Quaternionic continuity equation for charges

By Ir J.A.J. van Leunen
Retired physicist & software researcher
Location: Asten, the Netherlands
Website: http://www.crypts-of-physics.eu/
Communicate your comments to info at that site.
Last version: October 25, 2011

Abstract

The continuity equation is specified in quaternionic format. It means that the density and current of the considered "charge" is combined in a quaternionic probability amplitude distribution (PAD). Next, the Dirac equation is also put in quaternionic format. It is shown that it is a special form of continuity equation. Further it is shown that two other quaternionic continuity equations can be derived from the quaternionic Dirac equation. The square and the squared modulus of the PAD play an essential role in these new equations. Further, a whole series of equivalent equations of motions is derived from the possible flavor couplings. The corresponding particles are identified. The mass of the particles can be computed from their fields.
The interpretation of these extra equations leads to the insight that when fermions take a new position, they must step over a forbidden region. Finally, the role of the quaternionic covariant derivative is explained.

Quaternion flavors

Quaternion fields come in four flavors: ψ, ψ^*, $\psi^{\oplus}$ and $\psi^{\otimes}$. The flavors are determined by sign selections. Three sign selections play a role.
The background coordinate system has its own kind of flavor. The flavor of the background coordinate system can act as a reference for comparing quaternion field flavors.
A quaternionic field will stick with one and no more than one flavor. We will use the symbol ψ for the quaternionic field that has the same flavor as the local background coordinate system. The background coordinate system can be curved. In that case we use the local tangent space that acts as a quaternionic number space.
In the investigated continuity equations, pairs of field flavors will be treated that belong to the same field ψ.

$$\nabla\psi^x = m\,\psi^f \qquad (1)$$

The continuity equation will use one of the pair (ψ^x) as the analyzed field and the other pair member (ψ^f) as the source field. Each choice of the pair of field flavors will result in a different equation. The same equation may accept different basic fields (ψ). The standard model appears to use three different field configurations for ψ. Each of these configurations has its own set of coupling factors. This paper does not explain why these three field

configurations exist. Each ordered pair (ψ^x, ψ^f) represents an elementary particle type category. Each such pair corresponds to a specific continuity equation of motion, which is also an equation of motion.

Some categories appear in triplets. The members of the triplet are coupled to directions of imaginary base vectors. Apart for the categories for which the coupling factor equals zero, each category corresponds with three different coupling factors (m).

Flavor	Flip $\psi^x \Longrightarrow \psi^f$	Imaginary base vectors	Handedness	Isotropy
conjugation:	$\psi \Leftrightarrow \psi^*$	3	switch	isotropic
double flip:	$\psi \Leftrightarrow \psi^{\oplus}$	2	neutral	anisotropic
single flip:	$\psi \Leftrightarrow \psi^{\otimes}$	1	switch	anisotropic
No flip	$\psi = \psi$	0	neutral	isotropic
$\oplus\otimes$ flip	$\psi^{\oplus} \Leftrightarrow \psi^{\otimes}$	3	switch	anisotropic

The quaternionic nabla operator ∇ uses the flavor of the background coordinate system.

Continuity equation

When $\rho_0(q)$ is interpreted as a charge density distribution, then the conservation of the corresponding charge is given by the continuity equation:

Total change within *V* = flow into *V* + production inside *V* (1)

$$\frac{d}{dt}\int_V \rho_0 \, dV = \oint_S \hat{\boldsymbol{n}}\rho_0 \frac{\boldsymbol{v}}{c} \, dS + \int_V s_0 \, dV \tag{2}$$

$$\int_V \nabla_0 \rho_0 \, dV = \int_V \langle \boldsymbol{\nabla}, \boldsymbol{\rho} \rangle \, dV + \int_V s_0 \, dV \tag{3}$$

Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface *S*, $\boldsymbol{v}(t, \boldsymbol{q})$ is the velocity at which the charge density $\rho_0(t, \boldsymbol{q})$ enters volume *V* and s_0 is the source density inside *V*. In the above formula $\boldsymbol{\rho}$ stands for

$$\boldsymbol{\rho} = \rho_0 \boldsymbol{v}/c \tag{4}$$

It is the flux (flow per unit area and unit time) of ρ_0 .

The combination of $\rho_0(t, \boldsymbol{q})$ and $\boldsymbol{\rho}(t, \boldsymbol{q})$ is a quaternionic skew field $\rho(t, \boldsymbol{q})$ and can be seen as a probability amplitude distribution (PAD).

$$\rho \stackrel{\text{def}}{=} \rho_0 + \boldsymbol{\rho} \tag{5}$$

$\rho(t, \boldsymbol{q})\rho^*(t, \boldsymbol{q})$ can be seen as an overall probability density distribution (PDD). $\rho_0(t, \boldsymbol{q})$ is a charge density distribution. $\boldsymbol{\rho}(t, \boldsymbol{q})$ is the current density distribution.

Depending on their sign selection, quaternions come in four flavors. In a PAD the quaternion flavors do not mix. So, there are four PAD flavors.
Still these flavors can combine in pairs or in quadruples.

The quaternionic field $\rho(t,\boldsymbol{q})$ contains information on the distribution $\rho_0(t,\boldsymbol{q})$ of the considered charge density as well as on the current density $\boldsymbol{\rho}(t,\boldsymbol{q})$, which represents the transport of this charge density.

Where $\rho(t,\boldsymbol{q})\rho^*(t,\boldsymbol{q})$ can be seen as a probability density of finding the center of charge at position $\boldsymbol{q}$, the probability density distribution $\tilde{\rho}(t,\boldsymbol{p})\tilde{\rho}^*(t,\boldsymbol{p})$ can be seen as the probability density of finding the center of the corresponding wave package at location $\boldsymbol{p}$. $\tilde{\rho}(t,\boldsymbol{p})$ is the Fourier transform of $\rho(t,\boldsymbol{q})$.

The dimension of $\rho_0, \boldsymbol{\rho}$ and ρ is $[XTL^{-3}]$, the dimension of s_0 is $[XL^{-3}]$. The factor c has dimension $[T^{-1}L]$. $[X]$ is an arbitrary dimension. It attaches to the charge.

The conversion from formula (2) to formula (3) uses the Gauss theorem[1]. This results in the law of charge conservation

$$s_0(t,\boldsymbol{q}) = \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle\boldsymbol{\nabla}, \big(\rho_0(t,\boldsymbol{q})\boldsymbol{v}(t,\boldsymbol{q}) + \boldsymbol{\nabla}\times\boldsymbol{a}(t,\boldsymbol{q})\big)\rangle \tag{6}$$

$$= \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle\boldsymbol{\nabla}, \boldsymbol{\rho}(t,\boldsymbol{q}) + \boldsymbol{A}(t,\boldsymbol{q})\rangle$$

$$= \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle\boldsymbol{v}(t,\boldsymbol{q}), \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \mp \langle\boldsymbol{\nabla}, \boldsymbol{v}(t,\boldsymbol{q})\rangle\,\rho_0(t,\boldsymbol{q})$$

$$\mp\langle\boldsymbol{\nabla}, \boldsymbol{A}(t,\boldsymbol{q})\rangle$$

The blue colored ± indicates quaternionic sign selection through conjugation of the field $\rho(t,\boldsymbol{q})$. The field $\boldsymbol{a}(t,\boldsymbol{q})$ is an arbitrary differentiable vector function.

$$\langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\times\boldsymbol{a}(t,\boldsymbol{q})\rangle = 0 \tag{7}$$

$\boldsymbol{A}(t,\boldsymbol{q}) \stackrel{\text{def}}{=} \boldsymbol{\nabla}\times\boldsymbol{a}(t,\boldsymbol{q})$ is always divergence free. In the following we will neglect $\boldsymbol{A}(t,\boldsymbol{q})$.
In Fourier space the continuity equation becomes:

$$\tilde{s}_0(t,\boldsymbol{p}) = p_0\tilde{\rho}_0(t,\boldsymbol{p}) \mp \langle\boldsymbol{p}, \tilde{\boldsymbol{\rho}}(t,\boldsymbol{p})\rangle \tag{8}$$

This equation represents a balance equation for charge (or mass) density.
Here $\rho_0(q)$ is the charge distribution, $\boldsymbol{\rho}(q)$ is the current density.
This only treats the real part of the full equation. The full equation runs:

$$s(t,\boldsymbol{q}) = \nabla\rho(t,\boldsymbol{q}) = s_0(t,\boldsymbol{q}) + \boldsymbol{s}(t,\boldsymbol{q}) \tag{9}$$

$$= \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle\boldsymbol{\nabla}, \boldsymbol{\rho}(t,\boldsymbol{q})\rangle \pm \nabla_0\boldsymbol{\rho}(t,\boldsymbol{q}) + \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \pm \big(\pm\boldsymbol{\nabla}\times\boldsymbol{\rho}(t,\boldsymbol{q})\big)$$

$$= \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle\boldsymbol{v}(t,\boldsymbol{q}), \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \mp \langle\boldsymbol{\nabla}, \boldsymbol{v}(t,\boldsymbol{q})\rangle\,\rho_0(t,\boldsymbol{q})$$

[1] http://en.wikipedia.org/wiki/Divergence_theorem

$$\pm\nabla_0 \boldsymbol{v}(t,\boldsymbol{q}) + \nabla_0\rho_0(t,\boldsymbol{q}) + \ \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})$$

$$\pm\big(\pm(\rho_0(t,\boldsymbol{q})\ \boldsymbol{\nabla}\times\boldsymbol{v}(t,\boldsymbol{q}) - \boldsymbol{v}(t,\boldsymbol{q})\times\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\big)$$

$$s_0(t,\boldsymbol{q}) = 2\nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle \boldsymbol{v}(q), \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(t,\boldsymbol{q})\rangle\ \rho_0(t,\boldsymbol{q}) \tag{10}$$

$$\boldsymbol{s}(t,\boldsymbol{q}) = \pm\nabla_0\boldsymbol{v}(t,\boldsymbol{q}) \pm \ \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \tag{11}$$

$$\pm\Big(\pm\big(\rho_0(t,\boldsymbol{q})\ \boldsymbol{\nabla}\times\boldsymbol{v}(t,\boldsymbol{q}) - \boldsymbol{v}(t,\boldsymbol{q})\times\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\big)\Big)$$

The red sign selection indicates a change of handedness by changing the sign of one of the imaginary base vectors. (Conjugation also causes a switch of handedness). If temporarily no creation and no annihilation occur, then these equations reduce to equations of motion.

$$\nabla_0\rho_0(t,\boldsymbol{q}) \pm \nabla_0\boldsymbol{\rho}(t,\boldsymbol{q}) = \pm\langle\boldsymbol{\nabla}, \boldsymbol{\rho}(t,\boldsymbol{q})\rangle - \ \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \mp \big(\pm\boldsymbol{\nabla}\times\boldsymbol{\rho}(t,\boldsymbol{q})\big) \tag{12}$$

$$\nabla_0\rho(t,\boldsymbol{q}) = \langle\boldsymbol{\nabla}, \boldsymbol{\rho}(t,\boldsymbol{q})\rangle - \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \mp \boldsymbol{\nabla}\times\boldsymbol{\rho}(t,\boldsymbol{q}) \tag{13}$$

$$\nabla_0\rho_0(t,\boldsymbol{q}) = \mp\langle\boldsymbol{\nabla}, \boldsymbol{\rho}(t,\boldsymbol{q})\rangle \tag{14}$$

$$\nabla_0\boldsymbol{\rho}(t,\boldsymbol{q}) = \mp\ \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \mp \boldsymbol{\nabla}\times\boldsymbol{\rho}(t,\boldsymbol{q}) \tag{15}$$

The field $\rho(t,\boldsymbol{q})$ can be split in a (relative) stationary background $\rho_b(\boldsymbol{q})$ and the moving private field $\rho_p(t,\boldsymbol{q})$.
If $\boldsymbol{v}(t,\boldsymbol{q})$ is a constant then

$$s_0(t,\boldsymbol{q}) = 2\nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle \boldsymbol{v}, \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \tag{16}$$

$$\boldsymbol{s}(t,\boldsymbol{q}) = \pm\ \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \mp \big(\pm\boldsymbol{v}\times\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\big) \tag{17}$$

$$s(t,\boldsymbol{q}) = 2\nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle \boldsymbol{v}, \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \pm \ \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \mp \big(\pm\boldsymbol{v}\times\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\big) \tag{18}$$

The continuity equation has a direct relation to a corresponding conservation law[2]. The conserved quantity is $\rho_0(t,\boldsymbol{q})$ or its integral

$$Charge = \int_V \rho_0\, dV$$

Noether's theorem[3] provides the relation between conserved quantities, differentiable symmetries and the Lagrangian[4].

[2] http://en.wikipedia.org/wiki/Conservation_law
[3] http://en.wikipedia.org/wiki/Noether's_theorem
[4] http://en.wikipedia.org/wiki/Lagrangian#Lagrangians_in_quantum_field_theory

The Dirac equation

The PAD $\psi(\mathrm{t},\mathrm{q})$ can be used to define a charge probability density and probability current density. See http://www.vttoth.com/qt.htm.

$$\boldsymbol{\rho}(t,\boldsymbol{q}) = \boldsymbol{v}(t,\boldsymbol{q})\rho_0(t,\boldsymbol{q}) \tag{1}$$

The Dirac equation appears to be a special form of continuity equation.

The Dirac equation runs

$$\nabla_0\psi + \boldsymbol{\nabla\alpha}\psi = m\beta\psi \tag{2}$$

$\boldsymbol{\alpha}$ and β represent the matrices that implement the quaternion behavior including the sign selections of quaternions for complex fields.
We keep the sign selections of the background coordinate system $(t,\boldsymbol{q})$ fixed. Thus $\boldsymbol{\alpha}$ and β only influence the elements of **spinor** ψ.

$$\alpha_1 = \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{3}$$

$$\alpha_2 = \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{4}$$

$$\alpha_3 = \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{5}$$

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{6}$$

There exist also a relation between $\alpha_1,\alpha_2,\alpha_3$ and the Pauli[5] matrices $\sigma_1,\sigma_2,\sigma_3$:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{7}$$

$$1 \mapsto I, \quad \boldsymbol{i} \mapsto \sigma_1, \quad \boldsymbol{j} \mapsto \sigma_2, \quad \boldsymbol{k} \mapsto \sigma_3 \tag{8}$$

This combination is usually represented in the form of gamma matrices[6]. These matrices are not used in this paper. They are used when a complex Hilbert space must handle quaternionic behavior.

Transferring the matrix form of the Dirac equation into quaternionic format delivers two quaternionic fields ψ_R and ψ_L that couple two equations of motion.

$$\nabla_0\psi_R + \boldsymbol{\nabla}\psi_R = m\psi_L \tag{9}$$

$$\nabla_0\psi_L - \boldsymbol{\nabla}\psi_L = m\psi_R \tag{10}$$

[5] http://en.wikipedia.org/wiki/Pauli_matrices
[6] http://en.wikipedia.org/wiki/Gamma_matrices

The mass term m couples ψ_{L} and ψ_{R}. The fact $\mathrm{m} = 0$ decouples ψ_L and ψ_R.

$$\psi_R = \psi_L^* = \psi_0 + \boldsymbol{\psi} \tag{11}$$

Thus the fields are each other's quaternionic conjugate.
Reformulating the quaternionic equations gives

$$\nabla\psi = m\,\psi^* \tag{12}$$

$$\nabla_0(\psi_0 + \boldsymbol{\psi}) + \boldsymbol{\nabla}(\psi_0 + \boldsymbol{\psi}) = m(\psi_0 - \boldsymbol{\psi}) \tag{13}$$

For the conjugated field holds

$$\nabla\psi^* = m\,\psi \tag{14}$$

$$\nabla_0(\psi_0 - \boldsymbol{\psi}) - \boldsymbol{\nabla}(\psi_0 - \boldsymbol{\psi}) = m(\psi_0 + \boldsymbol{\psi}) \tag{15}$$

This implements the reverse flip. The corresponding particle is the antiparticle.

$$(\psi, \psi^*) \leftrightarrow\ (\psi^*, \psi) \tag{16}$$

Summing the equations gives via

$$\boldsymbol{\nabla\psi} = \boldsymbol{\nabla} \times \boldsymbol{\psi} - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle \tag{17}$$

The result

$$\nabla_0\psi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle = m\,\psi_0 \tag{18}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\psi} = 0 \tag{19}$$

$$\nabla_0\boldsymbol{\psi} + \boldsymbol{\nabla}\psi_0 = -m\,\boldsymbol{\psi} \tag{20}$$

An important outcome is that $\boldsymbol{\psi}$ is rotation free. Compare this with the continuity equations

$$\nabla_0\rho_0(t, \boldsymbol{q}) - \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(t, \boldsymbol{q}) \rangle = s_0(t, \boldsymbol{q}) \tag{21}$$

And (22)

$$\boldsymbol{s}(t, \boldsymbol{q}) = \nabla_0\boldsymbol{\rho}(t, \boldsymbol{q}) + \boldsymbol{\nabla}\rho_0(t, \boldsymbol{q}) + \boldsymbol{\nabla} \times \boldsymbol{\rho}(t, \boldsymbol{q})$$

This means that

$$s_0(t, \boldsymbol{q}) \mapsto m\,\psi_0(t, \boldsymbol{q}) \tag{23}$$

$$\boldsymbol{s}(t, \boldsymbol{q}) \mapsto -m\,\boldsymbol{\psi} \tag{24}$$

Thus in the Dirac equation the mass term is a source term that depends on the (conjugate) field.

The following definitions specify another continuity equation:

$$\rho_{Dirac} \stackrel{\text{def}}{=} \psi\psi = \psi_0\psi_0 - \langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle + 2\psi_0\boldsymbol{\psi} \tag{25}$$

$$\rho_{0_{Dirac}} = \psi_0\psi_0 - \langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle \tag{26}$$

$$\boldsymbol{\rho}_{Dirac} = 2\psi_0\boldsymbol{\psi} \tag{27}$$

$$\nabla\rho_{Dirac} = 2\psi\nabla\psi = 2m\psi\psi^* = 2m\varphi \tag{28}$$

$$\nabla\psi\psi^* = \nabla\varphi = 2\psi\nabla\psi = 2m\rho_{0_{Dirac}} \tag{29}$$

The field φ is real and non-negative and represents a probability density distribution. This result defines two new continuity equations. $\rho_{0_{Dirac}}$ has a Minkowski signature.

The interpretation of φ as the probability density distribution of **presence** leads to:

$$\int_V \varphi \, dV = 1 \tag{30}$$

$$\int_V \nabla\rho_{Dirac} \, dV = 2m \tag{31}$$

The coupling factor m for the antiparticle is the same.

The field ψ has an intrinsic spin[7]:

$$spin = \int_V (\boldsymbol{\nabla}\psi_0) \times \boldsymbol{\psi} \, dV = \int_V \boldsymbol{\nabla} \times (\psi_0\boldsymbol{\psi}) \, dV = ½ \int_V \boldsymbol{\nabla} \times \boldsymbol{\rho}_{Dirac} \, dV \tag{33}$$

The flavor flip reverses the spin.
The interaction free equation can be extended with interactions with other fields.

$$\nabla\psi = m\,\psi^* \tag{34}$$

$$\overleftrightarrow{\mathrm{D}}\psi = m\psi^* - e(A\psi + \psi B) + C \tag{35}$$

The field A is right covariant with ψ. The field B is left covariant with ψ. e is a coupling constant. Thus here $\overleftrightarrow{\mathrm{D}}$ is the two sided covariant derivative[8]. The field C represents a source.

For the interaction field A holds

[7] http://www.plasma.uu.se/CED/Book/EMFT_Book.pdf Section: Conservation of angular momentum, formula 4.70a
[8] See last paragraph

$$\square A = (\nabla_0^2 - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle) A = e(\psi\psi) = e\rho_{Dirac} \tag{36}$$

$\square$ is the d'Alembert operator

The wave equation for the electromagnetic field in vacuum is

$$\square A = 0 \tag{37}$$

The Majorana equation

The Majorana equation[9] differs from the Dirac equation in the way that the sign selection of the field ψ is changed.
In the Majorana equation the mass term contains the $\psi^{\oplus}$ flavor of the field ψ. In this case only two imaginary base vectors change their sign. This sign selection does not switch handedness.

$$\nabla\psi = m_n\psi^{\oplus} \tag{1}$$

The reverse equations is

$$\nabla\psi^{\oplus} = m_{nr}\psi \tag{2}$$

For the conjugated field holds:

$$\nabla\psi^{*} = m_z\psi^{*\oplus} = m_z\psi^{\otimes} \tag{3}$$

The sign selection $\psi^{\otimes}$ only switches a single imaginary base vector. Like the conjugation, it switches the handedness of ψ . Thus the conjugated equation does not switch the handedness.
The reverse of the conjugated equation is

$$\nabla\psi^{\otimes} = m_{zr}\psi^{*} \tag{4}$$

The conjugated equation and its reverse also handle neutral particles.

Neutrinos are supposed to obey the Majorana equation.

When the Majorana equation holds, then

$$\nabla\rho_{Dirac} = 2\psi\nabla\psi = 2m_n\psi\psi^{\oplus} \neq 2m_n\psi\psi^{*} \tag{6}$$

$$\int_V \psi\psi^{*}\, dV = 1 \tag{7}$$

But in general:

[9] http://en.wikipedia.org/wiki/Majorana_equation

$$\int_V \psi\psi^{\oplus}\, dV \neq 1 \tag{8}$$

$$2\psi^{\otimes}\nabla\psi = 2m_n\psi^{\otimes}\psi^{\oplus} = 2m_n\left|\psi^{\otimes}\right|^2 \tag{9}$$

$$\int_V \nabla\left(\psi\psi^{\otimes}\right) dV = 2m_n \int_V \psi^{\otimes}\psi^{\oplus}\, dV = 2m_n g \tag{10}$$

$$\int_V \psi^{\otimes}\psi^{\oplus}\, dV = \int_V \psi^{\oplus}\psi^{\otimes}\, dV = \int_V \left|\psi^{\otimes}\right|^2 dV = \int_V \left|\psi^{\oplus}\right|^2 dV = g \tag{11}$$

$$\int_V \nabla\left(\psi^*\psi^{\oplus}\right) dV = 2m_{nr} \int_V \psi\psi^*\, dV = 2m_{nr} \tag{12}$$

For the conjugated field holds:

$$2\psi^{\oplus}\nabla\psi^* = 2m_z\psi^{\oplus}\psi^{\otimes} = 2m_z\left|\psi^{\oplus}\right|^2 \tag{13}$$

$$\int_V \nabla\left(\psi^*\psi^{\oplus}\right) dV = 2m_z \int_V \psi^{\oplus}\psi^{\otimes}\, dV = 2m_z g \tag{14}$$

$$\int_V \nabla\left(\psi\,\psi^{\otimes}\right) dV = 2m_{zr} \int_V \psi\psi^*\, dV = 2m_{zr} \tag{15}$$

The third category flavor switch

Apart from the Dirac equation and the Majorana equation, a third category equation is possible. In this equation the mass term flips the sign of only one imaginary base vector. As a result the handedness flips as well. The flavor of the background coordinate system can act as a reference for comparing quaternion flavors. The quaternionic nabla operator uses that same flavor. With respect to the background flavor, three different possibilities for the choice of the flipped imaginary base vector exist. It will become clear that this category corresponds to quarks.

The corresponding equation is:

$$\nabla\psi_i = m_{ui}\psi_i^{\otimes} \tag{1}$$

The reverse equation is

$$\nabla\psi_i^{\otimes} = m_{uri}\psi_i \tag{2}$$

The index i runs over three color versions r, g and b.

For the conjugated field holds:

$$\nabla\psi_i^* = m_{di}\,\psi_i^{*\otimes} = m_{di}\psi_i^{\oplus} \tag{3}$$

The reverse equation is

$$\nabla\psi_i^{\oplus} = m_{dri}\psi_i^* \tag{4}$$

For each color i an up version ψ_i and a down version ψ_i^*exists. The up version obeys equation (1). The down version obeys equation (2). The down version is the conjugated version of the up version.

When the third category equation holds, then

$$\nabla\rho_{i-Dirac} = 2\psi_i\nabla\psi_i = 2m_{ui}\psi_i\psi_i^{\otimes} \neq 2m_{ui}\psi_i\psi_i^* \tag{5}$$

$$2\psi_i^{\oplus}\nabla\psi_i = 2m_{ui}\psi_i^{\oplus}\psi_i^{\otimes} = 2m_{ui}\left|\psi_i^{\otimes}\right|^2 \tag{6}$$

$$2\psi_i^{\otimes}\nabla\psi_i^* = 2m_{di}\psi_i^{\otimes}\psi_i^{\oplus} = 2m_{di}\left|\psi_i^{\oplus}\right|^2 \tag{7}$$

Again

$$\int_V \psi_i\psi_i^*\,dV = 1 \tag{8}$$

And

$$\int_V \psi_i^{\oplus}\psi_i^{\otimes}\,dV = \int_V \left|\psi_i^{\otimes}\right|^2 dV = g_i \tag{9}$$

$$\int_V \nabla\left(\psi_i\psi_i^{\oplus}\right)dV = 2m_{ui}\,g_i \tag{10}$$

$$\int_V \nabla\left(\psi_i^*\psi_i^{\oplus}\right)dV = 2m_{uri} \tag{11}$$

$$\int_V \nabla\left(\psi_i^*\psi_i^{\otimes}\right)dV = 2m_{di}\,g_i \tag{12}$$

$$\int_V \nabla\left(\psi_i\psi_i^{\otimes}\right)dV = 2m_{dri} \tag{13}$$

Thus the computation of mass is not as simple as in the case of the Dirac equation

In contrast to the Dirac equation the third category flavor equation is strongly anisotropic.

The three choices for the flipped imaginary base vector may be linked with color charges. Each imaginary base vector might have an up and a down version.
The antiparticles have anti-color. The particles and antiparticles may be linked with the color charges and the up and down versions of quarks. The fact that only one of the three, or with the conjugate field two of the three imaginary base vectors are flipped may account for the respective electrical charges, which are $-⅓$.or $+⅔$.

The cross-flavor equations

These equations describe the situation that a flip is made from a $\psi_i^{\otimes}$ field to a $\psi_i^{\oplus}$ field or vice versa. The direction i seems to play no role.

$$\nabla\psi_i^{\oplus} = m\psi_i^{\otimes} \tag{1}$$

$$\nabla\psi_i^{\otimes} = m\psi_i^{\oplus} \tag{2}$$

In fact equation 2 is the conjugated equation of equation 1.
The flavor switch affects three imaginary base vectors and flips the handedness. As a consequence the particles have a full electric charge. It concerns two particles, the W^- and the W^+ bosons. These bosons carry electrical charges.

$$2\psi_i^{\oplus}\,\nabla\psi_i^{\oplus} = 2m_{W_+}\psi_i^{\oplus}\,\psi_i^{\otimes} \tag{3}$$

$$\int_V \nabla\left(\psi_i^{\oplus}\psi_i^{\oplus}\right) dV = 2m_{W_+}\int_V \left(\psi_i^{\oplus}\,\psi_i^{\otimes}\right) dV = 2m_{W_+}g \tag{4}$$

$$2\psi_i^{\otimes}\,\nabla\psi_i^{\otimes} = 2m_{W\_}\psi_i^{\otimes}\,\psi_i^{\oplus} \tag{5}$$

$$\int_V \nabla\left(\psi_i^{\otimes}\psi_i^{\otimes}\right) dV = 2m_{W\_}\int_V \left(\psi_i^{\oplus}\,\psi_i^{\otimes}\right) dV = 2m_{W\_}g \tag{6}$$

The non-flavor flip category

In this category no switch is performed. The field couples with itself.
The corresponding equation is:

$$\nabla\psi = m\,\psi \tag{1}$$

For the antiparticle holds:

$$\nabla\psi^* = m\psi^* \tag{2}$$

And for the mass m holds

$$\int_V \nabla(\psi\psi^*)\, dV = 2m \int_V (\psi\psi^*)\, dV = 2m \tag{3}$$

The equation describes neutral particles. It is not clear that the corresponding particles are fermions. It might concern the Z boson.

However for the probability density no integral source or leakage exists. Thus this m is zero.

Fermion and boson equations

Three fermion equations exist. Their *interaction free forms* are:

$$\nabla\psi = m\,\psi^* \tag{1}$$

$$\nabla\psi = m\,\psi^{\oplus} \tag{2}$$

$$\nabla\psi = m\,\psi^{\otimes} \tag{3}$$

The fourth possibility

$$\nabla\psi = m\,\psi = m\,\psi^{**} \tag{4}$$

and the cross flavor equations

$$\nabla\psi_i^{\oplus} = m\psi_i^{\otimes} = m\psi_i^{\oplus *} \tag{5}$$

$$\nabla\psi_i^{\otimes} = m\psi_i^{\oplus} = m\psi_i^{\otimes *} \tag{6}$$

are supposed to be boson equations

The general form is:

$$\nabla\psi^x = m\,\psi^? \tag{7}$$

For all three holds:

$$\nabla_0\psi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle = m\,\psi_0 \tag{8}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\psi^x} + \boldsymbol{\nabla}\psi_0^x + \nabla_0\boldsymbol{\psi^x} = m\,\boldsymbol{\psi}^? \tag{9}$$

$$\int_V \psi\psi^*\, dV = 1 \tag{10}$$

Further, the equation for coupling factor m

$$\int_V \nabla(\psi^{?*}\psi^{x})\, dV = 2m \int_V (\psi^{?*}\psi^{?})\, dV = 2m \int_V \left|\psi^{?}\right|^2 dV$$

$$\nabla(\psi\psi) = 2m\psi\psi^{?} \tag{11}$$

$$\nabla(\psi\psi^{*}) = 2m\psi^{*}\psi^{?} \tag{12}$$

The Dirac equation explicitly has the extra restriction

$$\boldsymbol{\nabla} \times \boldsymbol{\psi} = 0 \tag{13}$$

In interactions that need not be true.

Survey of couplings

In the following table the attribution of particle names is speculative.

RLrl	e	Diff	Coupling	m		Particle	Multiplet
RL	-1	3	$\psi\,\psi^{*}$	m	fermion	electron	1
LR	1	3	$\psi^{*}\,\psi$	m	fermion	positron	1
Rl	-⅓	1	$\psi\,\psi^{\otimes}$	m_d	fermion	down-quark	3 colors
lR	⅓	1	$\psi^{\otimes}\,\psi$	m_{dr}	fermion	anti-down-quark?	3 colors
Lr	⅔	1	$\psi^{*}\,\psi^{\oplus}$	m_u	fermion	up-quark	3 colors
rL	-⅔	1	$\psi^{\oplus}\,\psi^{*}$	m_{ur}	fermion	anti-up-quark?	3 colors
Rr	0	2	$\psi\,\psi^{\oplus}$	m_n	fermion	neutrino	3?
rR	0	2	$\psi^{\oplus}\,\psi$	m_{nr}	fermion	anti-neutrino	3?
Ll	0	2	$\psi^{*}\,\psi^{\otimes}$	m_z	?	neutrino or Z?	3?
lL	0	2	$\psi^{\otimes}\psi^{*}$	m_{zr}	?	anti-neutrino or Z?	3?
rl	-1	1	$\psi^{\oplus}\,\psi^{\otimes}$	m_{w_-}	boson	W^{-}	3?
lr	1	1	$\psi^{\otimes}\,\psi^{\oplus}$	m_{w_+}	boson	W^{+}	3?
RR	0	0	$\psi\,\psi$	0	boson	photon	
LL	0	0	$\psi^{*}\,\psi^{*}$	0	boson	photon	
rr	0	0	$\psi^{\oplus}\,\psi^{\oplus}$	0	boson	gluon	3?
ll	0	0	$\psi^{\otimes}\,\psi^{\otimes}$	0	boson	gluon	3?

Colophon:
RLrl; switch by 3, 2 or 1 imaginary base vectors
e; electric charge of particle
Diff; number of imaginary base vectors difference
Coupling; the field flavors that are coupled
Fermion/boson;
Particle; elementary particle category
Multiplet; multiplet structure

The multiplicity of neutrinos, Z and W bosons is not (yet) observed.
It is not clear from this table whether what particle the $\psi^{*}\,\psi^{\oplus}$ flavor concerns. The only particle that did not get a place is the Z boson.
From the coupling it is not (yet) clear whether a particle is a fermion or a boson.
In the standard model three versions of fermion mass factors m exist. These versions are not (yet) explained by this model.

Coupling factors

The integral probability densities are:

$$\int_V (\psi\psi^{*})\,dV = \int_V |\psi|^2\,dV = 1$$

$$g = \int_V \left(\psi^{\otimes}\psi^{\oplus}\right) dV = \int_V \left(\psi^{\oplus}\psi^{\otimes}\right) dV = \int_V \left|\psi^{\otimes}\right|^2 dV = \int_V \left|\psi^{\oplus}\right|^2 dV$$

The coupling factors are:

Primary	Coupling factor	reverse	Coupling factor
ψ, ψ^*	$m = ½ \int_V \nabla\,(\psi\psi) dV$	ψ^*, ψ	$m = ½ \int_V \nabla\,(\psi\psi) dV$
$\psi, \psi^{\oplus}$	$m_n = \frac{1}{2g} \int_V \nabla(\psi\, \psi^{\otimes})\, dV$	$\psi^{\oplus}, \psi$	$m_{nr} = ½ \int_V \nabla(\psi^* \psi^{\oplus})\, dV$
$\psi^*, \psi^{\otimes}$	$m_z = \frac{1}{2g} \int_V \nabla(\psi^*\, \psi^{\oplus})\, dV$	$\psi^{\otimes}, \psi^*$	$m_{zr} = ½ \int_V \nabla(\psi\, \psi^{\otimes})\, dV$
$\psi_i, \psi_i^{\otimes}$	$m_{ui} = \frac{1}{2g_i} \int_V \nabla(\psi_i \psi_i^{\oplus})\, dV$	$\psi_i^{\otimes}, \psi_i$	$m_{uri} = ½ \int_V \nabla(\psi_i^* \psi_i^{\oplus})\, dV$
$\psi_i^*, \psi_i^{\oplus}$	$m_{di} = \frac{1}{2g_i} \int_V \nabla(\psi_i^* \psi_i^{\otimes})\, dV$	$\psi_i^{\oplus}, \psi_i^*$	$m_{dri} = ½ \int_V \nabla(\psi_i \psi_i^{\otimes})\, dV$
$\psi^{\oplus}, \psi^{\otimes}$	$m_{W_+} = \frac{1}{2g} \int_V \nabla(\psi_i^{\oplus} \psi_i^{\oplus})\, dV$	$\psi^{\otimes}, \psi^{\oplus}$	$m_{W_-} = \frac{1}{2g} \int_V \nabla(\psi_i^{\otimes} \psi_i^{\otimes})\, dV$

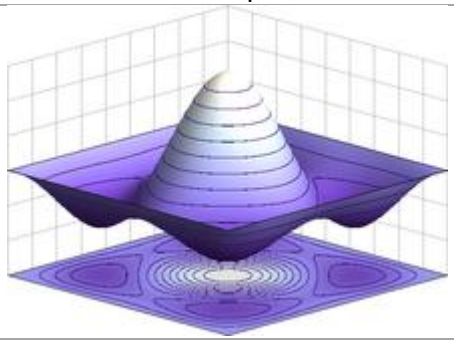
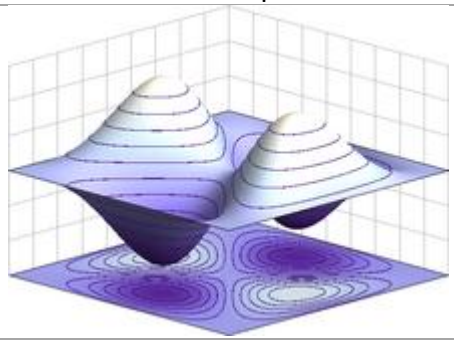
Some of the integral formulas are similar. Many are each other's conjugate. This does not mean that the integral is the same.
Most particle categories of the SM appear with three different coupling factors. This corresponds with three different configurations of ψ. This paper does not explain that extra diversity.

Forbidden region

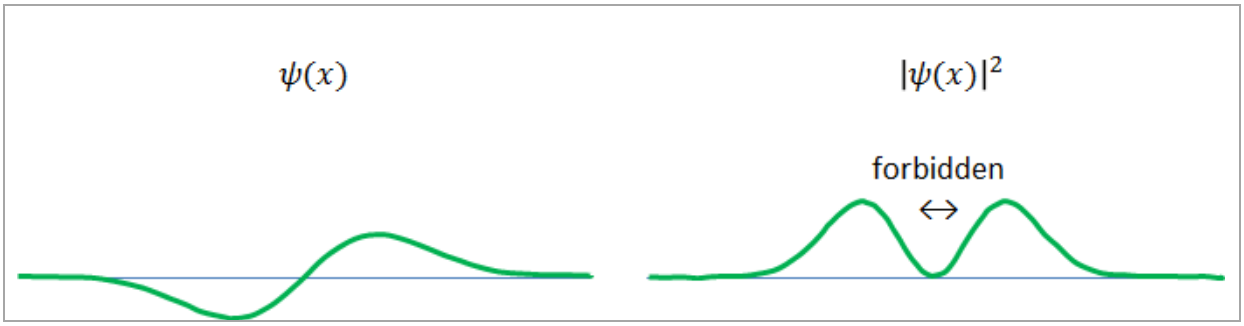
Fermions have asymmetric permutation wave functions. This fact has only significance when two or more states are considered. Let us consider the situation that the *two states are completely identical*[10] and are *nearly at the same location*. In that case the superposition of the two states is given by:

$$|\psi> = |n_1>|n_2> \pm |n_2>|n_1>$$

The plus sign holds for bosons and the minus sign holds for fermions. The images of the two cases are:

Boson pair	Fermion pair
	
Symmetric distribution	Asymmetric distribution

This is a two dimensional model, but it explains the general idea. Below the cut through the center of the asymmetric distribution is shown. When this is compared with the same cut of the squared modulus, then it reveals a forbidden region for the asymmetric distribution.



The particles were put at the closest possible position. Before the displacement occurs, the direction of the displacement is undefined. Thus the forbidden region has a spherical shape. When fermions go to their next position, they must step over the forbidden region. Bosons do not have that restriction.

Interpreting the flip event

The equations of motion indicate that a flip of field flavor occurs. The charge density distribution specifies the probability where this flip occurs. The current density distribution represents the transport of the location where the flip may occur.

[10] http://en.wikipedia.org/wiki/Identical_particles

The flip event can be observed. This is then the event of observing the corresponding quantum. The observation represents the interaction with another particle.

Photons and gluons are flipping at every progression step. That is why their coupling constant delivers zero.

Appendix

Covariant derivative

The covariant derivative plays a role in the Lagrangian and in the equation of motion.

The covariant derivative D is defined as

$$Df(q) = \nabla f(q) - \boldsymbol{A}(q)\, f(q) \tag{1}$$

This is interesting with respect to a gauge transformation of the form

$$f'(q) = G(q)\, f(q) \tag{2}$$

$$G^*(q)\ G(q) = 1 \tag{3}$$

$$\nabla G(q) = \boldsymbol{H}(q)\, G(q) \tag{4}$$

where with a corresponding vector potential transformation

$$\boldsymbol{A}'(q) = \boldsymbol{A}(q) + \boldsymbol{H}(q) \tag{5}$$

$$D' = \nabla - \boldsymbol{A}(q) - \boldsymbol{H}(q) \tag{6}$$

$$D'f'(q) = \boldsymbol{H}(q)\, G(q)\, f(q) + G(q)\, \nabla f(q) \tag{7}$$

$$-\boldsymbol{A}(q)G(q)f(q) - \boldsymbol{H}(q)G(q)f(q)$$

$$= G(q)\big(\nabla f(q) - \boldsymbol{A}(q)\, f(q)\big)$$

$$D'f'(q) = G(q)\, Df(q) \tag{8}$$

Thus with that transformation pair not only the modulus of the function stays invariant but also the modulus of the covariant derivative stays invariant. Further

$$f'^*(q)D'f'(q) = f^*(q)\, G^*(q)\ G(q)\, Df(q) \tag{9}$$

$$= f^*(q)\, Df(q)$$

Above the right sided covariant derivative D is defined

$$\vec{D}f(q) = \vec{\nabla} f(q) - \boldsymbol{A}(q)\, f(q) \tag{10}$$

The left sided covariant derivative is defined as:

$$f(q)\overleftarrow{D} = f(q)\overleftarrow{\nabla} - f(q)\,\boldsymbol{B}(q) \tag{11}$$

We will use $\overleftrightarrow{D}$ for both left sided and right sided covariant derivative:

$$\overleftrightarrow{D}f(q) = \frac{\vec{\nabla}f(q) + f(q)\overleftarrow{\nabla}}{2} - \boldsymbol{A}(q)\,f(q) - f(q)\,\boldsymbol{B}(q) \tag{12}$$

Multiplication with a unitary factor corresponds with a displacement in the canonical conjugate space, thus with a shift of the momentum of the field.

References

The contents of this paper is taken from part two of the Hilbert book model:
http://www.crypts-of-physics.eu/OntheoriginofdynamicsBoek2.pdf