THE INFINITY OF THE TWIN PRIMES

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ABSTRACT

The author had published a paper on the solutions for the twin primes conjecture in an international mathematics journal in 2003. This paper approaches the twin primes problem through the analysis of the composite numbers.

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Theorem:- The twin primes are infinite.

Proof:-

The integers or whole numbers comprise of both the even and odd integers, and, both the prime numbers and composite numbers, which are all infinite. (The primes had been proven to be infinite by Euclid long ago, and the composites of primes are also infinite, which is implied of course by the infinitude of the primes.) The even integers after 2 are composites of even, or, both even and odd, primes (there is only 1 even prime, i.e., 2, and it is always present in the composites which make up the even integers), e.g., 4 = 2 x 2, 8 = 2 x 2 x 2, 12 = 2 x 2 x 3, 30 = 2 x 3 x 5, 32 = 2 x 2 x 2 x 2 x 2, 36 = 2 x 2 x 3 x 3, etc., while the odd integers are comprised of both the primes, e.g., 3, 5, 7, 11, 13, 17, 19, etc., and composites of odd primes, e.g., 9 = 3 x 3, 15 = 3 x 5, 21 = 3 x 7, 27 = 3 x 3 x 3, 35 = 5 x 7, 57 = 3 x 19, etc..

All the primes in the infinite list of primes are separated by integers ranging from 2 (in the case of the twin primes), 4, 6, 8, 10, 12, and upwards, to infinity, as is shown in the appendix. The question is whether the twin primes, i.e., the primes pairs separated by 2 integers, are infinite. However, as is evident from the appendix, the list of the twin primes is not likely to be finite and can be expected to be infinite. We will proceed to prove this.

The Prime Number Theorem, which describes the distribution of the primes, is familiar. The primes have been known as the basic units, atoms or building-blocks of the integers, both even and odd ones. We will show the indispensable role of the twin primes, i.e., the primes pairs separated by 2 integers, as well as the primes pairs separated by 4 integers, 6 integers, 8 integers, 10 integers, 12 integers, and more, to infinity, and the other primes, in the formation or construction of the composite numbers. We will first of all do so by analysing the construction or composition of the composite numbers, both even and odd ones - we will do so by deconstructing a number of composite numbers. A Composite Number Theorem which describes the distribution of the composite numbers, as well as their building-blocks, the prime numbers, is a possibility. We note that each composite number, whether even or odd, is the unique product of a particular, unique set of prime numbers only, and, no product of a different set of primes can ever possibly result in the same composite number. We demonstrate this fact through some examples, which are as follows:-

Even Composite Numbers

(i) 10 = 2 x 5 (only)
(ii) 34 = 2 x 17 (only)
(iii) 106 = 2 x 53 (only)
(iv) 258 = 2 x 3 x 43 (only)
(v) Etc. to infinity

Odd Composite Numbers

(i) 21 = 3 x 7 (only)
(ii) 99 = 3 x 3 x 11 (only)
(iii) 325 = 5 x 5 x 13 (only)
(iv) 451 = 11 x 41 (only)
(v) Etc. to infinity

It is clear from these examples that no products of other primes can ever possibly produce the above-mentioned composites besides the products of the primes shown above, which are unique.

Lemma:

This is in accordance with the Fundamental Theorem of Arithmetic or Unique Factorisation Theorem, which states that there is only one possible combination of primes which will multiply together to produce any particular number, e.g., the only combination of primes which will produce the number 2,079 is: 3 x 3 x 3 x 7 x 11.

In the same manner, the following numbers are also uniquely factorised:

63 = 3 x 3 x 7 (only)
153 = 3 x 3 x 17 (only)
1,021,020 = 2 x 2 x 3 x 5 x 7 x 11 x 13 x 17 (only)

In other words, every positive whole number which is not prime can be broken up into prime factors, and, this can happen in only one way.

By this lemma, we can produce an infinitude of unique (only ones possible) composite numbers by "playing around" with or manipulating any list of prime numbers. For example, with the following list of primes:

2, 3, 5, 7, 11, 13, 17, 19

we obtain the following unique (only ones possible) composites:

39 = 3 x 13 (only)
78 = 2 x 3 x 13 (only)
182 = 2 x 7 x 13 (only)
2,261 = 7 x 17 x 19 (only)
3,230 = 2 x 5 x 17 x 19 (only)
46,189 = 11 x 13 x 17 x 19 (only)
62,985 = 3 x 5 x 13 x 17 x 19 (only)
746,130 = 2 x 3 x 5 x 7 x 11 x 17 x 19 (only)
Etc.

Similarly, with the following twin primes (in bold), and, other primes, we obtain the following unique (only ones possible) composites:

15 = **3** x **5** (only)
35 = **5** x **7** (only)
70 = 2 x **5** x **7** (only)
143 = **11** x **13** (only)
286 = 2 x **11** x **13** (only)
323 = **17** x **19** (only)
104,329 = **17** x **19** x **17** x **19** (only)
344,285 = 3 x 11 x **17** x **19** x **17** x **19** (only)
46,189 = **11** x **13** x **17** x **19** (only) (13 & 17 are a primes pair separated by 4 integers)
3,417,986 = 2 x **11** x **13** x **17** x **19** x 37 (only) (13 & 17 are a primes pair separated by 4 integers)
10,403 = **101** x **103** (only)
11,663 = **107** x **109** (only)
22,499 = **149** x **151** (only)
121,103 = **347** x **349** (only)
435 = **3** x **5** x 29 (only)
34,320 = 2 x 2 x 2 x 2 x **3** x **5** x **11** x **13** (only)
62,418 = 2 x 3 x **101** x **103** (only)
12,621,939 = 3 x 11 x 17 x **149** x **151** (only)
1,616,615 = **5** x **7** x **11** x **13** x **17** x **19** (only) (7 & 11 and 13 & 17 are each a primes pair separated by 4 integers)
74,364,290 = 2 x **5** x **7** x **11** x **13** x **17** x **19** x 23 (only) (7 & 11 and 13 & 17 are each a primes pair separated by 4 integers)
419,868 = 2 x 2 x 3 x 3 x **107** x **109** (only)
1,331,584 = 2 x 2 x 2 x 2 x 2 x 2 x 2 x **101** x **103** (only)
1,001,623 = 7 x **17** x **19** x 443 (only)
Etc. to infinity

The following is the product of primes for composite numbers in abbreviated form:-

$$c = \prod_{p \text{ prime}} p$$

In the formation or construction of these unique (only ones possible) composites described above, there will be an infinitude of combinations of the twin primes pairs, which we are here examining, with the other primes in the infinite list of the primes. As is shown above, a unique (only one possible) composite can be the product of a twin primes pair (e.g., 3 x 5). It can be the product of a twin primes pair with another prime or primes (e.g., 3 x 5 x 19 x 79). It can be the product of a twin primes pair with itself (e.g., 3 x

5 x 3 x 5). It can be the product of a twin primes pair with itself and another prime or primes (e.g., 3 x 5 x 3 x 5 x 23 x 89). It can be the product of a twin primes pair with another twin primes pair or other twin primes pairs (e.g., 11 x 13 x 227 x 229 x 461 x 463) It can be the product of a twin primes pair with another twin primes pair or other twin primes pairs and another prime or primes (e.g., 11 x 13 x 227 x 229 x 461 x 463 x 701 x 1,459 x 2,447). It is evident that due to the infinitude of the primes, the amount of such combinations will be infinite.

The reasoning to be brought up hereafter is subtle. There should be an infinitude of products of primes pairs separated by 2 integers (i.e., twin primes, e.g., 29 & 31 and 41 & 43), prime pairs separated by 4 integers (e.g., 7 & 11 and 13 & 17), prime pairs separated by 6 integers (e.g., 1,117 & 1,123 and 1,861 & 1,867), primes pairs separated by 8 integers (e.g., 2,459 & 2,467 and 4,289 & 4,297), primes pairs separated by 10 integers (e.g., 6,691 & 6,701 and 10,321 & 10,331), primes pairs separated by 12 integers (e.g., 9,649 & 9,661 and 11,399 & 11,411), and so on, to infinity, and other primes from the infinite list of the primes, involved in the formation or construction of the composite numbers, since all these primes pairs separated by intervals of various magnitudes or sizes (from 2 integers in the case of the twin primes, 4 integers, 6 integers, 8 integers, 10 integers, 12 integers, and so on, upwards to infinity), and the other primes from the infinite list of the primes are evidently the building-blocks of different configurations which are necessary for the construction of the infinite unique (only ones possible) composites through the unique (only one possible) product of primes pairs separated by intervals of various magnitudes or sizes and other primes as well, all these primes being the unique (only ones possible) factors of the unique (only ones possible) composites, in line with the above lemma. The formation or construction of such unique (only ones possible) composites, which are infinite, can be described, e.g., as follows:-

$$c = p \times p_1 \times p_2 \times p_3 \times p_4 \times \ldots \ldots \quad (p_1 \; \& \; p_2, \; p_3 \; \& \; p_4 \text{ being twin primes pairs})$$

It can thus be stated that the infinite list of the composite numbers comprises of the respective infinite lists of products of primes pairs separated by 2 integers (twin primes), primes pairs separated by 4 integers, primes pairs separated by 6 integers, primes pairs separated by 8 integers, primes pairs separated by 10 integers, primes pairs separated by 12 integers, and so on, upwards to infinity, and other primes.

All the respective lists of products relating to the primes pairs separated by intervals of various magnitudes or sizes and other primes which produce composites, which have been described above, are unique, the only ones of their kind possible, obtainable only through unique factorisation, which cannot be replaced or substituted through multiplying other prime numbers or factors (to produce the same composite number), the primes pairs separated by intervals of various magnitudes or sizes and the other primes being the building-blocks of different configurations which are necessary for the construction of the infinite unique (only ones possible) composites through the unique (only one possible) product of primes pairs separated by intervals of various magnitudes or sizes and other primes as well, which implies that an infinitude of them, since they are the building-blocks of different configurations, needs to exist in order that the list of the composites, of which they are the building-blocks of different configurations, is infinite, *more so* since they are unique (only ones possible) and cannot be replaced or substituted through multiplying other prime numbers or factors (to produce the same composite number). Since unique factorisation means that the products of all these primes pairs separated by intervals of various magnitudes or sizes and other primes as well will only produce unique (only ones possible) composites, in order for there to be larger and larger unique (only ones possible) composites to be produced we will need to get products of larger and larger primes pairs (i.e., primes pairs separated by 2 integers (twin primes), primes pairs separated by 4 integers, primes pairs separated by 6 integers, primes pairs separated by 8 integers, primes pairs separated by 10 integers, primes pairs separated by 12 integers, and so on, upwards to infinity) and larger and larger primes. To cater for the production of these larger and larger unique (only ones possible) composites to infinity, the supply of the primes pairs separated by 2 integers (twin primes), primes pairs separated by 4 integers, primes pairs separated by 6 integers, primes pairs separated by 8 integers, primes pairs separated by 10 integers, primes pairs separated by 12 integers, and so on, all the way upwards, and, the other primes, has also to be infinite, e.g., certain larger and larger unique (only ones possible) composites must necessarily have larger primes pairs separated by 2 integers (twin primes), larger primes pairs separated by 4 integers, larger primes pairs separated by 6 integers, larger primes pairs separated by 8 integers, larger primes pairs separated by 10 integers, larger primes pairs separated by 12 integers, and so on, upwards to infinity, and, other larger primes, as their *unique prime factors*, or, *only possible prime factors*, which implies that all these primes pairs separated by intervals of various magnitudes or sizes and the other primes need to be infinite in order to cater for the production of the infinite list of the larger and larger unique (only ones possible) composites. The infinite list of integers or whole numbers comprises of the infinite list of the prime numbers and the infinite list of the composite numbers, as is described above, which means that all these building-blocks of different configurations are also the building-blocks of the infinite integers or whole numbers.

What forms will these building-blocks of different configurations, these primes pairs separated by intervals of various magnitudes or sizes and the other primes, these *unique prime factors*, take, in the formation or construction of the unique (only ones possible) composites? Any primes pair in the group can be the product of one another. The primes pair can also be the product of itself with another prime or primes. It can also be the product of itself with itself. It can be the product of itself with itself and another prime or primes too. It can be the product of itself with another primes pair or other primes pairs as well. It can also be the product of itself with another primes pair or other primes pairs and another prime or primes. It is clear that due to the infinitude of the primes the amount of such combinations will be infinite. In this manner, an infinite quantity of larger and larger unique (only ones possible) composites can be produced, evidently owing to the fact that the primes involved are *unique prime factors* and larger primes. The product of a larger primes pair will always produce a larger composite number than the product of a smaller primes pair, e.g., the product of the larger twin primes pair 17 x 19 produces the larger unique (only one possible) composite number 323, while the product of the smaller twin primes pair 11 x 13 produces the smaller unique (only one possible) composite number 143. This fact implies that since the composites are infinite, the primes pairs separated by intervals of various magnitudes or sizes and the other primes, which are the composites' *unique prime factors*, will also be infinite, the infinite list of these primes pairs separated by intervals of various magnitudes and sizes and the other primes providing larger and larger prime factors for producing the larger and larger unique (only ones possible) composites. If we liken the infinite list of the integers or whole numbers to an infinitely large jigsaw puzzle, there will be an infinitude of "gaps of certain sizes", or, composites of certain values, that can only be filled up by the products of certain unique (only ones possible) primes from among the primes pairs separated by 2 integers (twin primes), primes pairs separated by 4 integers, primes pairs separated by 6 integers, primes pairs separated by 8 integers, primes pairs separated by 10 integers, primes pairs separated by 12 integers, and so on, upwards to infinity, and the other primes - these unique (only ones possible) primes are the *unique prime factors* of the unique (only ones possible) composites so constructed by the products of these unique (only ones possible) primes, as per the above lemma. This necessitates the infinitude of these unique (only ones possible) primes, i.e., the primes pairs separated by 2 integers (twin primes), primes pairs separated by 4 integers, primes pairs separated by 6 integers, primes pairs separated by 8 integers, primes pairs separated by 10 integers, primes pairs separated by 12 integers, and so on, upwards to infinity, and the other primes, for the purpose of filling up the "gaps of certain sizes", or, composites of certain values, which are infinite.

We bring up the following example of 10 larger and larger unique (only ones possible) composites which are the results of the products of 10 consecutive larger and larger twin primes pairs to show the "unique factorisation" effect:-

(i) 15 = 3 x 5 (only)
(ii) 35 = 5 x 7 (only)
(iii) 143 = 11 x 13 (only)
(iv) 323 = 17 x 19 (only)
(v) 899 = 29 x 31 (only)
(vi) 1,763 = 41 x 43 (only)
(vii) 3,599 = 59 x 61 (only)
(viii) 5,183 = 71 x 73 (only)
(ix) 10,403 = 101 x 103 (only)
(x) 11,663 = 107 x 109 (only)

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All the primes, or, building-blocks of different configurations, being *unique prime factors* of the unique (only ones possible) composites, are a necessity, a must-have, essential, compulsory, and, indispensable, for the construction of the latter, since there can never be any replacements or substitutes for them. The larger the unique (only ones possible) composites are, the larger their prime factors (i.e., these mentioned *unique prime factors*) therefore have to be. Since the unique (only ones possible) composites increase to infinity, it implies that these *unique prime factors* (which include the twin primes that are the subject of our examination), which are a necessity and are not substitutable, also have to increase to infinity. In the case of the twin primes, as in the case of the other primes in the infinite list of the primes, larger and larger twin primes, as well as larger and larger primes pairs separated by larger intervals than the twin primes' and other larger and larger primes, will be needed here and there for the production of larger and larger unique (only ones possible) composites. Thus, the twin primes, being *unique prime factors* of the infinite unique (only ones possible) composites, together with the other primes pairs separated by larger intervals and the other primes, are necessarily infinite.

In summing up, we state that since the twin primes, together with the other primes pairs separated by larger intervals and the other primes, are not replaceable or substitutable by other primes, and are a necessity, a must-have, essential, compulsory, and, indispensable, for the construction of the unique (only ones possible) composites, being the latter's atoms or building-blocks of different configurations, the latter's *unique prime factors* as per the above lemma, and since these unique (only ones possible) composites are infinite, it is implied that the twin primes, the other primes pairs separated by larger intervals and the other primes, being the latter's atoms or building-blocks of different configurations, also have to be infinite, larger and larger twin primes, primes pairs separated by larger intervals, and other primes, being essential, compulsory and necessary for the formation or construction of the larger and larger unique (only ones possible) composites and not replaceable or substitutable, becoming in effect the *unique prime factors*, or, *only possible prime factors*, of the latter, *more so* that they have to be around all the time, to be infinite, since they are essential, compulsory, necessary and cannot be replaced or substituted by other primes (to produce the same composite number) besides being the atoms or building-blocks of different configurations of the unique (only ones possible) composites.

APPENDIX

Anecdotal Evidence Of The Infinity Of The Twin Primes

TOP TWIN PRIMES IN 2000, 2001, 2007 & 2009

In the year 2000, 4648619711505 x $2^{60000} \pm 1$ (18,075 digits) had been the top twin primes pair which had been discovered. In the year 2001, it only ranked eighth in the list of top 20 twin primes pairs, with $318032361 \cdot 2^{107001} \pm 1$ (32,220 digits) topping the list. In the year 2007, in the list of top 20 twin primes pairs, $318032361 \cdot 2^{107001} \pm 1$ (32,220 digits) ranked eighth, while 4648619711505 x $2^{60000} \pm 1$ (18,075 digits) was nowhere to be seen; 2003663613*2^195000-1 and 2003663613*2^195000+1 (58,711 digits), which was discovered on January 15, 2007, by Eric Vautier (from France) of the Twin Prime Search (TPS) project in collaboration with PrimeGrid (BOINC platform), was at the top of the list. As at August 2009, $65516468355 \cdot 2^{333333}-1$ and $65516468355 \cdot 2^{333333}+1$ (100,355 digits) is at the top of the list of top 20 twin primes pairs, while $318032361 \cdot 2^{107001} \pm 1$ (32,220 digits) ranks 11th., and, 2003663613*2^195000-1 and 2003663613*2^195000+1 (58,711 digits) ranks second in this list.

We can expect larger twin primes than these extremely large twin primes, much larger ones, infinitely larger ones, to be discovered in due course.

LIST OF PRIMES PAIRS FOR THE FIRST 2,500 CONSECUTIVE PRIMES, 2 TO 22,307, RANKED ACCORDING TO THEIR FREQUENCIES OF APPEARANCE

S. No.	Ranking	Prime Pairs	No. Of Pairs	Percentage
(1)	1	primes pair separated by 6 integers	482	19.29 %
(2)	2	primes pair separated by 4 integers	378	15.13 %
(3)	3	primes pair separated by 2 integers (t. p.)	376	15.05 %
(4)	4	primes pair separated by 12 integers	267	10.68 %
(5)	5	primes pair separated by 10 integers	255	10.20 %
(6)	6	primes pair separated by 8 integers	229	9.16 %
(7)	7	primes pair separated by 14 integers	138	5.52 %
(8)	8	primes pair separated by 18 integers	111	4.44 %
(9)	9	primes pair separated by 16 integers	80	3.20 %
(10)	10	primes pair separated by 20 integers	47	1.88 %
(11)	11	primes pair separated by 22 integers	46	1.84 %
(12)	12	primes pair separated by 30 integers	24	0.96 %
(13)	13	primes pair separated by 28 integers	19	0.76 %
(14)	14	primes pair separated by 24 integers	16	0.64 %
(15)	15	primes pair separated by 26 integers	10	0.40 %
(16)	16	primes pair separated by 34 integers	9	0.36 %
(17)	17	primes pair separated by 36 integers	5	0.20 %
(18)	18	primes pair separated by 32 integers	2	0.08 %
(19)	18	primes pair separated by 40 integers	2	0.08 %
(20)	19	primes pair separated by 42 integers	1	0.04 %
(21)	19	primes pair separated by 52 integers	1	0.04 %

Total No. Of Primes Pairs In List: 2,498

It is evident in the above list that the primes pairs separated by 6 integers, 4 integers and 2 integers (twin primes), among the 21 classifications of primes pairs separated by from 2 integers to 52 integers (primes pairs separated by 38 integers, 44 integers, 46 integers, 48 integers & 50 integers are not among them, but, they are expected to appear further down in the infinite list of the primes), are the most dominant, important. There is a long list, an infinite list, of other primes pairs, besides those shown in the above list, which also play a part as the building-blocks of the infinite list of the integers.

The list of the integers is infinite. The list of the primes is also infinite. The infinite primes are the building-blocks of the infinite integers - the infinite odd integers are all either primes or composites of primes, and, the infinite even integers, except for 2 which is a prime, are all also composites of primes. Therefore, all the primes pairs separated by the integers of various magnitudes, as described above, can never all be finite. If there is any possibility at all for any of these primes pairs to be finite, there is only the possibility that a number of these primes pairs are finite (but never all of them). However, will it have to be the primes pairs separated by 2 integers or twin primes (which are the subject of our investigation here), which are the only primes pair, or, one among a number of primes pairs, which are finite? Why question only the infinity of the primes pairs separated by 2 integers, the twin primes? Are not the infinities of the primes pairs separated by 8 integers and more, whose frequencies of appearance are lower, as compared to those of the primes pairs which are separated by 6, 4 and 2 integers respectively, in the above list of primes pairs, more questionable? Why single out only the twin primes? (There are at least 18 other primes pairs, separated by from 8 integers to 52 integers, whose respective infinities should be more suspect, as is evident from the above list of primes pairs, if any infinities should be doubted. Evidently, the primes pairs separated by 2 integers (twin primes) are not that likely to be finite.)

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