# Sagittarius A*: a Compelling Case Against the Existence of a Supermassive Black Hole in the Center of Milky Way. I

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**Abstract**. The astrophysics literature tries to make a case for the existence of a supermassive black hole at the center of Milky Way, in the location of the radio source Sagittarius A*. We think that, with arguments of the very same nature, the evidence points quite to the contrary. While the observational data on the orbits of the starry objects around Sagittarius A*, being of a projective character, are entirely reliable, their physical explanation uses quite a particular type of Newtonian forces, namely those with magnitude depending exclusively on the distance between bodies. To begin with, this limitation assumes a priori that the bodies connected by such forces are special material points, viz. space positions endowed with mass. At space scales such as that of the galactic center region in discussion, this assumption is not realistic, and therefore, implicitly, such particular forces are themselves not quite realistic. Still using Newtonian forces in argument, strongly suggested by observational data as a matter of fact, one should allow, on such an occasion, their full generality. This means that we only need to assume that they are central forces with no other further constraints. Within the framework of the Newtonian theory of forces this freedom has important theoretical consequences discussed in the present work. Among these the chief one, from astrophysical point of view, is that the presence of a supermassive black hole in the center of Milky Way might not be a sustainable assumption. An alternative is presented.

**Key Words**: Sagittarius A*, Milky Way, central forces, Newtonian theory of forces, electromagnetic field, production of field, astrophysics, fundamental physics

## Introduction

The story starts with the discovery of the galactic radiosource called Sagittarius A* in the center of the Milky Way (**Balick, Brown, 1974**). The scientific consensus is that, physically, such a source should be correlated with the existence of a material body in that place. For, the fundamental physical notion is that the electromagnetic field is only created by the motion of matter. However, such a body is optically invisible at that position. And as, according to fundamental physical understanding, one cannot presume that the center of a spiral galaxy is simply an empty location emitting electromagnetic radiation, the astronomers got quite a mystery in their hands.

Then, later on, the adaptive optics stepped down from the military to scientific uses, and starting from about the beginning of the last decade of the previous century, it allowed to astrophysicists distinguishing starry isolated objects moving against the background of the center of the Milky Way. It was thus possible to notice coherent patterns in the motion paths of such stars, as projected on the canopy (for an outstanding review of the history, evidence and elimination of the many alternative physical explanations see **Reid, 2009** and the original literature cited there). One such object, called S02, even completed an elliptic path, under our eyes so to speak, in about 16 years, proving beyond any doubt that its motion is Keplerian. Further analysis revealed many such gravitating stars, whose paths are only partially accessible though. Nevertheless their observed positions are enough for allowing astrophysicists to infer that their complete orbits are ellipses.

The common feature of all these orbits is that they all contain the radiosource Sagittarius A* in one of their foci, therefore they should be Keplerian orbits, or at least very close to these. And as this source is dim in any kind of perturbations that can reach the Earth, one can easily suspect that not all radiation comes out from the source. First, the object is invisible. Therefore the optical part of the spectrum does not reach the Earth, and this can have a rational explanation: it is swallowed by the matter existing between the center of the Milky Way and the solar system. This seems only reasonable, inasmuch as the matter between the center of galaxy and the solar system dims the light by some 30 orders of magnitude. In other regions of spectrum we are luckier: infrared and radio waves are dimmed only by about three orders of magnitude. However, there is still a big discrepancy between the mass to be assigned to the body assumed to create the gravitational field responsible for the motions of those stars and the amount of radiation we are supposed to receive from such a body. This fact helped gradually built the conclusion that the central body works in the way in which a black hole is supposed to work. For, if one applies the Newtonian theory of forces in a classical way (see for instance **Gillessen et al, 2009**), the mass of an object that fits the requirement of being the source of such a gravitational field is about four million and a half solar masses: a supermassive black hole!

It is our opinion that the very theory of forces used to disentangle such a case is not completely adequate to the task, so the conclusion of the existence of a black hole in the center of Milky Way, or in the center of any other galaxy for that matter, might not be the appropriate one. In fact the observational data may be pointing out to the necessity of approaching the physics of the center of Milky Way with the ingenuity of Newton himself when he approached its prototype, the Keplerian synthesis of planets' motion. Thus, while we agree entirely with the statement that the Milky Way's center is "a laboratory for fundamental astrophysics and galactic nuclei" (**Ghez Et Al, 2010**), we think a little further, namely of 'a laboratory for *fundamental physics*' at large. For, the data itself may compel us to change the ideas about the fundamental forces as we claim to know them today. And that by doing nothing more than looking a little deeper into the history of those forces, and considering it face value.

Indeed, we are, here and now, in that unique situation in which the science was only once in its history. That was in the times when Newton, having at his disposal the Keplerian synthesis of Tycho Brahe's data on Mars, has invented the forces of which the physicists and astronomers speak today. Thus, on one hand, we have at our disposal the outstanding synthesis, allowed by the adaptive optics, of the motions of the stars in the very central part of our galaxy (see **Eckart Et Al, 2002**; **Ghez Et Al, 2005** and the earlier original works cited there). Like the old Kepler synthesis, the new synthesis points out to coherent motions, of stars this time, which projected on the canopy appear as elliptical motions. Therefore, in reality they cannot be but Keplerian motions, no matter of the orientation of their planes in space. Being based on projections on the canopy, the conclusion is by no means affected by the uncertainty in the galactic metric parameters (for a recent critical study of such uncertainties see for instance **McMillan, Binney, 2010**). It is therefore the most reliable conclusion one can draw based on observations. Now, when the orientation of those planes of motion is taken into consideration in extracting the orbits from the data, all these orbits reveal that Sagittarius A* is in one of their foci, just like the Sun in the Kepler's case. But unlike the planets of the solar system, the stars orbiting around Sagittarius A* are not in the same plane. All we can say is that as conic sections they belong to a family a quadrics having a common focus. This fact may, by itself, indicate that the case of the black hole is unsustainable.

For, on the other hand, the usual physical explanation of this observational synthesis stops at some quite particular class of forces that might not be appropriate to the task. These forces are assumed to be well known, being of the type which Newton used in order to explain the ideal Kepler motions, amended, on occasions, to account for the almost insignificant rotations of the orbits. In fact, with rare exceptions, the whole speculative physics today uses only such forces, distinguished by the fact that they are conservative and have the magnitude depending exclusively on the distance between the attracted and the attracting bodies. Provided, of course, these bodies can be considered material points in the classical sense, i.e. space positions endowed with physical properties (mass, charge, etc).

So, regarding the main physical argument used in explaining the observational data – the forces – we think that it calls for a more careful consideration. Specifically, we should go way deeper with the assumptions about the forces responsible for the contemporary Kepler motions in the center of Milky Way, at least as deep as Newton went in the prototypical case of the original Kepler motion. It appears therefore as only appropriate to start our present undertaking with the essentials of Newton's approach of his invention of central forces (see **Newton, 1995**, Book I, Sections II & III).

## The Newtonian Forces

In order to make our message more clear, let us rephrase the Corollary 3 of the Proposition VII from *Principia*, with reference to an arbitrary orbit, not just a Keplerian one. This confers maximum generality to the concept of Newtonian force and to its quantitative definition, pointing out the particular situation of the mass itself in the construction of force. This corollary is, in our opinion, the key of understanding the action of all forces in the universe. In a simplified expression, extracted from Newton's original (**Newton, 1995**, p.48), and adapted for our specific needs, it sounds like:

> The force by which a body P … revolves about a center of force S, is to the force by which the same body may revolve *in the same orbit, and the same periodic time*, about *another center of force* R, as the volume $SP \times RP^2$, … to the cube of the straight segment SG, drawn from the first center of force S, parallel to the

distance RP of the body from the second center of force R, and meeting the tangent PG of the orbit in G.

One can easily draw a figure in order to better assess the geometrical situation. The points S and R can occupy *any positions* with respect to the observed orbit in its plane.

To our knowledge, J. W. L. Glaisher appears to be the first one who properly put this statement into an analytical form, with no recourse whatsoever to dynamical principles, and with reference to the eliptic, therefore closer to Keplerian, form of the orbit (**Glaisher, 1878**). The theory goes, by and large, along the following lines. Assume that, in the Cartesian coordinates of the plane of motion, the equation of the observed orbit is the quadratic non-homogeneous equation

$$f(x,y) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0 \qquad (1)$$

Then the relation between the two forces expressed in the statement above can be translated into equation:

$$\text{FORCE toward } S = (1+\vec{\xi}\cdot\vec{\varepsilon})^3 \frac{r}{\sqrt{r^2+\varepsilon^2-2\vec{r}\cdot\vec{\varepsilon}}}\cdot(\text{FORCE toward } R) \qquad (2)$$

Here $\vec{\xi}$ is the vector of components

$$\frac{a_{11}x + a_{12}y + a_{13}}{a_{13}x + a_{23}y + a_{33}}, \quad \frac{a_{12}x + a_{22}y + a_{23}}{a_{13}x + a_{23}y + a_{33}} \qquad (3)$$

and $\vec{\varepsilon} \equiv \overrightarrow{SR}$. $\vec{r}$ denotes here the position vector of the moving point P with respect to S, and r is its magnitude – i.e. what we call here the distance, when not otherwise specified.

Equation (2) shows how to calculate analytically the force in P toward point S, when we happen to know the force in P toward the point R from the plane of motion. This is the basic mathematical principle of the Newtonian natural philosophy. It is not hard to see that it extends… naturally the observations related to the ‘working principle’ of sling shooting, whereby the force of gravity – i.e. the weight – acting vertically, is actually ‘compared’, by means of the sling itself, with the centrifugal force, acting horizontally or in any other direction.

Now, the force toward R can be taken as reference in measuring the force in the current point P of the orbit *in any other direction*. So, in this case, we are sort of compelled, so to speak, to a choice of R that makes the theory of forces universal, at least in the Keplerian situations. This leads, within modern theoretical physical views, to a calibration of Newtonian forces. In the cases where R occupies the position of *the center of the orbit*, Newton has inductively shown that the force toward it is directly proportional with the distance between R and P, i.e. with the distance from the center to the orbiting point. If we use this result in equation (2), then the force we need to know is

$$\text{FORCE toward } S = \mu\cdot r\cdot(1+\vec{\xi}\cdot\vec{\varepsilon})^3 \qquad (4)$$

where $\mu$ is a constant of proportionality, coming from the force toward the center of orbit. If, further on, we use the components (3) of the vector $\vec{\xi}$, we get:

$$\text{FORCE toward } S = \mu\cdot a_{33}^2\cdot\frac{r}{(a_{13}x + a_{23}y + a_{33})^3} \qquad (5)$$

in the reference frame where S is in origin. This result can be, of course, expressed in different manners, depending on the way of writing the equation of the conic representing the orbit. However, it carries an even more important message, at least from a geometrical point of view.

With the substantial help of the analytical geometry of conics, in words the result sounds: *the force toward a certain center by means of which a certain material point describes a conical orbit around that center, is directly proportional to the distance from the point to the center of force and inversely proportional with the third power of the distance from the point to the straight line conjugated to the center of force with respect to the orbit*. This is a theorem first given by W. R. Hamilton (**Hamilton, 1847**) on the "occasion of a study of *Principia*." Therefore, once again, the center of force can occupy any position with respect to the orbit, but in the case of conical orbits, and with a standard choice of the reference force – i.e. in modern terms, in a standard calibration or gauging of the forces – the definition of Newtonian force involves the very same elements as the definition of the orbit itself: the distances of the generic point of orbit from the center of force and from the polar line coresponding to that center of force.

This fact should be more obvious if we write the equation of force in the form:

$$f(x, y) = \frac{\mu r}{[(a_1 x + b_1 y + c_1)(a_2 x + b_2 y + c_2)]^{3/2}} \tag{6}$$

Here the equation of the orbit is understood in the form

$$(a_1 x + b_1 y + c_1)(a_2 x + b_2 y + c_2) - (a_{13} x + a_{23} y + a_{33})^2 = 0 \tag{7}$$

It expresses the fact that, geometrically, the orbit is determined by its tangent lines – real or imaginary – in the two points where the straight line polar to the center of force with respect to the orbit cuts it. The tangents are considered as having the equations:

$$a_1 x + b_1 y + c_1 = 0; \quad a_2 x + b_2 y + c_2 = 0 \tag{8}$$

while the polar itself of the center of force with respect to the orbit is taken as given by equation

$$a_{13} x + a_{23} y + a_{33} = 0 \tag{9}$$

Now, based on this general presentation, let's see where the limitation to the dependence of the magnitude of force exclusively on the distance enters the physics of gravitating systems. For, one can see from equation (6) that, in general, such a behavior of the magnitude of force is far from being the general case. Rather, the magnitude of the Newtonian force as we read it even in conical orbits, with no further specification of the position of the actual center of force, *depends also on the current direction of the orbiting body*.

The very idea of force in explaining celestial harmony started from the first of the Kepler laws: the planets describe elliptical orbits *with the Sun in one of their foci*. This last information is crucial. For, if the position of the center of force is in a focus of the ellipse, then the magnitude of force *cannot depend but on the distance*, and that even in a very special way. Indeed, we can then use the equation of the orbit referred explicitly to one of its foci. By the definition of a conic section, the ratio between the distances from the planet to one of the foci and from the planet to the corresponding directrix (the polar of focus) is constant: the eccentricity. This comes down analytically to the equation

$$r^2 = e^2 \cdot (a_{13}x + a_{23}y + a_{33})^2 \tag{10}$$

where ‘e’ is a number proportional to the eccentricity of the orbit. In this case, using the equation (7), equation (6) leads directly to:

$$f(x, y) = \frac{\mu}{r^2} \tag{11}$$

with $\mu$ – a constant. This is the force of ‘universal gravitation’ to which the classical physics makes always reference, with no mention though of the prerequisites of its expression: that the point of attraction and the point attracted have to be only… points, and furthermore, the point of attraction has to occupy a privileged position with respect to orbit – that of one of its foci. For, if the point of attraction occupies any other position in the plane of motion, with rare exceptions, we may be in the situation that the magnitude of force depends also on the direction from the center of force to the moving body. Thus the universally used Newtonian force of gravity is actually quite a particular choice among the possible forces responsible for the Kepler synthesis.

For the sake of completeness, let’s see what other cases may occur of dependence of the magnitude of Newtonian forces only on distance. The motion of planets is not the only one given to our experience, although we have to recognize that it is the one that stirred up everything. Thus, for instance, the immediate experience has certainly to do with with elastic forces too. These are the forces that ‘gauge’ – and that in a quite precise manner we should say (see **Mazilu, Agop, 2012**) – all the modern positive science, insofar as it needs to be submitted to experimental verification. And such forces are obtained, within the Newtonian program sketched above, in cases where the center of force coincides with the center of the orbit. In equation (7) this means $a_{13} = a_{23} = 0$, and therefore equation (6) becomes:

$$f(x, y) = \mu \cdot r \tag{12}$$

where $\mu$ is another constant, not necessarily that from equation (11). This might seem as a tautology – we started specifically from the idea that the force toward the center of the orbit is an elastic one – but, at a closer scrutiny we might have to change this opinion. First of all, equation (12) shows that the theory is not self-contradicting, and this is an important fact by itself. Secondly, this shows that the Newtonian formula works for the same point in the plane of orbit *in two different instances* – as material point and empty position – and this is most important conclusion for theoretical physics.

## The Concept of Field and the Modern Idea of Gauging

Indeed, this is the very essence of the *idea of field* in physics. For a better understanding, consider the situation of light. What we usually accept is that the Newtonian force is proportional to distance in the cases where the center of force *is material* and located in the center of the orbit. What about the cases when *that center of force is simply an empty position*? This is plainly the case of the Fresnel ellipse in the plane waves of light, perpendicular to the light ray: there is no material center of force on the light ray, and yet the light can be described as if there is one there. This fact reveals the real merit of the Newton’s definition of the force: it can be calculated with respect to any point, once the geometric setup is Keplerian! The force has indeed the characteristics of a true field, as these came down to us from the theoretical physics of the $19^{th}$ century. This would mean, for instance, that in the case of a light ray there is always a Newtonian central force *acting along the ray*, toward or away from the source of light. The idea was always rejected from the realm of physics, based exclusively on the fact that the force should be a vector, for which the formulas like (11) and (12) give the magnitude unconditionally. As Glaisher's analysis clearly shows, this was not at all the case when Newton has invented the forces.

Regarding the light phenomenon per se, this philosophy was materialized even from the times of Fresnel, by the 'gauging' proposal of James Mac Cullagh (**Mac Cullagh, 1831**) for the *representation of the light according to Newtonian view of forces*. Let us briefly see what Mac Cullagh's point of view is about. He was concerned with the elliptically polarized light, like the light passing through rock crystals. He found that this light can be represented by two harmonic vector processes in the same plane, like the processes invented by Fresnel to help explaining the light phenomenon, making a certain angle between them. Later on (**Mac Cullagh, 1836**) he noticed that the theory can be put in a space-time form by a system of coupled differential equations, which led him to the foundations of a theory of ether (**Mac Cullagh, 1839**) – improved afterwards by Lord Kelvin and Joseph Larmor – and finally to an exquisite explanation of the phenomenon of double refraction in quartz (**Mac Cullagh, 1840**). It is to be noticed that the veiled persuading argument of Mac Cullagh's feat seems to have been the faulty notion of describing the light by a displacement, advanced initially by Fresnel. Indeed, in the case of light – a continuum phenomenon – the mechanical displacement has no object, i.e. it is not referring to a material point, but simply to an empty position in space, for no matter as we know it is located there. This very fact made Newton's natural philosophy hardly relevant to the light, a detail corrected, as we see, in a brilliant way by Mac Cullagh. These observations explain, by and large, the almost explicit contribution of Mac Cullagh to the future electromagnetic theory of light (**Darrigol, 2002, 2010**). In hindsight though, Mac Cullagh's seems to us to be much more than an electromagnetic theory. It should be taken, indeed, as the very first specimen of a gauge theory (see **Mazilu, Agop, 2012** for a discussion of light in 'Mac Cullagh's gauge') of the kind that came into existence more than a century afterwards, in the form of the Yang-Mills theory (**Yang, Mills, 1954**).

Returning, for one last consideration, back to the equations (11) and (12), they reveal forces whose magnitude is exclusively dependent only on the distance between the points assumed to be physically correlated by them. These happen also to be the only forces that satisfy the Kepler geometry per se, i.e. the only ones having *closed orbits* (**Bertrand, 1873**). But, one can see from these examples that the dependence of the magnitude of force exclusively on distance is acquired, first of all, by the *special position of the center of force* with respect to the orbit. Secondly, and by no means less important, is the fact that the universal 'comparison force', the gauging force of the Newtonian procedure as it were, is the elastic force, which may or may not be actual after all, when referred to criteria dictated by our senses. For, the actuality of those forces depends on the circumstance that the center of orbit contains matter or not. And any comparison force, other than the elastic isotropic one, used in the Newtonian procedure of defining forces, would make the formula (6) more complicated, perhaps even prohibitively. This moment in the definition of force in the Newtonian philosophy turns out to have universal significance even through the modern idea of 'gauging' from theoretical physics. From this perspective it shows even more: nothing can be reproducibly described in physics, unless we have a gauge for it.

Nevertheless, it turns out that the equations (11) and (12) are used nowadays in theoretical physics with no reference to their Newtonian foundation, and therefore with no further qualifications about their very possibility at that. For instance there are occasions, and very often at that, when the force from equation (11) is considered a static force. This should entail special considerations, because originally the very existence of such force carries a precise identity: it is clearly related to a Keplerian motion, and moreover, the value (11) is referred to a gauging elastic isotropic force that might not even be actual. In hindsight though, judging by the success of such universal 'anonymous' forces in the history of physics, specifically in the theory of light and astrophysics, there seem to be no need at all for the existence of matter in the center of the conic, in order to ratify their actuality. If one needs a ratification anyway, this comes simply from the fact that the elastic forces are an expression of the existence of a privileged coordinate system – that of *harmonic coordinates* (**Mazilu, Agop, 2012**).

In the case of those two time-honored central forces with magnitude depending exclusively on distance, we have as centers of force the very center of the orbit and its focus. But these are by no means singular cases leading to a force with magnitude depending exclusively on distance. If, for instance, the motion is elliptic and the center of force is located on the orbit itself, the magnitude of the force accounting for this motion is inversely proportional with the fifth power of the distance. By the same token, if the orbit is of a special shape – other than a conic section – we may also have forces exclusively depending on distance. This is, for example, the case when the motion has the space form of a logarithmic spiral, like the arms of a galaxy. The force accounting for such a motion pulls toward the pole of the spiral with a magnitude inversely proportional with the third power of the distance from that center.

The case presented by Sagittarius A* is outstanding mostly from a special point of view of natural philosophy, that may induce us to reconsider the previous old natural philosophy founded by Newton. As we have already mentioned above, *the only criterion that validates the decision that a body is acted upon by a force pulling towards a certain point in space is the perceived matter in that point*. This can be actually quoted as the very first gauging criterion of physics, the 'zeroth principle' as it were. It turns out to work even today in full swing. According to this criterion, Sagittarius A* should be such a point, even though only 'partially perceived'. However, it does not satisfy itself to another criterion that historically became essential, but actually turned out to be quite arbitrary: *that of mass*. In order to define this criterion, and to recognize its true meaning, let's follow the evolution of Newtonian ideas along the theory of continuum material, leading to Poisson's equation. This is actually the route which led to the modern theoretical physics' idea of gauging in the first place.

## The Mass and the Newtonian Theory of Forces

Perhaps we would have never talked about the whole theoretical physics as it is today, if Newton would not have insisted on the idea that the force of gravity should be directly proportional with the ‘quantities of matter’ of the bodies involved in the interaction represented by that force. More precisely, in modern vector notation, the Newtonian force created by a body of mass M on a body of mass m, can be written as

$$\vec{f}(\vec{r}) = -G\frac{Mm}{r^2}\frac{\vec{r}}{r} \tag{13}$$

Here G is the so-called gravitational constant, and $\vec{r}$ is the position vector of m with respect to M, again, both *considered as material points*. Now this force can be thought of as existing by itself, separated from its roots so to speak, i.e. disregarding its Keplerian origin. As we have mentioned before, it can even be considered a static force. The physical problem now moves on to the realm of mechanics: can this force explain the observed motions? and how? The answer is well known, and resides in the principles of dynamics, put forward by Newton in order to be able to profitably use the force. This time though, the force is assumed to be the independent cause of the motion.

Our way of writing the force here points to the fact that the force is attractive, being opposite to the orientation of the position vector. Its magnitude does not depend but on the distance between the two material points assumed to be correlated by force, and this in a very special way, shown above in equation (11) as specific to a Keplerian setting involving an elliptical orbit with the center of force in one of its foci:

$$f(\vec{r}) = G\frac{Mm}{r^2} \tag{14}$$

Therefore $\mu \equiv GMm$ in formula (11) above.

Newton insisted at length on the fact that the force of gravitation should be proportional with the quantities of matter of both bodies involved in the interaction represented by force, otherwise nothing would make sense. This is a rather arbitrary assumption by itself (for *the only* critical approach of the issue, at least as far as we can judge, see **Poincaré, 1897**, and **Poincaré, 1921**, Ch. VII). It was like Newton was mindful of the fact that, some two hundred years after him, the general relativity would have to come into existence, and he ought to create its possibility. However, he was apparently guided in his insistence only by the fact that a force like that from equation (14) is able to offer physical support to the marvelous synthesis by Kepler of the motion of the planet Mars.

Indeed, in modern terms, in order to obtain the Kepler laws it is sufficient to solve the Newtonian equations of motion written in the form

$$\ddot{\vec{r}} + \frac{K}{r^2}\frac{\vec{r}}{r} = \vec{0} \tag{15}$$

as one proceeds routinely today. Here $K \equiv GM$, and so it is obvious that the description of motion by this equation is universal even in the more precise sense that it does not depend on the mass of the moving body. So, if the independence of the force (13) of its physical origin could still be counted as an arbitrary assumption, the undeniable success of the mathematics handling of equation (15) bestows upon it an equally incontestable actuality. For, this is the first moment where the idea of field, independent of matter, came up to science.

The Newtonian force from equation (13) is conservative, i.e. can be derived from a potential. The existence of a potential in the problem of classical gravitational field means however more than the mere law of conservation of the mechanical energy. It also opens the path of speculations regarding the structure of matter and of the characterization of a continuum from mechanical point of view, as initiated by Newton himself. In order to show this, let us notice that we can write the Newtonian force in the form

$$\vec{f}(\vec{r}) = m\nabla V_1(\vec{r}); \quad V_1(\vec{r}) \equiv \frac{GM}{r} \tag{16}$$

where $\nabla$ is the operation of gradient and $V_1$ is the potential energy. Considering only the force per unit mass, the force from equation (16) is:

$$\vec{f}(\vec{r}) = \nabla V_1(\vec{r}) \tag{17}$$

This force is therefore an intensity, characteristic of the space around the mass assumed to exert that force. It is this space, thus physically characterized, that came to be known as a field: *the gravitational field*. The force exists in every point of space, no matter of the other physical properties of that point: it can be simply a position in space, as well as the location of a material particle. Therefore the force is a continuous vector function of the position in space, and can very well be a characteristic of a material continuum filling the space. Which characteristic was not so long in coming to physical considerations, being, as it were, a necessity forced upon the mathematics of natural philosophy by the space expansion of material bodies.

Indeed, inasmuch as one can think of a physical structure of a material continuum, this gives us the right to calculate the flux of force around a certain point in space. Considering the force from equation (13) as a static force, we can use a spherical surface around origin of coordinates. If we calculate its flux through a spherical surface of radius r according to the usual formula

$$\oiint_{\text{Sphere}} \hat{n}\cdot\vec{f}(\vec{r})dA \tag{18}$$

where $\hat{n}$ is the unit normal to the sphere and dA is its elementary area, we get an interesting result. As the unit normal vector to the sphere is just the versor of the position vector, and $dA = r^2\sin\theta d\theta d\varphi$, we have

$$\oiint_{\text{Sphere}} \hat{n}\cdot\vec{f}(\vec{r})dA = 4\pi GM \tag{19}$$

Therefore the flux of force of the gravitational field is, up to a universal factor, the mass of the material point creating the field – appropriately called the source of field. Now, the mass of that source can be represented, according to Newton's definition of the density of matter (**Newton, 1995**, p. 9), by a volume integral:

$$M = \iiint_{\text{Volume}} \rho(\vec{r})d^3\vec{r} \tag{20}$$

where $\rho(\ )$ denotes the Newtonian density at the chosen location and $d^3\vec{r}$ is the volume element at the same location. Using the equations (17), (19) and (20) we get

$$\oiint_{\text{Sphere}} \hat{n}\cdot\nabla V_1(\vec{r})dA = 4\pi G\iiint_{\text{Volume}}\rho(\vec{r})d^3\vec{r} \tag{21}$$

where the 'Volume' is that of the corresponding 'Sphere'. Further on, using Gauss' theorem for the left hand side of this equation, we have

$$\iiint_{\text{Volume}}\left\{\nabla^2 V_1(\vec{r}) + 4\pi G\rho(\vec{r})\right\}d^3\vec{r} = 0 \tag{22}$$

In fairly general continuity conditions, the integrand of this equation should then be zero. This gives us the *Poisson's equation*, relating the Newtonian density of matter that generates the field to the field potential. Usually, the potential is taken without the gravitational constant, which comes to a simple redefinition: $V_1 = GV$. So, the equation of Poisson becomes the one we usually write today:

$$\nabla^2 V(\vec{r}) = -4\pi\rho(\vec{r}) \tag{23}$$

This is the equation which, from the perspective of general relativity for instance, is the fundamental equation of the classical mechanics. It is not usually considered quite by itself, but in conjunction with the implicit idea that *we are always able to know the density of matter*. It is therefore an equation giving us forces, when knowing that we have at our disposal the matter creating, as it were, these forces, provided that one can characterize this matter by a density in the Newtonian recognition. More than this, it contains, in the background at least, the idea that the material point of Newton is not simply a position in space, as actually the rigorous calculation requires: it should be endowed with a space expanse to be occasionally considered. It is in this general acceptance that the equation is used in characterizing the Sagittarius A* case.

Now, obviously, by equation (20), and therefore by equation (23), we actually describe the part of space *inside the matter*. Then, the Poisson's equation itself becomes part of fundamental physics. Indeed, it is really necessary in the description of nature, inasmuch as it provides us knowledge on the space inside matter. The hard part of the problem is that the Newtonian density is only a hypothesis, and quite unreliable at that, because the matter is not inherently homogeneous with respect to space, and we do not have access to its space details – at least not always. Nevertheless, within certain quite natural assumptions, that knowledge is inferrable, as it was actually the case all along the time. The most reliable, and the only one entirely realistic we should say, of these assumptions is offered by the particular case of the Poisson's equation, where the density of matter is zero, viz. *the Laplace equation*. Indeed, the Newtonian potential of the force from equation (13) above is actually a solution of Laplace equation, thus characterizing the situation in vacuum. And this fact is quite natural: the Newtonian force has not been invented otherwise, but specifically for describing the interaction between *material points in vacuum*. It is only its extension to the space inside matter – allowed by the equation of Poisson, *which in turn was allowed by the special assumption of Newton on the position of masses in the expression of the magnitude of force* – that creates the impression that the force depends physically on the density of matter. This line of thought was initiated indeed by Poisson in 1812, and put on mathematical firm grounds by Gauss in 1839 (see **Gauss, 1842**). It is along it, that Einstein found flaws to classical physics, and thus pressed forward the ideas that led to the general relativity. A brief history of the main points of development of the theory is perhaps in order.

In 1812 Poisson noticed that inside matter the law of attraction between different material points cannot be the law of Newton, because the potential cannot satisfy there a Laplace equation (**Poisson, 1812**). He has actually noticed that the Newtonian density of matter becomes instrumental there, and that the law of Newton corresponds in fact to a zero density (see also **Poisson, 1833**). Poisson might have thus sensed the possibility of still other forces, besides those going inversely with the square of distance, corresponding to nonzero density of matter. Only after the work of Gauss became it gradually clear that the force inside matter should be taken first and foremost as a flux, and therefore expressed by its divergence, rather than by its curl as in mechanics of a single material point. And this divergence has as *expression the Newtonian density of matter*. However, in this approach the matter has to have the essential property of vacuum, which turns out to be *the homogeneity* with respect to space, in order to be possible to correctly characterize it by a density. This desideratum is, nevertheless, far from being satisfied with no further qualifications, for the homogeneity is a matter of scale. As Einstein says somewhere, the universe is homogeneous only 'on average'. The physics of last century added to this the essential observation that the property of homogeneity 'on average' should be respected *at any scale*.

Indeed, 'on average' the density of matter is never zero, not even hypothetically. Although we can imagine some smearing out procedure in order to calculate a density, that doesn't mean that we hit the real thing. As a matter of fact, the evaluations of the density of matter in universe taken today into consideration as scientific figures, don't even represent the Newtonian density as required by the Poisson's equation, but numbers obtained from various rough evaluations, with the substantial contribution of some numerical densities in the sense of Hertz (**Hertz, 2003**). These are combined with even more arbitrary values of the volume of space where evaluations are made, assuming, still quite arbitrarily, that the matter should have a certain constitution in those regions of space. This is also the manner of evaluation of density for all the analyses related to the case of Sagittarius A*. However, with so many uncertainties in our hands, one can hardly think of a right quantitative appreciation of the density of matter! Useless to say, the very same is the case of evaluation of any density to be used into Poisson's equation.

## Back to the General Newtonian Forces

We can see therefore, from what was shown right above, that the development of differential calculus gradually spirited away the identity of force so to speak, i.e. the physical parameters representing the orbit from the expression of the magnitude of force. Indeed, the force could now be calculated as a solution of a differential equation in satisfactory limiting conditions. The force thus became a plain vector. And the most natural among the analytical conditions a force vector should satisfy, when referring it to a continuum, seem to be the classical ones, generalizing the properties of the Newtonian force from equation (13), which show that it *acts in vacuum* and *is conservative*:

$$\nabla \cdot \vec{f} = 0; \quad \nabla \times \vec{f} = \vec{0} \tag{24}$$

In fact, at some moment, the classical theory of forces even stipulates specifically that a certain force vector can be split into a sum of a divergence-free part and a curl-free part – the so-called *Helmholtz decomposition* – and that the decomposition is unique. The conditions (24) legalize, as it were, the two essential properties of the Newtonian gravitational force, only implicitly contained in the Poisson equation. The first condition says that the source of forces is the Newtonian density, but a vacuum force, like the Newtonian gravitational force, has no source; the second condition shows that the force is central, therefore conservative.

Let's say that we have obtained a formula for the magnitude of force in vacuum. The essential condition in order to be able to even use that formula is obviously that the force should satisfy the first equation (24). The second condition is only incidental, so to speak. However, if the magnitude of the force should depend exclusively on distance, then both conditions *are satisfied only for the Newtonian force* with magnitude inversely proportional with the square of the distance. Indeed, a central force with the magnitude depending exclusively on distance can be written in the form

$$\vec{f}(\vec{r}) = f(r)\frac{\vec{r}}{r} \tag{25}$$

where f(r) is the magnitude of force. The second condition from equation (24) is automatically satisfied, while the first condition amounts to

$$rf'(r) + 2f(r) = 0 \quad \therefore \quad f(r) = \frac{\mu}{r^2} \tag{26}$$

Here the accent denotes the derivative with respect to the variable. Now it becomes obvious that the central forces inside matter, with magnitude depending exclusively on distance, require also a certain behavior of the density of matter depending on that distance, otherwise it is not a possible force whithin matter. Such a property is hard to understand geometrically, but is easy to understand… parametrically, as it were. More specifically, it is hard to understand that a continuum has density decreasing in the same way in every direction starting from each one of its points. This would mean contradiction indeed, when we consider two different neighboring points. It is very easy to understand though, that a continuum has a certain density depending on the *distance between its points*, in cases where this distance can be defined.

However, if a force is central – therefore Newtonian – and has the magnitude dependent not only on distance but on direction too, then instead of equation (25) we must have

$$\vec{f}(\vec{r}) = \psi(x, y, z)\vec{r} \tag{27}$$

where x, y, z enter the expression of the magnitude of force by some algebraical combinations, other than the magnitude of the radius vector. In such a case the two conditions (24) boil down to

$$\sum x\frac{\partial \psi}{\partial x} + 3\psi = 0; \quad \vec{r} \times \nabla\psi = \vec{0} \tag{28}$$

Therefore the function ψ must be a homogeneous function of degree (–3), in the first place. If we limit our search to the functions derivable from the elliptic orbits of the planets, as Newton actually did, then such a function cannot be but of one of the following forms, also derivable from the second principle of dynamics (**Darboux, 1877**):

$$(\vec{a}\cdot\vec{r})^{-3};\quad (a_{ij}x^{i}x^{j})^{-3/2} \tag{29}$$

Here the vector $\vec{a}$ and the entries of matrix **a** are arbitrary constants, the coordinates are considered as contravariant, and the summation convention is respected. The expressions of forces are defined up to a multiplicative constant. We recognize in these the forces deriving from the Corollary 3 of the Proposition VII of Newton's *Principia*. Enforcing on them the second of conditions (28), shows that the first case is impossible, because the vector $\vec{a}$ would then have to be null identically. The second case works only if the matrix **a** is a multiple of the identity matrix. But this shows that the force is simply the Newtonian gravitational one, with the magnitude inversely proportional with the square of the distance. We thus find the Newtonian force as a property of field, with no reference whatsoever to motion, once it is conditioned by equations (24). As we already expressed it, the identity of orbit – and therefore of force itself – is lost. However, it comes back, only this time it does that through the initial conditions serving to solve the differential equation (15).

The inference about the existence of such particular forces in a problem of astrophysics should therefore be conditioned by the fulfillment of conditions (28), therefore of the conditions (24). Those conditions reduce the class of forces, as equations (29) show it in the most general case. Only the existence of Keplerian orbits would guarantee that these forces depend exclusively on distance, and moreover that their magnitudes are inversely proportional with the square of distance. This is a condition plainly satisfied by all of the results in the Sagittarius A* case: it came to attention of the scientific community by the very specific stellar orbits in the first place! And as Sagittarius A* is always in one of the foci of these orbits, which are of course elliptical, there can be no question of the reality of Newtonian force (13) in this case. Provided, of course, the matter exists in that place, *which is what the assumption of the existence of a black hole there brings about*. We are not quite so sure as to what extent, and in what particular conditions, the Kepler's second law, in its differential form, is satisfied for each one of those orbits. For, within Newtonian ideas, only the second of Kepler laws would be a clear indication of the presence and location of a center of force. As it happens though the theory of Newtonian forces works regardless of that law, and the conclusions of the present work should therefore remain theoretically valid (see **Mazilu, 2010**; **Mazilu, Agop, 2012**).

## The Variation of Orbit and the Production of Fields

One of the main reasons for which we must appeal to the original Newtonian theory of forces in problems of astrophysics, like the one presented by the Sagittarius A* case, is that such a theory uses, almost explicitly we should say, an analogy which transcends the space scale of the problems in which this kind of forces is involved. The initial analogy was the one already mentioned in passing before, between sling shooting and the motion of planets. Then, with the gradual introduction of classical dynamics, the equation (15) made its entrance into the mathematics related to mechanics. And as long as we consider this equation as fundamental, one can prove that the force given by equation (13) is the only one justified from the point of view of space scale transcendence. Indeed, the equation (15) transcends the space scale, and no other force introduced in it satisfies this condition (**Mariwalla, 1982**). Therefore we are entitled to use the classical dynamics in describing the central part of the Milky Way just as we are entitled to use it in the case of describing planetary motions, or to state that the the stars move around the galactic nucleus following Keplerian orbits. It is at this juncture though, that we need to pay close attention to the concept of force to be used in astrophysical matters, for it might indicate some other fundamental things if it is to consider the point of view of *space scale transcendence*.

One historically important, fundamental space scale transcendence is that allowing us to extend the conclusions of classical dynamics in the atomic realm. This means that the planetary – or nuclear – model of atom should be the only one entitled to close consideration from a theoretical physical point of view. This was indeed the case. Only, on this occasion we have learned that in the microscopic realm the model does not work the same way as the planetary system proper. For, the light gets in: insomuch as electromagnetic phenomenon it should be attached to the atom, due to the electrical properties of this last one. It is here the point where the contradictions started brewing, forcing us to assign the light to *the transitions between electronic orbits* (**Bohr, 1913**). While the original Bohr's work is refering to the simplest atomic model – the one for which the electronic orbit is a circle – there are strong reasons to believe that his conclusion is quite general: *the light* or, in general, any perturbation that can reach our eyes directly or through the intermediary of measurement devices, *is due to transitions between orbits*. The arms of spiral galaxies can thus be interpreted as geometric loci of such transitions points (**Mazilu, 2010**; **Mazilu, Agop, 2012**), whereby the stars, revolving along Keplerian orbits around galactic nucleus, conglomerate in stable structures.

Therefore, through the planetary model of atom, theoretical physics actually just enacted a status quo, naturally existing a priori by the space scale transcendence. However, Bohr's postulates show that it is quite precise in the choice of the terms of analogy so to speak: the atomic model from microcosmos is analogous to the galaxy from macrocosmos, rather than to the planetary system per se. Therefore one can say that, by quantum mechanics, the theoretical physics reinstated with full right the initial Newtonian forces, identified by the parameters representing the orbit from which they have been calculated. One might say that quantum mechanics of the atom was just a normal reaction of natural philosophy, which reclaimed the lost identity of the orbit in the expression of forces, or the lost identity of forces given by the orbit from which they were calculated.

## The Characteristic of Forces Transcending Mechanics

The second of conditions (24) precludes the Newtonian forces from transcending mechanics, for it is equivalent with the conservation of mechanical energy, to the extent this is defined by work. The general Newtonian forces do not have this restriction: *they are dissipative*. For instance, the force characterizing a material point describing a Kepler orbit is given, according to Glaisher, by equation (5). Without any loss of generality, it can be written in vector form as

$$\vec{f}(x,y) = \frac{\mu\vec{r}}{(a_{13}x + a_{23}y + a_{33})^3} \tag{30}$$

where (x, y) are the coordinates in the plane of motion. This force is of the form given in equation (29) with an obvious identification of function $\psi$, and for $a_{33} = 0$, in order to be considered a vacuum force. The general expression independent of the plane of motion is obviously the one using the first expression (29):

$$\vec{f}(x,y,z) = \frac{\mu\vec{r}}{(\vec{a}\cdot\vec{r})^3} \quad \therefore \quad \psi(x,y,z) = \frac{\mu}{(\vec{a}\cdot\vec{r})^3} \tag{31}$$

The first condition (29) is implicit, while the second condition is no more satisfied. In fact, we have

$$\nabla\times\vec{f} = -3\mu\frac{\vec{a}\times\vec{r}}{(\vec{a}\cdot\vec{r})^4} \tag{32}$$

Therefore, the elementary work of this force is not integrable in the ordinary sense. However it is integrable in the Frobenius sense, therefore in the thermodynamical sense, i.e. we have

$$\delta L \equiv \vec{f}\cdot d\vec{r} = w dF \tag{33}$$

for certain functions w and F. This can be proved directly by noticing that the Cartan integrability condition $\delta L\wedge(d\wedge\delta L) = 0$, where ‘$\wedge$’ is the sign of an ‘exterior’ operation (in this case differentiation) on differential forms, is satisfied in view of equations (31) and (32).

The classical motion sustained by the force from equation (31) is a Keplerian motion. This can be seen by solving the Binet’s equation of the Newtonian problem of motion

$$u'' + u = \frac{\mu/a^3}{\cos^3\theta} \tag{34}$$

where $u \equiv 1/r$ as usual, and the derivative is taken with respect to angle $\theta$ whose origin is the direction of the vector $\vec{a}$. The general solution of this equation

$$u(\theta) = (w_1 - \mu/a^3)\cos\theta + w_2\sin\theta + \frac{\mu/2a^3}{\cos\theta} \tag{35}$$

where $w_1$ and $w_2$ are some initial conditions of the problem. In the Cartesian coordinates $\xi$ and $\eta$ with respect to the center of force equation (35) becomes

$$(w_1 - \mu/2a^3)\xi^2 + w_2\xi\eta + (\mu/2a^3)\eta^2 - \xi = 0 \tag{36}$$

The center of the orbit has the coordinates

$$\xi_c = \frac{\mu a^3}{2w_1\mu a^3 - w_2^2 a^6 - \mu^2}; \quad \eta_c = \frac{w_2 a^6}{2w_1\mu a^3 - w_2^2 a^6 - \mu^2} \tag{37}$$

Therefore the physical parameters entering the force – the components of the vector $\vec{a}$ – determine also the characteristics of the Keplerian orbit induced by that force. This orbit is plane, with the plane determined by the initial conditions represented by the vector $\vec{w}$.

The work of the force performed on the orbiting body is not zero, as in the case of the forces with magnitude depending exclusively on distance, but can be recognized by a flux through the surface enclosed by the orbit, for we have

$$\oint_{Orbit} \vec{f}(\vec{r}) \cdot d\vec{r} = \iint_{Surface} (\nabla \times \vec{f}) \cdot d\vec{S} = -3\mu \iint_{Surface} (\vec{a} \cdot \vec{r})^{-4} (\vec{a} \times \vec{r}) \cdot d\vec{S} \tag{38}$$

What, though, if the orbit is not is not a closed curve?! In the Sagittarius A* case, for instance, all the data we have at our disposal, except the one referring to S02 of course, come only with segments of the whole orbit. Therefore, such data would only refer to the work done by force along an open segment of the orbit, and the proper question would be the to ask about the variation of this integral along the orbit. A solution to this problem is provided by the transport theorem (**Betounes, 1983**) in the form

$$\frac{d}{dt} \int_{\phi_t(\Gamma)} \vec{f} \cdot d\vec{r} + \int_{\phi_t(\Gamma)} [(\nabla \times \vec{f}) \times \vec{v}] \cdot d\vec{r} = \vec{f} \cdot \vec{v} \Big|_Q^P \tag{39}$$

Here we have a subtle understanding of things: $\Gamma$ is the segment of curve initially accessible. It evolves due to the motions of heavens – not only of the body on which we have concentrated our attention. The evolution is accounted for by a family $\phi_t$ of morphisms depending on time in the sense that the time is a continuous index of the family: for each moment of time there is a morphism mapping the initial segment $\Gamma$, between points P and Q of the orbit, to the current one denoted $\phi_t(\Gamma)$. The equation (39) can be reckoned as a continuity equation, showing how the power generated by force is dissipated.

It is a formula like equation (39) that may be able to explain the transitions between Keplerian orbits along the spiral arms of a galaxy for instance. However, we stop here for a moment, in order to frame our conclusions about forces that might relate to the Sagittarius A* case. Only after this rounding up of the conclusions shall we get into an explanation of the occurences of that case.

## Conclusions and Outlook

Sagittarius A* is an intricate case: we consider it an astrophysical subject of great theoretical interest for the whole physics. The theory of forces used in explanation of related occurences of the case, has to measure up with such a task, which turns out to be outstandingly complex, involving all of the aspects of theoretical physics known to us thus far, and perhaps much more. The main point of the work thus far is making clear what needs to be considered in the problem of forces, in order to make sure that these are able to be ‘equal to the task’.

The first general conclusion is that the classical Newtonian conservative force is much too poor a concept for the job at hand. The force to be considered should indeed be a central force, for we have no other possibility of classical explanation, but with no other restrictions upon it whatsoever. First, the conic shape per se, of the orbits of the stellar objects in the center of the Milky Way, is by no means indication that the center of force is in the focus of those orbits. Therefore, in their physical explanation we are not entitled to use the hypothesis of central force *with magnitude depending exclusively on the distance*. Secondly, the rich emission of electromagnetic energy from Sagittarius A*, may be an indication that the force is not necessarily conservative. In fact Sagittarius A* may be a *transformer of energy* as it were: the mechanical energy is transformed into electromagnetic energy in a kind of process like the one acting in the atom and explaining the Bohr's transitions at large. The explanation of the Sagittarius A* would then be crucial for the theoretical physics, for it gives clues regarding the mechanism of transition between orbits, and how the electromagnetic energy is created.

The whole theory thus far is based on the idea of material point in the classical acceptance: *a position endowed with physical properties*. Thus, the next step in our analysis seems to be a logically necessary one: what is the influence of the space expansion of the matter in the stars revolving around Sagittarius A*? For we are by no means entitled to consider the stars we observe by the means of adaptive optics as material points in a classical sense. The second part of the present work will be dedicated to this issue.

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