LASER THEORY

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Abstract: In this paper you can find not only a formal LASER Theory, but also useful appendixes which help you to understand the origins of all the equations used in the paper itself.

LASER THEORY:

The Laser is a device which is today used in a wide list of fields.

Laser is an acronim for Light Amplification by Stimulated Emission of Radiation.

The very first electromagnetic radiation amplifier was built with ammonia in the range of microwaves and was called Maser, for Microwave Amplification by Stimulated Emission of Radiation. After that they started building amplifiers in the visible range, from 1960 on, the word Laser was then introduced.

Let's consider a material immersed in radiation; we expect the following behaviours:

a) From the fundamental status of minimum energy E_i, the atom can absorb photons whose frequency is n , so that $hn = E_f - E_i$, where E_f is the energy of the excited state adjacent to E_i.

(a description of the Planck/Einstein equation $E=hn$ can be found in the appendix)

At the same time, the electron which absorbs the photon jumps from its fundamental state to the state E_f. This process is called stimulated absorption of radiation.

Albert Einstein supposed that the probability per unit of time that this happens is proportional, through a constant, to the energy density of the radiation, at the frequency n , that is $f(n)$ (see Planck's eq. in the appendix); so:

$$R_{if} = B_{if} \cdot f(n)$$; B_{if} is the B coefficient of Einstein.

The frequency of the stimulated events with transition i ----f referred to the unit of volume of the material will be:

$$N_f = n_i \cdot R_{if} = n_i \cdot B_{if} \cdot f(n)$$ 　　　　　　　　　　　　**[1.1]**

where n_i is the number of atoms, per unit of volume, which are in the fundamental state E_i (population of the fundamental state).


Stimulated absorption
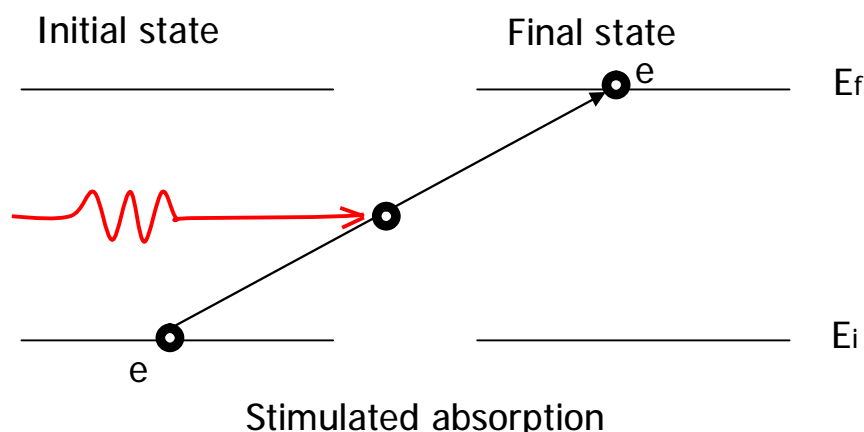
b) Once excited in the final state E_f, the atom tends to fall again in the fundamental state E_i. This can happen through several mechanisms; we are here interested to the following two radiative mechanisms:

-spontaneous emission of a photon, whose frequency is $n = (E_f - E_i) / h$ as well.

The mean life, per spontaneous emission, of the excited states, can change a lot from state to state: it can be from nanoseconds to milliseconds.

The spontaneous emission is the basis for all the standard light emitters, such as light bulbs.

-stimulated emission, when the atom which is in the excited state is struck from another photon whose frequency is $n = (E_f - E_i) / h$. In this case, the return to the initial state happens together with the emission of another photon, which will be together with the striking one and has, with respect to it, the same frequency, the same phase and the same propagation direction.

Spontaneous emission

Stimulated emission
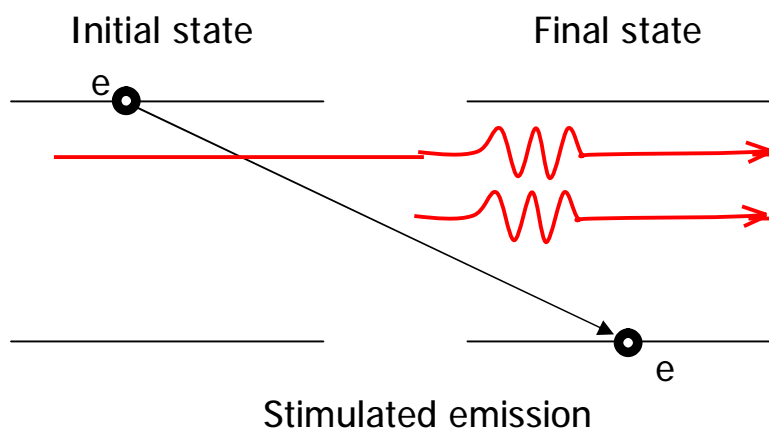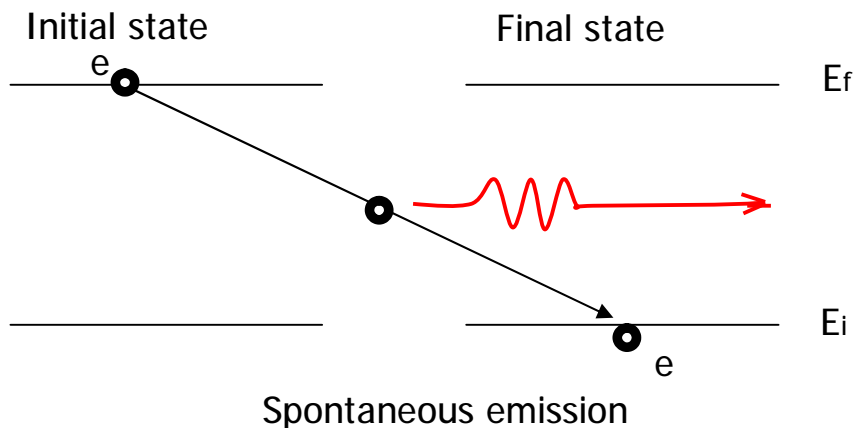
The probability R_{fi}, per unit of time, that the atom goes back to its fundamental initial state, can be written in this way:

$$R_{fi} = A_{fi} + B_{fi} \cdot f(n) \ ,$$

where A_{fi} is the Einstein's A coefficient, and represents the spontaneous emission term, while $B_{fi} f(n)$ is the stimulated emission term.

We see that the spontaneous emission term does not depend on the radiation $f(n)$, while the stimulated emission one, of course, does, and is proportional to the stimulating radiation density, that is, $f(n)$.

The emission frequency with transition f-----i, just like in the previous case, will be:

$$N_i = n_f \cdot R_{fi} = n_f[A_{fi} + B_{fi} \cdot f(n)] \qquad \text{[1.2]}$$

where n_f is the population of the excited state.

Let's start supposing that the material and the radiation are at balance. The number of photons emitted per unit of time must so be equal to the number of absorbed photons, that is:

$$N_f = N_i \rightarrow n_i \cdot B_{if} \cdot f(n) = n_f[A_{fi} + B_{fi} \cdot f(n)]$$

Resolving for $f(n)$, we get:

$$f(n) = \frac{A_{fi}\big/B_{fi}}{(n_i\big/n_f)(B_{if}\big/B_{fi}) - 1}$$

At the thermal equilibrium, for the Boltzmann's distribution (see appendix), we have:

$$n_i = n_0 e^{-E_i/kT} \quad , \qquad\qquad n_f = n_0 e^{-E_f/kT} \qquad , n_0 = \text{constant},$$

so:

$$n_i/n_f = e^{-E_i/kT}\Big/ e^{-E_f/kT} = e^{(E_f - E_i)/kT} = e^{hn/kT}$$

Therefore, for $f(n)$:

$$f(n) = \frac{A_{fi} / B_{fi}}{(B_{if} / B_{fi})e^{hn/kT} - 1}$$
[1.3]

As, at the thermal equilibrium, we have also Planck's equation on the blackbody emission (vedi appendice):

$$f(n) = \frac{8phn^3}{c^3}\left(\frac{1}{e^{hn/kT} - 1}\right)$$

we'll have, by comparison, the Einstein's relations:

$$B_{if} / B_{fi} = 1$$

$$A_{fi} / B_{fi} = 8phn^3/c^3$$

c is the speed of light in vacuum $\left(c = 3 \cdot 10^8 \, m/s\right)$.

These relations are necessary to assure the general conditions required at the thermal equilibrium and they have a general validity.

The first one says the B coefficient for the induced emission is equal to that for the induced absorption. The second one says the higher n is the more probable the spontaneous emission is, and so it's so high as high the difference $E_f - E_i$ in energy is.

That's why the making of X-rays Lasers (high n) is so problematic.

By putting $B_{if} = B_{fi}$ in equation [1.3], we get an equation which is valid at the thermal equilibrium and shows the ratio of the probability of a spontaneous decay and that of a stimulated decay:

$$\frac{A_{fi}}{B_{fi} \cdot f(n)} = \left(e^{hn/kT} - 1\right)$$
[1.4]

At ambient temperature, with a n value in the visible or infrared spectra, hn >>kT, and so such a ratio is >>1, that is, the spontaneous emission dominates on the stimulated one.

(We remind you that the Boltzmann's constant k is $1,38 \cdot 10^{-23} \, J/K$, while Planck's constant h is 6,625 $10^{-34} \, J \cdot s$; in order to have an interpretation of such constants, see the appendix).

A Laser is a device in which, through some methods, the populations n_i and n_f of those two states are controlled from the external side, bringing them to values different with respect to those at thermal equilibrium.

More exactly, one makes $n_f > n_i$, i.e., one keeps the population of the excited state higher than that of the fundamental one. We see that for small n (for instance.: microwaves), $hn << kT$; therefore, the left side of [1.4] would be small enough; in this situation, the ratio of [1.2] with [1.1] yields:

$$\frac{N_i}{N_f} = \frac{n_f}{n_i}\left(1 + \frac{A_{fi}}{B_{fi} \cdot f(n)}\right)$$ (the expression between brackets $\cong$ 1)

From this formula we see that if $n_f > n_i$, then $N_i > N_f$, so the radiation emitted is higher than the absorbed one.

By suitably shaping the geometry of the material, the radiation due to the stimulated emission is kept in the material, so the density f(n) becomes very high and the stimulated emission will dominate over the spontaneous one.

As with the stimulated emission the photon which is emitted is in phase with the stimulating one, we get a light beam which is monochromatic and very intense, coherent (photons in phase) and collimated.

n_f can be held higher than n_i by a pumping (optical, electrical, gasdynamical).

As photons in a Laser beam are all in phase (coherence), their effects (for instance: thermal ones) on a material where it strikes, sum up, so making the well known efficacy of the Laser beam itself.


OPTICAL PUMPING:

You choose a material with a state E_f which is metastable, that is, with a long mean life (for instance: $t = 10^{-3} s$) adjacent to a state $E_p > E_f$ with a short mean life (for instance: $t = 10^{-8} s$).

By putting this material under a radiation n = (E_p – E_i)/h (pumping radiation), electronic transitions towards E_p are induced. From there, the electrons spontaneously decay towards the state E_f, and as they have the possibility to stay there for a much longer time, the population can be increased to satisfaction.
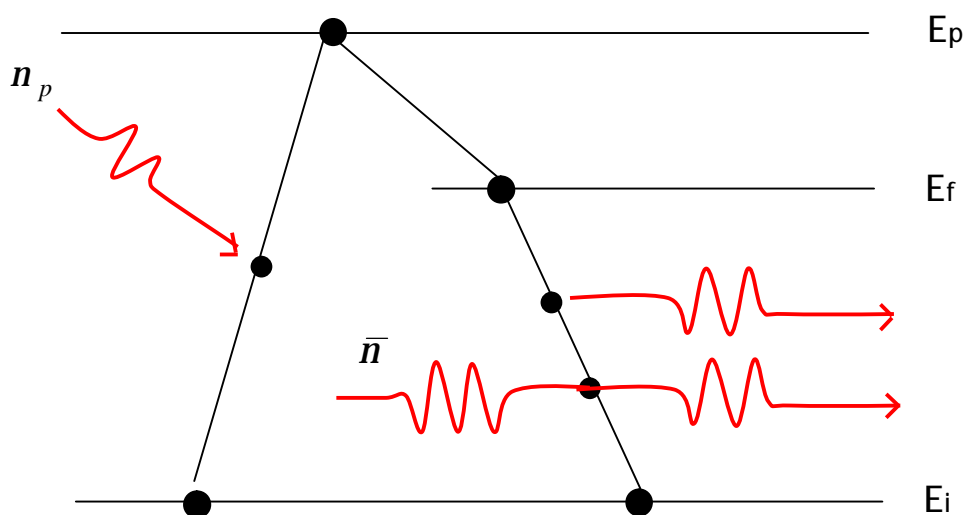
<p style="text-align:center"><u>AN EXAMPLE OF A RUBY LASER:</u></p>
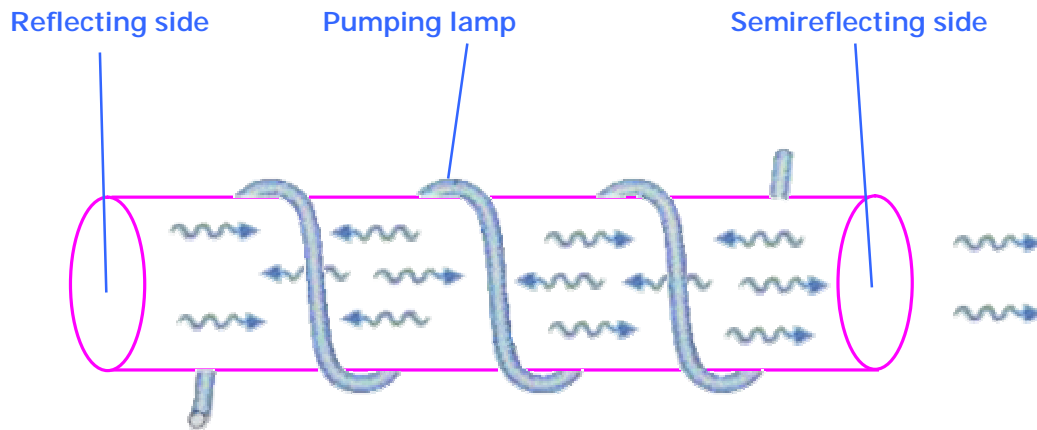
The ruby Laser is the first kind of Laser ever built; the ruby is a corundum crystal (Al_2O_3)
with impurities of ions Cr^{+++}. Ruby has got a strong absorption band in the green and it's
used to make the inversion of population.

Through a strong source of green light (high pressure mercury light), chromium ions are
excited to levels whose decay time to a lower level is very short. Such fast transitions are
caused by interactions of ions with vibrations of the crystal lattice.

The lower level becomes the active one through the laser action which takes place at a
wavelength of 694,3 nm.

The ruby crystal has got a cylindrical shape; one base is mirror-like, while the other is
semi mirror-like. Despite photons out of the axis, which can get out of the cylinder side
surface, the other photons moving along the axis make a long travel in the crystal, and so
they have a high probability to activate a stimulated emission in the crystal, so multiplying
the presence of photons all in phase and collimated.

Around the crystal, a spiral shaped lamp provides for the optical pumping; for the ruby,
the pumping radiation frequency is n = 5,455 10^{14} Hz. From the semi mirror-like side a
coherent and collimated beam comes out and its frequency is 4,321 10^{14} Hz.

<u>PUMPING METHODS:</u>

-optical pumping by standard sources (incandescent lamps, gas discharge, flash, Sun,
 etc), with Light Emitting Diodes or other Lasers.
-pumping through radiofrequency.
-pumping through electric discharge, if the material is a gas.
-pumping of solid material by collisions with electrons.
-chemical separation in excited states (chemical Lasers).
-thermodynamical processes (for instance: fast gas expansion).

<u>KINDS OF LASERS:</u>

The ruby Laser just described works on three levels $E_i(1)$, $E_f(2)$ and $E_p(3)$, and so is a
three level one, and the cycle is 1------3 -------2. There are also Lasers working on four
levels, in which level E_f is enriched by the transition 4-----3 from E_p to E_f, and,

simultaneously, level E_i is impoverished 2-----1 to a lower level. Four level Laser has got an efficiency higher than that of the three level one.

4
Fast transition
3
Excitation →
Laser emission
2
Fast transition
1
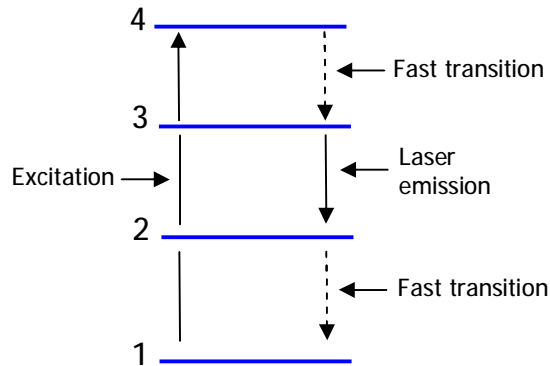
One of the most common gas Lasers is the He-Ne one and oscillates at the wavelength of 632,8 nm.
The Laser action takes place among the neon atom levels, while helium is used in the active mean to facilitate the pumping process.

For what the excitation method (pumping) is concerned, an electric discharge is held in the low pressure helium neon mixture; because of collisions among electrons and ions, also some metastable helium levels are populated, that is, those which have a very low probability to decay to lower levels. The population of those states becomes not negligible and, through collisions, their energy can be transferred to the neon atoms, in order to excite some proper levels. Therefore, the method of the inversion, in this case, is the following: population of a higher state of neon, through selected collisions; emptying, through transitions, towards the lower state; electric energy which is transformed directly into luminous energy.
We point out that the He-Ne Laser can work not only at the standard wavelenght of 632,8 nm, typical of all commercial Lasers, but also at further wavelenghts, in the visible range (for instance: 612 nm) , as well as in the infrared one (for instance: 1,15 µm,   3,39 µm).

Among Lasers working with a liquid active mean, the most known are the organic colouring matters.

Usually, it's about solutions of rhodamine, coumarin or similar substances, in water, alcohol or other suitable solvents. The molecules of colouring matters have a very complex structure, with a high density of energy levels. This permits a lot of transitions, so that the Laser emission can cover wide frequency bands, around 30/40 nm in λ. The working layout is a four level kind; the excitation at the highest state is obtained through flash lamps or through other visible Lasers, while the mean fast transitions are assured by strong interactions in liquids. The main interest to these Lasers is the possibility to have an emission tunable in frequency.

There are also many Lasers working on a pulsed basis, in which the inversion of population can be made for very short times, so that the emission, too, lasts just some nanoseconds. Among those, we remind you of some gas Lasers, like nitrogen Lasers, and excimer Lasers, that is, molecules made of halide of noble gases.

APPENDIX:

A)Boltzmann's Distribution Law:

Now we try to understand how changes, in a material, the number of molecules per unit of volume, when the energy changes.

Suppose to have a column of gas at a constant temperature, in a container and under the effect of the gravitational field.
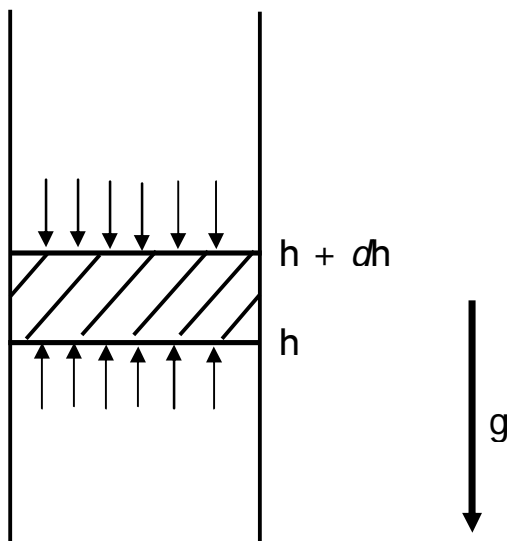
If this container has a volume V in which we have N gas particles, we define **n** as the number of particles per unit of volume.

With reference to the above figure, we examine a section S of the column of gas at the height h. The pressure P_h at the height h is obviously higher than that at the height h+dh, as at h the mass of gas pushing downwards is higher.

Being pressure P defined as dF/dS =(weight of the disc dh high and section S) / S, we have:

$$P_{h+dh} - P_h = dP = \frac{-m \cdot n \cdot S \cdot dh \cdot g}{S} = -mgndh \text{,}$$ [A.1]

where m is the mass of every single particle of gas, n is the number of particles per unit of volume, S dh is the volume of the disc, g is the gravitational acceleration end the negative sign tells us that dP is negative (P goes down while we go up).

We also know from thermodynamics that:

$$PV = n_{kmoles}RT = n_{kmoles}N_A \cdot \frac{R}{N_A}T = N \cdot k \cdot T \,,$$ [A.2]

where the first equality is the law of ideal gases (R=const), N_A is the number of particles in a kilomole, i.e. the Number of Avogadro, N= n_{kmoles} N_A= is the total number of gas particles (made of n_{kmoles}) and k=R/N_A is the Boltzmann's constant.

For a proof of the equation of state of ideal gases, see any of the books of general Physics.

From the previous equation, we have:

$$P = \frac{N}{V}kT = nkT \cdot$$

By differentiating this equation, we get:

$$dP = dnkT$$ [A.3]

By eq. [A.1] and [A.3] , we have:

$$\frac{dn}{n} = -\frac{mg}{kT}dh = -\frac{dE_p}{kT} \,,$$

where dE_p=mgdh is the differential of the potential energy of every particle.

The integration of this differential equation easily yields the following result:

$$n = n_0 e^{-E_p / kT} \,,$$ [A.4]

where n_0 is constant.

In case the particles are subject not to the gravitational field, but to any other conservative force, F_i (for instance, the intermolecular forces themselves), which we suppose it is oriented along x, in [A.4], instead of the potential energy E_p, we'll have the corresponding potential energy E_i coming from the force F_i, that is:

$$E_i = -\int F_i \cdot dx \,.$$

Finally:

$$n = n_0 e^{-Ei / kT}$$ [A.5]

The situation with non conservative forces is here not taken into account, as in this case it wouldn't be even possible to claim the thermal equilibrium.

In our opinion, the Boltzmann's equation [A.5] can be considered as proved and we want to remind you of what it means:

the probability to find molecules in a certain spatial disposition changes exponentially with the opposite of the potential energy of that disposition, divided by kT.

The same equation can be extended to our Laser on populations of energy levels.

B)Planck's Blackbody Spectrum:

Let's consider a cavity whose sides are at temperature T, uniform and constant.
Microscopic charges which makes the sides move because of the thermal agitation and, so doing, they radiate electromagnetic waves which fill the cavity; there is an energy transfer from the cavity sides to the electromagnetic field. Simultaneously, electromagnetic waves move into the cavity and hit the sides; so doing, they transfer energy from the field to the cavity sides. An equilibrium is so settled.
The **blackbody radiation spectrum** is the function $f(n)$ so that $f(n)dn$ is the *energy* had by the electromagnetic field in the unity of volume of the cavity, and with frequency between n and $n + dn$, that is:

$$f(n)dn = du \qquad [J/m^3]$$

Cavity sides emit and absorb radiation and can be held as made by small oscillating dipole. Moreover, we can assign the radiation in the cavity two degrees of freedom corresponding to two polarization planes which are perpendicular and independent each other and on which every electromagnetic wave can oscillate; in simpler words, an electromagnetic wave which propagates along z can oscillate transversally on both planes zx and zy.
We know from the kinetic theory of gases that for every particle, and so for every em wave emitted by the particles, and for every degree of freedom we can assign an energy equal to twice $\frac{1}{2}kT$, that is kT, as the total energy is made of a kinetic part and a potential part and their mean values are the same (see [A.12]).
For a proof of the fact that the total energy to be conferred is really kT/degree of freedom see any of the available general Physics books.
Now, suppose we have, out of simplicity, a cubic cavity whose electromagnetic radiation propagates along the three axis, so generating standing waves; moreover, we consider just one polarization plane per propagation axis (y), and we'll later take into account the
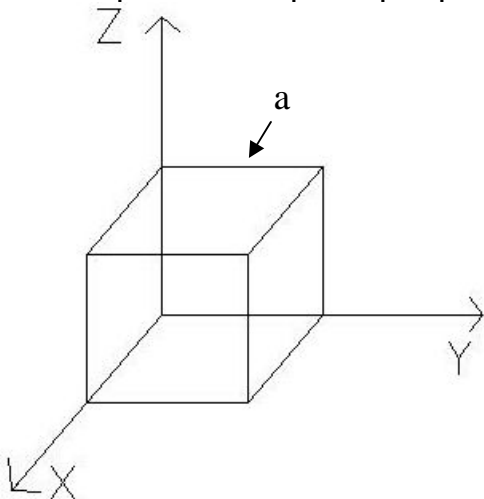
real existence of two degrees of freedom.
As the cavity is place of standing waves, and considering the x axis as the propagation one, we will write the following equation for a standing wave (see [A.13]):

$E_y(x,t) = E_{oy} \sin(kx) \cdot \sin(2pnt)$, k is the wave number $= \dfrac{2p}{l}$ and l is the wavelength.

We remind that : $c = l\,n$, and: $w = 2p / T = 2pn$.

As the standing wave must be zero in x = 0 and in x = a , we have:

$$ka = np \rightarrow n = 2a/l \rightarrow n = \frac{c}{l} = \frac{c \cdot n}{2a} \ \cdot$$

<u>n is positive and not zero, otherwise we don't have any wave.</u>

In general, for a wave propagating along a random direction, we have, component by component:

$$E_y(x,t) = E_{oy} \sin(k_x x) \cdot \sin(2pnt) \qquad k_x = (2p/l) \cdot \cos a$$
$$E_z(y,t) = E_{oz} \sin(k_y y) \cdot \sin(2pnt) \qquad k_y = (2p/l) \cdot \cos b$$
$$E_x(z,t) = E_{ox} \sin(k_z z) \cdot \sin(2pnt) \qquad k_z = (2p/l) \cdot \cos g$$

where the three direction cosines are the components of the versor $\hat{k}$ which indicates the direction of propagation of the wave.

Still by analogy with the single dimension case, we have:

$$k_x a = n_x p \qquad \rightarrow (2a/l)\cos a = n_x$$
$$k_y a = n_y p \qquad \rightarrow (2a/l)\cos b = n_y$$
$$k_z a = n_z p \qquad \rightarrow (2a/l)\cos g = n_z$$

$$n_x^{\,2} + n_y^{\,2} + n_z^{\,2} = (2a/l)^2 (\cos^2 a + \cos^2 b + \cos^2 g) = 4a^2/l^2 \ ,$$

from which:

$$n = \frac{c}{l} = \frac{c}{2a}\sqrt{n_x^{\,2} + n_y^{\,2} + n_z^{\,2}} \qquad\qquad \text{[A.6]}$$

With all values of n, we have all possible ways of vibration.

If we put such values n_x , n_y , n_z on three axes and considering the example

n_x , n_y , n_z = (1 , 2 , 2) , we see that the number of possible vibrations corresponding to terns n_x , n_y , n_z (n_x , n_y , $n_z \neq 0$, or we have a singularity case) are the vertexes of the following graph, where the n values are different from zero, so they are all the red spots.
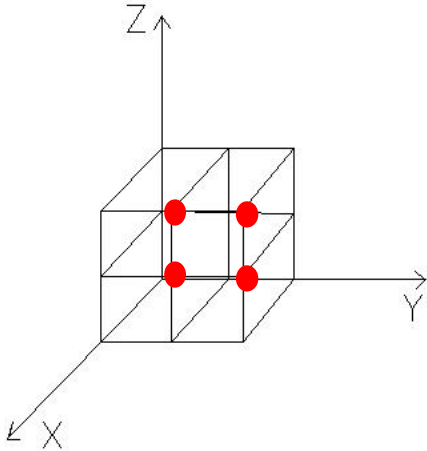
The fundamental thing we must take into account now (and this has a general validity) is that such *possible ways of vibration (●) correspond, in number, to the small unit side cubes (which are four, too).*

So: n . of possible ways of vibration= total volume V located by the tern n$_x$, n$_y$, n$_z$.

The radical in the expression [A.6] is just the radius of an octant of sphere located by the three components n$_x$, n$_y$, n$_z$ (of course, we consider just the octant where

n$_x$, n$_y$, n$_z$ are positive, as those must be positive and not zero).

The last remark makes us use the more suitable polar coordinates:

as the volume of an octant of a sphere is equal to $\frac{1}{8}\frac{4}{3}p \cdot r^3$, the number N of modes of

possible vibrations for a value of r between 0 and r is:

$$N = \frac{1}{8}\frac{4}{3}p \cdot r^3 \cdot$$

As a consequence, the number N(r) dr of possible modes of vibration for a value of r between r and r + dr can be obtained by differentiating the previous equation:

$$N(r)dr = \frac{p}{2}r^2 dr \cdot$$

Now, let's define an N(n) so that N(r)dr = N(n)dn =number of possible modes of vibration for frequencies between n and n +dn ; we see that, according to [A.6],
n = r c / (2 a) , and by differentiating the last equation, we have:

$$dn = \frac{c}{2a}dr \; ;$$ and then we get:

$$N(n)dn = \frac{p}{2}\left(\frac{2a}{c}\right)^3 n^2 dn = \frac{4p}{c^3}Vn^2 dn \;,$$ where $V = a^3 =$ volume of the cavity.

Now, in order to pass from the previous equation to f(n), and remembering that, according to the definition of f(n) itself we gave before, we have to:

-divide by V to refer to the unity of volume

-multiply by two to take into account the two possible states of polarization of the radiation (as per what was said on page 11/18)

-multiply by kT, that is, by the mean energy corresponding to each degree of freedom. Therefore:
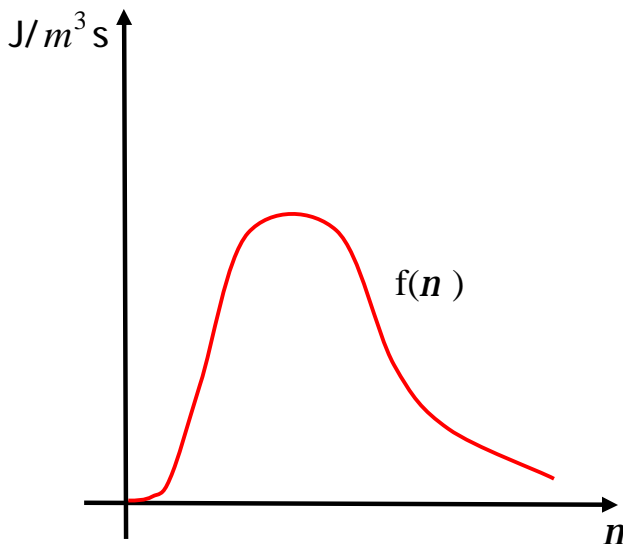
$$f(n)dn = \frac{8p}{c^3}kTn^2dn \quad ,$$

[A.7]

and this equation is known to be the Rayleigh-Jeans equation.

Of course:

$$f(n) = \frac{8p}{c^3}kTn^2$$

The graph of this equation is here below:

f(n)

n

Rayleigh-Jeans
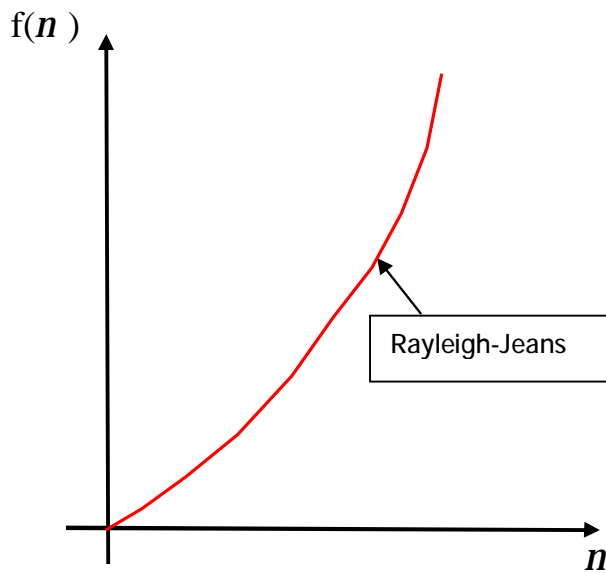
The experiments, on the contrary, show a different behaviour:

J/m^3s

f(n)

n

In the real situations, there is a peak, that is a value of frequency around which the emission of the blackbody concentrates.

Of course, the above curve is for a fixed temperature T and we'll see the more the temperature increases, the higher the frequency values are.

That's why, for instance, a piece of iron at ambient temperature emits an electromagnetic radiation in the range of the infrared waves, or around it, while if you heat it, it will emit visible radiation, at temperatures around some hundreds of centigrade degrees (white heat, red heat).

Similarly, you can find many characteristics of the surface of a star by just studying the frequency spectrum of the light the star irradiates.

Wavelength
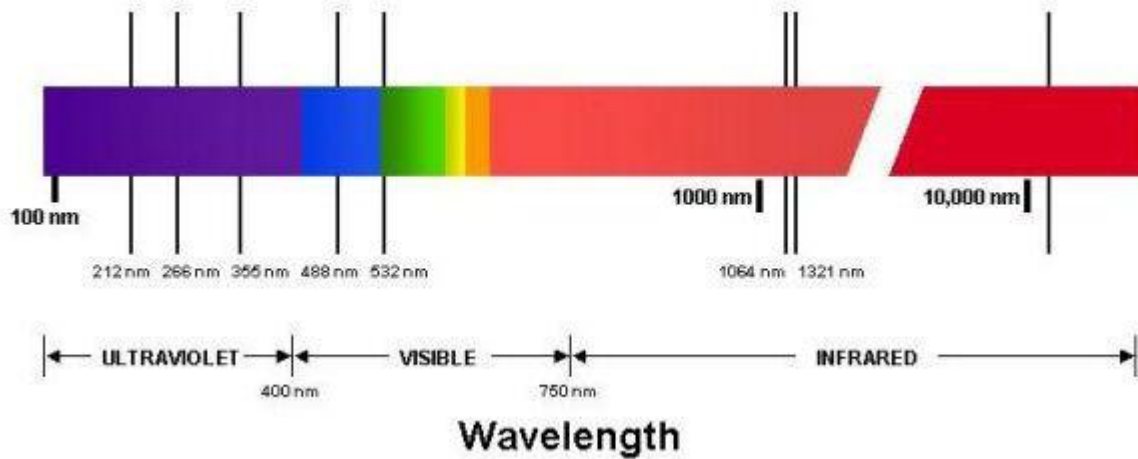
Nothing similar is shown by the Rayleigh-Jeans graph, which leads to an ultraviolet catastrophe. All this was the beginning of the chrisis of classic physics, and there was the need to bring new ideas and quantum hypotheses to make the theoretical deductions match the reality; as an example, we bring the Max Planck's supposition:

first of all, we see that if we want to figure out the mean energy $\overline{E}$ among all energies of the elements of a system, we can carry out a weighed average of all energies, which are distributed according to the already proved Boltzmann's formula [A.5] for n(E); therefore:

$$\overline{E} = \frac{\sum E_i \cdot n(E_i)}{\sum n(E_i)} = \frac{\sum E_i \cdot n_0 e^{-E_i/kT}}{\sum n_0 e^{-E_i/kT}} = \frac{\sum E_i \cdot e^{-E_i/kT}}{\sum e^{-E_i/kT}} \quad ; \qquad \text{[A.8]}$$

the numerator is the sum of all energies and each of them is weighed according to the number of components which have it, while the denominator is the total number of particles. For the moment, such an average value should be kT, and this is exactly the energy value we conferred to every constituent.

In order to jump from the Rayleigh-Jeans equation to one whose graph is that of the Planck's blackbody above reported, Planck supposed that for every value of frequency n, the energy of the system could have just discrete (quantized!) values:

$$E = hn, 2hn, \ldots\ldots, nhn \qquad \text{, (n integer).} \qquad \text{[Planck/Einstein equation]}$$

By such an assumption, [A.8] becomes (summation over n):

$$\overline{E} = \frac{\sum_0^\infty nhn \cdot e^{-nhn/kT}}{\sum_0^\infty e^{-nhn/kT}} \qquad .$$

The result is:

$$\overline{E} = \frac{hn}{e^{hn/kT} - 1};$$

In fact, by assuming that $\frac{hn}{kT} = z$, we have: $\overline{E} = kT \dfrac{\sum\limits_{0}^{\infty} nz \cdot e^{-nz}}{\sum\limits_{0}^{\infty} e^{-nz}}$; by defining:

$f(z) = \sum\limits_{0}^{\infty} e^{-nz}$, we have:

$-z \cdot df/dz = z \sum\limits_{0}^{\infty} n \cdot e^{-nz} = \sum\limits_{0}^{\infty} nz \cdot e^{-nz}$, so:

$$\overline{E} = -kTz \frac{df/dz}{f} = -kTz \frac{d}{dz} \ln z = -kTz \frac{d}{dz} \ln \sum\limits_{0}^{\infty} e^{-nz}.$$

Now, for Taylor's series, or for the study on geometrical series:

$$\sum\limits_{0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{and if we say:} \, e^{-z} = x, \, \text{we have:}$$

$$\overline{E} = -kTz \frac{d}{dz} \ln(1 - e^{-z})^{-1} = -kTz \frac{1}{(1-e^{-z})^{-1}} (1-e^{-z})^{-2} e^{-z} = kTz \frac{e^{-z}}{1-e^{-z}} = \frac{kTz}{e^z - 1} = \frac{hn}{e^{hn/kT} - 1}$$

that is, the assumption , after that we have taken into account the expression for z.

Therefore, Planck's news was to put in Rayleigh-Jeans' equation [A.7], the value of $\overline{E}$, just found, instead of the mean energy per component, that is, kT:

$$f(n)dn = \frac{8pn^2}{c^3} \frac{hn}{e^{hn/kT} - 1} dn \qquad\qquad \text{[A.9]}$$

and this is really the Planck's equation.
By dividing both sides by dn, we get an expression for f(n) which excellently describes the experimental graph above reported on the blackbody emission.

C) The Linear Armonic Oscillator

We consider a mass fixed to one end of a spring; the other end is fixed to a wall.

When the mass starts oscillating, as F=ma and, by Hooke, F=-kx, we can write the following differential equation:

$$ma + kx = m\frac{d^2x}{dt^2} + kx = 0 \text{ , whose solution is:}$$

$$x = x_0 \sin(wt + q) \text{ ,} \qquad\qquad\qquad\qquad\qquad \textbf{[A.10]}$$

where $\quad w = \sqrt{k/m}\quad$.

Now, we write the expression for the total energy E (which is the sum of the kinetic energy with the elastic potential one) of such an oscillating mass:

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 = E = E_k + E_p \text{ ;} \qquad\qquad\qquad \textbf{[A.11]}$$

 This is true because:

$$E_p = -\int F \cdot dx = \int kx \cdot dx = \frac{1}{2}kx^2 \text{ .}$$

Using [A.10] in [A.11] and taking into account the expression for W , we get:

$$E = \frac{m}{2}w^2 x_0{}^2 \cos^2(wt + q) + \frac{1}{2}kx_0{}^2 \sin^2(wt + q) =$$

$$= \frac{1}{2}kx_0{}^2[\cos^2(wt + q) + \sin^2(wt + q)] = \frac{1}{2}kx_0{}^2 \qquad\qquad \textbf{[A.12]}$$

As, from the previous expression, kinetic and potential components are the same, we have justified the reason why we assigned two identical values (½)kT for the total energy of the oscillators in the cavity (see page 11/18).

D) <u>Standing Waves</u>

If a wave S1 propagates in a limited mean, the superposition of it with its reflected one S2 generates a standing wave S:

$$S_1 = A\sin(kx - wt) \text{ ,} \qquad\qquad S_2 = A\sin(kx + wt) \quad .$$

The difference in sign in the arguments is due to the fact that those waves propagate in opposite directions; moreover, the term $wt = 2pnt$ tells us that if we fix a point x, we have an oscillation in time, while the term kx tells us that, if we fix a time t, we see an oscillation by moving along x.
Therefore, a propagating wave oscillates in time and also along the space through which it's propagating indeed.

$$S = S_1 + S_2 = 2A \cdot \sin kx \cdot \cos wt = 2A \cdot \sin \frac{2p}{l} x \cdot \cos 2pnt \quad ; \qquad \text{[A.13]}$$

after that we take into account the following trigonometric equality:

$$\sin a + \sin b = 2 \cos \frac{(a-b)}{2} \cdot \sin \frac{(a+b)}{2} \quad .$$

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