# Progress in Clifford Space Gravity

Carlos Castro
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, Georgia. 30314, perelmanc@hotmail.com

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**Abstract**

Clifford-space Gravity is revisited and new results are found. The Clifford space ($C$-space) generalized gravitational field equations are obtained from a variational principle and which is based on an extension of the Einstein-Hilbert-Cartan action. One of the main results of this work is that the $C$-space connection requires torsion in order to have consistency between the Clifford algebraic structure and the zero nonmetricity condition $\nabla_K g^{MN} = 0$. A discussion on the cosmological constant and bi-metric theories of gravity follows. We continue by pointing out the relations of Clifford space gravity to Lanczos-Lovelock-Cartan (LLC) higher curvature gravity with torsion. We finalize by pointing out that $C$-space gravity involves higher-spins beyond spin 2 and argue why one could view the LLC higher curvature actions, and other extended gravitational theories based on $f(R), f(R_{\mu\nu}), ...$ actions, for polynomial-valued functions, as mere $effective$ actions after integrating the $C$-space gravitational action with respect to all the poly-coordinates, except the vectorial ones $x^\mu$.

## 1 Introduction

In the past years, the Extended Relativity Theory in $C$-spaces (Clifford spaces) and Clifford-Phase spaces were developed [1], [2]. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory whose generalized coordinates are Clifford polyvector-valued quantities which incorporate the lines, areas, volumes, and hyper-volumes degrees of freedom associated with the collective dynamics of particles, strings, membranes, p-branes (closed p-branes) moving in a D-dimensional target spacetime background. C-space Relativity permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing. Our theory has 2 fundamental parameters : the speed of a light $c$ and a length scale which can be set to be equal to the Planck length. The role of "photons" in $C$-space is played by *tensionless* branes. An extensive review of the Extended Relativity Theory in Clifford spaces can be found in [1].

The poly-vector valued coordinates $x^\mu, x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, ...$ are now linked to the basis vectors generators $\gamma^\mu$, bi-vectors generators $\gamma_\mu \wedge \gamma_\nu$, tri-vectors generators $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$, ... of the Clifford algebra, including the Clifford algebra unit element (associated to a scalar coordinate). These poly-vector valued coordinates can be interpreted as the quenched-degrees of freedom of an ensemble of $p$-loops associated with the dynamics of closed $p$-branes, for $p = 0, 1, 2, ..., D-1$, embedded in a target $D$-dimensional spacetime background.

The $C$-space poly-vector-valued momentum is defined as $\mathbf{P} = d\mathbf{X}/d\Sigma$ where $\mathbf{X}$ is the Clifford-valued coordinate corresponding to the $Cl(1,3)$ algebra in four-dimensions

$$\mathbf{X} \;=\; \sigma\,\mathbf{1} + x^\mu\,\gamma_\mu + x^{\mu\nu}\,\gamma_\mu\wedge\gamma_\nu + x^{\mu\nu\rho}\,\gamma_\mu\wedge\gamma_\nu\wedge\gamma_\rho + x^{\mu\nu\rho\tau}\,\gamma_\mu\wedge\gamma_\nu\wedge\gamma_\rho\wedge\gamma_\tau \tag{1.1}$$

$\sigma$ is the Clifford scalar component of the poly-vector-valued coordinate and $d\Sigma$ is the infinitesimal $C$-space proper "time" interval which is *invariant* under $Cl(1,3)$ transformations which are the Clifford-algebra extensions of the $SO(1,3)$ Lorentz transformations [1]. One should emphasize that $d\Sigma$, which is given by the square root of the quadratic interval in $C$-space

$$(d\Sigma)^2 \;=\; (d\sigma)^2 \;+\; dx_\mu\,dx^\mu \;+\; dx_{\mu\nu}\,dx^{\mu\nu} \;+\; \ldots \tag{1.2}$$

is *not* equal to the proper time Lorentz-invariant interval $ds$ in ordinary spacetime $(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu = dx_\mu dx^\mu$. For extensive details and references we refer to [1].

The main purpose of this work is to revisit Clifford-space Gravity [1], [7] where new results are found. The Clifford space ($C$-space) generalized gravitational field equations are obtained from a variational principle and which is based on an extension of the Einstein-Hilbert-Cartan action. One of the main results of this work is that the $C$-space connection requires torsion in order to have consistency between the Clifford algebraic structure and the zero nonmetricity condition $\nabla_K g^{MN} = 0$. A discussion on the cosmological constant and bi-metric theories of gravity follows. We continue by pointing out the relations of Clifford space gravity to Lanczos-Lovelock-Cartan (LLC) higher curvature gravity with torsion. We finalize by pointing out that $C$-space gravity involves higher-spins beyond spin 2 and argue why one could view the LLC higher curvature actions, and other extended gravitational theories based on $f(R), f(R_{\mu\nu}), ...$ actions, for polynomial-valued functions, as mere $effective$ actions after integrating the $C$-space gravitational action with respect to all the poly-coordinates, except the vectorial ones $x^\mu$.

## 2 Geometry of Curved Clifford Space

Let the vector fields $\gamma_\mu$, $\mu = 1, 2, ..., n$ be a coordinate basis in $V_n$ satisfying the Clifford algebra relation

$$\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = g_{\mu\nu} \tag{2.1}$$

where $g_{\mu\nu}$ is the metric of $V_n$. In curved space $\gamma_\mu$ and $g_{\mu\nu}$ cannot be constant but necessarily depend on position $x^\mu$. An arbitrary vector is a linear superposition [3] $a = a^\mu \gamma_\mu$ where the components $a^\mu$ are *scalars* from the geometric point of view, whilst $\gamma_\mu$ are *vectors*.

Besides the basis $\gamma_\mu$ we can introduce the reciprocal dual basis $\gamma^\mu$ satisfying

$$\gamma^\mu \cdot \gamma^\nu \equiv \frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = g^{\mu\nu} \tag{2.2}$$

where $g^{\mu\nu}$ is the covariant metric tensor such that

$$g^{\mu\alpha}\ g_{\alpha\nu}\ =\ \delta^\mu{}_\nu,\ \ \{\gamma^\mu, \gamma_\nu\}\ =\ 2\delta^\mu{}_\nu,\ \ \gamma^\mu = g^{\mu\nu}\gamma_\nu \tag{2.3}$$

Let us now consider $C$-space [1], [4]. A basis in $C$-space is given by

$$E_A = \gamma,\ \gamma_\mu,\ \gamma_\mu \wedge \gamma_\nu,\ \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho,\ ... \tag{2.4}$$

where $\gamma$ is the unit element of the Clifford algebra that we label as $\mathbf{1}$ from now on. In (2.4) when one writes an $r$-vector basis $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge ... \wedge \gamma_{\mu_r}$ we take the indices in "lexicographical" order so that $\mu_1 < \mu_2 < .... < \mu_r$. An element of $C$-space is a Clifford number, called also *Polyvector* or *Clifford aggregate* which we now write in the form

$$X\ =\ X^A E_A\ =\ s\,\mathbf{1}\ +\ x^\mu \gamma_\mu\ +\ x^{\mu\nu}\gamma_\mu \wedge \gamma_\nu + ... \tag{2.5}$$

A $C$-space is parametrized not only by 1-vector coordinates $x^\mu$ but also by the 2-vector coordinates $x^{\mu\nu}$, 3-vector coordinates $x^{\mu\nu\alpha}$, ..., called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops,..., onto the coordinate planes. By $p$-loop we mean a closed $p$-brane; in particular, a 1-loop is closed string. In order to avoid using the powers of the Planck scale length parameter $L_p$ in the expansion of the poly-vector $X$ we can set set to unity to simplify matters.

In a *flat* $C$-space the basis vectors $E^A, E_A$ are *constants*. In a *curved* $C$-space this is no longer true. Each $E^A, E_A$ is a function of the $C$-space coordinates

$$X^A\ =\ s,\ x^\mu,\ x^{\mu_1\mu_2},\ .....\ x^{\mu_1\mu_2.....\mu_D} \tag{2.6}$$

which include scalar, vector, bivector,..., $r$-vector,... coordinates in the underlying $D$-dim base spacetime and whose corresponding $C$-space is $2^D$-dimensional since the Clifford algebra in $D$-dim is $2^D$-dimensional.

In curved $C$-space one introduces the $\mathbf{X}$-dependent basis generators $\gamma_M, \gamma^M$ defined in terms of the beins $E^A_M$, inverse beins $E^M_A$ and the flat tangent space generators $\gamma_A, \gamma^B$ as follows $\gamma_M = E^A_M(\mathbf{X})\gamma_A, \gamma^M = E^M_A(\mathbf{X})\gamma^A$.

The covariant derivative of $E^A_M(\mathbf{X}), E^M_A(\mathbf{X})$ involves the ordinary and spin connection and is defined as

$$\nabla_K E^A_M\ =\ \partial_K E^A_M\ -\ \Gamma^L_{KM}\ E^A_L\ +\ \omega^A_{KB}\ E^B_M \tag{2.7a}$$

$$\nabla_K E^M_A\ =\ \partial_K E^M_A\ +\ \Gamma^M_{KL}\ E^L_A\ -\ \omega_{KA}{}^B\ E^M_B \tag{2.7b}$$

If the nonmetricity is zero then $\nabla_K E^A_M = 0, \ \nabla_K E^M_A = 0$ in eqs-(2.7). If the nonmetricity is *not* zero one must include *extra* terms in the right hand side of eqs-(2.7). In this latter case $\nabla_K E^A_M \neq 0, \ \nabla_K E^M_A \neq 0$. To simplify matters we shall set the nonmetricity $Q_{KMN} = \nabla_K g_{MN} = 0$, to *zero* such that

$$\nabla_K g_{MN} \ = \ < (\nabla_K \gamma_M) \ \gamma_N > \ + \ < \gamma_M \ (\nabla_K \gamma_N) > \ =$$

$$\partial_K g_{MN} \ - \ \Gamma^L_{KM} \ g_{LN} \ - \ \Gamma^L_{KN} \ g_{LM} \ = \ 0 \tag{2.8}$$

There are two different choices of connections compatible with the zero nonmetricity conditions in eqs-(2.7, 2.8). One choice [5] requires the covariantly-constancy condition on the curved *and* tangent $C$-space basis generators

$$\nabla_K \gamma_M \ = \ \nabla_K (E^A_M \gamma_A) \ = \ E^A_M \nabla_K \gamma_A \ = \ 0 \tag{2.9}$$

since $\nabla_K E^A_M = 0$ when the nonmetricity is zero. From (2.9) one infers

$$\nabla_K \gamma_A \ = \ \partial_K \gamma_A + \omega_{KAB} \gamma^B = 0 \ \Rightarrow \ \partial_K \gamma_A \ = \ - \ \omega_{KAB} \gamma^B \tag{2.10a}$$

and

$$\nabla_K \gamma_M \ = \ \partial_K \gamma_M \ - \ \Gamma^L_{KM} \ \gamma_L \ = \ 0 \ \Rightarrow \ \partial_K \gamma_M \ = \ \Gamma^L_{KM} \ \gamma_L \tag{2.11a}$$

$$\nabla_K \gamma^M \ = \ \partial_K \gamma^M \ + \ \Gamma^M_{KL} \ \gamma^L \ = \ 0 \ \Rightarrow \ \partial_K \gamma^M \ = \ - \ \Gamma^M_{KL} \ \gamma^L \tag{2.11b}$$

Another choice which is also consistent with the vanishing nonmetricity condition in eqs-(2.7, 2.9) is to have truly constant $\gamma_A$ [7] such that now one has

$$\nabla_K \gamma_M \ = \ \nabla_K (E^A_M \gamma_A) \ = \ E^A_M \ \nabla_K \gamma_A \ = \ E^A_M \ \omega_{KAB} \gamma^B \tag{2.12}$$

after using $\nabla_K \gamma_A = \omega_{KAB} \gamma^B$ due to the fact that $\partial_K \gamma_A = 0$ when the tangent space gamma generators $\gamma_A$ are *constants*. Inserting eq-(2.12) into eq-(2.8) also yields a zero nonmetricity result as well

$$\nabla_K g_{MN} \ = \ < (\nabla_K \gamma_M) \ \gamma_N > \ + \ < \gamma_M \ (\nabla_K \gamma_N) > \ =$$

$$\omega_{KAB} \ ( \ E^A_M E^B_N \ + \ E^A_N E^B_M \ ) \ = \ 0 \tag{2.13}$$

because the last terms of (2.13) in the parenthesis (multiplying $\omega_{KAB}$) are symmetric under the index exchange $A \leftrightarrow B$, whereas the spin connection prefactor $\omega_{KAB} = -\omega_{KBA}$ is anti-symmetric. A parallel transport of $\gamma_A$ requires a (spin) connection and for this reason it is reasonable to set $\nabla_K \gamma_A = \omega_{KAB} \gamma^B$. Similarly one finds that $\nabla_K \eta_{AB} = 0$, after recurring to the definition $\eta_{AB} =< \gamma_A \gamma_B >$ and due to symmetry property of the constant tangent space metric $\eta_{AB}$ under the exchange of indices, and the anti-symmetry of $\omega_{KAB}$.

One may notice that eq-(2.12) can also be recast as a covariantly-constancy condition with respect to a new connection by moving the omega terms to the left-hand side and reabsorbing them into a redefinition of the connection as

$$\hat{\Gamma}^L_{KM} \ = \ \Gamma^L_{KM} \ + \ \omega_{KAB} \ E^A_M \ E^{BL} \ \Rightarrow$$

$$\hat{\nabla}_K \gamma_M \ = \ \partial_K \gamma_M \ - \ \hat{\Gamma}^L_{KM} \ \gamma_L \ = \ 0 \ \Rightarrow \ \hat{\nabla}_K g_{MN} \ = \ 0 \tag{2.14}$$

To simplify matters in the rest of this work we will simply choose to work with the connections appearing in eqs-(2.11) keeping in mind the possibility of working with the hatted connections (2.14) which are consistent with having truly constant tangent space generators $\gamma_A$.

If the connection is *symmetric* in the first two indices $\Gamma_{KMN} = \Gamma_{MKN}$ one can arrive at the torsionless Levi-Civita-like expression. This can be attained as usual by an index permutation of the zero nonmetricity condition

$$\partial_K \ g_{MN} \ - \ \Gamma_{KMN} \ - \ \Gamma_{KNM} \ = \ 0 \tag{2.15$a$}$$

$$\partial_M \ g_{KN} \ - \ \Gamma_{MKN} \ - \ \Gamma_{MNK} \ = \ 0 \tag{2.15$b$}$$

$$\partial_N \ g_{MK} \ - \ \Gamma_{NMK} \ - \ \Gamma_{NKM} \ = \ 0 \tag{2.15$c$}$$

after subtracting eqs-(2.15b, 2.15c) from eq-(2.15a) one arrives at the Levi-Civita-like expression for the connection

$$\Gamma_{MNK} \ = \ \frac{1}{2} \left( \partial_M g_{KN} \ + \ \partial_N g_{MK} \ - \ \partial_K g_{MN} \right) \tag{2.16}$$

due to the symmetry property $\Gamma_{KMN} = \Gamma_{MKN}$ in the first two indices.

In general, $C$-space admits torsion [1] and the connection $\Gamma^N_{KM} \neq \Gamma^N_{MK}$ is not symmetric. For example, if $\Gamma_{KMN} = \Gamma_{KNM}$ is symmetric in the *last* two indices, then from eq-(2.8) one can infer that there is a different connection $\Gamma_{KMN} = \frac{1}{2}\partial_K g_{MN} \neq \Gamma_{MKN} = \frac{1}{2}\partial_M g_{KN}$ which has torsion.

The *torsion* is defined as $T^N_{KM} = \Gamma^N_{KM} - \Gamma^N_{MK}$ in $C$-space, assuming the anholonomy coefficients $f^N_{KM}$ are zero, $[\partial_K, \partial_M] = f^N_{KM}\partial_N$. If the latter coefficients are not zero one must include $f^N_{KM}$ into the definition of Torsion as follows

$$T^N_{KM} \ = \ \Gamma^N_{KM} \ - \ \Gamma^N_{MK} \ - \ f^N_{KM} \tag{2.17}$$

In the case of nonsymmetric connections with torsion, the curvature obeys the following relations under the exchange of indices

$$\mathbf{R}_{MNJK} \ = \ - \ \mathbf{R}_{NMJK}, \ \ \mathbf{R}_{MNKJ} \ = \ - \ \mathbf{R}_{MNJK}, \ \ \textit{but} \ \ \mathbf{R}_{MNJK} \ \neq \ \mathbf{R}_{JKMN} \tag{2.18}$$

and is defined, when $f^J_{MN} = 0$, in terms of the connection components $\Gamma^L_{KM}$ as follows

$$\mathbf{R}_{MNJ}{}^K \ = \ \partial_M \ \Gamma^K_{NJ} \ - \ \partial_N \ \Gamma^K_{MJ} \ + \ \Gamma^K_{ML} \ \Gamma^L_{NJ} \ - \ \Gamma^K_{NL} \ \Gamma^L_{MJ} \tag{2.19}$$

If $f^K_{MN} \neq 0$ one must also include these anholonomy coefficients into the definition of curvature (2.19) by adding terms of the form $-f^L_{MN}\Gamma^K_{LJ}$.

The standard Riemann-Cartan curvature tensor in ordinary spacetime is *contained* in $C$-space as follows

$$\mathcal{R}_{\mu_1\mu_2\rho_1}{}^{\rho_2} \ = \ \partial_{\mu_1}\Gamma^{\rho_2}_{\mu_2\rho_1} \ - \ \partial_{\mu_2}\Gamma^{\rho_2}_{\mu_1\rho_1} \ + \ \Gamma^{\rho_2}_{\mu_1\sigma} \ \Gamma^{\sigma}_{\mu_2\rho_1} \ - \ \Gamma^{\rho_2}_{\mu_2\sigma} \ \Gamma^{\sigma}_{\mu_1\rho_1} \ \subset$$

$$\mathbf{R}_{\mu_1\mu_2\rho_1}{}^{\rho_2} = \partial_{\mu_1}\Gamma^{\rho_2}_{\mu_2\rho_1} - \partial_{\mu_2}\Gamma^{\rho_2}_{\mu_1\rho_1} + \Gamma^{\rho_2}_{\mu_1\ \mathbf{M}}\,\Gamma^{\mathbf{M}}_{\mu_2\rho_1} - \Gamma^{\rho_2}_{\mu_2\ \mathbf{M}}\,\Gamma^{\mathbf{M}}_{\mu_1\rho_1} \quad (2.20)$$

due to the contractions involving the poly-vector valued indices $\mathbf{M}$ in eq-(2.20) There is also the crucial difference that $\mathbf{R}^{\rho_2}_{\mu_1\mu_2\rho_1}(s, x^\nu, x^{\nu_1\nu_2}, ...)$ has now an *additional* dependence on all the $C$-space poly-vector valued coordinates $s, x^{\nu_1\nu_2}, x^{\nu_1\nu_2\nu_3}, ...$ besides the $x^\nu$ coordinates. The curvature in the presence of torsion does not satisfy the same symmetry relations when there is no torsion, therefore the Ricci-like tensor is no longer symmetric

$$\mathbf{R}_{MNJ}{}^{N} = \mathbf{R}_{MJ}, \quad \mathbf{R}_{MJ} \neq \mathbf{R}_{JM}, \quad \mathbf{R} = g^{MJ}\,\mathbf{R}_{MJ} \quad (2.21)$$

The $C$-space Ricci-like tensor is

$$\mathbf{R}_M^N = \sum_{j=1}^{D} \mathbf{R}_{M\ [\nu_1\nu_2...\nu_j]}^{\ N\ [\nu_1\nu_2...\nu_j]} + \mathbf{R}_{M\ \mathbf{0}}^{\ N\ \mathbf{0}} \quad (2.22)$$

and the $C$-space curvature scalar is

$$\mathbf{R} = \sum_{j=1}^{D}\sum_{k=1}^{D} \mathbf{R}_{[\mu_1\mu_2...\mu_j]\ [\nu_1\nu_2...\nu_k]}{}^{[\mu_1\mu_2...\mu_j]\ [\nu_1\nu_2...\nu_k]} + \sum_{j=1}^{D} \mathbf{R}_{[\mu_1\mu_2...\mu_j]\ \mathbf{0}}{}^{[\mu_1\mu_2...\mu_j]\ \mathbf{0}} \quad (2.23)$$

To finalize this section we add some remarks about the physical applications of $C$-space gravity to higher curvature theories of gravity [7]. One of the key properties of Lanczos-Lovelock-Cartan gravity (with torsion) is that the field equations do not contain higher derivatives of the metric tensor beyond the second order due to the fact that the action does not contain derivatives of the curvature, see [8], [12], [11] and references therein.

The $n$-th order Lanczos-Lovelock-Cartan curvature tensor is defined as

$$\mathcal{R}^{(n)}{}_{\mu_1\mu_2...\mu_{2n}}^{\ \rho_1\rho_2...\rho_{2n}} = \delta^{\rho_1\rho_2...\rho_{2n}}_{\tau_1\tau_2...\tau_{2n}}\,\delta^{\nu_1\nu_2...\nu_{2n}}_{\mu_1\mu_2...\mu_{2n}}\,\mathcal{R}_{\nu_1\nu_2}^{\ \tau_1\tau_2}\,\mathcal{R}_{\nu_3\nu_4}^{\ \tau_3\tau_4}\ \ldots\ \mathcal{R}_{\nu_{2n-1}\nu_{2n}}^{\ \tau_{2n-1}\tau_{2n}} \quad (2.24)$$

the $n$-th order Lovelock curvature scalar is

$$\mathcal{R}^{(n)} = \delta^{\nu_1\nu_2...\nu_{2n}}_{\tau_1\tau_2...\tau_{2n}}\,\mathcal{R}_{\nu_1\nu_2}^{\ \tau_1\tau_2}\,\mathcal{R}_{\nu_3\nu_4}^{\ \tau_3\tau_4}\ \ldots.\ \mathcal{R}_{\nu_{2n-1}\nu_{2n}}^{\ \tau_{2n-1}\tau_{2n}} \quad (2.25)$$

the above curvature tensors are antisymmetric under the exchange of any of the $\mu$ ($\rho$) indices. The Lanczos-Lovelock-Cartan Lagrangian density is

$$\mathcal{L} = \sqrt{g}\sum_{n=0}^{[\frac{D}{2}]} c_n\,\mathcal{L}_n, \quad \mathcal{L}_n = \frac{1}{2^n}\,\mathcal{R}^{(n)} \quad (2.26)$$

where $c_n$ are arbitrary coefficients; the first term corresponds to the cosmological constant. The integer part is $[\frac{D}{2}] = \frac{D}{2}$ when $D = even$, and $\frac{D-1}{2}$ when $D = odd$. The general Lanczos-Lovelock-Cartan (LLC) theory in $D$ spacetime dimensions is given by the action

$$S \;=\; \int d^D x\; \sqrt{|g|}\; \sum_{n=0}^{[\frac{D}{2}]} c_n\, \mathcal{L}_n, \tag{2.27}$$

A simple ansatz relating the LLC higher curvatures to $C$-space curvatures is based on the following contractions [7]

$$\frac{c_n}{2^n}\; \mathcal{R}^{(n)}{}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\nu_1\nu_2\ldots\nu_{2n}} \;=\; \sum_{k=1}^{D} \mathbf{R}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\nu_1\nu_2\ldots\nu_{2n}}{}_{\rho_1\rho_2\ldots\rho_k}{}^{\rho_1\rho_2\ldots\rho_k} \;+\; \mathbf{R}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\nu_1\nu_2\ldots\nu_{2n}}{}_{\mathbf{0}}{}^{\mathbf{0}} \tag{2.28}$$

where one must take a *slice* in $C$-space which requires to evaluate all the terms in the right hand side of eqs-(2.28) at the "points" $s = x^{\mu_1\mu_2} = \ldots = x^{\mu_1\mu_2\ldots\mu_D} = 0$, for all $x^\mu$, since the left hand side of eqs-(2.28) solely depends on the vector coordinates $x^\mu$.

After evaluating the $C$-space scalar curvature (2.23), setting the values of all the poly-vector coordinates to zero, except the $x^\mu$ coordinates, one can relate it to the LLC Lagrangian, up to the cosmological constant ( the $c_o$ term ), as follows

$$\mathbf{R}\; [x^\mu; s = x^{\mu_1\mu_2} = ..... = 0] \;=\; \sum_{n=1}^{[\frac{D}{2}]} \frac{c_n}{2^n}\; \mathcal{R}^{(n)}(x^\mu) \tag{2.29}$$

The curvature tensors and scalar curvature with torsion in Riemann-Cartan space appearing in the right hand side of (2.29) decompose into the standard Riemannian piece plus torsion squared terms and derivatives of torsion. For instance, $R = \hat{R} - \frac{1}{4} T_{abc} T^{abc}$ [10], where $\hat{R}$ is the Riemannian scalar curvature.

## 3 Clifford Algebraic Structure of Curved $C$-spaces

In this section we shall study the Clifford Algebraic Structure of Curved $C$-spaces. Without loss of generality we can facilitate matters enormously if one chooses a frame in the tangent $C$-space such that $E^A_M, E^M_A \neq 0$, if the grade of $M$ equals grade of $A$; and $E^A_M, E^M_A = 0$ if the grade of $M$ is *not* equal to the grade of $A$. Choosing such frame requires fixing some of the generalized Lorentz symmetries (poly-rotations) in the tangent $C$-space. Afterwards we will study the most general case scenario when there is a nontrivial grade mixing such that all the components of $E^A_M, E^M_A$ must be taken into account. The simpler "diagonal gauge" choice $E^A_M, E^M_A \neq 0$, if the grade of $M$ equals grade of $A$, permits us to begin with

$$\{\gamma^\mu, \gamma^\nu\} \;=\; E^\mu_a\; E^\nu_b\; \{\gamma^a, \gamma^b\} \;= 2\; E^\mu_a\; E^\nu_b\; \eta^{ab} \;=\; 2\; g^{\mu\nu} \tag{3.1}$$

and

$$[\gamma^\mu, \gamma^\nu] \;=\; E^\mu_a \; E^\nu_b \; [\gamma^a, \gamma^b] \;=\; 2 \; E^\mu_a \; E^\nu_b \; \gamma^{ab} \;=\; 2\gamma^{\mu\nu} \qquad (3.2)$$

one learns that

$$g^{\mu\nu} \;=\; E^\mu_a \; E^\nu_b \; \eta^{ab} \qquad (3.3)$$

and

$$\gamma^{\mu\nu} \;=\; E^{[\mu\nu]}_{[cd]} \gamma^{cd} \;=\; E^\mu_a \; E^\nu_b \; \gamma^{ab} \;=\; \frac{1}{2} \; E^\mu_{[a} \; E^\nu_{b]} \; \gamma^{ab} \qquad (3.4)$$

multiplying both sides of (3.4) by $\gamma_{mn}$ and taking the scalar parts $< \gamma^{cd}\gamma_{mn} >= c\delta^{cd}_{mn} = c\delta^c_{[m}\delta^d_{n]}$, where $c$ is a constant of proportionality that decouples from (3.4), one arrives at the crucial decomposition of

$$E^{[\mu\nu]}_{[cd]} \delta^{cd}_{mn} \;=\; E^{[\mu\nu]}_{[mn]} \;= \frac{1}{2} \; E^\mu_{[a} \; E^\nu_{b]} \; \delta^{ab}_{mn} \;=\; \frac{1}{2} \; E^\mu_{[m} \; E^\nu_{n]} \qquad (3.5)$$

in terms of anti-symmetrized products and where the (anti) symmetrization has a weight of 1.

The commutator

$$[\gamma^{ab}, \gamma^c] \;=\; -\, 2 \; (\eta^{ac}\gamma^b - \eta^{bc}\gamma^a) \qquad (3.6a)$$

yields

$$[\gamma^{\mu\nu}, \gamma^\rho] \;=\; [E^{[\mu\nu]}_{[ab]}\gamma^{ab}, E^\rho_c \gamma^c] \;=$$

$$-2 \; E^{[\mu\nu]}_{[ab]} \; E^\rho_c \; (\eta^{ac}\gamma^b - \eta^{bc}\gamma^a) \;=\; - \; (E^\mu_a E^\nu_b - E^\mu_b E^\nu_a) \; E^\rho_c \; (\eta^{ac}\gamma^b \; - \; \eta^{bc}\gamma^a) \;=$$

$$-2 \; (g^{\mu\rho} \; \gamma^\nu \; - \; g^{\nu\rho} \; \gamma^\mu) \qquad (3.6b)$$

hence we learned that the curved space commutator $[\gamma^{\mu\nu}, \gamma^\rho]$ has the *same* functional form as the tangent space commutator $[\gamma^{ab}, \gamma^c]$. Similarly, after some straightforward algebra one obtains

$$[\gamma^{\mu\nu}, \gamma^{\rho\tau}] \;=\; -\, 2 \; (g^{\mu\rho}\gamma^{\nu\tau} - g^{\nu\rho}\gamma^{\mu\tau} + .......) \qquad (3.7)$$

which has the same functional form as the commutator $[\gamma^{ab}, \gamma^{cd}]$ so the Jacobi identities are satisfied. For example, after using eqs-(3.2, 3.6,3.7) one still retains the vanishing condition

$$[\; \gamma^{\mu\nu}, \; [\gamma^\rho, \gamma^\tau] \;] \; + \; [\; \gamma^\rho, \; [\gamma^\tau, \gamma^{\mu\nu}] \;] \; + \; [\; \gamma^\tau, \; [\gamma^{\mu\nu}, \gamma^\rho] \;] \;=\; 0 \qquad (3.8)$$

The form of the (anti) commutators involving the curved space basis generators $\gamma^M$ will be *modified* considerably from those in the tangent space case if one did *not* set $E^M_A = 0$ for the mixed grade components. Nevertheless, in this more complicated case, the (graded) Jacobi identities will still be satisfied. The "diagonal gauge" conditions $E^M_A \neq 0$ if grade $A$ equals grade of $M$, simplifies enormously the form of the commutation relations $[\gamma^M, \gamma^N]$ of the curved space basis generators in such a way that they sill retain the *same* functional form as the flat tangent space commutators $[\gamma^A, \gamma^B]$, and as such, obey automatically the Jacobi identities. Similar conclusions apply to the anti-commutators.

Strictly speaking one does *not* have a Lie algebra because the metric $g^{\mu\rho}$ is no longer constant, hence one does not have structure constants in the right hand side of $[\gamma^M, \gamma^N] = f_L^{MN}\gamma^L$ but structure functions $f_L^{MN}$ instead

$$[\gamma^M, \gamma^N] = E_A^M \ E_B^N \ [\gamma^A, \gamma^B] = E_A^M \ E_B^N \ f_C^{AB} \ \gamma^C \ = \\ E_A^M \ E_B^N \ f_C^{AB} \ E_L^C \ \gamma^L \ = \ f_L^{MN} \ \gamma^L \quad (3.9a)$$

where the structure functions are defined by

$$E_A^M(\mathbf{X}) \ E_B^N(\mathbf{X}) \ f_C^{AB} \ E_L^C(\mathbf{X}) \ = \ f_L^{MN}(\mathbf{X}) \quad (3.9b)$$

The flat tangent $C$-space the metric $\eta^{AB}$ was defined by taking the scalar part of the Clifford geometric product of the tangent space generators $\eta^{AB} =< \gamma^A\gamma^B >$ and such that the tangent $C$-space metric $\eta^{AB}$ is not zero only when the grade of $A$ = grade of $B$. If one chooses a frame such that $E_A^M \neq 0$, if grade of $A$ equals the grade of $M$, then one arrives at $g^{MN} \neq 0$, if the grade of $M$ equals the grade of $N$. Whereas the mixed grade components of the curved $C$-space metric $g^{MN} =< \gamma^M\gamma^N >= E_A^M E_B^N \eta^{AB}$ are also zero. Consequently, if one uses now the expression for the connection with torsion given by

$$\Gamma_{MN}^K \ = \ \frac{1}{2} \ g^{KL} \ \partial_M g_{LN} \quad (3.10)$$

one will end up with the following non-vanishing values for those connection components of the form

$$\Gamma^{\mathbf{0}}_{M \ \mathbf{0}}, \ \Gamma^{\mu}_{M \ \sigma}, \ \Gamma^{[\mu_1\mu_2]}_{M \ [\sigma_1\sigma_2]}, \ .... \ \Gamma^{[\mu_1\mu_2....\mu_D]}_{M \ [\sigma_1\sigma_2......\sigma_D]} \quad (3.11)$$

this will simplify enormously the calculations because the derivatives of $\gamma^M$ given by $\partial_K\gamma^M \ = \ -\Gamma_{KL}^M\gamma^L$ will only involve the contribution of those non-vanishing connection components (3.11). Namely, those where the grade of $L$ equals the grade of $M$.

For example, taking derivatives of eq-(3.2) with respect to $x^\rho$ and after using eq-(3.11) gives

$$\Gamma^\sigma_{\rho\mu} \ g_{[\sigma\nu] \ [\alpha\beta]} \ - \ \Gamma^\sigma_{\rho\nu} \ g_{[\sigma\mu] \ [\alpha\beta]} \ = \ \Gamma_{\rho \ [\mu\nu] \ [\alpha\beta]} \quad (3.12)$$

If one were to use the torsionless Levi-Civita-like connection expression in (2.16), rather than the connection in eq-(3.10), for $\Gamma^\sigma_{\rho\mu}, \Gamma^\sigma_{\rho\nu}, \Gamma_{\rho \ [\mu\nu] \ [\alpha\beta]}$, after taking the derivatives of eq-(3.2) one will arrive at the same equation (3.12), in addition to extra equations resulting from the additional terms stemming from the mixed grade components appearing in $\partial_K\gamma^M = -\Gamma_{KL}^M\gamma^L$.

One can verify that if one uses the Levi-Civita connection (2.12) in eq-(3.12, while performing the following decomposition of the metric

$$g_{[\alpha\beta] \ [\mu\nu]} \ = \ g_{\alpha\mu} \ g_{\beta\nu} \ - \ g_{\alpha\nu} \ g_{\beta\mu} \quad (3.13)$$

and after setting to zero the mixed-grade components of the metric, it leads to a differential constraint among the derivatives of the metric of the form

$$g_{\nu\beta}(\partial_\mu g_{\alpha\rho} - \partial_\alpha g_{\rho\mu}) \ - \ g_{\nu\alpha}(\partial_\mu g_{\beta\rho} - \partial_\beta g_{\rho\mu}) \ +$$

$$g_{\mu\alpha}(\partial_\nu g_{\beta\rho} - \partial_\beta g_{\rho\nu}) \ - \ g_{\mu\beta}(\partial_\nu g_{\alpha\rho} - \partial_\alpha g_{\rho\nu}) \ = \ 0 \qquad (3.14)$$

One can *avoid* this differential constraint (3.14) if one does *not* recur to the torsionless Levi-Civita-like connection expression (2.16) but instead one uses the following expression for the connection with torsion in eq-(3.10)

$$\Gamma^{\sigma}_{\rho\mu} \ = \ \frac{1}{2} \ g^{\sigma\tau} \ \partial_\rho g_{\mu\tau}; \quad \Gamma_{\rho \ [\mu\nu] \ [\alpha\beta]} \ = \ \frac{1}{2} \ \partial_\rho g_{[\alpha\beta] \ [\mu\nu]} \qquad (3.15)$$

In this case, eq-(3.12), after recurring to the decomposition (3.13), reduces then to a mere *identity* between the left and right hand sides in such a way that there are *no* longer differential constraints imposed among the first derivatives of the metric, like they occurred in (3.14). Therefore we have found a good reason why one must choose a connection with torsion of the form given by eqs-(3.10, 3.15). It is dictated to us by the Clifford algebraic structure. The apparent differential constraints become mere identities, upon a closer inspection, when the connection with torsion (3.10, 3.15) is chosen and when the metric components are decomposed into its irreducible factors (3.13). We shall provide a few more examples of why this is true below.

Another example of how an apparent differential constraint turns into a mere identity is by taking derivatives on both sides of (3.7) with respect to $\tau$, for example. After evaluating the commutators one arrives at

$$-\Gamma^{[\mu\nu]}_{\tau \ \sigma_1\sigma_2}(g^{\sigma_1\rho}\gamma^{\sigma_2} \ - \ g^{\sigma_2\rho}\gamma^{\sigma_1}) \ - \ \Gamma^{\rho}_{\tau \ \sigma}(g^{\mu\sigma} \ \gamma^{\nu} \ - \ g^{\nu\sigma} \ \gamma^{\sigma_\mu}) \ =$$

$$(\partial_\tau g^{\mu\rho}) \ \gamma^{\nu} \ - \ (\partial_\tau g^{\nu\rho}) \ \gamma^{\mu} \ - \ g^{\mu\rho} \ \Gamma^{\nu}_{\tau\alpha} \ \gamma^{\alpha} \ + \ g^{\nu\rho} \ \Gamma^{\mu}_{\tau\alpha} \ \gamma^{\alpha} \qquad (3.16)$$

In the Appendix we shall explicitly show how after contracting the gammas, by multiplying by $\gamma_\alpha$ on both sides of eq-(3.16) and taking the scalar parts, one arrives at an strict *identity*, thus avoiding the introduction of spurious differential constraints involving derivatives of the metric. One may verify as well that after taking derivatives of the anti-commutators leads to apparent differential constraints which become mere identities after using the expression for the connection with torsion (3.10), the zero mixed-grade conditions $g_{MN} = 0$ for the metric, and when the metric components $g^{[\mu_1\mu_2...\mu_k] \ [\nu_1\nu_2...\nu_k]}$ are decomposed into its irreducible factors as

$$det \begin{pmatrix} g^{\mu_1\nu_1} & \dots & \dots & g^{\mu_1\nu_k} \\ g^{\mu_2\nu_1} & \dots & \dots & g^{\mu_2\nu_k} \\ ------------ & & ------------- & \\ g^{\mu_k\nu_1} & \dots & \dots & g^{\mu_k\nu_k} \end{pmatrix} \qquad (3.13')$$

The calculations are very tedious as one can see in the Appendix. Even more so when one evaluates the remaining (anti) commutators and takes their derivatives.

To prove this in the most general case for all the (anti) commutators of the Clifford algebra, *without* having to recur to the zero mixed-grade conditions for the metric, beins, inverse beins; without having to decompose the same grade metric components into their irreducible pieces, and without having to perform tedious calculations, one begins with the structure constants associated with the flat tangent space Clifford algebra

$$\gamma^A\gamma^B = c_C^{AB}\gamma^C,\ [\gamma^A,\gamma^B] = f_C^{AB}\gamma^C,\ \{\gamma^A,\gamma^B\} = f_C^{AB}\gamma^C \tag{3.17a}$$

the structure functions associated with the curved space basis generators are

$$\gamma^M\gamma^N = c_L'^{MN}\gamma^L,\ \ [\gamma^M,\gamma^N] = f_L'^{MN}\gamma^L,\ \ \{\gamma^M,\gamma^N\} = d_L'^{MN}\gamma^L \tag{3.17b}$$

where

$$c_L'^{MN} = E_A^M\ E_B^N\ E_L^C\ c_C^{AB} \tag{3.18a}$$

$$f_L'^{MN} = E_A^M\ E_B^N\ E_L^C\ f_C^{AB} \tag{3.18b}$$

$$d_L'^{MN} = E_A^M\ E_B^N\ E_L^C\ d_C^{AB} \tag{3.18c}$$

we use primes in the left hand side of eqs-(3.18) to emphasize that in the most general case, when one does *not* longer choose the zero mixed-grade conditions (a "diagonal" gauge) , the functional form of the (anti) commutators for the curved basis Clifford generators will not be the same as in the tangent space case. Denoting by $h_L'^{MN}$ any one of the three structure functions $c_L'^{MN}, f_L'^{MN}, d_L'^{MN}$, and after taking ordinary derivatives on each single one of the terms in eqs-(3.18), one can see that one arrives precisely at the $covariantly-constancy$ condition on the structure functions $h_L'^{MN}$

$$\partial_K h_L'^{MN}\ +\ \Gamma_{KQ}^M\ h_L'^{QN}\ +\ \Gamma_{KQ}^M\ h_L'^{MQ}\ -\ \Gamma_{KL}^Q\ h_Q'^{MN}\ =\ 0\ \Rightarrow\ \nabla_K(h_L'^{MN})\ =\ 0 \tag{3.19}$$

which is obeyed by a metric compatible connection satisfying

$$\nabla_K E_M^A\ =\ \partial_K E_M^A\ -\ \Gamma_{KM}^L\ E_L^A\ +\ \omega_K^A{}_B\ E_M^B\ =\ 0,$$
$$\nabla_K E_A^M\ =\ \partial_K E_A^M\ +\ \Gamma_{KL}^M\ E_A^L\ -\ \omega_{KA}{}^B\ E_B^M\ =\ 0 \tag{3.20}$$

To show that eq-(3.19) is satisfied it is important to notice once again the two different choices discussed in section **2** for the connection $\Gamma$ appearing in (3.20) and the hatted connections $\hat\Gamma$ defined by eq-(2.14). When one uses the $\Gamma$'s, one has a covariantly-constancy condition imposed on the tangent space and curved space basis generators. Thus the *covariant* derivatives of the structure "constants" $h_C^{AB}$ are zero if one wishes to maintain the tangent space Clifford algebra intact. Also zero are the covariant derivatives of the beins, and inverse beins in (3.20). Thus one has automatically $\nabla_K h_L'^{MN} = 0$. On the other hand, if one were to use the $\hat\Gamma$ connection instead, the covariant derivatives are then

$$\hat{\nabla}_K h_C^{AB} = \omega_K{}^A{}_D\, h_C^{DB} + \omega_K{}^B{}_D\, h_C^{AD} - \omega_{KC}{}^D\, h_D^{AB} \tag{3.21}$$

since the ordinary derivative of a true constant is $\partial_K h_C^{AB} = 0$. The covariant derivatives (with respect to the hatted connections) of the beins and inverse beins are no longer zero zero, but instead are

$$\hat{\nabla}_K E_M^A = -\,\omega_K{}^A{}_B\, E_M^B, \quad \hat{\nabla}_K E_A^M = \omega_{KA}{}^B\, E_B^M \tag{3.22}$$

By recurring to the definitions of the structure functions (3.18), and the action of the hatted covariant derivatives described by eqs-(3.21, 3.22), one can verify the *covariantly−constancy* condition (with respect to the hatted connection $\hat{\Gamma}$) on the structure functions $h_L'^{MN}$. To see this one regroups the 6 terms obtained after taking the covariant derivatives of eqs-(3.18) into three pairs. The three pairs originate from taking covariant derivatives on the beins, inverse beins and the structure constants in eqs-(3.21, 3.22). The first pair, after raising and lowering indices, is indeed zero

$$\omega_{KAD}\ [\ E^{MD}\ E_B^N\ E_L^C\ h_C^{AB} + E^{MA}\ E_B^N\ E_L^C\ h_C^{DB}\ ] = 0 \tag{3.23}$$

because the term inside the bracket is *symmetric* under the exchange of $A \leftrightarrow D$ indices, while the spin connection $\omega_{KAD}$ is anti-symmetric. Similar findings occur to the remaining two pairs. Therefore, $\hat{\nabla}_K (h_L'^{MN}) = 0$ and the covariant-constancy condition on the structure functions is obeyed for both connections $\Gamma, \hat{\Gamma}$. Namely, the key point is that the choice of the metric compatible connections has to be *consistent* with the Clifford algebraic structure of the curved $C$-space.

We continue this section by adding some important remarks about the zero mixed-grade condition (”diagonal” gauge choice) which simplified the calculations. Since under (poly) coordinate transformations the metric transforms as

$$g'_{JK} = g_{MN}\ \frac{\partial X^M}{\partial X'^J}\ \frac{\partial X^N}{\partial X'^K} \tag{3.24}$$

one can realize that if the mixed-grade components are zero in one coordinate system this does *not* mean that they are also zero in another coordinate system. In order to preserve the zero mixed-grade condition on the metric components, one must *restrict* the coordinate transformations such that $g'_{JK} = 0$ if the grade of $J$ is not equal to the grade of $K$. This in turn requires that the coordinate transformations must be restricted to be grade-preserving as well, namely one must have coordinate transformations of the form

$$x'^\mu = x'^\mu(x^\nu), \quad x'^{\mu\nu} = x'^{\mu\nu}(x^{\rho\tau}), \quad s' = s'(s), ..... \tag{3.25}$$

so that under the restricted coordinate transformations (3.25) one has that $g'_{JK} \neq 0$, if grade of $J$ equals grade of $K$, when $g_{MN} \neq 0$, if grade of $M$ equals grade of $N$. Therefore, if one wishes to preserve the conditions $E_A'^M \neq 0$, if grade of $M$ equals grade of $A$, and $E_A'^M = 0$ if grade of $M$ is *not* equal to the grade of $A$, one must *restrict* the poly-coordinate transformations and generalized Lorentz transformations (poly-rotations affecting the tangent $C$-space indices $A, B, C...$ in $E_M^A, E_A^M$ ) to be *grade*-preserving. This also applies to the metric when the mixed-grade components of $g_{MN}$ are zero. Only a restricted set of poly-coordinate transformations (generalized Lorentz transformations in the tangent space) will preserve such zero mixed-grade condition on $g_{MN}$ and $E_M^A, E_A^M$.

Of course in the most general case we are not confined to perform restricted poly-coordinate transformations and restricted generalized Lorentz transformations. Hence one should allow for grade-mixing transformations. In particular, the connection does *not* transform homogeneously under poly-coordinate transformations because it is *not* a (poly) tensor, like the metric. The connection transforms as

$$\Gamma'^L_{MN} \;=\; \Gamma^P_{QR}\,(\frac{\partial X^Q}{\partial X'^M})\,(\frac{\partial X^R}{\partial X'^N})\,(\frac{\partial X'^L}{\partial X^P}) \;+\; +(\frac{\partial^2 X^P}{\partial X'^M \partial X'^N})\,(\frac{\partial X'^L}{\partial X^P}) \quad (3.26)$$

where the last terms are the inhomogeneous pieces.

We have shown explicitly in this section that when the zero mixed-grade condition was imposed, and when the diagonal metric components were decomposed into its irreducible components, the Levi-Civita connection was *not* satisfactory because it furnishes spurious differential constraints among the first derivatives of the metric. Whereas the connection choice with torsion in eq-(3.10) was satisfactory because it rendered the apparent differential constraints into mere identities. An important question to ask now is whether or not in a different coordinate system the Levi-Civita connection might turn out to be satisfactory.

To answer this question we must again recur to the covariantly-constancy conditions on *both* the metric and structure functions $\nabla_K h'^{MN}_L = 0,\ \nabla_K g^{MN} = 0$. Such conditions are covariant in $C$-space, as they should. From the zero nonmetricity condition one obtains a connection which is determined in terms of the metric and for this reason we may write it symbolically as $\Gamma[g]$. From the other condition $\nabla_K(h'^{MN}_L) = 0$ we obtain a connection that we may write as $\Gamma[h']$. Since the covariant derivatives were defined in terms of the *same* connection $\Gamma$ , we must have $\Gamma[g] = \Gamma[h']$. This last functional equality is very *restrictive* as we have seen above when the "diagonal gauge" choice was taken : the Levi-Civita connection was *not* satisfactory, whereas the connection with torsion given by eq-(3.10) was.

Under coordinate transformations, in the new frame of reference denoted by a tilde, we will have : $\tilde{\Gamma}[\tilde{g}] = \tilde{\Gamma}[\tilde{h}']$. Since torsion transforms as a tensor under coordinate transformations, if there is torsion in one coordinate system one cannot eliminate it in the new coordinate system. Therefore the new $\tilde{\Gamma}$ must have torsion (contorsion) components as well, and as such, it cannot coincide with the torsionless Levi-Civita connection. One of the main results of this section is that $C$-space has torsion which is required for the connection in order to have a consistent system of simultaneous equations $\nabla_K h'^{MN}_L = 0,\ \nabla_K g^{MN} = 0$.

To summarize, after studying the algebraic conditions imposed by the Clifford algebra in curved $C$-space we found : (i) in a given coordinate system (generalized Lorentz frame) the mixed-grade components of the metric $g_{MN}, g^{MN}$, and beins $E^A_M$, inverse beins $E^M_A$, can be set to zero in order to considerably simplify the calculations; namely due to the very large diffeomorphism symmetry in $C$-space, one may choose a frame ("diagonal gauge") such that the mixed grade components of the metric, beins, inverse beins are zero. (ii) In this case, the Clifford algebra associated to the curved space basis generators assumes the *same* functional form as it does in the flat tangent space, and obeys the (graded) Jacobi identities. (iii) The metric, beins, inverse beins, admit a decomposition into their irreducible pieces ; (iv) only a restricted set of poly-coordinate transformations (generalized Lorentz transformations in the tangent space) will preserve such zero mixed-grade condition; (v) the connection has torsion and is given by eq-(3.10) $\Gamma^K_{MN} = \frac{1}{2} g^{KL} \partial_M g_{LN}$.

These conditions allowed us to convert the *apparent* differential constraints among the first derivatives of the metric, resulting from the Clifford algebraic structure associated with the curved $C$-space basis generators, into strict identities as we have explicitly shown in this section and the appendix.

In the most general case, when the mixed grade components of the metric, beins and inverse beins are *not* set to zero; and when their diagonal components do *not* necessarily decompose into antisymmetrized sums of products of their irreducible pieces, we have found that the metric compatible connection $\nabla_K g^{MN} = 0$ must be consistent with the Clifford algebraic structure if $\nabla_K h'^{MN}_L = 0$. This consistency condition singles out an specific family of connections (orbits) obtained by performing coordinate transformations of the fiducial connection with torsion given by eq-(3.10).

An example of the most general case (when the diagonal gauge is not chosen) is that now the $C$-space metric component (written in bold font)

$$\mathbf{g}^{\mu\nu} \ = \ E^\mu_0 \ E^\nu_0 \ \eta^{00} \ + \ E^\mu_a \ E^\nu_b \ \eta^{ab} \ + \ E^\mu_{a_1 a_2} \ E^\nu_{b_1 b_2} \ \eta^{a_1 a_2 \ b_1 b_2} \ + \ .......... \quad (3.27)$$

is given by a *sum* of many pieces. The ordinary spacetime metric can naturally be embedded into the term $g^{\mu\nu}_{(1)} = E^\mu_a E^\nu_b \eta^{ab}$ (which has also a dependence on *all* of the poly-vector coordinates $\mathbf{X}$) and is just *one* piece of the $C$-space metric element $\mathbf{g}^{\mu\nu}$. This is not farfetched, bi-metric theories of gravity, for example, have been known for a long time since the work of Rosen. The "zero" term, corresponding to the scalar-scalar components, is denoted by $g^{\mu\nu}_{(0)} = E^\mu_0 E^\nu_0 \eta^{00}$ and the others will be denoted by $g^{\mu\nu}_{(n)}$ where $n = 1, 2, 3, ....., D$. The reason why the diagonal gauge choice of setting the mixed-grade components of $E^A_M, E^M_A$ to zero is very physical is because $\mathbf{g}^{\mu\nu}(\mathbf{X})$ reduces then to the standard metric $g^{\mu\nu}_{(1)}(\mathbf{X})$.

The upshot of breaking $\mathbf{g}^{\mu\nu}$ into several pieces, is that the quantity

$$\mathbf{g}^{\mu\rho} \ \delta^\nu_\tau \ R^\tau_{\mu\nu\rho}[\mathbf{g}, \Gamma] \ = \ R_{(0)} \ + \ R_{(1)} \ + \ ........... \quad (3.28)$$

admits a splitting into several terms. In the case of *constant* curvature backgrounds one may relate the "zero" term $R_{(0)} = g^{\mu\rho}_{(0)} R_{\mu\rho}[\mathbf{g}, \Gamma]$ with the very large Planck scale vacuum energy contribution (a very large cosmological constant in $C$-space), whereas the second term $R_{(1)} = g^{\mu\rho}_{(1)} R_{\mu\rho}[\mathbf{g}, \Gamma]$ could be related to the extremely small observed cosmological constant in ordinary spacetime. When one chooses the diagonal gauge by setting the mixed-grade components of $E^A_M, E^M_A$ to zero, one has $\mathbf{g}^{\mu\nu} = g^{\mu\nu}_{(1)} = E^\mu_a E^\nu_b \eta^{ab}$ and the expression in eq-(3.28) reduces then to the standard scalar curvature with the inclusion of torsion terms $R[g, \Gamma] = \hat{R} + T^2 + \nabla T$. In a pedestrian way one has "gauged away" the very large cosmological constant which resided in the term $R_{(0)}$. More work needs to be done to explore the validity of this possibility.

## 4 Clifford Space Gravitation

One may construct an Einstein-Hilbert-Cartan like action based on the $C$-space curvature scalar. There are *two* approaches to this process. One approach requires the use of hyper-determinants of hyper-matrices. And the other approach requires ordinary determinants of square matrices in $2^D$-dimensions.

The hyper-determinant of a hyper-matrix [13] can be recast in terms of discriminants [14]. In this fashion one can define the hyper-determinant of $g_{MN}$ as products of the hyper-determinants corresponding to the hyper-matrices[1]

$$g_{[\mu_1\mu_2]\ [\nu_1\nu_2]}, \ \ldots, \ g_{[\mu_1\mu_2\ldots\mu_k]\ [\nu_1\nu_2\ldots\nu_k]}, \quad for \ \ 1 < k < D \tag{4.1}$$

and construct a suitable measure of integration $\mu_{\mathbf{m}}(s, x^\mu, x^{\mu_1\mu_2}, \ldots, x^{\mu_1\mu_2\ldots\mu_D})$ in $C$-space which, in turn, would allow us to build the $C$-space version of the Einstein-Hilbert-Cartan action with a cosmological constant

$$\frac{1}{2\kappa^2} \int ds \prod dx^\mu \prod dx^{\mu_1\mu_2} \ \ldots \ dx^{\mu_1\mu_2\ldots\mu_D} \ \mu_{\mathbf{m}}(s, x^\mu, x^{\mu_1\mu_2}, \ldots) \ (\mathbf{R} - 2\Lambda) \tag{4.2}$$

$\kappa^2$ is the $C$-space gravitational coupling constant. In ordinary gravity it is set to $8\pi G_N$, with $G_N$ being the Newtonian coupling constant.

The measure must obey the relation

$$[\mathbf{DX}] \ \mu_{\mathbf{m}}(\mathbf{X}) \ = \ [\mathbf{DX}'] \ \mu'_{\mathbf{m}}(\mathbf{X}') \tag{4.3}$$

under poly-vector valued coordinate transformations in $C$-space. The $C$-space metric transforms as

$$g'_{JK} \ = \ g_{MN} \ \frac{\partial X^M}{\partial X'^J} \ \frac{\partial X^N}{\partial X'^K} \tag{4.4}$$

[1]The hyper-determinant of a product of two hyper-matrices is *not* equal to the product of their hyper-determinants. However, one is not multiplying two hyper-matrices but decomposing the hyper-matrix $g_{MN}$ into its different blocks.

but now one has that

$$\sqrt{hdet\ g'}\ \neq\ \sqrt{hdet\ g}\ hdet\left(\frac{\partial X^M}{\partial X'^N}\right) \tag{4.5}$$

due to the multiplicative "anomaly" of the product of hyper-determinants. So the measure $\mu_m$ does not coincide with the square root of the hyper-determinant. It is a more complicated function of the hyper-determinant of $g_{AB}$ and obeying eq-(4.3).[2] One could write $hdet(X\cdot Y) = Z_A hdet(X) hdet(Y)$, where $Z_A \neq 1$ is the multiplicative anomaly and in this fashion eq-(2.30) leads to an *implicit* definition of the measure $\mu_m(hdet g_{AB})$.

The ordinary determinant $g = det(g_{\mu\nu})$ obeys

$$\delta\sqrt{-g} = -\ \frac{1}{2}\ \sqrt{-g}\ g_{\mu\nu}\ \delta g^{\mu\nu} \tag{4.6}$$

which was fundamental in the derivation of Einstein equations from a variation of the Einstein-Hilbert action. However, when hyper-determinants of the $C$-space metric $g_{AB}$ are involved it is *no* longer true that the relation (4.6) holds anymore in order to obtain the $C$-space gravity field equations in the presence of torsion and a cosmological constant.

Using the relation $\delta \mathbf{R}_{MN} = \nabla_J \delta\hat{\Gamma}^J_{MN} - \nabla_N \delta\hat{\Gamma}^J_{JM}$, a variation of the action

$$\frac{1}{2\kappa^2}\int ds \prod dx^\mu \prod dx^{\mu_1\mu_2}\ \ldots\ dx^{\mu_1\mu_2\ldots\mu_D}\ \mu_m(|hdet\, g_{MN}|)\ (\mathbf{R}-2\Lambda) + S_{matter} \tag{4.7}$$

with respect to the $C$-space metric $g_{MN}$ yields the $C$-space field equations

$$\mathbf{R}_{(MN)}\ +\ (\mathbf{R}-2\Lambda)\ \frac{\delta ln(\mu_m(|hdet\ g_{MN}|))}{\delta g^{MN}}\ =\ \kappa^2\ \mathbf{T}_{MN} \tag{4.8}$$

If, and only if,

$$\frac{\delta ln(\mu_m(|hdet\ g_{MN}|))}{\delta g^{MN}}\ =\ -\ \frac{1}{2}\ g_{MN} \tag{4.9}$$

then the field equations (4.8) would coincide with the $C$-space extension of Einstein's equations with a cosmological constant. One should note that the field equations (4.8) *contain torsion* since $\mathbf{R}_{(MN)}, \mathbf{R}$ are defined in terms of the nonsymmetric connection $\Gamma^J_{MN} \neq \Gamma^J_{NM}$ . The field equations (4.8), for example, are very different from those found in [17] based on a fourth-rank symmetric metric tensor.

The hyper-determinant of the $C$-space metric $g_{MN}$ (a hyper matrix) involving all the components of the same and different grade is defined as

$$hdet\ (g_{MN}) \equiv g_{\mathbf{00}}\ \det(g_{\mu\nu})\ hdet(g_{\mu_1\mu_2\ \nu_1\nu_2})\ hdet(g_{\mu\ \nu_1\nu_2})\cdots\cdots$$

[2]There is no known generalization of the Binet-Cauchy formula $\det(AB) = \det(A)\det(B)$ for 2 arbitrary hypermatrices. However, in the case of particular types of hypermatrices, some results are known. Let X, Y be two hypermatrices. Suppose that Y is a $n\times n$ matrix. Then, a well-defined hypermatrix product XY is defined in such a way that the hyperdeterminant satisfies the rule $hdet(X\cdot Y) = hdet(X)hdet(Y)^{N/n}$. There, $n$ is the degree of the hyperdeterminant and $N$ is a number related to the format of the hypermatrix X.

$$hdet(g_{\mu_1...\mu_{D-1}\nu_1...\nu_{D-1}})\ g_{\mu_1...\mu_D\nu_1...\nu_D} \tag{4.10}$$

where the hyper-determinant of $g_{\mu\nu}$ coincides with the ordinary determinant of $g_{\mu\nu}$. Notice once more that the hyper-determinant of a product of two hyper-matrices is *not* equal to the product of their hyper-determinants. However, in (4.10) one is not multiplying two hyper-matrices $g_{AB}, g'_{AB}$, but decomposing the hyper-matrix $g_{AB}$ into different blocks. Hyperdeterminants have found physical applications in the black-hole/qubit correspondence [15].

One can avoid the use of hyperdeterminants by working in a *blockwise* fashion, when dealing with poly-vector valued indices, rather than dealing with each one of the indices of their associated hypermatrices individually. The $C$-space metric $g_{MN}$ associated with a Clifford algebra in $D$-dimensions has a one-to-one correspondence with an ordinary metric $g_{ij}$ in $2^D$-dimensions. In particular, the metric $g_{ij}$ is a square $2^D \times 2^D$ symmetric matrix with $\frac{1}{2}2^D(2^D-1)$ independent components. The determinant of the square matrix $g_{ij}$ is defined as usual in terms of epsilon tensors, where the indices range is $i,j = 1,2,3,....,2^D$.

The poly-vector coordinates $\mathbf{X} = s, x^\mu, x^{\mu_1\mu_2}, ....$, and their derivatives, have also a one-to-one correspondence with the coordinates $y^i = y^1, y^2, ......, y^{2^D}$, and their derivatives, of the associated $2^D$-dim space. Thus, one has a correspondence of the action (4.2) in $C$-space with the ordinary Einstein-Cartan action, with a cosmological constant $\lambda$, in $2^D$-dimensions

$$\frac{1}{2\kappa^2} \int d^{2^D}y \sqrt{|det\ g_{ij}|}\ (\ \mathcal{R} - 2\lambda\ ) \tag{4.11}$$

However, having a *correspondence* between the actions in (4.2) and (4.11) does not mean that they are physically *equivalent*, even if one *replaces* the measure in eq-(4.2) by $\sqrt{|det\ g_{ij}|}$. The reason being that the Clifford algebraic structure imposes very strong constraints on the allowed $C$-space connection and on the metric components $g_{MN}$, when the zero grade-mixing condition $g_{MN} = 0$ is chosen. As we have shown in section **2**, the same grade metric components decompose into their irreducible pieces as described in eqs-(3.13, 3.13'). In order to attain an equivalence one would have to add to the action (4.11) extra terms involving Lagrange multipliers enforcing the decomposition conditions (constraints) in eqs-(3.13, .13').

Another way of implementing those conditions (3.13, 3.13') is by writing the variation of the action (4.2) as

$$\delta S\ =\ \frac{\delta S}{\delta g^{00}}\ \delta g^{00}\ +\ \frac{\delta S}{\delta g^{\mu\nu}}\ \delta g^{\mu\nu}\ +$$

$$\frac{\delta S}{\delta g^{[\mu_1\mu_2]\ [\nu_1\nu_2]}}\ \frac{\delta g^{[\mu_1\mu_2]\ [\nu_1\nu_2]}}{\delta g^{\mu\nu}}\ \delta g^{\mu\nu}\ +\ .... \tag{4.12}$$

leading to the $C$-space gravitational field equations

$$\frac{\delta S}{\delta g^{00}} = 0,\quad \frac{\delta S}{\delta g^{\mu\nu}}\ +\ \frac{\delta S}{\delta g^{[\mu_1\mu_2]\ [\nu_1\nu_2]}}\ \frac{\delta g^{[\mu_1\mu_2]\ [\nu_1\nu_2]}}{\delta g^{\mu\nu}}\ +\ .......\ =\ 0 \tag{4.13}$$

The $C$-space scalar curvature $\mathbf{R}$ after setting all the poly-coordinates to zero, except $x^\mu$, was postulated to be related to the LLC Lagrangian, up to a cosmological constant, as provided by eq-(2.29). Instead of setting/truncating these poly-coordinates to zero one could view the LLC action as an $effective$ action after integrating the $C$-space action (4.2) with respect to all the poly-coordinates, except the $x^\mu$

$$\int d^D x \sqrt{|g(x^\mu)|} \mathcal{L}_{LLC}(x^\mu) =$$

$$\int ds\, dx^{\mu\nu} \ldots\ldots \mu_{\mathbf{m}}(s, x^\mu, x^{\mu_1\mu_2}, \ldots) (\mathbf{R} - 2\Lambda) \tag{4.14}$$

This possibility warrants further investigation. The plausible relation to extended gravitational theories based on $f(R), f(R_{\mu\nu}), \ldots$ actions for polynomial-valued functions, and which obviate the need for dark matter, warrants also further investigation [16]. For instance, instead of reproducing the LLC action as an effective action in (4.14) one may generate instead an $f(R,T)$ action with torsion.

To conclude, one should add that by decomposing the same grade metric components into their irreducible pieces (3.13, 3.13') one is introducing higher spins beyond spin 2. Higher spin theories $s = 2, 3, ...., \infty$ in Anti de Sitter backgrounds have been extensively studied in the past decades [19]. The higher spins corresponding to the higher grade metric components $g_{MN}$ will have an upper bound determined by the dimension $D$.

The introduction of matter terms for the gravitational action in $C$-space is straightforward. Besides ordinary fermions one has spinor-tensors $\Psi_\alpha^{[\mu_1\mu_2\ldots\mu_n]}$ fields which contribute to the stress energy tensor. Introducing nonmetricity furnishes higher curvature extensions of metric affine theories of gravity [9]. An immediate question arises, does the Palatini formalism work also in $C$-spaces? Namely, does a variation of the action (4.2) with respect to the $C$-space connection $(\delta S/\delta\Gamma^J_{MN}) = 0$ yield the *same* connections as those described by eq-(3.10) ? This and other remaining questions need to be answered. The most important is how $C$-space gravity will improve the quantization program. Noncommutative Clifford spaces based on noncommuting $\mathbf{X}$ poly-coordinates were considered in [18].

## APPENDIX

In the first part of this appendix we will write down the (anti) commutators involving the flat tangent space Clifford basis generators in $D$ dimensions. In the second part of this appendix we will verify that eq-(3.16) is an identity instead of an apparent differential constraint. Similar conclusions follow for more complicated (anti) commutators involving curved space Clifford basis generators.

The Clifford geometric product corresponding to the tangent space generators can be written as

$$\gamma_A \ \gamma_B \ = \ \frac{1}{2} \ \{\gamma_A, \gamma_B\} \ + \frac{1}{2} \ [\gamma_A, \gamma_B] \qquad (A.1)$$

The commutators $[\gamma_A, \gamma_B]$ for $pq = odd$ one has [6]

$$[ \ \gamma_{b_1 b_2 \ldots b_p}, \ \gamma^{a_1 a_2 \ldots a_q} \ ] \ = \ 2\gamma^{a_1 a_2 \ldots a_q}_{b_1 b_2 \ldots b_p} \ -$$

$$\frac{2p!q!}{2!(p-2)!(q-2)!} \ \delta^{[a_1 a_2}_{[b_1 b_2} \ \gamma^{a_3 \ldots a_q]}_{b_3 \ldots b_p]} \ + \ \frac{2p!q!}{4!(p-4)!(q-4)!} \ \delta^{[a_1 \ldots a_4}_{[b_1 \ldots b_4} \ \gamma^{a_5 \ldots a_q]}_{b_5 \ldots b_p]} \ - \ \ldots \qquad (A.2)$$

for $pq = even$ one has

$$[ \ \gamma_{b_1 b_2 \ldots b_p}, \ \gamma^{a_1 a_2 \ldots a_q} \ ] \ = \ - \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \ \delta^{[a_1}_{[b_1} \ \gamma^{a_2 a_3 \ldots a_q]}_{b_2 b_3 \ldots b_p]} \ -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \ \delta^{[a_1 \ldots a_3}_{[b_1 \ldots b_3} \ \gamma^{a_4 \ldots a_q]}_{b_4 \ldots b_p]} \ + \ \ldots \qquad (A.3)$$

The anti-commutators for $pq = even$ are

$$\{ \ \gamma_{b_1 b_2 \ldots b_p}, \gamma^{a_1 a_2 \ldots a_q} \ \} \ = \ 2\gamma^{a_1 a_2 \ldots a_q}_{b_1 b_2 \ldots b_p} \ -$$

$$\frac{2p!q!}{2!(p-2)!(q-2)!} \ \delta^{[a_1 a_2}_{[b_1 b_2} \ \gamma^{a_3 \ldots a_q]}_{b_3 \ldots b_p]} \ + \ \frac{2p!q!}{4!(p-4)!(q-4)!} \ \delta^{[a_1 \ldots a_4}_{[b_1 \ldots b_4} \ \gamma^{a_5 \ldots a_q]}_{b_5 \ldots b_p]} \ - \ \ldots \qquad (A.4)$$

and the anti-commutators for $pq = odd$ are

$$\{ \ \gamma_{b_1 b_2 \ldots b_p}, \ \gamma^{a_1 a_2 \ldots a_q} \ \} \ = \ - \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \ \delta^{[a_1}_{[b_1} \ \gamma^{a_2 a_3 \ldots a_q]}_{b_2 b_3 \ldots b_p]} \ -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \ \delta^{[a_1 \ldots a_3}_{[b_1 \ldots b_3} \ \gamma^{a_4 \ldots a_q]}_{b_4 \ldots b_p]} \ + \ \ldots \qquad (A.5)$$

Eqs-(A.1-A.5) allows to construct explicitly the Clifford geometric product of the curved $C$-space basis generators $\gamma_M \gamma_N = E^A_M E^B_N \gamma_A \gamma_B$ via the introduction of the $C$-space beins.

We turn now to verify that eq-(3.16) is an identity instead of an apparent differential constraint. Multiplying by $\gamma_\alpha$ both sides of (3.16) and taking the scalar part of the Clifford geometric product yields

$$-\Gamma^{[\mu\nu]}_{\tau \ \sigma_1 \alpha}(g^{\sigma_1 \rho} \ + \ \Gamma^{[\mu\nu]}_{\tau \ \alpha \sigma_2} \ g^{\sigma_2 \rho} \ + \ \Gamma^{\rho}_{\tau \ \sigma}(g^{\nu\sigma} \ \delta^\mu_\alpha \ - \ g^{\mu\sigma} \ \delta^\sigma_\nu) \ =$$

$$(\partial_\tau g^{\mu\rho}) \ \delta^\nu_\alpha \ - \ (\partial_\tau g^{\nu\rho}) \ \delta^\mu_\alpha \ - \ g^{\mu\rho} \ \Gamma^\nu_{\tau\alpha} \ + \ g^{\nu\rho} \ \Gamma^\mu_{\tau\alpha} \qquad (A.6)$$

Given the connection defined as

$$\Gamma^{[\mu\nu]}_{\tau \ \sigma\alpha} \ = \ \frac{1}{2} \ g^{\mu\nu \ \beta\gamma} \ \partial_k (g_{\beta\gamma \ \sigma\alpha}), \quad \Gamma^\rho_{\tau\sigma} \ = \ \frac{1}{2} g^{\rho\beta} \partial_\tau (g_{\beta\sigma}) \qquad (A.7)$$

and the decomposition of the bivector-bivector metric components

$$g_{\beta\gamma\ \sigma\alpha} \ = \ g_{\beta\sigma}\ g_{\gamma\alpha} \ - \ g_{\gamma\sigma}\ g_{\beta\alpha}$$

$$g^{\mu\nu\ \beta\gamma} \ = \ g^{\mu\beta}\ g_{\nu\gamma} \ - \ g_{\nu\beta}\ g_{\mu\gamma} \qquad (A.8)$$

and moving the derivatives as follows

$$g^{\nu\beta}\ (\partial_\tau g_{\beta\sigma}) \ = \ \partial_\tau (g^{\nu\beta}\ g_{\beta\sigma}) \ - \ (\partial_\tau g^{\nu\beta})\ g_{\beta\sigma} \ =$$

$$\partial_\tau (\delta^\nu_\sigma) \ - \ (\partial_\tau g^{\nu\beta})\ g_{\beta\sigma} \ = \ - \ (\partial_\tau g^{\nu\beta})\ g_{\beta\sigma} \qquad (A.9)$$

allow us to verify that eq- (A.6) becomes an identity after recurring to eqs-(A.7-A.9). For instance, the particular terms in the left hand side of (A.6)

$$\frac{1}{2} g^{\sigma\rho}\ g^{\mu\gamma}\ g^{\nu\beta}\ (\partial_\tau g_{\beta\sigma})\ g_{\gamma_\alpha} \ = \ \frac{1}{2} g^{\sigma\rho}\ \delta^\mu_\alpha\ g^{\nu\beta}\ (\partial_\tau g_{\beta\sigma}) \ =$$

$$- \ \frac{1}{2} g^{\sigma\rho}\ \delta^\mu_\alpha\ (\partial_\tau g^{\nu\beta})\ g_{\beta\sigma} \ = \ - \ \frac{1}{2}\ \delta^\rho_\beta\ \delta^\mu_\alpha\ (\partial_\tau g^{\nu\beta}) \ = \ - \ \frac{1}{2}\ \delta^\mu_\alpha\ (\partial_\tau g^{\nu\rho}) \qquad (A.10)$$

can be combined with the terms

$$\frac{1}{2} g^{\rho\beta}\ (\partial_\tau g_{\beta\sigma})\ g^{\nu\sigma}\ \delta^\mu_\alpha \ = \ - \ \frac{1}{2} (\partial_\tau g^{\rho\beta})\ g_{\beta\sigma}\ g^{\nu\sigma}\ \delta^\mu_\alpha \ =$$

$$- \ \frac{1}{2} (\partial_\tau g^{\rho\beta})\ \delta^\nu_\beta\ \delta^\mu_\alpha \ = \ - \ \frac{1}{2} (\partial_\tau g^{\rho\nu})\ \delta^\mu_\alpha \qquad (A.11)$$

such that after adding the right hand sides of eqs-(A.10, A.11) gives $-(\partial_\tau g^{\rho\nu})\ \delta^\mu_\alpha$ which is precisely the term appearing in the right hand side of (A.6).

The particular term in the left hand side of (A.6)

$$-\frac{1}{2}\ g^{\sigma\rho}\ g^{\mu\beta}\ g^{\nu\gamma}\ (\partial_\tau g_{\gamma\alpha})\ g_{\beta\sigma} \ = \ -\frac{1}{2}\ g^{\sigma\rho}\ g^{\nu\gamma}\ \delta^\mu_\sigma\ (\partial_\tau g_{\gamma\alpha}) \ = \ - \ g^{\mu\rho}\ \Gamma^\nu_{\tau\alpha} \qquad (A.12)$$

becomes precisely the same term in the right hand side of eq-(A.6). Repeating similar calculations with the remaining terms of eq-(A.6), one can show that indeed eq-(A.6) is an identity rather than a differential constraint among first derivatives of the metric. It was crucial to recur to the eqs-(A.7-A.9) in order to attain this finding.

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