On Point Mass Sources, Null Naked Singularities and Euclidean Gravitational Action as Entropy

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Abstract

It is rigorously shown how the static spherically symmetric solutions of Einstein's equations can furnish a *null* naked singularity associated with a point mass source at $r = 0$. The construction relies in the possibility of having a metric *discontinuity* at the location of the point mass. This result should be contrasted with the spacelike singularity described by the textbook black hole solution. It has been argued by some authors why one cannot get any information from the null naked singularity so it will not have any undesirable physical effect to an outside far away observer and cannot cause a breakdown of predictability. In this way one may preserve the essence of the cosmic censorship hypothesis. The field equations due to a delta-function point-mass source at $r = 0$ are solved and the Euclidean gravitational action (in $\hbar$ units) corresponding to those solutions is evaluated explicitly. It is found that it is precisely equal to the black hole entropy (in Planck area units). This result holds in any dimensions $D \geq 3$. We finalize by arguing why the Noncommutative Gravity of the spacetime tangent (co-tangent) bundle is the proper arena to study point masses.

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1 Static Spherically Symmetric Solutions and Null Naked Singularities

To prove how null naked singularities can be associated to point mass sources, let us begin by writing down the class of static spherically symmetric (SSS) *vacuum* solutions of Einstein's equations [1] in $D = 4$ and given by a family of metrics parametrized by the *area* radial functions $R(r)$ (in $c = 1$ units)

$$(ds)^2 \;=\; (1 - \frac{2GM}{R})\;(dt)^2\; - (1 - \frac{2GM}{R})^{-1}\;(dR)^2 \;-\; R^2(r)\;(d\Omega)^2. \qquad (1.1)$$

where $(dR)^2 = (dR/dr)^2(dr)^2$ and the solid angle infinitesimal element is $(d\Omega)^2 = (d\phi)^2 + sin^2(\phi)(d\theta)^2$. This expression of the metric is given in terms of the *area* radial function $R(r)$ (a radial gauge) and does *not* violate Birkoff's theorem since the metric (1.1) expressed in terms of the area radial function $R(r)$ has exactly the same functional form as that required by Birkoff's theorem. The values of r span the region $0 \leq r \leq \infty$.

There are two interesting cases to study based on the boundary conditions obeyed by $R(r)$: (i) the Hilbert textbook (black hole) solution [4] when $R(r) = r$ obeying $R(r = 0) = 0$, $R(r \to \infty) \to r$. And : (ii) the Abrams-Brillouin [3] radial gauge $R(r) = r + 2GM$ based on choosing the cutoff $R(r = 0) = 2GM$ such that $g_{tt}(r = 0) = 0$ which apparently seems to "eliminate" the horizon and $R(r \to \infty) \to r$. This was also the salient feature behind the *original* solution of 1916 found by Schwarzschild. However, the choice $R(r = 0) = 2GM$ has a serious $flaw$ and is : How is it *possible* for a point-mass at $r = 0$ to have a non-zero area $4\pi(2GM)^2$ and a *zero* volume *simultaneously* ?; so it seems that one is forced to choose the Hilbert gauge $R(r = 0) = 0$. Nevertheless it was shown in [8] how by slecting a $judicious$ choice of $R(r)$ one can cure this flaw and have the correct boundary condition $R(r = 0) = 0$, while displacing the horizon from $r = 2GM$ to a location *arbitrarily* close to $r = 0$ as one desires, $r_h \to 0$, and where stringy geometry and Quantum Gravitational effects begin to take place.

There are two ways to shift the horizon away from the known $2GM$ value. One way is by assigning an infinite family of everywhere smooth area radial functions $R(r)$ such that $R(r = 0) = 0$; $R(r \to \infty) \to r$, so the value of the shifted horizon r_h, defined by the condition $R(r = r_h) = 2GM$, obeys $0 < r_h \leq 2GM$. In this case one has an infinite family of metric solutions that are diffeomorphic to the Hilbert text book solution defined by $R(r) = r$.

There is another choice for the area radial function $R(r)$ leading to a metric solution which is *not* diffeomorphic to the above Hilbert text book solution because $R(r)$ and the metric is *discontinuous* at $r = 0$. This was attained in [8] when the area radial function, was chosen to be $R(r) = r + 2G|M|\Theta(r)$, where the Heaviside Step function is defined $\Theta(r) = 1$ when $r > 0$, $\Theta(r) = -1$ when $r < 0$ and $\Theta(r = 0) = 0$ (the arithmetic mean of the values at $r > 0$ and $r < 0$). The area radial function becomes $R \sim r$ when $r >> 2GM$ and one recovers the correct Newtonian limit in the weak field limit regime. It is now, via the Heaviside step function, that we may maintain the correct behaviour $R(r = 0) = 0$, when $r = 0$, consistent with our intuitive notion that the spatial area and spatial volume of the point mass at $r = 0$ *has* to be *zero*. The area radial function $R(r) = r + 2G|M|\Theta(r)$ also obeys $R(r < 0) = -R(r > 0) < 0$ such that the solutions with $r < 0,\ M > 0$ have a one-to-one correspondence to the solutions with $r > 0,\ M < 0$ ("white hole") because $|-M| = |M|$. The latter $M < 0$ repulsive gravity regime is what it is called a "white" hole.

The metric in eq-(1.1) associated with our choice of the areal radial function $R(r) =$

$r + 2G|M|\Theta(r)$ is continuous everywhere except at the singularity $r = 0$, the location of the point mass source. Because this is an infinitely compact source there is nothing *wrong* with having a *discontinuity* of the metric at $r = 0$. The study of Einstein equations and the joining of discontinuous metrics when these are discontinuous across the joining (hyper) surface was studied by [6] in the static spherically symmetric case. These discontinuous metrics obey Einstein equations with an energy-momentum tensor which has a delta function type of singularity on the (hyper) surface of discontinuity. It was found that a surface tension is always associated to the cases where the metrics are discontinuous. The kind of metric discontinuity which follows by our choice of the areal radial function $R(r)$ above is of a $different$ type than the ones studied by [6]. In section **2** we shall study explicitly the case where it is a delta function type of singularity for the energy-momentum tensor (mass density and pressure) associated with the point mass which is the source of a curvature discontinuity at $r = 0$.

Due to the *discontinuity* of the metric at $r = 0$, the location $r = 0, R(r = 0) = 0$ corresponds to a *spacelike* singularity since $g_{tt}(r = 0) = -\infty < 0$: it changes sign. Whereas $g_{rr}(r = 0) = 0$ because the quantity $r(1+2GM\delta(r))^2 = 0$, when $r = 0$, due to the fact that it is an *odd* function of r so the latter expression vanishes at $r = 0$. Furthermore, because $g_{tt}(r = 0) = -\infty < 0$ has changed sign, and $g_{rr} = 0$, the displacement $ds^2 < 0$ is now *spacelike*, so there is *no violation* of the cosmic censorship conjecture (that rules out *timelike* singularities).

Secondly, despite that the metric is not continuous nor differentiable at $r = 0$, it is explicitly shown in the Appendix that despite the derivatives $\frac{dR}{dr} = 1 + 2G|M|\delta(r)$ and $(d^2R/dr^2) = 2G|M|\delta'(r)$ are *singular* at $r = 0$, there is an exact and precise *cancellation* of these singular derivatives (involving delta functions) in the evaluation of the Ricci curvature tensor components yielding a zero Ricci tensor $\mathcal{R}_{\mu\nu} = 0$ and a zero Ricci scalar $\mathcal{R} = 0$. What is *not* zero is the Riemann curvature tensor $\mathcal{R}_{\mu\nu\rho\tau}$. Therefore, the conditions $\mathcal{R}_{\mu\nu} = 0$ and $\mathcal{R} = 0$ are satisfied for *any* area radial function $R(r)$, irrespective if it has singular derivatives at $r = 0$ or not.

Since $r = \pm\sqrt{x^2 + y^2 + z^2}$, a negative r branch is mathematically possible and is compatible with the *double* covering inherent in the Fronsdal-Kruskal-Szekeres [5] analytical continuation in terms of the U, V coordinates. Each point of spacetime inside $r < 2GM$ is represented *twice* (black hole and white hole picture). However there is a $fundamental$ difference (besides others) with the Fronsdal-Kruskal-Szekeres extension into the interior of $r = 2GM$, their metric description is *no* longer *static* in $r < 2GM$, whereas in our case the metric *is static* for *all* values of r.

To sum up, because $R(r) = \epsilon + 2G|M|$, when $r = \epsilon > 0$, the horizon to be can the be displaced from $r = 2G|M|$ to a location as *arbitrarily* close to $r = 0$ as desired $r_{Horizon} \rightarrow 0$. To be more precise, the horizon occurs at $r = 0^+$ and at $r = 0$ one hits the singularity due to the discontinuity of the metric. In the r-coordinates picture there is a *discontinuity* of the metric (and scalar curvature) at $r = 0$, the location of the point mass source. In the R-coordinate picture, due to the correct condition $R(r = 0) = 0$ consistent with the fact that a point must have zero area (since $\Theta(r = 0) = 0$), one can interpret the discontinuity of the metric as if the region of $0 < R < 2GM$ were eliminated from the spacetime manifold to make the surface at $R = 2GM$ a boundary of the spacetime while leaving the singularity at $r = 0$ behind.

Having $R(r = 0) = 0$ and $R(r = 0^+) = 2GM$ (we shall omit the absolute symbol in M for simplicity), our solutions can be described by focusing on the right and left regions (quadrants) of the Rindler-wedge formed by the straight (null) lines $U = \pm V$, corresponding to $r = 0^+$, $t = \pm\infty$, and whose slope is $+45, -45$ degrees respectively. This is attained after performing the Fronsdal-Kruskal-Szekeres change of coordinates [5] in the exterior region $R > 2GM$

$$U = (\frac{R}{2GM} - 1)^{\frac{1}{2}} e^{R/4GM} cosh (\frac{t}{4GM}), \quad V = (\frac{R}{2GM} - 1)^{\frac{1}{2}} e^{R/4GM} sinh (\frac{t}{4GM}); \ R > 2GM \tag{1.2}$$

and the change of coordinates in the interior region $R < 2GM$

$$U = (1 - \frac{R}{2GM})^{\frac{1}{2}} e^{R/4GM} sinh (\frac{t}{4GM}), \quad V = (1 - \frac{R}{2GM})^{\frac{1}{2}} e^{R/4GM} cosh (\frac{t}{4GM}); \ R < 2GM \tag{1.3}$$

In the overlap $R = 2GM$, one has that $U = \pm V$ and $t = \pm\infty$; and $U = V = 0$ for $finite$ t. The coordinate transformations lead to a well behaved metric (except at $R(r = 0) = 0$)

$$ds^2 \ = \ \frac{4(2GM)^3}{R(U,V)} \ e^{-R(U,V)/2GM} \ (dV^2 \ - dU^2 \ ) \ - R(U,V)^2 (d\Omega)^2. \tag{1.4}$$

the functional form $R(U,V)$ is defined implicitly by the equation

$$U^2 \ - \ V^2 \ = \ (1 - \frac{R}{2GM}) \ e^{R/42GM} \ \Rightarrow \ \frac{R}{2GM} \ = \ 1 \ + \ W(\frac{V^2 - U^2}{e}) \tag{1.5}$$

where W is the Lambert function defined implicitly by $z = W(z) e^{W(z)}$. When $R = 2GM$ and $d\Omega = 0$, the above interval displacement $ds^2 = 0$ is *null* along the lines $U = \pm V \Rightarrow dU = \pm dV$. It is singular at $R(r = 0) = 0$ along the (spacelike) lines $V^2 - U^2 = 1 \Rightarrow dV \neq \pm dU$.

The picture proposed in [8] was that a radially incoming photon, starting at point P in the right region (quadrant) of the Rindler wedge, moves upwards parallel to the -45 degrees ($U = -V$) null-line and reaches the null-line branch ($U = V$), given by $R(r = 0^+) = 2GM$ and $t = \infty$, at point P'. Then it "tunnels" through the interior region $R < 2GM$ and reaches the spacelike singularity $r = 0, R(r = 0) = 0$ at point P'' and whose value of $t(P'')$ is $finite$. This "tunneling" behaviour from P' to P'' is a direct consequence of the *discontinuity* of the metric at $r = 0$ resuting in a *separation* between the points $P' = (r = 0^+, t = \infty)$ and $P'' = (r = 0, t = finite)$.

In essence, the singularity $r = 0, R(r = 0) = 0$, for all values of t, has been spliced-off from the rest of spacetime by *carving out* the future and past regions (quadrants) of the Rindler wedge (creating a spacetime void) leaving only the right and left regions (quadrants) bounded by the null lines $U = \pm V$, corresponding to $R(r = 0^+) = 2GM$ at $t = \pm\infty$. The fact that we end up only with the left and right regions of the Rindler wedge might have some relationship to the factor of *two* discrepancy of the Hawking radiation temperature which appears when working with the left-right versus the future-right regions of the Rindler wedge [10].

The purpose of this letter is to considerable *improve* the findings in [8] due to the fact that this "tunneling" behavior through the interior region $R < 2GM$ described above is not fully satisfactory since there is a discontinuity/disconnect in the geodesics resulting in the separation of the singularity from the points of the geodesic lines inside the region $R \geq 2GM$. The correct procedure goes as follows. Firstly, one retains the Kruskal-Szekeres change of coordinates in the exterior region $R \geq 2GM$, but one *replaces* the change of coordinates in the *interior* region $R < 2GM$ in eqs-(1.3) by the following one

$$V \ = \ (\frac{R}{2GM})^{\frac{1}{2}} \ cosh(\frac{t}{4GM}); \quad U \ = \ (\frac{R}{2GM})^{\frac{1}{2}} \ sinh(\frac{t}{4GM}); \quad R < 2GM \qquad (1.6)$$

leading to $V^2 - U^2 = \frac{R}{2GM}$ and $\frac{U}{V} = tanh(t/4GM)$. In doing so one has that the points $R(r=0) = 0$ and $t = \pm\infty$ are mapped to the straight lines $U = \pm V$ with a ± 45 degree slope, respectively. While $R(r=0) = 0$ is mapped to the origin of coordinates $U = V = 0$ for arbitrary but $finite$ values of t. In this fashion there is geodesic completeness and there are no disconnected points along the geodesics. The incoming radial null geodesics (and future-oriented time like geodesics) all end up in the *null* singularity described now by the straight line $U = V$, instead of the (spacelike) hyperbola $V^2 - U^2 = 1$, and without "*tunneling*" through the interior region $R < 2GM$.

To show that now one has a *null* singularity at $U = \pm V$ one inserts the above change of coordinates (1.6) for the region $R < 2GM$ into the metric (1.1), such that it leads to a $different$ expression for the metric than in eq-(1.4) and given by

$$ds^2 \ = \ g_{UU} \ dU^2 \ + \ g_{VV} \ dV^2 \ + \ 2 \ g_{UV} \ dU \ dV \ + \ R^2(U,V) \ d\Omega^2, \quad R < 2GM \qquad (1.7)$$

where

$$g_{UU} \ = \ (1 - \frac{1}{V^2 - U^2}) \ (\frac{4GMV}{V^2 - U^2})^2 \ - \ (1 - \frac{1}{V^2 - U^2})^{-1} \ (4GMU)^2 \qquad (1.8a)$$

$$g_{VV} \ = \ (1 - \frac{1}{V^2 - U^2}) \ (\frac{4GMU}{V^2 - U^2})^2 \ - \ (1 - \frac{1}{V^2 - U^2})^{-1} \ (4GMV)^2 \qquad (1.8b)$$

$$g_{UV} \ = \ g_{VU} \ = \ - \ (1 - \frac{1}{V^2 - U^2}) \ (\frac{4GMV}{V^2 - U^2}) \ (\frac{4GMU}{V^2 - U^2}) \ + (4GM)^2 \ (1 - \frac{1}{V^2 - U^2})^{-1} \ U \ V \qquad (1.8c)$$

Despite the different expression for the metric components in eqs-(1.7) from those in eq-(1.4) , one still has a *null* interval displacement $ds^2 = 0$ along the lines $U = \pm V$, and which correspond to the values $R(r=0) = 0$ and $t = \pm\infty$, respectively. Therefore, one has now a *null singularity* along the lines $U = \pm V$ instead of a spacelike singularity along the hyperbola $V^2 - U^2 = 1$. One can verify explicitly that when $U = \pm V, dU = \pm dV$ there is an *exact* cancellation of the singular terms

$$2\,\frac{(4GM)^2UV}{(V^2-U^2)^3}\,dUdV\;-\;\frac{(4GM)^2U^2}{(V^2-U^2)^3}\,dV^2\;-\;\frac{(4GM)^2V^2}{(V^2-U^2)^3}\,dU^2 \qquad (1.9a)$$

and

$$-2\,\frac{(4GM)^2UV}{(V^2-U^2)^2}\,dUdV\;+\;\frac{(4GM)^2U^2}{(V^2-U^2)^2}\,dV^2\;+\;\frac{(4GM)^2V^2}{(V^2-U^2)^2}\,dU^2 \qquad (1.9b)$$

in the above infinitesimal interval ds^2 of eqs-(1.7,1.8). Whereas there is also an *exact* cancellation of the non-singular terms when $U = \pm V, dU = \pm dV$. Since $R(r = 0) = 0$, one obtains a net zero value for the displacement $ds^2 = 0$ in eq-(1.7) furnishing then a *null* interval. [1] Because the curvature-squared Kretschmann invariant blows up $\mathcal{R}_{\mu\nu\rho\tau}\mathcal{R}^{\mu\nu\rho\tau} \sim (2GM)^2/R(r)^6 \to \infty$ when $R(r) = 0$ at $r = 0$, one has then a *null* singularity at $r = 0$, as opposed to a *spacelike* singularity in the traditional solutions.

The physical insignificance of null naked singularities within the context of Penrose's cosmic censorship conjecture was analyzed by [12] in the study of gravitational collapse of general forms matter in the most general of spacetimes. It was shown that the energy is completely trapped inside the null singularity and therefore these null singularities cannot be experimentally observed and cannot cause a breakdown of predictability. This conclusion strongly supports and preserves the essence of the cosmic censorship hypothesis. A timelike singularity is in principle likely to be visible to an outside observer as the redshift is always finite for the light rays emerging from it. For the null singularity surface, the redshift basically diverges as the proper time goes to zero on the null surface. It was argued by [12] that despite that the null singularity is geometrically naked (null geodesics can come out of it) essentially it is not physically visible (naked) as no energy can come out of it due to the infinite redshift. Because one cannot get any information from the null naked singularity it will not have any undesirable physical effect to an outside observer.

Having made these remarks one should emphasize that we have a point-mass source at $r = 0$ and which may, or may not, arise from gravitational collapse. Secondly, the point-mass singularity is described by a null world-line, instead of a null surface. One could study the scenario where gravitational collapse might end up in the null singularity described in this work. This would require, in particular, abandoning the condition that the metric is smooth everywhere because we have a discontinuity of the area radial function $R(r)$ at $r = 0$ furnishing a class of spacetime metrics that have *not* been considered before in the singularity theorems literature, to our knowledge. As of today, we do *not* have any proof or any specific mathematically rigorous formulation of the Cosmic Censorship Conjecture available within the framework of gravitation theory [19]. Therefore, we still cannot ascertain with absolute certainty that spacetime singularities formed after gravitational collapse would always be covered by black hole horizons. For this reason we believe that the null naked singularity solution associated with point mass sources deserves to be investigated further.

To sum up, the choice of the coordinate transformations in eq-(1.6), valid for $R < 2GM$, leads to the expected proper *discontinuity* of the area radial function $R(r)$ at

[1] One may verify also that $ds^2 = 0$ in eq-(1.7) when $d\Omega^2 = 0$ and $V^2 - U^2 = 1 \Leftrightarrow R = 2GM$, which follows only if one uses eq-(1.6), and after inserting the values for the metric components given by eqs-(1.8) due to an exact cancellation of the singular terms when $V^2 - U^2 = 1 \Leftrightarrow dV = UdU/V$.

$r = 0$; i.e. $R(r)$ jumps from the value of $2GM$ to 0 along the null lines $U = \pm V$, whereas the values of t remain properly fixed at $\pm\infty$, respectively, as they should. On the other hand, the Kruskal-Szekeres coordinate transformation in eq-(1.2), valid for $R \geq 2GM$, is duly consistent with the value of $R = 2GM$ at $r = 0^+$, and which also corresponds to the null lines $U = \pm V$ and $t = \pm\infty$, which can be verified by a mere application of eqs-(1.2). Infalling particles and photons will reach the null singularity at an infinite coordinate time $t = \infty$ as measured by an external observer in the asymptotically flat region at infinity due to the infinite redshift factor.

Furthermore, since there is *no* longer an *interior* region between $R = 0$ and $R = 2GM$, due to the chosen discontinuity of the area radial function $R(r)$ at $r = 0$, there is no longer an spatial volume inside. This could be relevant in explaining why the gravitational entropy does not depend on the volumes but depends on the areas. Hawking radiation can also emerge from naked singularities. In particular, under certain adiabatic conditions, a Planck-distributed flux of Hawking-like radiation can emerge from evolving black holes for which *no* horizon has been formed yet, or even will ever form [13]. On the other hand, it has been argued that until an understanding of quantum gravity is made, in at least some regimes, no compelling theoretical case for, or against radiation, by black holes is likely to be made, see [14] and references therein. For this reason it is beyond the scope of this work to try to answer the question whether or not the findings of this work will be useful in resolving the information paradox and which differs from the main approaches to the solution of the paradox based on the AdS/CFT correspondence, Black hole complementarity, Fuzzballs in string theory, the Holographic principle, etc....

2 Point Mass Sources and Euclidean Gravitational Action as Entropy

A rigorous correct treatment of point mass distributions in General Relativity has been provided based on Colombeau's [7] theory of nonlinear distributions, generalized functions and nonlinear calculus. This permits the proper multiplication of distributions since the old Schwarz theory of linear distributions is invalid in nonlinear theories like General Relativity. Colombeau's nonlinear distributional geometry supersedes the no-go results of Geroch and Traschen [16] stating that there is no proper framework to study distributions of matter of co-dimensions higher than two (neither points nor strings in $D = 4$) in General Relativity. Colombeau's theory of Nonlinear Distributions (and Nonstandard Analysis) is the proper way to deal with point-mass sources in nonlinear theories like Gravity and where one may rigorously solve the problem without having to introduce a boundary of spacetime at $r = 0$.

Nevertheless one may still arrive at some interesting physical results by recurring to the ordinary Dirac delta functions. In order to generate $\delta(r)$ terms in the right hand side of Einstein's equations in the presence of a point-mass source, it was argued in [8] that one must replace everywhere $r \rightarrow |r|$ as required when point-mass sources are inserted.

The Newtonian gravitational potential (in three dimensions) due to a point-mass source at $r = 0$ is given by $-G_N M/|r|$. It is consistent with Poisson's law which states that the non-zero Laplacian of the Newtonian potential $\nabla^2(-GM/|r|) = 4\pi G\rho$ is proportional to the mass density distribution $\rho = (M/4\pi r^2)\delta(r)$. However, the Laplacian in spherical coordinates of $(1/r)$ is identically *zero*. For this reason, there is a *fundamental* difference in dealing with expressions involving absolute values $|r|$ like $1/|r|$ from those which depend on r like $1/r$. This is a direct consequence of the *discontinuity* of the derivatives of the function $|r|$ at $r = 0$.

In particular, after rewriting the metric components in the form

$$g_{tt} = 1 - \frac{2GM}{|r|} = 1 - \frac{2GM}{r} \frac{r}{|r|} = 1 - \frac{2GM}{r} f(r); \quad f(r) \equiv \frac{r}{|r|}. \tag{2.1}$$

$$g_{rr} = - \frac{1}{g_{tt}}. \tag{2.2}$$

such that the derivatives

$$f'(r) = \frac{df(r)}{dr} = \delta(r); \quad f''(r) = \frac{d^2 f(r)}{dr^2} = \delta'(r). \tag{2.3}$$

reveals that the *nonvanishing* $\mathcal{R}$ is given by :

$$\mathcal{R} = - 2GM \left[\frac{f''(r)}{r} + 2 \frac{f'(r)}{r^2} \right] =$$

$$- 2GM \left[\frac{\delta'(r)}{r} + 2 \frac{\delta(r)}{r^2} \right] = - 8\pi G \, T \tag{2.4}$$

where T is the trace of the stress energy tensor $g^{\mu\nu}T_{\mu\nu}$ in the Einstein's field equations due to the presence of matter and the signature chosen is $(+, -, -, -)$. The scalar curvature (2.4) is $\mathcal{R} = 0$ for $r > 0$ and it is singular at $r = 0$. Whereas the scalar curvature $\mathcal{R}$ and Ricci tensor $\mathcal{R}_{\mu\nu}$ associated with the standard Schwarzschild (Hilbert) solutions, involving r instead of $|r|$, are identically *zero* for *all* values of r, including $r = 0$. [2]

The non-trivial Einstein-Hilbert action associated with a point-mass source is

$$S = - \frac{1}{16\pi G} \int \mathcal{R} \, 4\pi r^2 \, dr \, dt = \frac{1}{16\pi G} \int 2GM \left[\frac{\delta'(r)}{r} + 2 \frac{\delta(r)}{r^2} \right] 4\pi r^2 \, dr \, dt. \tag{2.5}$$

Integrating by parts yields

$$\frac{1}{16\pi G} \int 8\pi GM \, [\, 2\delta(r) - \delta(r) \,] \, dr \, dt = \frac{1}{16\pi G} \int 8\pi G \left(\frac{M \, \delta(r)}{4\pi r^2} \right) 4\pi r^2 \, dr \, dt =$$

[2]One may notice that by choosing $f(r) = \kappa/r$ in eq-(2.4) for κ= constant, it yields $\mathcal{R} = 0$ which implies a zero trace for the stress energy tensor $T = 0$, as it happens in Electromagnetism due to the conformal invariance of Maxwell equations in $D = 4$. The Reisner-Nordstrom solutions (in the massless case) have for temporal metric component $g_{tt} = 1 - e^2/r^2$, which has the same functional form as $g_{tt} = 1 - (2GM/r)f(r) = 1 - 2GM\kappa/r^2 \leftrightarrow 1 - e^2/r^2$.

$$\frac{1}{2} \int M \ dt \tag{2.6}$$

The Euclideanized Einstein-Hilbert action associated with the non-trivial scalar curvature (2.4) is obtained after a compactification of the temporal direction along a circle S^1 whose net Euclidean time integration interval is $2\pi t_E$. The latter interval can be defined in terms of the Hawking temperature T_H and Boltzman constant k_B as $2\pi t_E = (1/k_B T_H) = 8\pi GM$. Integrating with respect to the Euclidean temporal coordinate, the Euclidean gravitational action becomes then

$$S_E = (\frac{M}{2}) \ (2\pi t_E) \ = \ 4\pi \ G \ M^2 = \ \frac{1}{4} \ \frac{4\pi(2GM)^2}{G_N} \ = \ \frac{Area}{4 \ L_P^2}. \tag{2.7}$$

which is precisely the black bole Entropy in Planck area units $G = L_P^2$ ($\hbar = c = 1$).

This result that the Euclideanized gravitational action (associated with a non-trivial scalar curvature involving delta functions due to point-mass sources) is the same as the black hole entropy can be generalized to higher dimensions. In the Reissner-Nordsrom (massive-charged) and Kerr-Newman black hole case (massive-rotating-charged) we gave shown also [8] that the Euclidean action in a bulk domain bounded by the inner and outer horizons is the same as the black hole entropy. These findings should be compared to Verlinde's enthropic gravity proposal [18] based on the holographic principle.

Replacing the *area* radial function R for is absolute value $|R|$ in eq-(1.1) gives

$$(ds)^2 \ = \ (1 - \frac{2GM}{|R|}) \ (dt)^2 \ - (1 - \frac{2GM}{|R|})^{-1} \ (dR)^2 \ - \ R^2(r) \ (d\Omega)^2. \tag{2.8}$$

and it leads to a non-trivial scalar curvature

$$\mathcal{R} = \ - \ 2GM \ [\ \frac{\delta'(R)}{R} \ + \ 2 \ \frac{\delta(R)}{R^2} \ ]; \ \ \delta'(R) = \frac{\partial(\delta(R))}{\partial R} \tag{2.9}$$

that is singular at $R(r = 0) = 0$ and vanishing for $r > 0 \Rightarrow R(r) > 0$. The non-trivial Einstein-Hilbert action becomes, after integrating by parts,

$$S = \ - \frac{1}{16\pi G} \ \int \mathcal{R} \ 4\pi R^2 \ dR \ dt = \ \frac{1}{2} \ \int \ M \ dt \tag{2.10}$$

The end result after integration is the same as in eq-(2.7) as one would expect. The main difference in using the area radial function $R(r) = r + 2G|M|\Theta(r)$ is that the horizon is now displaced to $r = 0^+$.

As discussed in detail in [8] we can model the mass distribution by a *smeared* delta function [17], by starting with the following field equations associated with the signature $(+,-,-,-)$

$$\mathcal{R}_{00} \ - \frac{1}{2} \ g_{00} \ \mathcal{R} \ = \ 8\pi G \ T_{00} \ = \ g_{00} \ 8\pi G \ \rho(r), \quad \mathcal{R}_{ij} \ - \frac{1}{2} \ g_{ij} \ \mathcal{R} \ = \ 8\pi G \ T_{ij} \tag{2.11}$$

where $\rho(r)$ is a smeared delta function given by the Gaussian, and the T_{ij} elements are comprised of a radial and tangential pressures of a self-gravitating anisotropic fluid [17]

$$\rho(r) \ = \ M \ \frac{e^{-r^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}}, \quad p_r \ = \ - \rho(r), \quad p_{tan} \ = \ p_\theta \ = \ p_\phi \ = -\rho(r) - \frac{r}{2} \ \frac{d\rho}{dr}. \quad (2.12)$$

The radial pressure $p_r = -\rho$ is negative pointing towards the center $r = 0$ consistent with the self-gravitating picture of the droplet. The radial dependence of the mass distribution is explicitly given in terms of the incomplete Gamma function $\gamma[a, r]$ as

$$\mathcal{M}(r, \sigma) = \ M \ \int_0^r \ \frac{e^{-r^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}} \ 4\pi r^2 \ dr \ = \ \frac{2M}{\sqrt{\pi}} \ \gamma[\frac{3}{2}, \ \frac{r^2}{4\sigma^2}]. \quad (2.13)$$

The metric solution to the Einstein's equations (2.11) are [17]

$$(ds)^2 \ = \ (1 - \frac{2G\mathcal{M}(r, \sigma)}{r}) \ (dt)^2 \ - (1 - \frac{2G\mathcal{M}(r, \sigma)}{r})^{-1} \ (dr)^2 \ - \ r^2 \ (d\Omega)^2. \quad (2.14)$$

In the limit $\sigma^2 \to 0$ one recovers the delta function

$$lim_{\sigma \to 0} \ \frac{e^{-r^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}} \ \to \frac{\delta(r)}{4\pi r^2}. \quad (2.15)$$

and the incomplete Gamma function reduces to the ordinary Gamma function $\Gamma(\frac{3}{2}) = (\sqrt{\pi}/2)$ such that $\mathcal{M}(r, \sigma \to \infty)$ tends to M. The stress energy tensor for a point mass source is given explicitly by the zero-width limit of the Gaussian in the right hand side of eqs-(2.12), as shown explicitly in [8]. It involves both density *and* pressure terms. One one can verify the validity of eq- (2.4), in the zero width limit, after taking the trace of the stress energy tensor $T^\mu_\nu = diagonal \ (\rho, -p_r, -p_\theta, -p_\phi)$ and whose components in the $\sigma \to 0$ limit are

$$\rho(r) \ = \ - p_r(r) \ = \ M \frac{\delta(r)}{4\pi r^2}, \quad p_\theta(r) \ = \ p_\phi(r) \ = \ - M \frac{\delta'(r)}{8\pi r} \quad (2.16)$$

The generation of source terms in General Relativity and the appearance of naked singularities due to the choices of differential structures in exotic four-dim manifolds has been studied extensively by [15]. The discontinuity of $R(r)$ and its derivatives at the singularity $r = 0$ (location of the point mass source) that lead to null naked singularities might be relevant to the existence of different diffeomorphic structures in exotic four-dimensional manifolds (and higher dimensions). This warrants further investigation.

We finalize by adding some remarks [8] about how a $fuzzy$ point mass may admit a short distance cut-off of the Brillouin form $R(r = 0) = 2GM$ (instead of $zero$) if one has a Noncommutative spacetime coordinates algebra $[x^\mu, x^\nu] = i\Sigma^{\mu\nu}, \ [p^\mu, p^\nu] = 0, \ [x^\mu, p^\nu] = i\hbar\eta^{\mu\nu}$ where $\Sigma^{\mu\nu}$ are c-numbers of $(Planck \ length)^2$ magnitude. A change of coordinates in phase space $x'^\mu = x^\mu + \frac{1}{2}\Sigma^{\mu\nu} \ p_\nu$ leads to commuting coordinates x'^μ and allows to define $r'(r) = \sqrt{(x^i + \frac{1}{2} \ \Sigma^{i\rho} \ p_\rho) \ (x_i + \frac{1}{2} \ \Sigma_{i\tau} \ p^\tau)}$. One can select $\Sigma^{\mu\nu}$ such that $r'(x^i = 0) = r'(r = 0) = 2GM$, after using the on-shell condition $p_\mu p^\mu = M^2$.

Therefore one recovers the cut-off corresponding to the Brillouin area radial function $R(r) = r + 2GM \Rightarrow R(r=0) = 2GM$. Thus a fuzzy point mass has non-zero area and volume. Finsler geometry (Lagrange-Finsler manifolds) associated with the spacetime tangent bundle and the Hamilton-Cartan geometry of phase spaces, *is* the proper arena where one can study point masses within the context of Noncommutative Gravity of the spacetime tangent (co-tangent) bundle. For a nice review of the physical applications of Finsler geometry see [20].

Another Planck scale cut-off can be derived in terms of noncommutative Moyal star products $f(x) * g(x)$ simply by replacing $r \rightarrow r_* = \sqrt{r * r} = \sqrt{r^2 + \Sigma^{ij} x_i x_j / r^2 + ...}$ so $r_*(x^i = 0) \neq 0$, and it receives Planck scale corrections. A point is fuzzy and delocalized, henceforth it has a non-zero fuzzy area and fuzzy volume. An open problem is to verify whether or not Schwarzschild deformed metrics of the form

$$g_{tt}(r_*) = 1 - \frac{2GM}{r_*}, \quad g_{rr} = -g_{tt}^{-1}, \quad r_* = \sqrt{r * r} = \sqrt{r^2 + \Sigma^{ij} x_i x_j / r^2 + ...}\ . \qquad (4.2a)$$

with the angular part $r_* \ * \ r_* \ (d\Omega)^2$, solve the Noncommutative Gravity field equations to *all* orders in the noncommutative parameter $\Sigma^{\mu\nu}$. This is a very difficult problem.

APPENDIX : Schwarzschild-like solutions in $D > 3$

In this Appendix we follow closely our prior calculations [9] to the static spherically symmetric *vacuum* solutions to Einstein's equations in any dimension $D > 3$. Let us start with the line element with signature $(-, +, +, +, ..., +)$

$$ds^2 = -e^{\mu(r)}(dt)^2 + e^{\nu(r)}(dr)^2 + R^2(r)\tilde{g}_{ij} d\xi^i d\xi^j. \qquad (A.1)$$

Here, the metric $\tilde{g}_{ij}$ corresponds to a homogeneous space and $i, j = 3, 4, ..., D-2$ and the temporal and radial indices are denoted by $1, 2$ respectively. In our text we denoted the temporal index by 0. The only non-vanishing Christoffel symbols are given in terms of the following partial derivatives with respect to the r variable and denoted with a prime

$$\begin{array}{lll} \Gamma^1_{21} = \frac{1}{2}\mu', & \Gamma^2_{22} = \frac{1}{2}\nu', & \Gamma^2_{11} = \frac{1}{2}\mu' e^{\mu-\nu}, \\ \Gamma^2_{ij} = -e^{-\nu} R R' \tilde{g}_{ij}, & \Gamma^i_{2j} = \frac{R'}{R}\delta^i_j, & \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}, \end{array} \qquad (A.2)$$

and the only nonvanishing Riemann tensor are

$$\begin{array}{ll} \mathcal{R}^1_{212} = -\frac{1}{2}\mu'' - \frac{1}{4}\mu'^2 + \frac{1}{4}\nu'\mu', & \mathcal{R}^1_{i1j} = -\frac{1}{2}\mu' e^{-\nu} R R' \tilde{g}_{ij}, \\ \mathcal{R}^2_{121} = e^{\mu-\nu}(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 - \frac{1}{4}\nu'\mu'), & \mathcal{R}^2_{i2j} = e^{-\nu}(\frac{1}{2}\nu' R R' - R R'')\tilde{g}_{ij}, \\ \mathcal{R}^i_{jkl} = \tilde{R}^i_{jkl} - R'^2 e^{-\nu}(\delta^i_k \tilde{g}_{jl} - \delta^i_l \tilde{g}_{jk}). & \end{array} \qquad (A.3)$$

The vacuum field equations are

$$\mathcal{R}_{11} = e^{\mu-\nu}(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 - \frac{1}{4}\mu'\nu' + \frac{(D-2)}{2}\mu'\frac{R'}{R}) = 0, \qquad (A.4)$$

$$\mathcal{R}_{22} = -\frac{1}{2}\mu'' - \frac{1}{4}\mu'^2 + \frac{1}{4}\mu'\nu' + (D-2)(\frac{1}{2}\nu'\frac{R'}{R} - \frac{R''}{R}) = 0, \qquad (A.5)$$

and

$$\mathcal{R}_{ij} = \frac{e^{-\nu}}{R^2}(\frac{1}{2}(\nu' - \mu')RR' - RR'' - (D-3)R'^2)\tilde{g}_{ij} + \frac{k}{R^2}(D-3)\tilde{g}_{ij} = 0, \qquad (A.6)$$

where $k = \pm 1$, depending if $\tilde{g}_{ij}$ refers to positive or negative curvature. From the combination $e^{-\mu+\nu}\mathcal{R}_{11} + \mathcal{R}_{22} = 0$ we get

$$\mu' + \nu' = \frac{2R''}{R'}. \qquad (A.7)$$

The solution of this equation is

$$\mu + \nu \ = \ \ln R'^2 \ + \ C, \qquad (A.8)$$

where C is an integration constant that one sets to *zero* if one wishes to recover the flat Minkowski spacetime metric in spherical coordinates in the asymptotic region $r \to \infty$.

Substituting (A.7) into the equation (A.6) we find

$$e^{-\nu} \ (\ \nu' RR' - 2RR'' - (D-3)R'^2 \ ) \ = \ - k(D-3) \qquad (A.9)$$

or

$$\gamma' RR' + 2\gamma RR'' + (D-3)\gamma R'^2 = k(D-3), \qquad (A.10)$$

where

$$\gamma = e^{-\nu}. \qquad (A.11)$$

The solution of (A.10) for an ordinary D-dim spacetime (one temporal dimension) corresponding to a $D-2$-dim sphere for the homogeneous space can be written as

$$\gamma = (1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2} R^{D-3}}) \ (\frac{dR}{dr})^{-2} \Rightarrow$$

$$g_{rr} = e^{\nu} = (1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2} R^{D-3}})^{-1} \ (\frac{dR}{dr})^2. \qquad (A.12)$$

where Ω_{D-2} is the appropriate solid angle in $D-2$-dim and G_D is the D-dim gravitational constant whose units are $(length)^{D-2}$. Thus $G_D M$ has units of $(length)^{D-3}$ as it should. When $D = 4$ as a result that the 2-dim solid angle is $\Omega_2 = 4\pi$ one recovers from eq-(A.12) the 4-dim Schwarzchild solution. The solution in eq-(A.12) is consistent with Gauss law and Poisson's equation in $D-1$ spatial dimensions obtained in the Newtonian limit.

For the most general case of the $D-2$-dim homogeneous space we should write

$$-\nu = \ln(k - \frac{\beta_D G_D M}{R^{D-3}}) - 2\ln R'. \qquad (A.13)$$

β_D is a constant equal to $16\pi/(D-2)\Omega_{D-2}$, where Ω_{D-2} is the solid angle in the $D-2$ transverse dimensions to r, t and is given by $(D-1)\pi^{(D-1)/2}/\Gamma[(D+1)/2]$.

Thus, according to (A.8) we get

$$\mu = \ln(k - \frac{\beta_D G_D M}{R^{D-3}}) \ + \ constant. \qquad (A.14)$$

we can set the constant to zero, and this means the line element (A.1) can be written as

$$ds^2 = -(k - \frac{\beta_D G_D M}{R^{D-3}})(dt)^2 \ + \ \frac{(dR/dr)^2}{(k - \frac{\beta_D G_D M}{R^{D-3}})}(dr)^2 \ + \ R^2(r)\tilde{g}_{ij}d\xi^i d\xi^j \ =$$

$$-(k - \frac{\beta_D G_D M}{R^{D-3}})(dt)^2 \ + \ \frac{1}{(k - \frac{\beta_D G_D M}{R^{D-3}})}(dR)^2 \ + \ R^2(r)\tilde{g}_{ij}d\xi^i d\xi^j \qquad (A.15)$$

One can verify, that the equations (A.4)-(A.6),leading to eqs-(A.9)-(A.10), do *not* determine the form $R(r)$. These equations are satisfied even if $R(r)$ has *singular* derivatives at $r=0$, like those appearing in $dR/dr = 1 + 2G|M|\delta(r)$. It is also interesting to observe that the only effect of the homogeneous metric $\tilde{g}_{ij}$ is reflected in the $k = \pm 1$ parameter, associated with a positive (negative) constant scalar curvature of the homogeneous $D-2$-dim space. $k=0$ corresponds to a spatially flat $D-2$-dim section. The metric solution in eq-(1.1) is associated to a different signature than the one chosen in this Appendix, and corresponds to $D=4$ and $k=1$.

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