THE NAVIER-STOKES EQUATION:
The Queen of Fluid Dynamics.
A proof simple, but complete.

Leonardo ***Rubino***
leonrubino@yahoo.it
September 2010 – Rev. 00
For www.vixra.org

Abstract: in this paper you will find a simple demonstration of the Navier-Stokes equation, while, most of times, in books, you find it broken into its vectorial components whose proofs are usually not so clear, so getting confused on the topic. Moreover, in the appendixes, you can also find an original proof of the Stokes' (rotor) theorem, by the author of this paper.

The Navier-Stokes Equation in the case o fan incompressibile fluid, that is $r = const$ and $\vec{\nabla} \cdot \vec{v} = 0$:
(this situations is about most of practical cases)

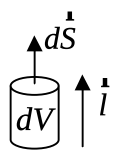
$\boxed{r[\frac{\partial \vec{v}}{\partial t} + \vec{\Omega} \times \vec{v} + \frac{1}{2}\vec{\nabla} v^2] = -\vec{\nabla} p - r\vec{\nabla} f + h\nabla^2 \vec{v}}$ where $\vec{\Omega} = \vec{\nabla} \times \vec{v}$ (vorticity), η (viscosity), f (gravitational potential), ρ (density), $\vec{v}$ (velocity), t (time).

Proof:

-Let's start from the Continuity Equation $\frac{\partial r}{\partial t} + \vec{\nabla}(r\vec{v}) = 0$, and we prove it:

$r\vec{v} = \vec{J}$ is the mass current density $[\frac{kg/s}{m^2}]$ (dimensionally obvious)

$M = \int_V r \cdot dV$ (held obvious)



We have: $\frac{\partial}{\partial t} M = \frac{\partial}{\partial t} \int_V r \cdot dV = \int_V \frac{\partial r}{\partial t} \cdot dV = -\int_S r\vec{v} \cdot d\vec{S}$,in fact, in terms of dimensions:

$dV = \vec{l} \cdot d\vec{S}$ and so $\frac{\partial}{\partial t} dV = d\vec{S} \frac{\partial \vec{l}}{\partial t} = d\vec{S} \cdot \vec{v}$ and sign – is in case of "escaping" mass.

So: $\int_V \frac{\partial r}{\partial t} dV = -\int_S (r\vec{v}) \cdot d\vec{S} = -\int_V \vec{\nabla} \cdot (r\vec{v}) \cdot dV$, after having used the Divergence Theorem in the last equality.

Therefore: $\int_V [\frac{\partial r}{\partial t} + \vec{\nabla} \cdot (r\vec{v})] dV = 0$, from which we get the Continuity Equation.

-and let's also start from the Euler's Equation ($\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\frac{\vec{\nabla}p}{\rho}$), and we also prove this:
(p is the pressure; moreover, this equation is a sketch of the Navier-Stokes Equation, whereas we're not yet taking into account the gravitational field and the viscous forces)
The force acting on a small fluid volume dV is $d\vec{f}=-p\cdot d\vec{S}$, with sign -, as we are dealing with a force towards the small volume. Moreover:
$\vec{f}=-\int_S p\cdot d\vec{S}=-\int_V \vec{\nabla}p\cdot dV$, after having used a dual of the Divergence theorem (a Green's formula), to go from the surface integral to the volume one.

We also have: $\frac{\partial \vec{f}}{\partial V}=\frac{\partial}{\partial V}[-\int_V \vec{\nabla}p\cdot dV]=-\vec{\nabla}p$, but, in terms of dimensions, it's simultaneously true that:
$\frac{\partial \vec{f}}{\partial V}=\frac{d}{dV}[M\frac{d\vec{v}}{dt}]=\frac{dM}{dV}\frac{d\vec{v}}{dt}=\rho\frac{d\vec{v}}{dt}=$ and from these two equations, we have:

$$\rho\frac{d\vec{v}}{dt}=-\vec{\nabla}p\,. \qquad (1.1)$$

Now we remind that: $d\vec{l}=(dx,dy,dz)$, $\vec{\nabla}=(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz})$ and $\vec{v}=(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt})$, so we can easily write that:

$\vec{v}\cdot ...$

$$\frac{d\vec{v}}{dt}=\frac{\partial \vec{v}}{\partial t}+\boxed{\frac{\partial \vec{v}}{\partial x}}\boxed{\frac{dx}{dt}}+\boxed{\frac{\partial \vec{v}}{\partial y}}\boxed{\frac{dy}{dt}}+\boxed{\frac{\partial \vec{v}}{\partial z}}\boxed{\frac{dz}{dt}}=\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=\frac{d\vec{v}}{dt}$$ and for (1.1) we finally have:

$...\vec{\nabla})\vec{v}$

$\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\frac{\vec{\nabla}p}{\rho}$ that is the Euler's Equation, indeed.

Now, the terms of this Euler's Equation have the dimensiono f an acceleration $\vec{a}$; so, if we want to take into account the gravitational field, too, on the right side we can algebraically add the gravitational acceleration $\vec{g}$, with a negative sign, as it's downwards.
But we know that the gradient of the potential φ is really $\vec{g}$ ($\vec{\nabla}\varphi=\vec{g}$), so:

$\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\frac{\vec{\nabla}p}{\rho}-\vec{\nabla}\varphi$. As the following vectorial identity is in force:
$(\vec{v}\cdot\vec{\nabla})\vec{v}=(\vec{\nabla}\times\vec{v})\times\vec{v}+\frac{1}{2}\vec{\nabla}(\vec{v}\cdot\vec{v})$, and if we take the expression for the vorticity, on page 1, ($\vec{\Omega}=\vec{\nabla}\times\vec{v}$), we have:
$\frac{\partial \vec{v}}{\partial t}+\vec{\Omega}\times\vec{v}+\frac{1}{2}\vec{\nabla}v^2=-\frac{\vec{\nabla}p}{\rho}-\vec{\nabla}\varphi$ and, so far, we have also taken into account the gravitational field.

In the most general case where we have to do with a <u>viscous</u> fluid , we'll also add a viscous force component:

$$\frac{\partial \vec{v}}{\partial t}+\vec{\Omega}\times\vec{v}+\frac{1}{2}\vec{\nabla}v^{2}=-\frac{\vec{\nabla}p}{\rho}-\vec{\nabla}\varphi+\frac{\vec{f}_{visc}}{\rho} \qquad (1.2)$$

whereas $\vec{f}_{visc}$ is divided by the density because of the dimension compatibilità with other terms in that equation.
(1.2) is already the Navier-Stokes Equation, whereas the viscous force $\vec{f}_{visc}$ is still to be evaluated.
We will evaluate $\vec{f}_{visc}$ in the case of incompressibile fluids, that is fluids with $\rho = const$, >> $\frac{\partial \rho}{\partial t}=0$ so, for the Continuity Equations, $\vec{\nabla}(\rho\vec{v})=0$, >> $\vec{\nabla}\cdot\vec{v}=0$.

<u>Calculation of</u> $\vec{f}_{visc}$:

<u>VISCOSITY:</u>

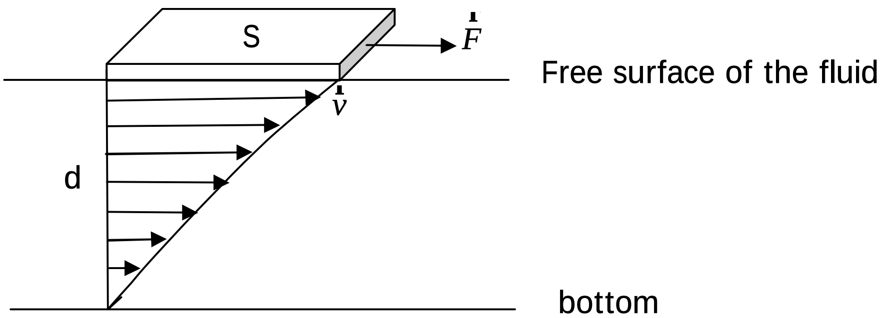


Fig. 1.

We know from general physics that: $\frac{\vec{F}}{S}=\eta\frac{\vec{v}}{d}$, (1.3)

That is, in order to drag the slab whose base surface is S, over the fluid, at a d distance from the bottom, and drag it at a $\vec{v}$ speed, we need a force $\vec{F}$
Now, let's write down (1.3) in a differential form, for stresses $\vec{\tau}$ and for components: (x)
$\tau_x=\frac{F_x}{S}=\eta\frac{\partial u}{\partial y}$, having set $\vec{v}=(u,v,w)$, and so:

$$F_x=\eta\frac{\partial u}{\partial y}\cdot S \qquad (1.4)$$

We now use (1.4) on a small fluid volume dV in Fig. 2:

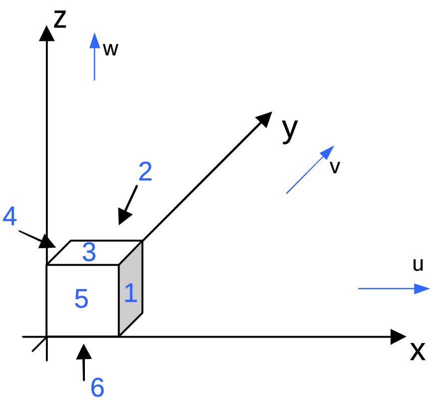


Fig. 2: Smal volume of fluid dV.

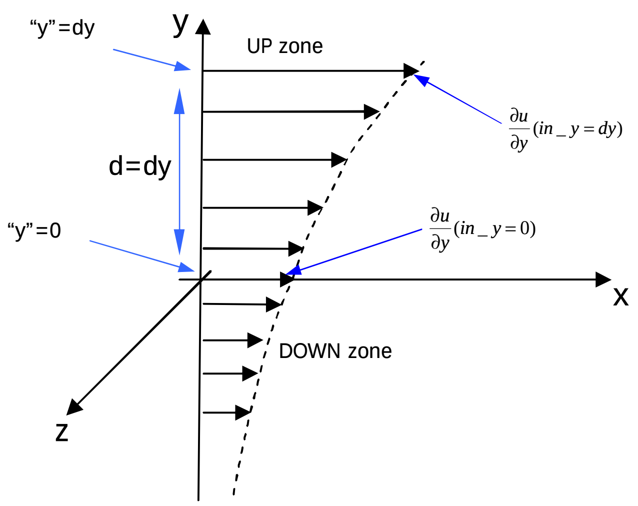


Fig. 3: Axis y, faces 2 and 5.

In Fig. 3 we have reproduced what shown in Fig. 1, but in a three-dimension context.

<u>Faces 2 and 5:</u>

so, with reference to Fig. 3, let's figure out the viscous forces (due to variations of u) on faces 2 and 5 of the small volume, that is those we meet when moving along the y axis, by using (1.4):

<u>Viscous shear stress on face 2</u> $= +h[\frac{\partial u}{\partial y}(in\_y = dy)]dxdz$

$S\_in\_(1.3)$

$\frac{v}{d}in\_(1.3)$

This force acting on face 2 is positive (+) because the fluid over the point where it's figured out (UP zone) has got a higher speed (longer horizontal arrows) which drags S along the positive x.

On face 5, on the contrary, we'll have a (-) negative sign, because the fluid under such S surface has got a lower speed (down) and want to be dragged, so making a resistance, that is a negative force:

<u>Viscous shear stress on face 5</u> $= -h[\frac{\partial u}{\partial y}(in\_y = 0)]dxdz$

The resultant on x is the difference between the two equations, or better, the algebraic sum:

$$F_{x(y)} = h[\frac{\partial u}{\partial y}(y = dy) - \frac{\partial u}{\partial y}(y = 0)]dxdz = h\frac{[\frac{\partial u}{\partial y}(y = dy) - \frac{\partial u}{\partial y}(y = 0)]}{dy}dxdydz = h\frac{\partial^2 u}{\partial y^2}dV\ ,$$ after

having multiplied numerator and denominator by dy. Therefore:

$$F_{x(y)} = h \frac{\partial^2 u}{\partial y^2} dV$$ (viscous force on x due to variations of u along y) (1.5)

Faces 3 and 6:

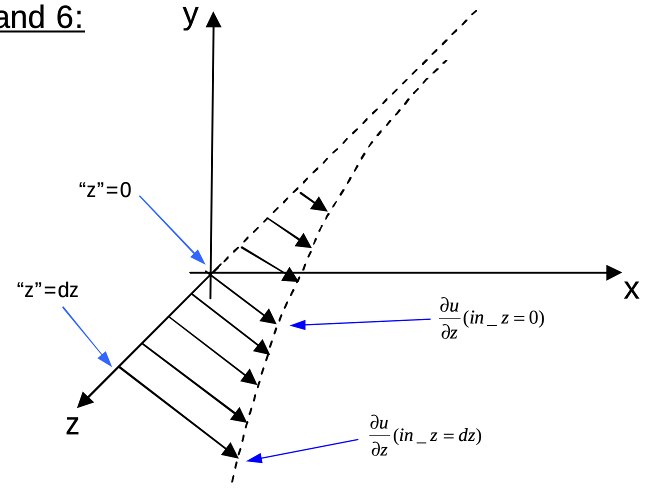


Fig. 4: Axis z, faces 3 and 6.

Similarly to the previous case, we have, as a result:

$$F_{x(z)} = h \frac{\partial^2 u}{\partial z^2} dV$$ (viscous force on x due to variationd of u along z) (1.6)

Faces 1 and 4:

For what case $F_{x(x)}$ is concerned, that is the viscous force on x due to variations of u (which is a component on x) along x itself, we will not talk about shear stresses, as, in such a case, the relevant force is still about x, but acts on S=dydz, which is orthogonal to x; so, it's about a NORMAL force, a tensile/compression one, and we refer to Fig. 5 below:

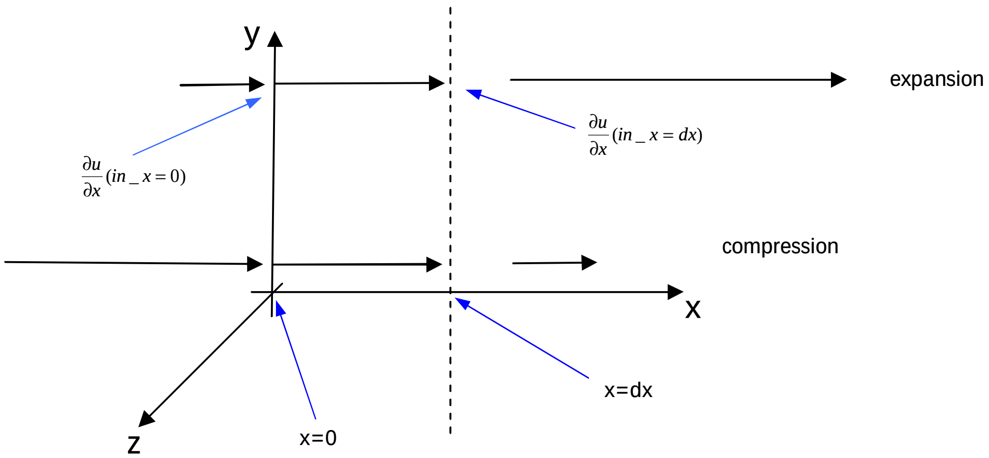


Fig. 5: Axis x, faces 1 and 4.

Anyway, nothing changes with numbers, with respect to previous cases, and we have:

$$F_{x(x)} = h\frac{\partial^2 u}{\partial x^2}dV$$ (viscous force on x due to variations of u along x itself) (1.7)

Now that we have three components of the viscous forces acting along x (that is those due to variations of the u component (comp. x) of speed $\vec{v}$, with respect to y, z and x itself), let's sum them up and get F_{x-visc} :

$$F_{x-visc} = h\frac{\partial^2 u}{\partial y^2}dV + h\frac{\partial^2 u}{\partial z^2}dV + h\frac{\partial^2 u}{\partial x^2}dV = h\cdot dV\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2}\right) = h\cdot dV\cdot\nabla^2 u$$, and we rewrite it below:

$$F_{x-visc} = h\cdot dV\cdot\nabla^2 u \tag{1.8}$$

Now we carry out the same reasonings fora n evaluation of F_{y-visc} and of F_{z-visc}, and obviously get ($\vec{v} = (u, v, w)$):

$$F_{y-visc} = h\cdot dV\cdot\nabla^2 v \tag{1.9}$$

$$F_{z-visc} = h\cdot dV\cdot\nabla^2 w \tag{1.10}$$

from which, finally, by adding (1.8), (1.9), and (1.10), we have:

$$\vec{F}_{visc} = F_{x-visc}\hat{x} + F_{y-visc}\hat{y} + F_{z-visc}\hat{z} = h\cdot dV[\hat{x}\nabla^2 u + \hat{y}\nabla^2 v + \hat{z}\nabla^2 w] = h\cdot dV\cdot\nabla^2\vec{v}$$ che riscriviamo:

$$\vec{F}_{visc} = h\cdot dV\cdot\nabla^2\vec{v} \tag{1.11}$$

Now, such a $\vec{F}_{visc}$ must be used in (1.2), after having divided it by r and by dV (that is, for $M = r\cdot dV$), as both sides of (1.2) have got the dimensiono f a force per a mass, indie, so:

$$\frac{\partial\vec{v}}{\partial t} + \vec{\Omega}\times\vec{v} + \frac{1}{2}\vec{\nabla}v^2 = -\frac{\vec{\nabla}p}{r} - \vec{\nabla}f + \frac{h}{r}\cdot\nabla^2\vec{v} \tag{1.12}$$

And therefore, finally, the **Navier-Stokes Equation**, and we write it better again:

$$\frac{\partial\vec{v}}{\partial t} + \vec{\Omega}\times\vec{v} + \frac{1}{2}\vec{\nabla}v^2 = -\frac{\vec{\nabla}p}{r} - \vec{\nabla}f + \frac{h}{r}\cdot\nabla^2\vec{v}$$

$$\frac{\vec{F}}{M} = \frac{\partial\vec{v}}{\partial t} + \frac{\partial}{\partial}........$$

General acceleration

Pressure forces

Gravity forces

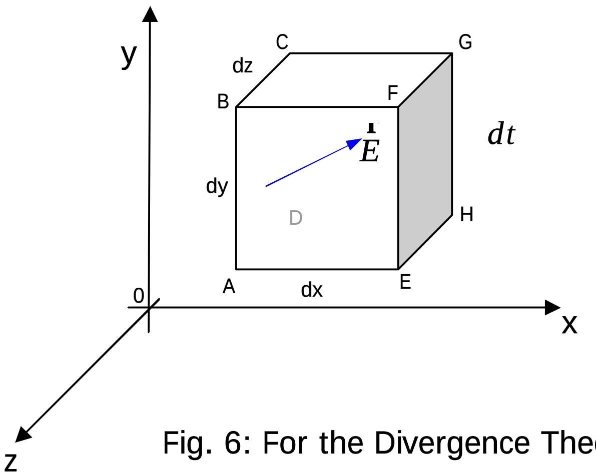
Viscous forces

Appendixes:

Appendix 1) Compressible fluids – very rare cases:

for those cases, $r \neq const$, $>> \frac{\partial r}{\partial t} \neq 0$, $>> \vec{\nabla}(r\vec{v}) \neq 0$, and to (1.12) we have to add the following term: $+\frac{(h+h')}{r}\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$, but (1.12) already enclose a big series of practical cases...

Appendix 2) Divergence Theorem (practical proof):



$$\boxed{\int_S \vec{E}\cdot d\vec{S} = \int_V div\vec{E}\cdot dV}$$

Fig. 6: For the Divergence Theorem.

Name f the flux of the vector $\vec{E}$; we have:

$df_{ABCD} = \vec{E}\cdot d\vec{S} = -E_x(x,\bar{y},\bar{z})dydz$ ($\bar{y}$ means y "mean")

$df_{EFGH} = E_x(x+dx,\bar{y},\bar{z})dydz$, but we obviously know that also: (as a development):

$E_x(x+dx,\bar{y},\bar{z}) = E_x(x,\bar{y},\bar{z}) + \frac{\partial E_x(x,\bar{y},\bar{z})}{\partial x}dx$ so:

$df_{EFGH} = E_x(x,\bar{y},\bar{z})dydz + \frac{\partial E_x(x,\bar{y},\bar{z})}{\partial x}dxdydz$ and so:

$df_{ABCD} + df_{EFGH} = \frac{\partial E_x}{\partial x}dV$. We similarly act on axes y and z:

$df_{AEHD} + df_{BCGF} = \frac{\partial E_y}{\partial y}dV$

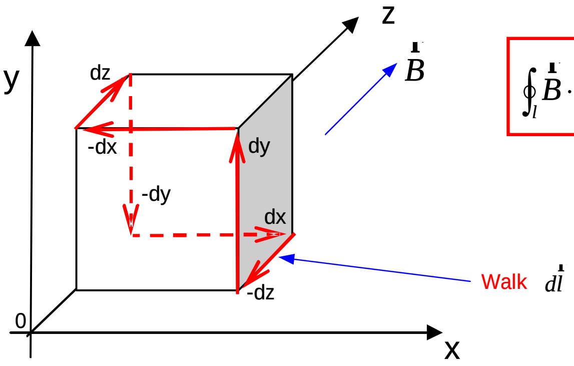
$df_{ABFE} + df_{CGHD} = \frac{\partial E_z}{\partial z}dV$

And then we sum up the fluxes so found, having totally:

$df = (\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z})dV = (div\cdot\vec{E})dV = (\vec{\nabla}\cdot\vec{E})dV$ therefore:

$f_S(\vec{E}) = \int_f df = \int_S \vec{E}\cdot d\vec{S} = \int_V div\vec{E}\cdot dV = \int_V (\vec{\nabla}\cdot\vec{E})\cdot dV$ that is the statement.

Appendix 3) <u>Rotor or Stokes' Theorem (practical proof-**by Rubino!**)</u>:



$$\boxed{\oint_l \vec{B}\cdot d\vec{l} = \int_S rot\vec{B}\cdot d\vec{S} = \int_S \vec{\nabla}\times\vec{B}\cdot d\vec{S}}$$

Fig. 7: For the Rotor Theorem (proof by Rubino).

Let's figure out $\vec{B}\cdot d\vec{l}$:

On dz B is B_z; on dx B is B_x; on dy B is B_y;

on -dz B is $B_z + \frac{\partial B_z}{\partial x}dx - \frac{\partial B_z}{\partial y}dy$, for 3-D Taylor's development and also because to go from the center of dz to that of –dz we go up along x, then we go down along y and nothing along z itself.

Similarly, on -dx B is $B_x - \frac{\partial B_x}{\partial z}dz + \frac{\partial B_x}{\partial y}dy$ and on -dy B is $B_y - \frac{\partial B_y}{\partial x}dx + \frac{\partial B_y}{\partial z}dz$.

By summing up all contributions:

$$\vec{B}\cdot d\vec{l} = B_z dz - (B_z + \frac{\partial B_z}{\partial x}dx - \frac{\partial B_z}{\partial y}dy)dz + B_x dx - (B_x - \frac{\partial B_x}{\partial z}dz + \frac{\partial B_x}{\partial y}dy)dx + B_y dy -$$

$$+(B_y - \frac{\partial B_y}{\partial x}dx + \frac{\partial B_y}{\partial z}dz)dy = (\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z})dydz + (\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x})dxdz + (\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y})dxdy =$$

$= rot\vec{B}\cdot d\vec{S} = \vec{\nabla}\times\vec{B}\cdot d\vec{S}$ whereas here $d\vec{S}$ has got components $[\hat{x}(dydz), \hat{y}(dxdz), \hat{z}(dxdy)]$

that is, the statement: $\oint_l \vec{B}\cdot d\vec{l} = \int_S rot\vec{B}\cdot d\vec{S} = \int_S \vec{\nabla}\times\vec{B}\cdot d\vec{S}$, after having reminded of:

$$rot\vec{B} = \vec{\nabla}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} .$$

Appendix 4) <u>The Bernoulli's Equations: </u> $\frac{1}{2}rv^2 + p + rgz = 0$

If we are in a stationary situation, whereas $\vec{v} \neq f(t) >> \frac{\partial \vec{v}}{\partial t} = 0$, and then $r = const$, and where there's no viscous forces, the Navier-Stokes Equation for sure reduce sto the Euler's one (but added with the gravitational component):

$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{r} - \vec{g}$, and, better, as we said that $\frac{\partial \vec{v}}{\partial t} = 0$, we have:

$$(\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{r} - \vec{g} \ . \qquad (1.13)$$

If now we consider the divergence and the gradient in terms of directional derivative, on direction $d\vec{l}$, specifically, then we have in (1.13): $\frac{dv}{dl}$ instead of $\vec{\nabla} \cdot \vec{v}$, and $\frac{dp}{dl}$ instead of $\vec{\nabla} p$ and then, still in (1.13), the gravitational acceleration $\vec{g}$ (which exerts along z, downwards) must be projected along $d\vec{l}$ ($\frac{dz}{dl}$ is the relevant direction cosine), and so (1.13) becomes:

$v\frac{dv}{dl} = -\frac{1}{r}\frac{dp}{dl} - g\frac{dz}{dl}$, from which: $vdv + \frac{1}{r}dp + gdz = 0$ and by integrating it:

$\frac{1}{2}v^2 + \frac{p}{r} + gz = 0$, and by multiplying by the density r , we get: $\frac{1}{2}rv^2 + p + rgz = 0$
that is, really the statement!

Bibliography:

1) *(C. Mencuccini and S. Silvestrini)* FISICA I - Meccanica Termodinamica, Liguori.

2) *(* Y. Nakayama*)* INTRODUCTION TO FLUID MECHANICS - Butterworth Heinemann.

3) *(L. D. Landau & E. M. Lifshitz)* FLUID MECHANICS **-** Pergamon Press.

4) ME 563 - INTERMEDIATE FLUID DYNAMICS (Lectures).

5) *(R. Feynman)* THE FEYNMAN PHYSICS II – Zanichelli.

6) *(L. Rubino)* Publications on physics in the Italian physics website fisicamente.net.
