Product of Distributions Applied to Discrete Differential Geometry

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Abstract

A method for dealing with the product of step discontinuous and delta function is proposed. A new space of generalised function, extending the space D', is constructed. The new space of generalised functions is used to show why it is not possible to define the most general product, among steps, deltas and delta derivatives. The new space of generalized function is used also to prove interesting equalities involving products among elements of D'.

A standard method, for applying the above defined product of distributions to polyhedron vertices, is analysed and the method is applied to a special case where the famous defect angle formula, for the discrete curvature of polyhedra, is derived using the tools of tensor calculus.

Key Words: distribution theory, product of distributions, discrete differential geometry.

1 Introduction

Products of distributions are quite common in several fields of both mathematics and physics. Examples arise naturally in quantum field theory, gravitation and, in partial differential equation theory such as shock wave solutions in hydrodynamics, (see [1]). An important issue, related to product of distributions, is the fact that the product, in the general case, is not associative, issue known as the Schwartz impossibility result (see [1] §1.3) and that only the product between a smooth function and a distribution is well defined.

Discrete differential geometry is a rather new field of mathematics which borrows concepts and ideas from both differential geometry and discrete mathematics. Main applications are concerned with the discrete version of several classical concepts of differential geometry such as discrete curvature, minimal surfaces, geodesics coordinates, minimal paths, surfaces of constant curvature, curvature line parametrisation and the discrete version of continuous functionals (see [2]). At the moment, discrete differential geometry uses many tools of discrete mathematics while the classical tools of differential geometry (e.g. tensor calculus) are difficult to be applied. This leads to an ambiguous definition of

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the various operators (see [3]) which are instead well defined in the continuous counterpart of the theory.

In this paper, we propose a method for evaluating the product of step discontinuous functions and Dirac delta functions, related each other by an integrable function. Moreover, the method is applied to a special class of non differentiable varieties for which, the classical idea of curvature, together with all tools of differential geometry, needs to be redefined in terms of distribution functions. In particular, the class of varieties analysed is the one composed of a collection of several Riemannian varieties glued in such a way the final surface is not differentiable on the resulting edges and vertices. In this case, it is possible to show that vertices and edges carry a concentrated discrete curvature which gives a contribution to the total curvature of the surface, contribution that has to be taken into account in order for the Gauss-Bonnet theorem to work.

For vertices, an important results was already known since the time of Descartes which proved, in the first half of the 17th century, its defect angle theorem for polyhedra. That idea, using the modern concept of curvature and applied to the class of surfaces defined above, can be stated by saying that the discrete total curvature of a vertex is equal to 2π minus the sum of the angles between edges.

For edges, using the Gauss-Bonnet theorem, it is easy to see that the discrete curvature carried by an edge L_{ij} is given by:

$$k_{L_{ij}} = \int_{L_{ij}} (k_{g_i} + k_{g_j}) ds \tag{1}$$

where k_{g_i} and k_{g_j} are the geodesic curvatures, evaluated on the edge L_{ij}, of the two variety S_i and S_j for which L_{ij} is the boundary. If the surface is differentiable on L_{ij}, then k_{g_i} and k_{g_j} are opposite and the integral vanishes. If the surface is not differentiable, the integral (1) gives in general a finite result witch corresponds to the discrete curvature concentrated on L_{ij} and $(k_{g_i} + k_{g_j})$ is the discrete curvature for unit length of the surface on L_{ij}.

This kind of surfaces, characterised by a step discontinuous metric, are typical of problems ranging from theoretical physics up to computer graphics, where the usual way to proceed is to brake down the problem and to define boundary conditions (with conserved quantities) in order to keep the whole problem definition consistent (see [4]) or to use methods of discrete mathematics to define the relevant operators (see [3]). The approach proposed in this paper is to use the tensor calculus where all the derivatives are performed according to the rules of distributions and to use the above mentioned method to evaluate the products of step discontinuities and Dirac delta functions present in the coefficients of the various differential quantities.

2 Product of steps and delta functions

Proposition 1. *Let $g(x)$ be a function discontinuous in 0 and defined as follows:*

$$g(x) = \begin{cases} a & for\, x < 0 \\ b & for\, x > 0 \end{cases} \tag{2}$$

with $a, b \in \mathbb{R}$, and let $f(x)$ be any function integrable in $A \supseteq [a, b]$ (or $[b, a]$ if $b < a$). Also let $(b-a)\delta(x)$ be the derivative of $g(x)$. Then:

$$f(g(x))\delta(x) = \frac{1}{b-a}\left(\int_a^b f(x)dx\right)\delta(x) \tag{3}$$

where the product $(b-a)f(g(x))\delta(x)$ has to be intended as the $\lim_{n\to\infty} f(g_n)g'_n$ for any sequence g_n such that, $\lim_{n\to\infty} g_n = g$ and $\forall n$, the g_n have value in $B \subseteq A$.

Proof.[1] The prove is given for $a < b$ only, changes to the proof, for the case $b < a$, are trivial. First, we write down an useful equality. Let $h(x)$ be a function having the following characteristics:

$$\begin{array}{l} 1)\ h(x)\ is\ continuous\ \forall x \in \mathbb{R} \\ 2)\ \lim_{x\to-\infty} h(x) = a \\ 3)\ \lim_{x\to+\infty} h(x) = b \end{array} \tag{4}$$

we have:

$$\int_{-\infty}^{+\infty} f(h(x))h'(x)dx = \int_{-\infty}^{+\infty} \frac{d}{dx}F(h(x))dx = F(b) - F(a) \tag{5}$$

where $F(x)$ is the primitive of $f(x)$.

Note that $F(h(-\infty)) = F(a)$ and $F(h(+\infty)) = F(b)$ have been used. It is easy to see that the (5) is independent from the function h(x) since it is depending only on $F(x)$, a and b.

Then, let $g_n(x)$ be a sequence of locally integrable functions (inducing regular distributions, the same symbol $g_n(x)$ will be used for both the functions and the induced regular distributions) having the characteristics (4) and having in addition the following characteristics:

$$\begin{array}{l} 1)\ g_n(x)\ is\ monotone\ \forall x \in R \\ 2)\ g_n(x)\ constant \Rightarrow g_n^{'}(x) = 0\ \forall x \notin [-\frac{1}{n}, \frac{1}{n}] \\ 3)\ \lim_{n\to\infty} g_n(x) = g(x) \Rightarrow \lim_{n\to\infty} g'_n(x) = (b-a)\delta(x) \end{array} \tag{6}$$

Let also $\phi(x)$ be a test function. Since $g_n^{'}(x)$ vanishes outside the interval $[-\frac{1}{n}, \frac{1}{n}]$ and taking into account the (5) it is possible to write:

$$\left|\int_{-\infty}^{+\infty} f(g_n(x))g'_n(x)\phi(x)dx - [F(b) - F(a)]\phi(0)\right|$$

$$= \left|\int_{-1/n}^{1/n} f(g_n(x))g'_n(x)[\phi(x) - \phi(0)]dx\right|$$

[1]The proof will be given only for a restricted class of sequences, composed of functions having characteristics (6), and not for any sequence, as stated by the proposition. However, the (6) define quite general and nice functions to be used for constructing sequences which have, as a limit, step functions. For the above reason, the given partial proof does not lead to any limitation for practical applications. The full prof of proposition 1, which is a little more involved but not conceptually difficult, will not be given in this paper.

$$\leq \int_{-1/n}^{1/n} |f(g_n(x))g_n'(x)|\,|\phi(x)-\phi(0)|dx \tag{7}$$

Since ϕ is a test function, it is continuous at $x = 0$. By definition of continuity, given any $\epsilon > 0$, it is possible to find $\delta > 0$ such that whenever $|x| < \delta$, $|\phi(x) - \phi(0)| < \epsilon$. So if we choose any $n > \frac{1}{\delta}$,

$$\int_{-1/n}^{1/n} |f(g_n(x))g_n'(x)|\,|\phi(x)-\phi(0)|dx \leq \epsilon \int_{-1/n}^{1/n} |f(g_n(x))g_n'(x)|dx \tag{8}$$

Given the (6), for which $g_n'(x) \geq 0$, the (5) and writing $f(x) = f(x)_+ - f(x)_-$ as the sum of its positive and negative part (both integrable) we have:

$$\int_{-1/n}^{1/n} |f(g_n(x))g_n'(x)|dx$$

$$= \int_{-1/n}^{1/n} f_+(g_n(x))g_n'(x)dx + \int_{-1/n}^{1/n} f_-(g_n(x))g_n'(x)dx = M > 0 \tag{9}$$

where M is independent from $g_n(x)$. Given the (8) and the (9) we have:

$$\left| \int_{-1/n}^{1/n} f(g_n(x))g_n'(x)\phi(x)dx - [F(b) - F(a)]\phi(0) \right| \leq \epsilon M \tag{10}$$

This proves that:

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} f(g_n(x))g_n'(x)\phi(x)dx = [F(b) - F(a)]\phi(0) \tag{11}$$

Now, if we call $(b - a)f(g(x))\delta(x)$ the limit of the sequence of distributions $f(g_n(x))g_n'(x)$, the (11) proves the following:

- the limit exists
- the limit is a Dirac delta function
- the amplitude of the delta function is given by the (3) □

Proposition 1 can be further generalised as for the following lemma:

Lemma. *Let $s(x)$ be a locally integrable function with a step discontinuity in x_0 with $s(x_0^-) = a$, $s(x_0^+) = b$ where $a, b \in \mathbb{R}$ and let $f(x)$ be any function integrable in $A \supseteq [a, b]$ (or $[b, a]$ if $b < a$). Also let $(b - a)\delta(x - x_0)$ be the derivative of $s(x)$ at x_0. Then:*

$$f(s(x))\delta(x - x_0) = \frac{1}{b-a}\left(\int_a^b f(x)dx\right)\delta(x - x_0) \tag{12}$$

where the product has to be intended as for proposition 1.

Proof. The prove is given for $a < b$ only, changes to the proof, for the case $b < a$, are trivial. Any locally integrable function $s(x)$, with a step discontinuity in x_0, can always be decomposed in the sum $s(x) = \tilde{s}(x)+g(x)$ where $\tilde{s}(x)$ is a function continuous in x_0 and $g(x)$ is defined as follows:

$$g(x) = \begin{cases} a & for\, x < x_0 \\ b & for\, x > x_0 \end{cases} \tag{13}$$

In this case, let $s_n(x) = \tilde{s}_n(x) + g_n(x)$ be any succession of distributions with $g_n(x)$ constant for $x \notin [x_0 - 1/n, x_0 + 1/n]$, $\tilde{s}(x) = 0$ for $x \in [x_0 - 1/n, x_0 + 1/n]$ and with $\lim_{n\to\infty} \tilde{s}_n(x) = \tilde{s}(x)$ and $\lim_{n\to\infty} g_n(x) = g(x)$ (note that the limit of $s_n(x)$, for n that goes to infinity, is $s(x)$). Let also $\phi(x)$ be a test function. We have:

$$\int_{-\infty}^{+\infty} f(\, \tilde{s}_n(x) + g_n(x)\,)[\tilde{s}_n^{'}(x) + g_n'(x)]\phi(x)dx$$

$$= \int_{-\infty}^{+\infty} f(s_n(x))\tilde{s}_n^{'}(x)\phi(x)dx + \int_{x_0-1/n}^{x_0+1/n} f(g_n(x))g_n^{'}(x)\phi(x)dx \tag{14}$$

Now, if we take the limit for n that goes to infinity, clearly the first term of the (14) is the regular distribution induced by $f(s(x))\tilde{s}^{'}(x)$ (note that $\tilde{s}_n^{'}(x) = s_n^{'}(x)$ for $x \neq x_0$). Moreover we already know how to treat the second term, which is simply the (11) shifted to a coordinate x_0. We have therefore:

$$f(s(x))s^{'}(x - x_0) = f(s(x))\tilde{s}^{'}(x) + (b - a)f(g(x))\delta(x - x_0) \tag{15}$$

Finally, from the (15) we can easily prove the lemma. □

Note that, even in the case where $F(a) = F(b)$ and therefore there is no step in the discontinuity, proposition 1 and its lemma are essential to evaluate the product of the discontinuity with a related delta function. For example, is easy to show that $sign^2(x)\delta(x) = \frac{1}{3}\delta(x)$.

3 The multidimensional case

Proposition 2. *Let $g_1(x)$ and $g_2(y)$ be two functions discontinuous in 0 and defined as follows:*

$$g_1(x) = \begin{cases} a & for\, x < 0 \\ b & for\, x > 0 \end{cases} \tag{16}$$

$$g_2(y) = \begin{cases} c & for\, y < 0 \\ d & for\, y > 0 \end{cases} \tag{17}$$

with $a, b, c, d \in \mathbb{R}$ and let $f(x, y)$ be any function integrable in $A \supseteq [a, b] \times [c, d]$ (if $b < a$ and/or $d < c$ the definition of A has to be changed accordingly). Also let $(b - a)(c - d)\delta(x, y)$ be the product of the derivatives of $g_1(x)$ and $g_2(y)$. Then:

$$f(g_1(x), g_2(y))\delta(x, y) = \frac{1}{(b - a)(d - c)} \left(\int_c^d dy \int_a^b f(x, y)dx \right) \delta(x, y) \tag{18}$$

where the product $(b - a)(c - d)f(g_1(x), g_2(y))\delta(x, y)$ has to be intended as the $\lim_{n\to\infty} f(g_{1n}, g_{2n})g_{1n}'g_{2n}'$ *for any pair of sequences g_{1n},g_{2n} such that,* $\lim_{n\to\infty} g_{1n} =$

g_1, $\lim_{n\to\infty} g_{2n} = g_2$ *and* $\forall n$, g_{1n} *has value in* B_1, g_{2n} *has value in* B_2 *and* $B_1 \times B_2 \subseteq A$.

Obviously, we can interchange the roles of x and y since we may integrate first with respect of y and then with respect of x. Note that the discontinuity $f(g_1(x), g_2(y))$ addressed by this proposition is not the most general step discontinuity we may have in two dimensions.

As for proposition 1, in order to prove the above proposition, we first need to prove an useful equality. Let $h_1(x), h_2(y)$ be two functions which have the following characteristics:

$$\begin{array}{l} 1)\ h_1(x), h_2(y)\ \textit{are continuous}\ \forall x, y \in \mathbb{R} \\ 2)\ \lim_{x\to-\infty} h_1(x) = a;\ \lim_{x\to+\infty} h_1(x) = b \\ 3)\ \lim_{y\to-\infty} h_2(y) = c;\ \lim_{y\to+\infty} h_2(y) = d \end{array} \tag{19}$$

and let F(x,y) be a function such that $F_{xy} = F_{yx} = f(x, y)$. We have:

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} f(h_1(x), h_2(y)) h_1^{'}(x) h_2^{'}(y) dx$$
$$= \int_{-\infty}^{+\infty} dy \frac{\partial}{\partial y} \int_{-\infty}^{+\infty} \frac{\partial}{\partial x} F(h_1(x), h_2(y)) dx \tag{20}$$

where, to prove the (20), we have taken the symbol $\frac{\partial}{\partial y}$ inside the integral (for the linearity of integrals) and applied the definition of $F(x, y)$. It is easy to see that the (20) is independent from the $h_{1,2}$ and is equal to $F(b, d) - F(a, d) - F(b, c) + F(a, c)$. From this point on, it is possible to prove proposition 2 by following similar steps to the ones used for proving proposition 1. As we did for the monodimensional case (se lemma of proposition 1), a generalization to locally integrable functions, may also be given in this case.

Note that proposition 2 gives a clear path on the possible way to generalise the idea of products of step discontinuities and delta functions to the case with as many dimensions as we like.

4 Further discussions on product of distribution

So far, we have been mainly interested in step discontinuities and Dirac delta functions. Each discontinuity of this kind can be defined by means of the limit of an infinite number of successions of distributions, all different each other and having, in a distributional sense, the same limit in D'. For the purpose of this paper, we define the structure of such discontinuities to be the specific succession we use to define them. In general, since we want a distribution to be as generic as possible, we never define its own structure and we leave it indeterminate. However, the concept of structure of a discontinuity is essential to this paper as it will be clear shortly.

Moreover, for products of distributions, every time we define the product in a point x_0, where the distributions are discontinuous, we always want the discontinuities to have each other structure related by a well known law so that, if the structure of one distribution in x_0, which is unknown to us, changes, the structure of all other distributions in the same point will change accordingly.

Since we never want to define the structures of the distributions, we are mostly interested in products of distributions, like the ones of proposition 1 and 2, which work regardless their underlying structures. This is why in proposition 1 and 2 we define the product as the limit of any possible succession (i.e. structure) and we want this limit to be independent from it.

The idea that a particular distribution may have an infinite number of different structures is very similar to the notion of associated distributions present in the Colombeau theory (see [1] §3.2), where the product make sense if it is independent from the particular representative of the involved generalised functions (see [1] §3.1).

As a final remark, note that proposition 1 and 2 is valid for f integrable in, at least, respectively $A = [a, b]$ and $A = [a, b] \times [c, d]$. An important example, where we use integrable functions f, which go to infinity in a point of the integration set, is shown in paragraph 9 and 10 of this paper.

5 The need for new generalised functions.

So far, we have focused our attention only on the structure of step discontinuities and the way they are modified (by composition with an integrable function f). When it comes to Dirac delta functions, it is possible to show that they change their own structure by means of multiplication by step discontinuous functions. Let us consider the function $f(g(x))$ where g is a step discontinuous function defined as in (2) and f is integrable in $[a, b]$. We have:

$$Df(g(x)) = (b-a)f'(g(x))\delta(x) \tag{21}$$

from which we see that by multiplying a delta function having structure $g(x)$ (i.e. derivative of a step discontinuous function $g(x)$) by $f'(g(x))$ we get a delta function with structure $f(g(x))$.

We have seen that, in a product of distributions, if we change the structure of a term we get a different result. In order to overcame this limitation, we want now to extend the space of distributions D' by adding to it, as separate generalised functions, additional elements representing any possible discontinuity structure needed for describing products of step and delta functions.

We will assume now that all step discontinuous and delta functions, we are dealing with, are all related to the same Heaviside function and their structure can be described by the way they are related to it. From this new point of view, the function f, which before was used to relate distribution structures, became now the structure itself of the distribution. We will say that a step discontinuity has structure f if it is of the form $f(u(x))$. We will say that a delta function has structure f if it is the derivative of a step discontinuous function of structure f. We will consider steps and delta functions, with different structures, separate generalised functions.

We will use the following notation:

$$\begin{array}{lll} u_{[f(x)]} = f(u(x)) & \textit{step function having structure } f \\ \delta_{[f'(x)]} = f'(u(x))\delta(x) & \textit{delta function having structure } f \end{array} \tag{22}$$

where $u_{[f(x)]}$ and $\delta_{[f(x)]}$ are not normalised (i.e they may have amplitude different from 1) and $u_{[x]} = u(x) \in D'$, $\delta_{[1]} = \delta(x) \in D'$. We will show, with an

example at the end of the paragraph, that the above defined generalised functions have components outside D' and therefore there is a need for defining a larger space of generalised functions including D'. We will do that in the next paragraphs. Note that by defining new generalised functions by means of the (22), we have done something analogous to the Colombeau theory where an extended space is build adding to D', among others, the generalised functions $u^n(x)$ as separate functions. The new thing here is that we explicitly add to D' also the generalised functions $\delta_{[f'(x)]}$.
Using the (22), we define the multiplication as follows:

$$u_{[f_1]}u_{[f_2]} \cdot \ \ldots \ \cdot u_{[f_n]}\delta_{[f_{n+1}]} = \delta_{[f_1 f_2 \cdot \ \ldots \ \cdot f_n f_{n+1}]} \tag{23}$$

Finally we define a projector operator P_0, which project any generalised function of the kind (22), on the space D'. For step discontinuous functions the way P_0 works is trivial (e.g. $u^2(x)$ goes to $u(x)$). For delta functions, we apply the theory developed in paragraph 2 and, by using proposition 1, we have:

$$P_0\left(\delta_{[f_1 f_2 \cdot \ \ldots \ \cdot f_n f_{n+1}]}\right) = \left(\int_0^1 f_1 f_2 \cdot \ \ldots \ \cdot f_n f_{n+1} dx\right)\delta(x) \in D' \tag{24}$$

where the integration is performed between 0 and 1, the values between which $u(x)$, our reference step discontinuity, jumps. Note that the (23) and (24) provide a well defined product of the (22) which is fully coherent with the theory developed in the previous paragraphs. The product is also commutative and associative since commutative and associative is the product of the f_i functions used in the definition of the (23).

Let us make an example. Consider the product of distributions $sign^2(x)\delta(x)$ (compare with [5] §1.1 ex. iii). By using proposition 1 we find easily that:

$$sign^2(x)\delta(x) = \frac{1}{3}\delta(x) \tag{25}$$

Let us check associativity by using, once again, proposition 1:

$$sign^2(x)\delta(x) = sign(x)[sign(x)\delta(x)] = sign(x) \cdot 0 = 0 \tag{26}$$

we conclude that, in D', our product is not associative. Let us see what happen using the (23):

$$sign(x)[sign(x)\delta(x)] = sign(x)\delta_{[(2x-1)\cdot 1]} = sign(x)[\delta_{[2x]} - \delta_{[1]}] \tag{27}$$

In D', $\delta_{[1]} = \delta$ and $P_0(\delta_{[2x]}) = \delta$. However, as generalised function of the kind (22), they are separate objects and they do not cancel each other. We have eventually:

$$sign^2(x)\delta(x) = P_0\left(\delta_{[(2x-1)^2]}\right) = \frac{1}{3}\delta(x) \tag{28}$$

6 New generalised functions

Definition 1. *We define the generalised function $\eta^{p,q}$ to be the limit of the following succession of distributions:*

$$\eta^{p,q}(x) = \lim_{n\to\infty} n^{q-1}\sum_{k=0}^{p}(-1)^k\binom{p}{k}\delta(x-\frac{k}{n}) \ \ with\, p,q \geq 0 \tag{29}$$

It is easy to see that:

$$\eta^{p,p+1}(x) = \delta^{(p)}(x) \tag{30}$$

What kind of generalised function is $\eta^{p,q}$? If the succession of functions f_n converges to $\delta^{(p)}$, then $\frac{f_n}{n^{p-q+1}}$ converges to $\eta^{p,q}$. So, with an abuse of notation, we may say that:

$$\eta^{p,q} = \frac{\delta^{(p)}}{n^{p-q+1}} \tag{31}$$

The $\eta^{p,q}$ are therefore the limit of successions of functions that are shaped like $\delta^{(p)}$ and that, when we take the limit, grow at a lower or faster rate (according to the sign of p-q+1).

The $\eta^{p,q}$ can be defined by means of the limit of a succession of functions $f_n(x)$. In this paper we will deal only with generalised functions defined by means of the limit of a succession of the form:

$$\lim_{n\to\infty} n^q f(nx) \tag{32}$$

Note that the above successions are not the most general way to define distributions. For example, there is not sequence of the form (32) converging to $\delta + \delta'$. We will call $f(x)$ the generating function, $n^q f(nx)$ the generating sequence and q the growing rate of the generalised function defined by the (32).

Next, we define $\mathbb{F}$ to be the set of all the function $f(x)$ having the following characteristics.

$$\begin{array}{l} 1)\ f(x) \in C^\infty \\ 2)\ \lim_{x\to-\infty} f(x)x^k = 0\ \ for\ any\ k \in \mathbb{N} \\ 3)\ \lim_{x\to+\infty} f(x)x^k = 0\ \ for\ any\ k \in \mathbb{N} \end{array} \tag{33}$$

Now, let us see how to determine all the $\eta^{p,q}$ components of a generalised function defined by means of the (32) and having generating function $f(x) \in \mathbb{F}$. First of all, we note that all the components of the distribution (32) have the same growing rate q and therefore are of the form $\eta^{p,q}$. We have:

$$d = \lim_{n\to\infty} n^q f(nx) = \sum_{p=0}^{\infty} a_p \eta^{p,q} \tag{34}$$

which contains one distribution $\eta^{q-1,q}(x) = \delta^{(q-1)}(x) \in D'$.

For the distribution defined by the (34) we can determine the a_p by using the Schwartz theory of distribution. Let ϕ be a test function, we have (see appendix):

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} n^q f(nx)\phi(x)dx = \sum_{p=0}^{\infty} (-1)^p n^{q-p-1} a_p \phi^{(p)}(0) \tag{35}$$

To better evaluate all a_p we decide to use a test function ϕ that has all derivatives $\phi^{(i)}(0) = 0$ for $i \neq p$. A test function with this characteristic is $\phi(x) = x^p$. Of course a test function should vanish outside a compact interval and x^p does not. However, since $f(x)$ goes to 0 for $|x|$ going to infinity, this is not a problem. We have:

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} n^q f(nx)x^p dx = (-1)^p n^{q-p-1}\, a_p\, p! \tag{36}$$

where $p!$ is the value of the p^{th} derivatives of x^p. From the (36) we can easily evaluate the a_p as follows:

$$a_p = \lim_{n\to\infty} \frac{(-1)^p\, n^{p-q+1}}{p!} \int_{-\infty}^{+\infty} n^q f(nx) x^p dx =$$

$$= \lim_{n\to\infty} \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} n\, f(nx)(nx)^p dx \tag{37}$$

We note that the right part of the (37), for n that goes to infinity, in the (x, y) plane, shrinks (along x) and grows (along y) like n, which leaves the integral unchanged. For the above reason, the limit of the (37) is simply the value of the integrals for any n. We may as well evaluate it for n=1. We have:

$$a_p = \frac{(-1)^p}{p!} \int_{-\infty}^{+\infty} f(x) x^p dx \tag{38}$$

We are now ready to define our new space of generalised functions. We define, for now, $\mathbb{G}^\eta$ to be the space of generalised function composed of D' plus all generalised functions which are the limit of succession of the kind (32) with $q \geq 1$ and $f \in \mathbb{F}$. We will add more elements to $\mathbb{G}^\eta$ in the following paragraph. We define also $\mathbb{A}$ to be the set of all sequences of coefficient $a_f = (a_0, a_1, \ldots)$ associated by the (38) to the generating function f.

Let us see an example. If we choose $f(x)$ be a Gaussian distribution as follows:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{39}$$

and we choose q=1, from the (38) we find:

$$\lim_{n\to\infty} n\, f(nx) = d \in \mathbb{G}^\eta = \delta(x) + \frac{1}{2}\eta^{2,1} + \frac{1}{8}\eta^{4,1} + R\left(\eta^{6,1}\right) \tag{40}$$

where $R(\eta^{6,1})$ means that, to have the above equality exact, you need to add components of growing rate 1 and order ≥ 6.

7 Main generating functions

In the paragraphs above, we have defined the concept of structure of a discontinuity. We note that, for a generalised function $d \in \mathbb{G}^\eta$, if q is the growing rate, $f \in \mathbb{F}$ is the generating functions and $a_f \in \mathbb{A}$ are the coefficients of the $\eta^{p,q}$, we can fully characterize the structure of a discontinuity (i.e. fully define the relevant generalised function) by providing either couples (f, q) or (a_f, q). Moreover, if $a_f = (a_0, a_1, \ldots)$, then $a_{f'} = (0, a_0, a_1, \ldots)$ and therefore, in $\mathbb{G}^\eta$, the derivative of $d = (f, q)$ is $d' = (f', q + 1)$.

Let $f(x) \in \mathbb{F}$ be a generating function for δ in D' (trough the generating sequence $n\, f(nx)$). If we evaluate the coefficients $a_f \in \mathbb{A}$, we know that $a_1 = 1$. We also know that all the others coefficient can have any value (see example in the previous paragraph). We are interested , among all the $f \in \mathbb{F}$, to the ones for which a_f is of the form $a_0 = 1$ and $a_p = 0$ for $p > 1$.
We give the following definitions:

Definition 2. *Let $\xi(x) \in \mathbb{F}$. If $\xi(x)$ verifies the following equations:*

$$\int_{-\infty}^{+\infty} \xi(x) x^p dx = \begin{cases} 1 & for\ p = 0 \\ 0 & for\ p > 0 \end{cases} \tag{41}$$

then we call ξ a main generating function for $\delta(x)$.

We have:

$$\lim_{n \to \infty} n^{p+1} \xi^{(p)} = \delta^{(p)}(x) \ \ in\ \mathbb{G}^\eta \tag{42}$$

The (42) states that, if we use main generating functions, we can define delta and delta derivatives that have no components outside D'. In a few word, if we accept generalised function $\eta^{p,q}$ to be real things (i.e. we work in $\mathbb{G}^\eta$), we have also to accept that only sequences $n\xi(nx)$ composed of main generating functions converge to δ.

The following figure is a plot of a $\xi(x)$ evaluated numerically:

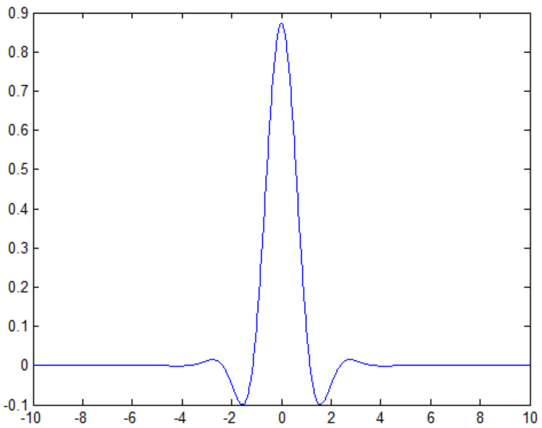

Figure 1: ξ function

Let $\xi \in \mathbb{F}$ be a generating function for δ. Then we define the following function:

$$\chi(x) = \int_{-\infty}^{x} \xi(t) dt \tag{43}$$

to be a generating function for $u(x)$, the Heaviside function where we use a growing rate $q = 0$.

Now we are ready to further enlarge $\mathbb{G}^\eta$. Given $g \in C^{(0)}$, and χ a generating functions for $u(x)$, we add to $\mathbb{G}^\eta$ all the generalised functions μ^g defined as follows.

$$\mu^g = \lim_{n \to \infty} g(\chi(nx)) \tag{44}$$

This completes our definition of $\mathbb{G}^\eta$

Given $f \in \mathbb{F}$, there is only one element $a_f \in \mathbb{A}$. On the contrary, given $a_f \in \mathbb{A}$, there exist at least one generating function $g \in \mathbb{F}$, with $f \neq g$, such that $a_f = a_g$. In particular if $\xi \in \mathbb{F}$ is a main generating function for δ, then $\alpha\xi(\alpha x)$

with $\alpha \neq 0$ is also a main generating function for δ. Moreover if $\xi^{(p)} \in \mathbb{F}$ is a main generating function for $\delta^{(p)}$ then $\alpha^{p+1}\xi^{(p)}(\alpha x)$ with $\alpha \neq 0$ is also a main generating function for $\delta^{(p)}$.

8 Product of generalised functions in $\mathbb{G}^\eta$.

Let us see now, how to use the theory developed in the previous paragraphs to evaluate the product of steps, deltas and delta derivatives.

We say that $d \in \mathbb{G}^\eta$ is homogeneous if it is composed of generalised functions all of the same growing rate. An homogeneous generalised function is always the limit of a generating sequence of the kind (32) and, conversely, the limit of a succession of the kind (32) is always homogeneous. Now, given ξ, a main generating function for δ, there is a one to one correspondence between generalised functions in $\mathbb{G}^\eta$ and generating sequences of the kind (32).

Given n homogeneous generalised functions in $\mathbb{G}\eta$, we define the generating sequence of their product to be the product of their generating sequences.

If the resulting generating sequence has growing rate $q = 0$, then we are facing a trivial case (i.e. no deltas and delta derivatives in the product). If the resulting sequence has growing rate $q > 0$ then we can use the (38) to evaluate the relevant $\eta^{p,q}$ components.

Commutativity, associativity and applicability of the Leibniz rule in $\mathbb{G}^\eta$, for the product defined in this paragraph, is ensured by the commutativity, associativity and applicability of the Leibniz rule for the relevant generating sequences.

Unfortunately, the above defined product is ξ dependent. As a matter of fact, if we choose $\alpha\xi(\alpha x)$ as main generating function for delta, then we get:

$$d = \sum_{p=0}^{\infty} (\alpha n)^{q-p-1} a_p(\xi) \eta^{p,q} \tag{45}$$

as a general result of a product of generalised functions. The result depends on α and therefore on the main generating function chosen for δ. However, the product we have just defined is not completely useless for the following reasons:

- The above developed theory shows us why the most general product of distributions in D' is not well defined (i.e. it is ξ dependant in $\mathbb{G}^\eta$).
- By using the above defined product, we can prove interesting equalities involving products among elements of D'. For example, we can remove some ξ dependant higher order terms from the (45) by linearly combining several separate products of generalised functions. Also, we can apply the Leibniz rule (that we know to be applicable) to derive new equalities.

We will show that last point with an example. Note that, in the following example we will use the notation introduced in (31) and, since we do not have ξ in a closed form, the coefficients of the $\eta^{p,q}$ will be evaluated numerically.
We want to evaluate $u(x)\delta'(x)$:

$$u(x)\delta'(x) \to (\alpha n)^2 \, \chi(\alpha n x)\xi'(\alpha n x) \tag{46}$$

From which we have:

$$u(x)\delta'(x) = \alpha a_0(\xi) n\delta(x) + \frac{1}{2}\,\delta'(x) + \frac{a_2(\xi)}{\alpha}\,\frac{\delta^{(2)}}{n} + R\left(\frac{\delta^{(4)}}{n^3}\right) \tag{47}$$

We want to remove the $n\delta$ term. To do that, we evaluate the product $\delta^2(x)$:

$$\delta^2(x) \rightarrow (\alpha n)^2\,\xi^2(\alpha n x) \tag{48}$$

From which we have:

$$\delta^2(x) = \alpha b_0(\xi) n\delta(x) + \frac{b_2(\xi)}{\alpha}\,\frac{\delta^{(2)}}{n} + \frac{b_4(\xi)}{\alpha^3}\,\frac{\delta^{(4)}}{n^3} + R\left(\frac{\delta^{(6)}}{n^5}\right) \tag{49}$$

Where b_3 and b_5, evaluated numerically, are smaller, in module, then 10^{-15}. For any ξ, $a_0 = -b_0$, since $a_0 + b_0$, evaluated numerically is smaller, in module, then 10^{-14}. By substituting the value $n\delta$ from the (49) in the (47), we have eventually (compare with [6]):

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) + R\left(\frac{\delta^{(2)}}{n}\right) \tag{50}$$

or, as an equality among products of elements of D' (i.e. ignoring the higher order terms):

$$u(x)\delta'(x) = -\delta^2(x) + \frac{1}{2}\,\delta'(x) \tag{51}$$

We can get to the same results by using the Leibniz rule. We evaluate the product of $u(x)\delta(x)$. We have:

$$u(x)\delta(x) \rightarrow \alpha n\;\chi(\alpha n x)\xi(\alpha n x) \tag{52}$$

From which we have:

$$u(x)\delta(x) = \frac{1}{2}\delta(x) + R\left(\frac{\delta'}{n}\right) \tag{53}$$

by taking the derivatives of both sides we have:

$$\delta^2(x) + u(x)\delta'(x) = \frac{1}{2}\delta'(x) + R\left(\frac{\delta^{(2)}}{n}\right) \tag{54}$$

as expected. More examples can be found in the appendix.

9 Metrics for a polyhedron vertex

The product of of step and delta functions, developed in paragraphs 2 and 3, may be applied to a number of fields of both physics and mathematics where the product of step discontinuity and Dirac delta function arise naturally from the theory. Among all, we have decided to focus our attention to applications related to differential geometry and, in particular, to the evaluation of the curvature for those varieties, described in the introduction, having step discontinuous metric.

As mentioned in the introduction, this kind of variety may have discrete curvature concentrated on edges and vertices. In both cases, Christoffel symbols, Riemann and Ricci tensors, curvature as well as a number of different differential operators, may only be expressed by means of product of step and delta functions. In this case, the relationship between the structures of the step discontinuities and the delta functions codify the geometrical aspects of the non-differentiable point of the surface and proposition 1 (for edges) and proposition 2 (for vertices) turn up to be very useful in finding an expression for the differential quantity of interest

As an example, in this paragraph we will show a convenient and standard way to define a step discontinuous metric for vertices of polyhedra with 3 or 4 concurrent edges, which are very common in many applications, and in paragraph 10 we will show how to use these metrics to evaluate the curvature of that polyhedron in the vertices. Even thought this paragraph is focused on curvatures, the same method can be applied to evaluate any kind of differential parameters and operators (e.g. Laplace-Beltrami operators).

Before we proceed, we need to introduce a definition. For the purpose of this paper, we will call a 2d-step function any function defined as follows:

$$s(x_1,x_2) = \begin{cases} r_1 & for\, x_1 > 0, x_2 > 0 \\ r_2 & for\, x_1 < 0, x_2 > 0 \\ r_3 & for\, x_1 < 0, x_2 < 0 \\ r_4 & for\, x_1 > 0, x_2 < 0 \end{cases} \tag{55}$$

where $r_i \in \mathbb{R}$ and $s(x_1, x_2)$ is not defined on the axis (x_1, x_2). Any function of the kind (55) can always be expressed in the form:

$$s(x_1, x_2) = s_0 + s_1(x_1)s_2(x_2) \tag{56}$$

where $s_0 \in \mathbb{R}$ and s_1, s_2 are defined as follows:

$$s_1(x) = \begin{cases} a & for\, x_1 < 0 \\ b & for\, x_1 > 0 \end{cases} \tag{57}$$

$$s_2(y) = \begin{cases} c & for\, x_2 < 0 \\ d & for\, x_2 > 0 \end{cases} \tag{58}$$

and where there is always one degree of freedom in the parameters (s_0, a, b, c, d). Conversely any function of the form (56) is always a 2d-step function.

Now, let V be a vertex of a polyhedron whit 4 edges and angles between edges α, β, γ and η. Let also S be the surface composed of the vertex, the 4 edges and the relevant 4 faces. We can always open S on a (x_1, x_2) plane by stretching each face by a different amount so that each of the 4 edges lies on one of the semi-axes of the plane. By doing so, we basically map each face of S to a specific sector of the plane (x_1, x_2). It is easy to see that the metric of S is:

$$g_{ij} = \begin{pmatrix} 1 & s(x_1, x_2) \\ s(x_1, x_2) & 1 \end{pmatrix} \tag{59}$$

where $s(x_1, x_2)$ is a 2d-step function for which the amplitude, in each sector of the (x_1, x_2) plane, is a function of one of the angles $\alpha, \beta, \gamma, \eta$ and the parameters

(s_1, a, b, c, d) are defined as follows:

$$s(x_1, x_2) = \begin{cases} cos(\alpha) = s_0 + bd & for\ x_1 > 0, x_2 > 0 \\ -cos(\beta) = s_0 + ad & for\ x_1 < 0, x_2 > 0 \\ cos(\gamma) = s_0 + ac & for\ x_1 < 0, x_2 < 0 \\ -cos(\eta) = s_0 + bc & for\ x_1 > 0, x_2 < 0 \end{cases} \tag{60}$$

The (60) define at the same time $s(x_1, x_2)$ and the equation to determine its parameters. The minus signs in the (60) is to take into account that we are in a sector with one of the two dx_i negative and therefore the angle to consider in the metrics is the one between dx_1 and dx_2 positive which is equal to π minus the angle of the relevant polyhedron face for that sector. Since $cos(\pi - x) = -cos(x)$ a minus sign is needed.

As far as vertices with 3 concurrent edges are concerned, we can apply the same procedure by adding a 4th face with angle between edges equal to ϵ and then take the limit for $\epsilon \to 0$. This is equivalent to cut the surface along one of the edges, open the surface on the plane so that each face corresponds to a sector of the axis (x_1, x_2) while the 4th sector remains uncovered and, finally, assign a null metric to that sector (i.e. $s(x_1, x_2) = 1$). This obviously will lead to an infinity inverse metric in the sector. This is not a problem since we are mainly interested in evaluating the curvature in the discontinuity and not the curvature on the surface (which we know to vanish).

An infinity inverse metric will lead to a function $f(x, y)$, of proposition 2 above, which is continuous in $A =]a, b[\times]c, d[$ and that goes to infinity in one of the point of the border of A (the one related to the null metric). Since proposition 2 works also for function which have discontinuities where $f(x, y)$ goes to infinity in A, as long as the function is integrable in the same set, this is not really an issue.

10 Vertex curvature and defect angle formula

Given the metric of a vertex defined as for the previous paragraph, we will see now how to evaluate its curvature by means of proposition 2. To do that, we will evaluate all the classical differential parameters, and eventually the curvature, as distributions. First of all we evaluate the $g^{i,j}$. From the (59) we have:

$$g^{ij} = \frac{1}{1 - s^2} \begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix} \tag{61}$$

The derivatives of the metric are:

$$\Delta_1 = \frac{\partial g_{12}}{\partial x_1} = \frac{\partial g_{21}}{\partial x_1} = (b - a)\delta(x_1)s_2(x_2) \tag{62}$$

$$\Delta_2 = \frac{\partial g_{12}}{\partial x_2} = \frac{\partial g_{21}}{\partial x_2} = (d - c)s_1(x_1)\delta(x_2) \tag{63}$$

all other derivatives vanish. We proceed by evaluating the Christoffel symbol of the first kind. We have:

$$\Gamma_{112} = \frac{1}{2}(-0 + \Delta_1 + \Delta_1) = (b - a)\delta(x_1)s_2(x_2) \tag{64}$$

$$\Gamma_{221} = \frac{1}{2}(-0 + \Delta_2 + \Delta_2) = (d-c)s_1(x_1)\delta(x_2) \tag{65}$$

all other coefficients of the Christoffel symbol of the first kind vanish. For our purpose we need to evaluate only one of the coefficients of the Christoffel symbol of the second kind:

$$\Gamma_{22}^{2} = g^{21}\Gamma_{221} + g^{22}\Gamma_{222} = -\frac{(d-c)s}{1-s^2}s_1(x_1)\delta(x_2) \tag{66}$$

We have now all the elements we need to evaluate the Riemann tensor:

$$R_{1212} = \frac{(b-a)(d-c)}{1-s^2}(1-s^2+s\,s_1s_2)\delta(x_1,x_2) \tag{67}$$

for surfaces and given the Riemann tensor, a classical formula for evaluating the curvature is the following:

$$k = \frac{R_{1212}}{g_{11}g_{22}-g_{12}g_{21}} = \frac{R_{1212}}{1-s^2} \tag{68}$$

as expected the curvature is a Dirac delta function in (0,0). The total curvature can be evaluated by integrating the curvature on S:

$$\begin{aligned} k_T &= \iint_S k\sqrt{1-s^2}dx_1dx_2 = \iint_S R_{1212}\frac{\sqrt{1-s^2}}{1-s^2}dx_1dx_2 \\ &= (b-a)(d-c)\iint_S (1-s^2+s\,s_1s_2)(1-s^2)^{-\frac{3}{2}}\delta(x_1,x_2)dx_1dx_2 \end{aligned} \tag{69}$$

since the integrand is impulsive, it is clear that the total curvature is equal to the amplitude of the impulse, which can be evaluated using proposition 2. We have:

$$s_1(x_1) = x;\ s_2(x_2) = y;\ s(x_1,x_2) = s_0 + xy; \tag{70}$$

by using the (70) in the (18) we get the final expression for the total curvature:

$$k_T = \int_a^b dy \int_c^d (1-s_0^2-s_0xy)\left[1-s_0^2-2s_0xy-x^2y^2\right]^{-\frac{3}{2}}dx \tag{71}$$

integrating, first with respect of x and then with respect of y, we obtain the primitive $F(x,y)$:

$$F(x,y) = \arctan\left(\frac{s_0+xy}{\sqrt{1-(s_0+xy)^2}}\right) \tag{72}$$

Let us see how to use the (72) by checking, for example, the value of $F(x,y)$ in (b,d). Given the (60) we have:

$$F(b,d) = \arctan\left(\frac{s_0+bd}{\sqrt{1-(s_0+bd)^2}}\right) = \arctan\left(\frac{\cos\alpha}{\sin\alpha}\right) = \frac{\pi}{2} - \alpha \tag{73}$$

where we have used the plus sign of the square root. The minus sign corresponds to the case where we swap all the signs in the (60). This is equivalent to choosing a different mapping, between faces and sectors, of the surface on (x_1, x_2). From the (71) we evaluate our final results:

$$k_T = F(b,d) - F(a,d) - F(b,c) + F(a,c) = 2\pi - \alpha - \beta - \gamma - \eta \tag{74}$$

which is, as expected, the defect angle formula. It is remarkable that, by means of proposition 2, we have derived the defect angle formula, in a non-differentiable point, by using the tools of differential geometry.

Taking the limit for one of the angles going to zero, we get the example, mentioned at the end of the previous paragraph, of a null metric and an infinite inverse metric in a sector. As anticipated above, in this case the function $f(x, y)$ of proposition 2 goes to infinity (compare with the integrand of (71) above) in a point of the integration set. However, the function is still integrable as clearly shown by the (72) where the primitive is finite in the same point.

Appendix

A.1 Proof of the (3) using Colombeau coefficients

We will prove the (3) by using the Colombeau coefficients. For simplicity, the proof will be given for $g(x) = u(x)$, the Heaviside function, and for $f \in C^\infty$.

Proof. Colombeau coefficients are defined as follows (see [1] §3.3):

$$u^n(x)\delta(x) = \frac{1}{n+1}\delta(x) \tag{75}$$

we have:

$$f(u(x))\delta(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} u^n(x)\delta(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!(n+1)}\delta(x) \tag{76}$$

where we have used the (75). With the substitution $k = n + 1$ we have:

$$f(u(x))\delta(x) = \sum_{k=1}^{\infty} \frac{f^{(k-1)}(0)}{k!}\delta(x) = \sum_{k=1}^{\infty} \frac{F^{(k)}(0)}{k!}\delta(x) \tag{77}$$

where F is the primitive of f. We have eventually:

$$f(u(x))\delta(x) = \left[-F(0) + \sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{k!}(1)^k\right]\delta(x) = [F(1) - F(0)]\delta(x) \tag{78}$$

□

A.2 Schwartz functional applied to $n^p f(nx)$ sequences

We will try to justify equation (35) with some specific examples. Let us consider the following sequence of distributions (i.e. $\in D'$):

$$x^q\, f_n(x) = n^{q-1}\left[\delta\left(x\right) - \delta\left(x - \frac{1}{n}\right)\right] \tag{79}$$

where we have x^{q-1} in the left part of the (79) because f_n is supposed to have growing rate $q = 0$ while the $\delta(x)$ has growing rate $q = 1$. We have:

$$\lim_{n\to\infty} x^q\, f_n(x) = n^{q-1}\, \delta^{(1)}(x) = \eta^{1,q} \tag{80}$$

Note that, in this case, the order of the generalised function is p=1. If we apply the Schwartz functional to the (79) and we take into account that $\phi(x)\delta(x - x_0) = \phi(x_0)$ we have:

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} x^q\, f_n \phi dx = \lim_{n\to\infty} -n^{q-1-1} \frac{\phi\left(x - \frac{1}{n}\right) - \phi(x)}{\frac{1}{n}} = -n^{q-1-1}\phi^{(1)}(0) \tag{81}$$

In the same way, for the following sequence, we have:

$$x^q\, f_n(x) = n^{q-1} \left[n\delta\,(x) - 2n\delta \left(x - \frac{1}{n} \right) + n\delta \left(x - \frac{2}{n} \right) \right] \tag{82}$$

for which we have:

$$\lim_{n\to\infty} x^q\, f_n(x) = n^{q-2}\, \delta^{(2)}(x) = \eta^{2,q} \tag{83}$$

Where the order of the generalised function is p=2. Applying the Schwartz functional we have:

$$\lim_{n\to\infty} +n^{q-1-2} \frac{\phi(x) - 2\phi\left(x - \frac{1}{n}\right) + \phi\left(x - \frac{2}{n}\right)}{\frac{1}{n^2}} = n^{q-1-2}\phi^{(2)}(0) \tag{84}$$

Eventually, from the above examples we see that if:

$$\lim_{n\to\infty} x^q\, f_n(x) = a_p \eta^{p,q} \tag{85}$$

then we have:

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} x^q\, f_n(x)\phi(x)dx = (-1)^p n^{q-p-1} a_p \phi^{(p)}(0) \tag{86}$$

A.3 Examples of product of distributions

All the a_p coefficients, of the following products of distributions, are evaluated numerically. We will use the notation introduced in (31).
Example 1:

$$\delta(x)\delta'(x) \tag{87}$$

By taking twice the derivative of both sides of the (53), and rearranging the terms we get:

$$\delta(x)\delta'(x) = \frac{1}{6}\delta^{(2)}(x) - \frac{1}{3}u(x)\delta^{(2)}(x) + R\left(\frac{\delta^{(3)}}{n}\right) \tag{88}$$

Example 2: (compare with paragraph 5 above)

$$sign^2(x)\delta(x) \to \alpha n\,(2\chi(\alpha n x) - 1)^2 \xi(\alpha n x) \tag{89}$$

from which we have:

$$sign^2(x)\delta(x) = \frac{1}{3}\delta(x) + R\left(\frac{\delta^{(2)}}{n^2}\right) \tag{90}$$

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