum_{p=0}^{\infty} \frac{a_p}{\alpha^{p+1}} \eta^{p,0} \tag{58}$$

From the (56) follows that the representation of $f(g(x))$ in $\mathbb{G}^\eta$ is not unique. Since we know that $f(g(x))$ has unique representation in D', from the above result we can say that, although $f(g(x))$ has not unique representation in $\mathbb{G}^\eta$, it can be projected in an unique way in D'.

8 Product of generalised functions in $\mathbb{G}^\eta$.

Let us see now, how to use the theory developed in the previous paragraphs to evaluate the product of steps, deltas and delta derivatives.

We say that $d \in \mathbb{G}^\eta$ is homogeneous if it is composed of generalised functions all of the same growing index. An homogeneous generalised function is always the limit of a generating sequence of the form (35) and, conversely, the limit of a sequence of the form (35) is always homogeneous. Now, given ξ, a main generating function for δ, there is a one to one correspondence between generalised functions in $\mathbb{G}^\eta$ and induced generating sequences (induced by the choice of ξ, see previous paragraph) of the form (35).

Definition 4. *Given ξ, a generating function for δ, and k homogeneous generalised functions $d_i \in \mathbb{G}^\eta$ with generating sequences $n^{q_i} f_i(nx)$ (where the f_i are the generating functions induced by ξ), we define d, the product of the d_i, to be the limit of the product of the generating sequences $n^{q_i} f_i(nx)$:*

$$d = \lim_{n\to\infty} n^{q_1 \cdot \ldots \cdot q_k} f_1(nx) \cdot \ldots \cdot f_k(nx) \tag{59}$$

Commutativity, associativity and applicability of the Leibniz rule in $\mathbb{G}^\eta$, for the product defined above, is ensured by the commutativity, associativity and applicability of the Leibniz rule for the relevant generating sequences.

Unfortunately, the above defined product is ξ dependent. As a matter of fact, if we choose $\alpha\xi(\alpha x)$ as main generating function for delta, by using the (53) we get, as a general result of a product of generalised functions, the following:

$$d = \sum_{p=0}^{\infty} \alpha^{q-p-1} a_p(\xi)\eta^{p,q} \tag{60}$$

The result depends on α and therefore on the main generating function $\alpha\xi(\alpha x)$ chosen for δ.

We define $\mathbb{G}^{\eta/\xi}$ to be the classes of equivalence in $\mathbb{G}^\eta$ of the generalised functions which, when written in the form given by the (60), differ only by a different choice of α and therefore of the main generating function $\alpha\xi(\alpha x)$. In $\mathbb{G}^{\eta/\xi}$, the above defined product, is associative, commutative, compliant with the Leibniz rule and it does not depend from any choice of the generating sequences used. It is therefore a well defined product.

Also, we will call $\mathbb{G}^{\eta^q/\xi} \subset \mathbb{G}^{\eta/\xi}$, the subset of element of $\mathbb{G}^{\eta/\xi}$ with grooving index q. The product of two generalised functions $d_1 \in \mathbb{G}^{\eta^{q_1}/\xi}$ and $d_2 \in \mathbb{G}^{\eta^{q_2}/\xi}$ is always an element of $\mathbb{G}^{\eta^{(q_1+q_2)}/\xi}$

Given any function $f \in C^\infty$ and a step discontinuous function defined as in (3), all elements of $\mathbb{G}^{\eta^1/\xi}$ can be written as:

$$f(g(x))\delta(x) = \sum_{p=0}^{\infty} \frac{a_p}{\alpha^p}(\xi)\eta^{p,1} \tag{61}$$

If f is chosen such that $f(a) = f(b) = 0$, all elements of $\mathbb{G}^{\eta^0/\xi}$ can be written as (compare with the (56) in the previous paragraph):

$$f(g(x)) = \sum_{p=0}^{\infty} \frac{a_p}{\alpha^{p+1}}(\xi)\eta^{p,0} \tag{62}$$

We note that, when we apply the Schwartz functional to an element of $\mathbb{G}^{\eta/\xi}$, to project it in D', all the components $\eta^{p,q}$ with $q < p+1$ get filtered out.

The set $\mathbb{G}^{\eta^0/\xi}$ has no components with $q \geq p+1$ and in $\mathbb{G}^{\eta^1/\xi}$ the only component with $q \geq p+1$ ($\eta^{0,1}(x) = \delta(x)$) has its own coefficient, as shown by the (61), independent from ξ. This is the reason why both $f(g(x))\delta(x) \in \mathbb{G}^{\eta^1/\xi}$ and $(f(g(x)) - \beta - \gamma u(x)) \in \mathbb{G}^{\eta^0/\xi}$ (compare with (56)) are the only elements of $\mathbb{G}^{\eta/\xi}$ that can be always uniquely projected in D' (elements of $\mathbb{G}^{\eta^0/\xi}$ are projected in the 0 function). This is also the reason why, products of a step discontinuous with delta functions, as the ones defined in proposition 1, are the only products among the generalised functions defined in this paper, which can be uniquely projected in D'.

In defining $\mathbb{G}^{\eta/\xi}$, we have done something similar to what is done in Colombeau theory where an algebra of generalised functions is defined by means of a factor space. Both products, the one defined in the Colombeau theory and the one defined in this paper, are products modulo some negligible nets of functions. From this point of view they may be completely equivalent.

By using the above defined product, we can prove interesting equalities involving products among elements of D'. For example, we can remove some ξ dependant higher order terms from the (60) by linearly combining several separate products of generalised functions. Also, we can apply the Leibniz rule (that we know to be applicable) to derive new equalities.

We will show that with an example. Note that, in the following example we will use the notation introduced in (34) and, since we do not have ξ in a closed form, the coefficients of the $\eta^{p,q}$ will be evaluated numerically.
We want to evaluate $u(x)\delta'(x)$:

$$u(x)\delta'(x) \to (\alpha n)^2 \, \chi(\alpha n x)\xi'(\alpha n x) \tag{63}$$

From which we have:

$$u(x)\delta'(x) = \alpha a_0(\xi) n\delta(x) + \frac{1}{2}\,\delta'(x) + \frac{a_2(\xi)}{\alpha}\,\frac{\delta^{(2)}}{n} + R\left(\frac{\delta^{(4)}}{n^3}\right) \tag{64}$$

We want to remove the $n\delta$ term. To do that, we evaluate the product $\delta^2(x)$:

$$\delta^2(x) \to (\alpha n)^2 \, \xi^2(\alpha n x) \tag{65}$$

From which we have:

$$\delta^2(x) = \alpha b_0(\xi) n\delta(x) + \frac{b_2(\xi)}{\alpha}\,\frac{\delta^{(2)}}{n} + \frac{b_4(\xi)}{\alpha^3}\,\frac{\delta^{(4)}}{n^3} + R\left(\frac{\delta^{(6)}}{n^5}\right) \tag{66}$$

Where b_3 and b_5 vanish (evaluated numerically, are smaller, in module, then 10^{-15}). For any ξ, $a_0 = -b_0$ ($a_0 + b_0$, evaluated numerically, is smaller, in module, then 10^{-14}). By substituting the value $n\delta$ from the (66) in the (64), we have eventually (compare with [6]):

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) + R\left(\frac{\delta^{(2)}}{n}\right) \tag{67}$$

or, as an equality among products of elements of D' (i.e. ignoring the higher order terms):

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) \tag{68}$$

We can get to the same results by using the Leibniz rule. We evaluate the product of $u(x)\delta(x)$. We have:

$$u(x)\delta(x) \to \alpha n \, \chi(\alpha n x)\xi(\alpha n x) \tag{69}$$

From which we have:

$$u(x)\delta(x) = \frac{1}{2}\delta(x) + R\left(\frac{\delta'}{n}\right) \tag{70}$$

by taking the derivatives of both sides we have:

$$\delta^2(x) + u(x)\delta'(x) = \frac{1}{2}\delta'(x) + R\left(\frac{\delta^{(2)}}{n}\right) \tag{71}$$

as expected. More examples can be found in the appendix.

9 Metrics for a polyhedron vertex

The product of step and delta functions, developed in paragraphs 2 and 3, may be applied to a number of fields of both physics and mathematics where the product of step discontinuity and Dirac delta function arise naturally from the theory. Among all, we have decided to focus our attention to applications related to differential geometry and, in particular, to the evaluation of the curvature for those varieties, described in the introduction, having step discontinuous metric.

As mentioned in the introduction, this kind of variety may have discrete curvature concentrated on edges and vertices. In both cases, Christoffel symbols, Riemann and Ricci tensors, curvature as well as a number of different differential operators, may only be expressed by means of product of step and delta functions. In this case, the relationship between the structures of the step discontinuities and the delta functions codify the geometrical aspects of the non-differentiable point of the surface and proposition 1 (for edges) and proposition 2 (for vertices) turn up to be very useful in finding an expression for the differential quantity of interest

As an example, in this paragraph we will show a convenient and standard way to define a step discontinuous metric for vertices of polyhedra with 3 or 4 concurrent edges, which are very common in many applications, and in paragraph 10 we will show how to use these metrics to evaluate the curvature of that polyhedron in the vertices. Even thought this paragraph is focused on curvatures, the same method can be applied to evaluate any kind of differential parameters and operators (e.g. Laplace-Beltrami operators).

Before we proceed, we need to introduce a definition. For the purpose of this paper, we will call a 2d-step function any function defined as follows:

$$s(x_1,x_2)=\begin{cases} r_1 & for\, x_1>0, x_2>0\\ r_2 & for\, x_1<0, x_2>0\\ r_3 & for\, x_1<0, x_2<0\\ r_4 & for\, x_1>0, x_2<0 \end{cases} \tag{72}$$

where $r_i \in \mathbb{R}$ and $s(x_1, x_2)$ is not defined on the axis (x_1, x_2). Any function of the kind (72) can always be expressed in the form:

$$s(x_1,x_2)=s_0+s_1(x_1)s_2(x_2) \tag{73}$$

where $s_0 \in \mathbb{R}$ and s_1, s_2 are defined as follows:

$$s_1(x_1)=\begin{cases} a & for\, x_1<0\\ b & for\, x_1>0 \end{cases} \tag{74}$$

$$s_2(x_2)=\begin{cases} c & for\, x_2<0\\ d & for\, x_2>0 \end{cases} \tag{75}$$

and where there is always one degree of freedom in the parameters (s_0, a, b, c, d). Conversely any function of the form (73) is always a 2d-step function.

Now, let V be a vertex of a polyhedron whit 4 edges and angles between edges α, β, γ and θ. Let also S be the surface composed of the vertex, the 4 edges and the relevant 4 faces. We can always open S on a (x_1, x_2) plane by stretching each face by a different amount so that each of the 4 edges lies on one of the semi-axes of the plane. By doing so, we basically map each face of S to a specific sector of the plane (x_1, x_2). It is easy to see that the metric of S is:

$$g_{ij} = \begin{pmatrix} 1 & s(x_1, x_2) \\ s(x_1, x_2) & 1 \end{pmatrix} \tag{76}$$

where $s(x_1, x_2)$ is a 2d-step function for which the amplitude, in each sector of the (x_1, x_2) plane, is a function of one of the angles $\alpha, \beta, \gamma, \theta$ and the parameters (s_0, a, b, c, d) are defined as follows:

$$s(x_1, x_2) = \begin{cases} cos(\alpha) = s_0 + bd & for\; x_1 > 0, x_2 > 0 \\ -cos(\beta) = s_0 + ad & for\; x_1 < 0, x_2 > 0 \\ cos(\gamma) = s_0 + ac & for\; x_1 < 0, x_2 < 0 \\ -cos(\theta) = s_0 + bc & for\; x_1 > 0, x_2 < 0 \end{cases} \tag{77}$$

The (77) define at the same time $s(x_1, x_2)$ and the equation to determine its parameters. The minus signs in the (77) is to take into account that we are in a sector with one of the two dx_i negative and therefore the angle to consider in the metrics is the one between dx_1 and dx_2 positive which is equal to π minus the angle of the relevant polyhedron face for that sector. Since $cos(\pi - x) = -cos(x)$ a minus sign is needed.

As far as vertices with 3 concurrent edges are concerned, we can apply the same procedure by adding a 4th face with angle between edges equal to ϵ and then take the limit for $\epsilon \to 0$. This is equivalent to cut the surface along one of the edges, open the surface on the plane so that each face corresponds to a sector of the axis (x_1, x_2) while the 4th sector remains uncovered and, finally, assign a null metric to that sector (i.e. $s(x_1, x_2) = 1$). This obviously will lead to an infinity inverse metric in the sector. This is not a problem since we are mainly interested in evaluating the curvature in the discontinuity and not the curvature on the surface (which we know to vanish).

An infinity inverse metric will lead to a function $f(x, y)$, of proposition 2 above, which is continuous in $A =]a, b[\times]c, d[$ and that goes to infinity in one of the point of the border of A (the corner related to the null metric). Since proposition 2 works also for this kind of functions, as long as the function is integrable, this is not really an issue.

10 Vertex curvature and defect angle formula

Given the metric of a vertex defined as for the previous paragraph, we will see now how to evaluate its curvature by means of proposition 2. To do that, we will evaluate all the classical differential parameters, and eventually the curvature, as distributions. First of all we evaluate the $g^{i,j}$. From the (76) we have:

$$g^{ij} = \frac{1}{1 - s^2} \begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix} \tag{78}$$

The derivatives of the metric are:

$$\Delta_1 = \frac{\partial g_{12}}{\partial x_1} = \frac{\partial g_{21}}{\partial x_1} = (b - a)\delta(x_1)s_2(x_2) \tag{79}$$

$$\Delta_2 = \frac{\partial g_{12}}{\partial x_2} = \frac{\partial g_{21}}{\partial x_2} = (d - c)s_1(x_1)\delta(x_2) \tag{80}$$

all other derivatives vanish. We proceed by evaluating the Christoffel symbol of the first kind. We have:

$$\Gamma_{112} = \frac{1}{2}(-0 + \Delta_1 + \Delta_1) = (b-a)\delta(x_1)s_2(x_2) \tag{81}$$

$$\Gamma_{221} = \frac{1}{2}(-0 + \Delta_2 + \Delta_2) = (d-c)s_1(x_1)\delta(x_2) \tag{82}$$

all other coefficients of the Christoffel symbol of the first kind vanish. For our purpose we need to evaluate only one of the coefficients of the Christoffel symbol of the second kind:

$$\Gamma_{22}^2 = g^{21}\Gamma_{221} + g^{22}\Gamma_{222} = -\frac{(d-c)s}{1-s^2}s_1(x_1)\delta(x_2) \tag{83}$$

We have now all the elements we need to evaluate the Riemann tensor:

$$R_{1212} = \frac{(b-a)(d-c)}{1-s^2}(1 - s^2 + s\, s_1 s_2)\delta(x_1, x_2) \tag{84}$$

for surfaces and given the Riemann tensor, a classical formula for evaluating the curvature is the following:

$$k = \frac{R_{1212}}{g_{11}g_{22} - g_{12}g_{21}} = \frac{R_{1212}}{1-s^2} \tag{85}$$

as expected the curvature is a Dirac delta function in (0,0). The total curvature can be evaluated by integrating the curvature on S:

$$\begin{aligned} k_T &= \iint_S k\sqrt{1-s^2}dx_1dx_2 = \iint_S R_{1212}\frac{\sqrt{1-s^2}}{1-s^2}dx_1dx_2 \\ &= (b-a)(d-c)\iint_S (1 - s^2 + s\, s_1 s_2)(1-s^2)^{-\frac{3}{2}}\delta(x_1,x_2)dx_1dx_2 \end{aligned} \tag{86}$$

since the integrand is impulsive, it is clear that the total curvature is equal to the amplitude of the impulse, which can be evaluated using proposition 2. We have:

$$s_1(x_1) = x;\ s_2(x_2) = y;\ s(x_1, x_2) = s_0 + xy; \tag{87}$$

by using the (87) in the (20) we get the final expression for the total curvature:

$$k_T = \int_a^b dy \int_c^d (1 - s_0^2 - s_0 xy)\left[1 - s_0^2 - 2s_0 xy - x^2y^2\right]^{-\frac{3}{2}} dx \tag{88}$$

integrating, first with respect of x and then with respect of y, we obtain the primitive $F(x, y)$:

$$F(x,y) = \arctan\left(\frac{s_0 + xy}{\sqrt{1-(s_0+xy)^2}}\right) \tag{89}$$

Let us see how to use the (89) by checking, for example, the value of $F(x, y)$ in (b, d). Given the (77) we have:

$$F(b,d) = \arctan\left(\frac{s_0 + bd}{\sqrt{1-(s_0+bd)^2}}\right) = \arctan\left(\frac{\cos\alpha}{\sin\alpha}\right) = \frac{\pi}{2} - \alpha \tag{90}$$

where we have used the plus sign of the square root. The minus sign corresponds to the case where we swap all the signs in the (77). This is equivalent to choosing a different mapping, between faces and sectors, of the surface on (x_1, x_2). From the (88) we evaluate our final results:

$$k_T = F(b,d) - F(a,d) - F(b,c) + F(a,c) = 2\pi - \alpha - \beta - \gamma - \theta \tag{91}$$

which is, as expected, the defect angle formula. It is remarkable that, by means of proposition 2, we have derived the defect angle formula, in a non-differentiable point, by using the tools of differential geometry.

Taking the limit for one of the angles going to zero, we get the example, mentioned at the end of the previous paragraph, of a null metric and an infinite inverse metric in a sector. As anticipated above, in this case the function $f(x,y)$ of proposition 2 goes to infinity (compare with the integrand of (88) above) in a point of the integration set. However, the function is still integrable as clearly shown by the (89) where the primitive is finite in the same point.

Appendix

A.1 Relationship between the (4) and Colombeau theory

We show now the relationship between the (4) and the Colombeau theory. What follows cannot be taken as a formal proof of the (4) for two main reasons:

- The relation (92) below is not true with equality in the Colombeau algebra, but only in the sense of association.
- It is not possible to find a well defined notion of convergence for the series (93) below.

For simplicity, we will use $g(x) = u(x)$, the Heaviside function, and $f \in C^\infty$.

Colombeau coefficients are defined as follows (see [1] §3.3):

$$u^n(x)\delta(x) = \frac{1}{n+1}\delta(x) \tag{92}$$

we have:

$$f(u(x))\delta(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} u^n(x)\delta(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!(n+1)}\delta(x) \tag{93}$$

where we have used the (92). With the substitution $k = n+1$ we have:

$$f(u(x))\delta(x) = \sum_{k=1}^{\infty} \frac{f^{(k-1)}(0)}{k!}\delta(x) = \sum_{k=1}^{\infty} \frac{F^{(k)}(0)}{k!}\delta(x) \tag{94}$$

where F is the primitive of f. We have eventually:

$$f(u(x))\delta(x) = \left[-F(0) + \sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{k!}(1)^k\right]\delta(x) = [F(1) - F(0)]\delta(x) \tag{95}$$

If the above prof is not formally correct, why does it work? According to the theory defined in this paper, the $u^n(x)$ are elements of $\mathbb{G}^{\eta/\xi}$. The (92) represents the projection of u^n in D'. We can define a notion of convergence in $\mathbb{G}^{\eta/\xi}$ by defining a convergence criteria for the relevant generating functions in $\mathbb{F}$. If the (93) converges in $\mathbb{G}^{\eta/\xi}$, then convergence in D' is automatically ensured.

A.2 Examples of product of distributions

All the a_p coefficients, of the following products of distributions, are evaluated numerically. We will use the notation introduced in (34).

Example 1:

$$\delta(x)\delta'(x) \tag{96}$$

By taking twice the derivative of both sides of the (70), and rearranging the terms we get:

$$\delta(x)\delta'(x) = \frac{1}{6}\delta^{(2)}(x) - \frac{1}{3}u(x)\delta^{(2)}(x) + R\left(\frac{\delta^{(3)}}{n}\right) \tag{97}$$

Example 2: (compare with paragraph 5 above)

$$sign^2(x)\delta(x) \to \alpha n\,(2\chi(\alpha n x) - 1)^2\xi(\alpha n x) \tag{98}$$

from which we have:

$$sign^2(x)\delta(x) = \frac{1}{3}\delta(x) + R\left(\frac{\delta^{(2)}}{n^2}\right) \tag{99}$$

References

[1] J. F. Colombeau. *Multiplication of Distributions.* Springer-Verlag (1992)

[2] A. I. Bobenko, P. Schröder, J. M. Sullivan, G. M. Ziegler. *Discrete Differential Geometry.* Oberwolfach Seminars, Volume 38.

[3] Mark Meyer, Mathieu Desbrun, Peter Schröder, and Alan H. Barr. *Discrete Differential-Geometry Operators for Triangulated 2-Manifolds.* Vis-Math (2002), Berlin.

[4] Jacek Jezierski, Jerzy Kijowski, Ewa Czuchry. *Geometry of null-like surfaces in General Relativity and its application to dynamics of gravitating matter.* Reports on Math. Physics 46 (2000) 399-418.

[5] M. Grosser, M. Kunzinger, M. Oberguggenberger, R. Steinbauer. *Geometric Theory of Genralized Functions with Applications to General Relativity.* Mathematics and its Applications, Vol. 537, Kluwer Academic Publishers, Dordrecht (2001).

[6] B. P. Damyanov. *Multiplication of Schwartz Distributions and Colombeau Generalized Functions.* Journal of Applied Analysis Vol. 5, No. 2 (1999), pp. 249-260.