# Clifford Space Gravitational Field Equations and Dark Energy *

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**Abstract**

We continue with the study of Clifford-space Gravity and analyze further the Clifford space ($C$-space) generalized gravitational field equations which are obtained from a variational principle based on the generalization of the Einstein-Hilbert-Cartan action. One of the main features is that the $C$-space connection requires *torsion* in order to have consistency with the Clifford algebraic structure associated with the curved $C$-space basis generators. Hence no spin matter is required to induce torsion since it already exists in the vacuum. The field equations in $C$-spaces associated to a Clifford algebra in $D$-dimensions are *not* equivalent to the ordinary gravitational equations with torsion in higher $2^D$-dimensions. The most physically relevant conclusion, besides the presence of torsion in the vacuum, is the contribution of the *higher* grade metric components $g^{\mu_1\mu_2\ \nu_1\nu_2}, g^{\mu_1\mu_2\mu_2\ \nu_1\nu_2\nu_3}$, ..... of the $C$-space metric to dark energy/dark matter.

## 1 Introduction

In the past years, the Extended Relativity Theory in $C$-spaces (Clifford spaces) and Clifford-Phase spaces were developed [1], [2]. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory whose generalized coordinates are Clifford polyvector-valued quantities which incorporate the lines, areas, volumes, and hyper-volumes degrees of freedom associated with the collective dynamics of particles, strings, membranes, p-branes (closed p-branes) moving in a D-dimensional target spacetime background. C-space Relativity permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing. Our theory has 2 fundamental parameters : the speed of a light $c$ and a length scale which can be

*dedicated to the memory of Adam Christopher Bowers set equal to the Planck length. The role of "photons" in $C$-space is played by *tensionless* branes. An extensive review of the Extended Relativity Theory in Clifford spaces can be found in [1].

The polyvector valued coordinates $x^\mu, x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, ...$ are now linked to the basis vectors generators $\gamma^\mu$, bi-vectors generators $\gamma_\mu \wedge \gamma_\nu$, tri-vectors generators $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$, ... of the Clifford algebra, including the Clifford algebra unit element (associated to a scalar coordinate). These polyvector valued coordinates can be interpreted as the quenched-degrees of freedom of an ensemble of $p$-loops associated with the dynamics of closed $p$-branes, for $p = 0, 1, 2, ..., D-1$, embedded in a target $D$-dimensional spacetime background [6].

The $C$-space poly-vector-valued momentum is defined as $\mathbf{P} = d\mathbf{X}/d\Sigma$ where $\mathbf{X}$ is the Clifford-valued coordinate corresponding to the $Cl(1,3)$ algebra in four-dimensions, for example,

$$\mathbf{X} \ = \ s\,\mathbf{1} + x^\mu\ \gamma_\mu + x^{\mu\nu}\ \gamma_\mu\wedge\gamma_\nu + x^{\mu\nu\rho}\ \gamma_\mu\wedge\gamma_\nu\wedge\gamma_\rho + x^{\mu\nu\rho\tau}\ \gamma_\mu\wedge\gamma_\nu\wedge\gamma_\rho\wedge\gamma_\tau \quad (1.1)$$

it can be generalized to any dimensions, including $D = 0$.

The component $s$ is the Clifford scalar component of the polyvector-valued coordinate and $d\Sigma$ is the infinitesimal $C$-space proper "time" interval which is *invariant* under $Cl(1,3)$ transformations which are the Clifford-algebra extensions of the $SO(1,3)$ Lorentz transformations [1]. One should emphasize that $d\Sigma$, which is given by the square root of the quadratic interval in $C$-space

$$(d\Sigma)^2 \ = \ (ds)^2 \ + \ dx_\mu\ dx^\mu \ + \ dx_{\mu\nu}\ dx^{\mu\nu} \ + \ \ldots \quad (1.2)$$

is *not* equal to the proper time Lorentz-invariant interval $d\tau$ in ordinary spacetime $(d\tau)^2 = g_{\mu\nu}dx^\mu dx^\nu = dx_\mu dx^\mu$. In order to match units in all terms of eqs-(1.1,1.2) suitable powers of a length scale (say Planck scale) must be introduced. For convenience purposes it is set to unity. For extensive details of the generalized Lorentz transformations (poly-rotations) in flat $C$-spaces and references we refer to [1].

Let us now consider $C$-space [1]. A basis in $C$-space is given by

$$E_A = \gamma,\ \gamma_\mu,\ \gamma_\mu\wedge\gamma_\nu,\ \gamma_\mu\wedge\gamma_\nu\wedge\gamma_\rho,\ ... \quad (1.3)$$

where $\gamma$ is the unit element of the Clifford algebra that we label as $\mathbf{1}$ from now on. In (1.3) when one writes an $r$-vector basis $\gamma_{\mu_1}\wedge\gamma_{\mu_2}\wedge ... \wedge\gamma_{\mu_r}$ we take the indices in "lexicographical" order so that $\mu_1 < \mu_2 < .... < \mu_r$. An element of $C$-space is a Clifford number, called also *Polyvector* or *Clifford aggregate* which we now write in the form

$$X \ = \ X^A E_A \ = \ s\,\mathbf{1} \ + \ x^\mu\gamma_\mu \ + \ x^{\mu\nu}\gamma_\mu\wedge\gamma_\nu + ... \quad (1.4)$$

A $C$-space is parametrized not only by 1-vector coordinates $x^\mu$ but also by the 2-vector coordinates $x^{\mu\nu}$, 3-vector coordinates $x^{\mu\nu\alpha}$, ..., called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops,..., onto the coordinate planes [6]. By $p$-loop we mean a closed $p$-brane;

in particular, a 1-loop is closed string. In order to avoid using the powers of the Planck scale length parameter $L_p$ in the expansion of the polyvector $X$ (in order to match units) we can set it to unity to simplify matters.

In a *flat* $C$-space the basis vectors $E^A, E_A$ are *constants*. In a *curved* $C$-space this is no longer true. Each $E^A, E_A$ is a function of the $C$-space coordinates

$$X^A \ = \ \{ \ s, \ x^\mu, \ x^{\mu_1\mu_2}, \ ....., \ x^{\mu_1\mu_2.....\mu_D} \ \} \tag{1.5}$$

which include scalar, vector, bivector,..., $p$-vector,... coordinates in the underlying $D$-dim base spacetime and whose corresponding $C$-space is $2^D$-dimensional since the Clifford algebra in $D$-dim is $2^D$-dimensional.

In curved $C$-space one introduces the $\mathbf{X}$-dependent basis generators $\gamma_M, \gamma^M$ defined in terms of the beins $E^A_M$, inverse beins $E^M_A$ and the flat tangent space generators $\gamma_A, \gamma^A$ as follows $\gamma_M = E^A_M(\mathbf{X})\gamma_A, \gamma^M = E^M_A(\mathbf{X})\gamma^A$. The curved $C$-space metric expression $g_{MN} = E^A_M E^B_N \eta_{AB}$ also agrees with taking the scalar part of the Clifford geometric product $< \gamma_M\gamma_N >= g_{MN}$.

The covariant derivative of $E^A_M(\mathbf{X}), E^M_A(\mathbf{X})$ involves the generalized connection and spin connection and is defined as

$$\nabla_K E^A_M \ = \ \partial_K E^A_M \ - \ \Gamma^L_{KM} \ E^A_L \ + \ \omega^A_{KB} \ E^B_M \tag{1.6a}$$

$$\nabla_K E^M_A \ = \ \partial_K E^M_A \ + \ \Gamma^M_{KL} \ E^L_A \ - \ \omega_{KA}{}^B \ E^M_B \tag{1.6b}$$

If the nonmetricity is zero then $\nabla_K E^A_M = 0, \ \nabla_K E^M_A = 0$ in eqs-(1.6).

One of the salient features in [3] is that the $C$-space connection requires torsion in order to have consistency between the Clifford algebraic structure and the zero nonmetricity condition $\nabla_K g^{MN} = 0$. In the case of nonsymmetric connections with torsion, the curvature obeys the following relations under the exchange of indices

$$\mathbf{R}_{MNJK} \ = \ - \ \mathbf{R}_{NMJK}, \ \ \mathbf{R}_{MNKJ} \ = \ - \ \mathbf{R}_{MNJK}, \ \ \textit{but} \ \mathbf{R}_{MNJK} \ \neq \ \mathbf{R}_{JKMN} \tag{1.7}$$

and is defined in terms of the connection components $\Gamma^L_{KM}$ as follows

$$\mathbf{R}_{MNJ}{}^K \ = \ \partial_M \ \Gamma_{NJ}{}^K \ - \ \partial_N \ \Gamma_{MJ}{}^K \ + \ \Gamma^K_{ML} \ \Gamma^L_{NJ} \ - \ \Gamma^K_{NL} \ \Gamma^L_{MJ} \tag{1.8}$$

If the anholonomy coefficients$f^K_{MN} \neq 0$ one must also include them into the definition of curvature (1.8) by adding terms of the form $-f^L_{MN}\Gamma^K_{LJ}$.

The standard Riemann-Cartan curvature tensor in ordinary spacetime is *contained* in $C$-space as follows

$$\mathcal{R}_{\mu_1\mu_2\rho_1}{}^{\rho_2} \ = \ \partial_{\mu_1}\Gamma^{\rho_2}_{\mu_2\rho_1} \ - \ \partial_{\mu_2}\Gamma^{\rho_2}_{\mu_1\rho_1} \ + \ \Gamma^{\rho_2}_{\mu_1\sigma} \ \Gamma^{\sigma}_{\ \mu_2\rho_1} \ - \ \Gamma^{\rho_2}_{\mu_2\sigma} \ \Gamma^{\sigma}_{\ \mu_1\rho_1} \ \subset$$

$$\mathbf{R}_{\mu_1\mu_2\rho_1}{}^{\rho_2} \ = \partial_{\mu_1}\Gamma^{\rho_2}_{\mu_2\rho_1} \ - \ \partial_{\mu_2}\Gamma^{\rho_2}_{\mu_1\rho_1} \ + \ \Gamma^{\rho_2}_{\mu_1 \ \mathbf{M}} \ \Gamma^{\mathbf{M}}_{\mu_2\rho_1} \ - \ \Gamma^{\rho_2}_{\mu_2 \ \mathbf{M}} \ \Gamma^{\mathbf{M}}_{\mu_1\rho_1} \tag{1.9}$$

due to the contractions involving the polyvector valued indices $\mathbf{M}$ in eq-(1.9) There is also the crucial difference that $\mathbf{R}^{\rho_2}_{\mu_1\mu_2\rho_1}(s, x^\nu, x^{\nu_1\nu_2}, ...)$ has now an *additional* dependence on all the $C$-space polyvector valued coordinates $s, x^{\nu_1\nu_2}, x^{\nu_1\nu_2\nu_3}, \ldots$ besides the $x^\nu$ coordinates. The curvature in the presence of torsion does not satisfy the same symmetry relations when there is no torsion, therefore the Ricci-like tensor is no longer symmetric

$$\mathbf{R}_{MNJ}{}^{N} \ = \ \mathbf{R}_{MJ}, \quad \mathbf{R}_{MJ} \ \neq \ \mathbf{R}_{JM}, \quad \mathbf{R} \ = \ g^{MJ} \ \mathbf{R}_{MJ} \tag{1.10}$$

Denoting the Clifford scalar $s$ component by the index 0, and that must *not* be confused with the temporal coordinate $t$, the $C$-space Ricci-like tensor is

$$\mathbf{R}_M^N \ = \ \sum_{j=1}^{D} \ \mathbf{R}_{M\ [\nu_1\nu_2\ldots\nu_j]}^{\ N\ [\nu_1\nu_2\ldots\nu_j]} \ + \ \mathbf{R}_{M\ \mathbf{0}}^{\ N\ \mathbf{0}} \tag{1.11}$$

and the $C$-space curvature scalar is

$$\mathbf{R} = \sum_{j=1}^{D}\sum_{k=1}^{D} \mathbf{R}_{[\mu_1\mu_2\ldots\mu_j]\ [\nu_1\nu_2\ldots\nu_k]}{}^{[\mu_1\mu_2\ldots\mu_j]\ [\nu_1\nu_2\ldots\nu_k]} + \sum_{j=1}^{D} \mathbf{R}_{[\mu_1\mu_2\ldots\mu_j]\ \mathbf{0}}{}^{[\mu_1\mu_2\ldots\mu_j]\ \mathbf{0}} \tag{1.12}$$

The physical applications of $C$-space gravity to higher curvature theories of gravity were studied in [3]. One of the key properties of Lanczos-Lovelock-Cartan gravity (with torsion) is that the field equations do not contain higher derivatives of the metric tensor beyond the second order due to the fact that the action does not contain derivatives of the curvature, see [10], [13], [12] and references therein.

One may construct an Einstein-Hilbert-Cartan like action based on the $C$-space curvature scalar

$$\frac{1}{2\kappa^2} \int ds \ \prod dx^\mu \ \prod dx^{\mu_1\mu_2} \ \ldots \ dx^{\mu_1\mu_2\ldots\mu_D} \ \mu_m(g_{MN}) \ \ \mathbf{R} \tag{1.13}$$

where $\mu_m(g_{MN})$ is a suitable integration measure.

There are *two* approaches to construct measures in $C$-spaces. One approach requires the use of hyper-determinants of hyper-matrices. And the other approach requires ordinary determinants of square matrices in $2^D$-dimensions. The hyper-determinant of a hyper-matrix [14] can be recast in terms of discriminants [15]. Hyper-determinants have been found to play a key role in the black-hole/qubits correspondence [16]. Because the hyper-determinant of a product of two hyper-matrices is *not* equal to the product of their hyper-determinants this complicates matters.

One can avoid the use of hyper-determinants by working in a *blockwise* fashion, when dealing with polyvector valued indices, rather than dealing with each one of the indices of their associated hyper-matrices individually. The $C$-space metric $g_{MN}$ associated with a Clifford algebra in $D$-dimensions has a one-to-one correspondence with an ordinary metric $g_{ij}$ in $2^D$-dimensions. In particular, the metric $g_{ij}$ is a square $2^D \times 2^D$ symmetric matrix with $\frac{1}{2}2^D(2^D+1)$

independent components. The determinant of the square matrix $g_{ij}$ is defined as usual in terms of epsilon tensors, where the indices range is $i, j = 1, 2, 3, ...., 2^D$.

The polyvector coordinates $\mathbf{X} = s, x^\mu, x^{\mu_1\mu_2}, ....$, and their derivatives, have also a one-to-one correspondence with the coordinates $y^i = y^1, y^2, ......, y^{2^D}$, and their derivatives, of the associated $2^D$-dim space. Thus, one has a correspondence of the action (1.13) in $C$-space with the ordinary Einstein-Cartan action in $2^D$-dimensions

$$\frac{1}{2\kappa^2} \int d^{2^D} y \sqrt{|det\ g_{ij}|}\ \ \mathcal{R} \tag{1.14}$$

However, having a *correspondence* between the actions in (1.13) and (1.14) does not mean that they are physically *equivalent*, even if one *replaces* the measure in eq-(1.13) by $\sqrt{|det\ g_{ij}|}$. The reason being that the Clifford algebraic structure imposes very strong conditions on the allowed $C$-space connections, and in particular, on the metric components $g_{MN}$, when the zero grade-mixing condition (gauge) $g_{MN} = 0$ is chosen [3]. These conditions are $\nabla_K C^{MNL} = 0$ where $C^{MNL}$ are the curved Clifford space structure functions. This will be the main subject of the next section : to study the field equations in $C$-spaces and to show why they are *not* equivalent to ordinary gravity in higher $2^D$-dimensions.

Among some of the main results in [3] were that $C$-space gravity, in general, involves spacetime multi-metrics $g^{\mu\nu}_{(n)}, n = 1, 2, 3, ...., D$; higher-spins beyond spin 2 [18] ; the Lanczos-Lovelock-Cartan higher curvature gravitational actions, and other extended gravitational theories based on $f(R), f(R_{\mu\nu}), f(R, T)$... actions [17], for polynomial-valued functions of curvatures and torsion, could be obtained as $effective$ actions after integrating the $C$-space gravitational action with respect to all the polyvector-valued coordinates, except the vectorial ones $x^\mu$.

To conclude this introduction, we shall choose for $C$-space measure the following function $\mu_m(g_{MN})$ in (1.13) such that it has the same properties as $\sqrt{|g|} = \sqrt{|det\ g_{ij}|}$ in (1.14) in so far as the differentation/variation is concerned

$$\delta\mu_m(g_{MN}) = \frac{1}{2}\ \mu_m(g_{MN})\ g^{MN}\ \delta\ g_{MN}\ =\ -\ \frac{1}{2}\ \mu_m(g_{MN})\ g_{MN}\ \delta\ g^{MN};\ \ \mu_m(g_{MN}) \leftrightarrow \sqrt{|g|} \tag{1.15}$$

In this fashion the Clifford space ($C$-space) generalized gravitational field equations are obtained, next, from a variational principle and which is based on an extension of the Einstein-Hilbert-Cartan action.

# 2 Clifford Space Gravitational Field equations

In this section we shall provide the Clifford Space Gravitational Field equations. When there is *torsion*, the Palatini variation of the curvature tensor in ordinary spaces is given by

$$\delta R^{\lambda}_{\alpha\mu\beta} \;=\; \nabla_{\mu}\delta\Gamma^{\lambda}_{\alpha\beta} \;-\; \nabla_{\beta}\delta\Gamma^{\lambda}_{\alpha\mu} \;+\; T^{\rho}_{\mu\beta}\;\delta\Gamma^{\lambda}_{\alpha\rho} \tag{2.1}$$

contracting indices one has the variation of the Ricci tensor

$$\delta R_{\alpha\beta} \;=\; \nabla_{\mu}\delta\Gamma^{\mu}_{\alpha\beta} \;-\; \nabla_{\beta}\delta\Gamma^{\mu}_{\alpha\mu} \;+\; T^{\rho}_{\mu\beta}\;\delta\Gamma^{\mu}_{\alpha\rho} \tag{2.2}$$

Generalizing these results to $C$-space gives

$$\delta\mathbf{R}_{MN} = \nabla_J\delta\Gamma^J_{MN} - \nabla_N\delta\Gamma^J_{MJ} \;+\; T^L_{JN}\;\delta\Gamma^J_{ML} \tag{2.3}$$

The variation of the action is

$$\int\;[D^{2^D}X]\;\delta\;(\sqrt{|g|}\;g^{MN}\;\mathbf{R}_{MN})\;=$$

$$\int\;[D^{2^D}X]\,[\delta\,(\sqrt{|g|}\,g^{MN}\,)\mathbf{R}_{MN} + \sqrt{|g|}\,g^{MN}(\nabla_J\delta\Gamma^J_{MN} - \nabla_N\delta\Gamma^J_{MJ} + T^L_{JN}\,\delta\Gamma^J_{ML})\,] \tag{2.4}$$

If the contorsion and nonmetricity were zero, the last terms (2.4) yield a total derivative due to the fact that the divergence operator, associated to a symmetric (torsionless) metric compatible Levi-Civita connection, can be written as

$$\nabla^{\{\}}_M\;(g^{NJ}\;\delta\Gamma^M_{NJ})\;=\;\frac{1}{\sqrt{|g|}}\;\partial_M(\sqrt{|g|}\;g^{NJ}\;\delta\Gamma^M_{NJ}) \tag{2.6}$$

and such that one recovers a total derivative of the form

$$\int\;[D^{2^D}X]\;\sqrt{|g|}\;\nabla^{\{\}}_M\;(g^{NJ}\;\delta\Gamma^M_{NJ})\;=\;\int\;[D^{2^D}X]\;\partial_M(\;\sqrt{|g|}\;g^{NJ}\;\delta\Gamma^M_{NJ}\;) \tag{2.7}$$

As usual, despite that $\Gamma$ does not transform as a tensor, the variation $\delta\Gamma$ does.

However, when the contorsion is *not* zero, there are *extra* torsion terms in the right hand side of the covariant divergence expression (2.6) given by

$$\nabla_M\;(g^{NJ}\;\delta\Gamma^M_{NJ})\;=\;\frac{1}{\sqrt{|g|}}\;\partial_M(\sqrt{|g|}\;g^{NJ}\;\delta\Gamma^M_{NJ})\;+\;T_M\;(\;g^{NJ}\;\delta\Gamma^M_{NJ}) \tag{2.8}$$

where the torsion vector is $T^L_{ML} = T_M$ and which lead to the final variation of the action, up to total derivatives terms, of the form

$$\int\;[D^{2^D}X]\;[\;\delta\;(\sqrt{|g|}\;g^{MN}\;)\;\mathbf{R}_{MN}\;+\;\sqrt{|g|}\;(g^{MN}\;T_J\;-\;T^N\delta^M_J\;+\;T^{MN}_J\;)\;\delta\Gamma^J_{MN}\;] \tag{2.9}$$

A Palatini variation of the action (2.3) with respect to the $C$-space metric $g_{MN}$, and the connection $\Gamma^J_{MN}$ separately, furnishes the vacuum field equations

$$\frac{\delta S}{\delta g^{MN}}\;=\;0\;\Rightarrow\;\mathbf{R}_{(MN)}\;-\frac{1}{2}\;g_{MN}\;\mathbf{R}\;\;=\;0 \tag{2.10}$$

and

$$\frac{\delta S}{\delta \Gamma^J_{MN}} \;=\; 0 \;\Rightarrow\; g^{MN}\;T_J \;-\; T^N \delta^M_J \;+\; T^{MN}_J \;=\; 0 \tag{2.11}$$

When there is matter one should add the stress energy tensor contribution to the right hand side of (2.10)

$$\kappa^2 \;\mathbf{T}_{MN} \;=\; -\;\frac{2\kappa^2}{\sqrt{|g|}}\;\frac{\delta(\sqrt{|g|}\;\mathcal{L}_{matter})}{\delta g^{MN}} \tag{2.12}$$

and replace (2.11) with

$$\frac{\delta(\sqrt{|g|}\;\mathcal{L})}{\delta \Gamma^J_{MN}} \;=\; \frac{\sqrt{|g|}}{2\kappa^2}\;(\;g^{MN}\;T_J \;-\; T^N \delta^M_J \;+\; T^{MN}_J\;) \;=\; -\;\frac{\delta(\sqrt{|g|}\;\mathcal{L}_{matter})}{\delta \Gamma^J_{MN}} \tag{2.13}$$

The anholonomic version of the field equations in affine theories of gravity with torsion and nonmetricity, and Weyl-Cartan gravity can be found in [11], [8], among others. See [9] and references therein as well. Eqs-(2.10-2.13) are the $C$-space extension of the Einstein-Cartan field equations with zero nonmetricity. One could include a cosmological constant term $\Lambda g_{MN}$ to (2.10) if one wishes. One should note that the field equations (2.10) *contain torsion* since $\mathbf{R}_{(MN)}, \mathbf{R}$ are defined in terms of the connection involving the torsionless Levi-Civita connection and the contorsion. $\mathbf{R}_{MN} = \mathbf{R}_{(MN)} + \mathbf{R}_{[MN]}; \mathbf{R}$ can be decomposed into the standard Riemannian pieces involving the Levi-Civita connection plus torsion squared terms and covariant derivatives of torsion.

A key difference between Einstein-Cartan gravity in $2^D$-dimensions and $C$-space gravity in $D$-dimensions is that one *must supplement* the above field equations (2.6) with the Clifford algebraic compatibility conditions on the connection [3]

$$\nabla_K C^{MNL} \;=\; 0 \tag{2.14}$$

where $C^{MNL}$ are the curved Clifford space structure functions. Such additional equation results in an over-determined system of differential equations. It is important to emphasize that one is implementing the conditions (2.14) $after$ the variation of the action is performed. We shall discuss below the case when the conditions are implemented $before$ a variation of the action is performed. One can view Clifford space as a Clifford group manifold, and as such, strong restrictions on the connections are imposed. Invariant, metric-compatible connections with torsion on homogeneous spaces $G/H$ and their applications to Kaluza-Klein theories were described by [4].

In the vacuum case, if the nonmetricity tensor is set to zero, from eq-(2.11) one can infer that the torsion is zero (on-shell) and one recovers the Levi-Civita connection as expected in the Palatini formulation. However, there is a problem because the torsionless Levi-Civita connection is *not* consistent with the Clifford algebraic compatibility condition of the connection $\nabla_K C^{MNL} = 0$, unless additional differential constraints are imposed on the first derivatives of the metric [3].

One could *add* spin matter sources (like spin fluids) to the action so that the equations (2.13) with spin matter sources generate (non-propagating) torsion terms. Another possibility is to include nonzero nonmetricity $\nabla_M g_{NK} \neq 0$ such that eq-(2.11) is modified by the inclusion of extra nonmetricity terms. One must now go back to the variation of the action and take into account that $\nabla_M(\sqrt{g}\ g_{NK}) \neq 0$ when one integrates by parts. After straightforward algebra one arrives at

$$\nabla_L\ (\sqrt{|g|}\ \delta^M_J\ g^{LN}) - \nabla_J(\sqrt{|g|}\ g^{MN}) + \sqrt{|g|}\ (\ g^{MN}\ T_J\ -\ T^N \delta^M_J\ +\ T^{MN}_J\ )\ =$$

$$-2\kappa^2\ \frac{\delta(\sqrt{|g|}\ \mathcal{L}_{matter})}{\delta\Gamma^J_{MN}} \tag{2.15}$$

Eqs-(2.15) are the $C$-space extension of the equations displayed in the monograph [9], up to an overall minus sign. The first two terms in (2.15) contain nonmetricity terms like $\sqrt{|g|}\ Q^{MN}_J,\ \sqrt{|g|}\ \delta^M_J\ Q^N$.

The most general connection, in a holonomic coordinate basis, is comprised of the torsionless Levi-Civita connection, the contorsion and nonmetricity tensors, respectively

$$\hat\Gamma^L_{MN}\ =\ \{^L_{MN}\}\ +\ K^L_{MN}\ -\frac{1}{2}\ Q^L_{MN} \tag{2.16}$$

In this case one must use this $\hat\Gamma$ connection (2.16) in the field equations for the $C$-space metric *and* also in the Clifford algebraic compatibility condition $\hat\nabla_K C^{MNL} = 0$. Once again, one arrives at an over-determined system of equations that may not have nontrivial solutions. In the rest of this work we will explore the procedure of how to avoid an over-determined system of differential equations

For simplicity we shall set the nonmetricity $Q^L_{MN}$ to zero from now on, so that

$$\Gamma^L_{MN}\ =\ \{^L_{MN}\}\ +\ K^L_{MN} \tag{2.17}$$

In [3] it was shown that a metric compatible connection (zero nonmetricity case) which is consistent with eq-(2.14) is given by

$$\Gamma^K_{MN}\ =\ \frac{1}{2}\ g^{KL}\ \partial_M g_{NL},\quad \Gamma_{MNL}\ =\ \frac{1}{2}\ \partial_M g_{NL}, \tag{2.18}$$

and has torsion $T^K_{MN} = \Gamma^K_{MN} - \Gamma^K_{NM}$. The contorsion tensor is in this case

$$K_{MNK} = \Gamma_{MNK} - \{_{MNK}\}\ =\ \frac{1}{2}(\partial_K g_{NM} - \partial_N g_{KM})\ =\ -\frac{1}{2}\ \Gamma_{[NK]M}\ =\ -\frac{1}{2}\ T_{NKM} \tag{2.19}$$

The contorsion tensor is defined in terms of the components of the torsion tensor as

$$K_{JMN}\ =\ \frac{1}{2}\ (T_{JMN}\ -\ T_{MNJ}\ +\ T_{NJM}) \tag{2.20a}$$

$$K^L_{MN} \ = \ g^{LJ} \ K_{JMN}, \ \ K_{JMN} \ = \ - \ K_{JNM}, \ \ T_{JMN} \ = \ - \ T_{MJN} \quad (2.20b)$$

One can verify that the contorsion $K_{MNK}$ found above in eq-(2.19) is indeed *consistent* with the definition of the contorsion in eqs-(2.20) after plugging-in directly the expression for the connection given by eq-(2.18). $K_{MNK}$ is antisymmetric in the last two indices while $T_{NKM}$ is antisymmetric in its first two indices. In an ordinary manifold, the contorsion is a one-form while the torsion is a two-form. This can more easily be seen in the anholonomic basis. The contorsion one-form is $K^{ab}_\mu dx^\mu$ and the torsion two-form is given in terms of the bein $e^a = e^a_\mu dx^\mu$, and spin connection $\omega^{ab}_\mu dx^\mu$ as $T^a_{\mu\nu} dx^\mu \wedge dx^\nu = \mathbf{d}e^a + \omega^a_b \wedge e^b$. One should add that the decomposition

$$\Gamma_{MNK} \ = \ \Gamma_{(MN)K} \ + \ \Gamma_{[MN]K} \quad (2.21)$$

of the metric compatible connection $\Gamma_{MNK} = \frac{1}{2}\partial_M g_{NK}$ is not desirable because the connection piece $\Gamma_{(MN)K}$ is *not* metric compatible, whereas the Levi-Civita connection $\{_{MNK}\}$ is. The correct decomposition is the one displayed in eqs-(2.17).

The results (2.18) were obtained in the so-called "diagonal gauge" [3] where in a given coordinate system (generalized Lorentz frame) the mixed-grade components of the metric $g_{MN}, g^{MN}$, and beins $E^A_M$, inverse beins $E^M_A$, can be set to zero in order to considerably simplify the calculations; namely due to the very large diffeomorphism symmetry in $C$-space, one may choose a frame ("diagonal gauge") such that the mixed grade components of the metric, beins, inverse beins are zero. In this case, the Clifford algebra associated to the curved space basis generators $\gamma_M = E^A_M \gamma_A$ assumes the *same* functional form as it does in the flat tangent space, and obeys the (graded) Jacobi identities. The metric, beins, inverse beins, admit a decomposition into their irreducible pieces like in eq-(2.22) below. Only a restricted set of poly-coordinate transformations (generalized Lorentz transformations in the tangent space) will preserve such zero mixed-grade condition, namely the grade-preserving transformations.

The same-grade metric components $g^{[\mu_1\mu_2\ldots\mu_k] \ [\nu_1\nu_2\ldots\nu_k]}$ can be decomposed into its irreducible factors as antisymmetrized sums of products of $g^{\mu\nu}$ as follows

$$det \begin{pmatrix} g^{\mu_1\nu_1} & \ldots & \ldots & g^{\mu_1\nu_k} \\ g^{\mu_2\nu_1} & \ldots & \ldots & g^{\mu_2\nu_k} \\ ----------- & & --------------- & \\ g^{\mu_k\nu_1} & \ldots & \ldots & g^{\mu_k\nu_k} \end{pmatrix} \quad (2.22)$$

One can avoid an over-determined system of differential equations by implementing the compatibility conditions (2.14), in the diagonal gauge, *before* performing a variation of the action; i.e. to go ahead and plug-in directly the connection (2.18) in terms of the metric and derivatives (no longer following a Palatini variation procedure), and to insert the metric decomposition (2.22), before performing the variation of the action, leading to

$$\delta S \ = \ \frac{\delta S}{\delta g^{00}} \ \delta g^{00} \ + \ \frac{\delta S}{\delta g^{\mu\nu}} \ \delta g^{\mu\nu} \ +$$

$$\frac{\delta S}{\delta g^{[\mu_1\mu_2]\ [\nu_1\nu_2]}}\ \frac{\delta g^{[\mu_1\mu_2]\ [\nu_1\nu_2]}}{\delta g^{\mu\nu}}\ \delta g^{\mu\nu}\ +\ .... \tag{2.23}$$

A variation of the connection (2.18) gives

$$\delta\Gamma^J_{MN}\ =\ \frac{1}{2}\ (\partial_M g_{NK})\ \delta g^{JK}\ +\ \frac{1}{2}\ g^{JK}\ \partial_M(\delta g_{NK}) \tag{2.24}$$

Defining the poly-tensor density which appears in the variation of the action displayed in (2.9) as

$$P^{MN}_J\ \equiv\ \sqrt{|g|}\ \mathcal{P}^{MN}_J\ \equiv\ \sqrt{|g|}\ (\ g^{MN}\ T_J\ -\ T^N\delta^M_J\ +\ T^{MN}_J\ ) \tag{2.25$a$}$$

$$P^M_{JN}\ \equiv\ \sqrt{|g|}\ \mathcal{P}^M_{JN}\ \equiv\ \sqrt{|g|}\ (\ \delta^M_N\ T_J\ -\ \delta^M_J\ T_N\ +\ T^M_{JN}\ ) \tag{2.25$b$}$$

and after using the relations

$$(\Gamma^N_{MN}\ -\ \Gamma^N_{NM})\ P^M_{JK}\ =\ T_M\ P^M_{JK} \tag{2.26}$$

combined with

$$(\ \Gamma_{MJN}\ P^{MN}_K\ -\ \Gamma_{MKN}\ P^{MN}_J\ )\ \delta g^{JK}\ =\ 0 \tag{2.27}$$

due to the symmetry of $\delta g^{JK}$ and antisymmetry under the exchange of $JK$ indices of the terms inside the parenthesis, one can show after some algebra and integrating by parts, that the explicit torsion terms contribution to the variation of the $C$-space gravitational Lagrangian density is

$$P^{MN}_J\ \delta\Gamma^J_{MN}\ =\ -\frac{1}{2}\ (\ T_M\ P^M_{JK}\ +\ \nabla_M P^M_{JK}\ )\ \delta g^{JK} \tag{2.28$a$}$$

so the vacuum field equations become

$$\mathbf{R}_{(JK)}\ -\frac{1}{2}\ g_{JK}\ \mathbf{R}\ -\frac{1}{2}\ (T_L\ \mathcal{P}^L_{(JK)}\ +\ \nabla_L\mathcal{P}^L_{(JK)})\ =\ \mathbf{R}_{(JK)}\ -\frac{1}{2}\ g_{JK}\ \mathbf{R}\ =\ 0 \tag{2.28$b$}$$

because $\mathcal{P}^L_{(JK)} = 0$ due to the antisymmetry of the tensor $\mathcal{P}^L_{JK}$ in the $JK$ indices as described in (2.25b). For antisymmetric metrics the terms in (2.28a) will not be zero. For symmetric metrics (2.28a) is zero due to the symmetry of $\delta g^{JK}$ and antisymmetry of $\mathcal{P}^L_{JK} = 0$.

Decomposing the curvatures in (2.28b) into their Riemmanian-like pieces plus torsion terms yield

$$\mathbf{R}_{JK}(\{\})\ -\frac{1}{2}\ g_{JK}\ \mathbf{R}(\{\})\ +\frac{1}{2}\ T^{MN}_{(J}\ T_{K)MN}\ -\frac{3}{8}\ g_{JK}\ T_{LMN}\ T^{LMN}\ =\ 0 \tag{2.28$c$}$$

where we denote the Riemannian parts by $\mathbf{R}_{JK}(\{\}) = \mathbf{R}_{KJ}(\{\}); \mathbf{R}(\{\})$.

There is a caveat because one must not forget that the higher grade $g^{MN}$ metric components are *not* independent functions of $g^{\mu\nu}$ due to the decomposition in (2.22) that result after imposing the Clifford algebraic compatibility conditions (2.14) in the diagonal gauge [3]. For this reason instead of having the field equations (2.28b, 2.28c), the $C$-space vacuum gravitational field equations in the diagonal gauge become, after renaming indices,

$$\frac{\delta(\sqrt{|g|\mathcal{L})}}{\delta g^{00}} - \partial( \frac{\delta(\sqrt{|g|\mathcal{L})}}{\delta\ \partial g^{00}} ) = 0 \tag{2.29}$$

and

$$\frac{\delta(\sqrt{|g|\mathcal{L})}}{\delta g^{\mu\nu}} - \partial( \frac{\delta(\sqrt{|g|\mathcal{L})}}{\delta\ \partial g^{\mu\nu}} ) +$$

$$\left( \frac{\delta(\sqrt{|g|\mathcal{L})}}{\delta g^{[\mu_1\mu_2]\ [\nu_1\nu_2]}} - \partial( \frac{\delta(\sqrt{|g|\mathcal{L})}}{\delta\ \partial g^{[\mu_1\mu_2]\ [\nu_1\nu_2]}} \right) \frac{\delta g^{[\mu_1\mu_2]\ [\nu_1\nu_2]}}{\delta g^{\mu\nu}} + ....... = 0 \tag{2.30}$$

with

$$\frac{\delta(\sqrt{|g|\mathcal{L})}}{\delta g^{\mu\nu}} - \partial(.......) \leftrightarrow \mathbf{R}_{(\mu\nu)} - \frac{1}{2}\ g_{\mu\nu}\ \mathbf{R} \tag{2.31}$$

the other remaining contributions of the polyvector-components denoted by the hatted indices give

$$\frac{\delta(\sqrt{|g|\mathcal{L})}}{\delta g^{\hat{M}\hat{N}}} - \partial(.........) \leftrightarrow \mathbf{R}_{(\hat{M}\hat{N})} - \frac{1}{2}\ g_{\hat{M}\hat{N}}\ \mathbf{R} \tag{2.32}$$

such that the $g_{\mu\nu}$ field equations in (2.30) acquire now the *extra* terms given by

$$\Delta_{\mu\nu} = \left( \mathbf{R}_{(\hat{M}\hat{N})} - \frac{1}{2}\ g_{\hat{M}\hat{N}}\ \mathbf{R} \right) \frac{\delta g^{\hat{M}\hat{N}}}{\delta g^{\mu\nu}} \tag{2.33}$$

These extra terms (2.33) to eqs-(2.31) have the same role as an *effective* stress energy tensor term $\kappa^2 \mathbf{T}^{eff}_{\mu\nu}$ contribution, up to a minus sign. The advantage of using the connection (2.18) and the metric decomposition (2.22) *before* the variation of the action takes place, in the diagonal gauge, is that one does *not* longer have an over-determined system of differential equations. Therefore, the effective field equations for the ordinary metric $g^{\mu\nu}$ in the so-called "diagonal" gauge described earlier become

$$\mathbf{R}_{(\mu\nu)} - \frac{1}{2}\ g_{\mu\nu}\ \mathbf{R} = - \Delta_{\mu\nu} = \kappa^2\ \mathbf{T}^{eff}_{\mu\nu} \tag{2.34}$$

where the effective $\mathbf{T}^{eff}_{\mu\nu}$ is defined in terms of the higher grade metric components as shown in eq-(2.33). The relevance of having one single set of equations in (2.34), instead of setting to zero, separately, both equations in (2.31, 2.32), is that one does not longer have an over-determined system of differential equations for $g^{\mu\nu}$. One should notice once more that the higher grade $g^{\hat{M}\hat{N}}$ metric components are functions of $g^{\mu\nu}$ due to the decomposition in (2.22). For this reason if one were to set to zero both eqs-(2.31, 2.32) this would lead to an over-determined system of equations for the ordinary metric $g^{\mu\nu}$.

Furthermore, the most salient and physically relevant feature is that the effective $\mathbf{T}^{eff}_{\mu\nu}$ may be interpreted as dark energy/dark matter. It has been known for a long time that the torsion terms inside $\mathbf{R}_{(\mu\nu)}, \mathbf{R}$ may contribute to dark energy/dark matter as well [11], [8]. What is *novel* in $C$-space gravity, to our knowledge, besides having torsion is :

(i) the contribution of the *higher* grade metric components $g^{\hat{M}\hat{N}}$ of the $C$-space metric to dark energy/dark matter as described in eqs-(2.33, 2.34).

(ii) No spin matter is required to induce torsion because torsion is imposed by the Clifford algebraic structure itself and which is reflected in (2.14) leading to a connection with torsion (2.18). Even in the absence of matter one has torsion in $C$-space. Even in the case for those metrics which might obey the restricted condition $\partial_M g_{NL} = \partial_N g_{ML}$ constraining the torsion to zero, we *still* have the contribution $\Delta_{\mu\nu}$ of the *higher* grade metric components to the field equations (2.34) mimicking the role of dark energy/dark matter.

A model of dilaton matter based on particles which are endowed with intrinsic spin and dilaton charge (a dilaton-spin fluid) has been considered as the source of the gravitational field in a Weyl-Cartan spacetime [11], [8]. Dark matter was modeled in terms of the dilaton and inflation-like and accelerating expansion solutions for the universe were obtained. Since $C$-space gravity encodes Weyl scalings as shown in [1] the latter dilaton-spin fluid model can naturally be incorporated within the framework of $C$-space gravity. However as stated above, *no* spin matter is required to induce torsion in $C$-space.

Furthermore, no dilaton matter charge is required either because the Clifford *scalar* component $g^{00}$ of the $C$-space metric is a scalar field which can replace the dilaton. It is tempting to speculate whether the Clifford *scalar* $g^{00}$ might be related to the Higgs. The Clifford pseudo-scalar component $g^{\mu_1\mu_2....\mu_D \ \nu_1\nu_2....\nu_D}$ of the $C$-space metric is the pseudoscalar counterpart and due to the decomposition (2.22) it is a composite field made out of $g^{\mu\nu}$.

One must include also the additional equation to (2.34)

$$\frac{\delta(\sqrt{|g|}\mathcal{L})}{\delta g^{00}} \ - \ \partial( \ \frac{\delta(\sqrt{|g|}\mathcal{L})}{\delta \ \partial g^{00}} \ ) = 0 \ \Rightarrow \ \mathbf{R}_{00} \ - \frac{1}{2} \ g_{00} \ \mathbf{R} \ = \ 0 \qquad (2.35)$$

Because there is the residual grade-preserving symmetry transformations of the polyvector valued coordinates in $C$-space, and which is left over after the diagonal gauge is imposed [3], this is the symmetry operating in the effective field equations (2.34, 2.35) for $g^{\mu\nu}$ and $g^{00}$, respectively.

In forthcoming work we shall freeze the dependence on all polyvector coordinates except the vectorial ones $x^\mu$ in the field equations (2.34, 2.35), to simplify the calculations, and explore the correspondence between our numerical results and current astrophysical observations. Another project which warrants investigation is the introduction of Finsler geometric structures in Clifford spaces. Clifford-Finsler Algebroids and Nonholonomic Einstein-Dirac Structures have been introduced by [7].

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