Born's Reciprocal Gravity in Curved Phase-Spaces and the Cosmological Constant

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Abstract

The main features of how to build a Born's Reciprocal Gravitational theory in curved phase-spaces are developed. The scalar curvature of the $8D$ cotangent bundle (phase space) is explicitly evaluated and a generalized gravitational action in $8D$ is constructed that yields the observed value of the cosmological constant and the Brans-Dicke-Jordan Gravity action in $4D$ as two special cases. It is found that the geometry of the momentum space can be linked to the observed value of the cosmological constant when the curvature in *momentum* space is very large, namely the small size of P is of the order of $(1/R_{Hubble})$. More general $8D$ actions can be developed that involve sums of 5 distinct types of torsion squared terms and 3 distinct curvature scalars $\mathcal{R}, \mathcal{P}, \mathcal{S}$. Finally we develop a Born's reciprocal complex gravitational theory as a local gauge theory in $8D$ of the $deformed$ Quaplectic group that is given by the semi-direct product of $U(1,3)$ with the $deformed$ (noncommutative) Weyl-Heisenberg group involving four *noncommutative* coordinates and momenta. The metric is complex with symmetric real components and antisymmetric imaginary ones. An action in $8D$ involving 2 curvature scalars and torsion squared terms is presented.

1 Introduction : Born's Reciprocal Relativity in Phase Space

Born's reciprocal ("dual") relativity [1] was proposed long ago based on the idea that coordinates and momenta should be unified on the same footing, and consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality). The generalized velocity and acceleration boosts (rotations) transformations of the $8D$ Phase space, where $X^i, T, E, P^i; i = 1,2,3$ are *all* boosted (rotated) into each-other, were given by [2] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$.

The $U(1,3) = SU(1,3) \otimes U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -\, dt \wedge dp^0 + \delta_{ij} dx^i \wedge dp^j; i,j = 1,2,3$ and also the following Born-Green line interval in the $8D$ phase-space (in natural units $\hbar = c = 1$)

$$(d\sigma)^2 \;=\; (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 + \frac{1}{b^2} \left((dE)^2 \;-\; (dp_x)^2 \;-\; (dp_y)^2 \;-\; (dp_z)^2 \right) \tag{1.1}$$

the rotations, velocity and force (acceleration) boosts leaving invariant the symplectic 2-form and the line interval in the $8D$ phase-space are rather elaborate, see [2] for details. These transformations can be simplified drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions y, z, p_y, p_z intact. There is now a subgroup $U(1,1) = SU(1,1) \otimes U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 \;=\; (dT)^2 \;-\; (dX)^2 \;+\; \frac{(dE)^2 \;-\; (dP)^2}{b^2} \;=$$
$$(d\tau)^2 \left(1 \;+\; \frac{(dE/d\tau)^2 \;-\; (dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \left(1 \;-\; \frac{F^2}{F^2_{max}} \right) \tag{1.2}$$

where one has factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in (2.2). The proper force interval $(dE/d\tau)^2 - (dP/d\tau)^2 = -F^2 < 0$ is "spacelike" when the proper velocity interval $(dT/d\tau)^2 - (dX/d\tau)^2 > 0$ is timelike. The analog of the Lorentz relativistic factor in eq-(2.2) involves the ratios of two proper *forces*.

If (in natural units $\hbar = c = 1$) one sets the maximal proper-force to be given by $b \equiv m_P A_{max}$, where $m_P = (1/L_P)$ is the Planck mass and $A_{max} = (1/L_p)$, then $b = (1/L_P)^2$ may also be interpreted as the maximal string tension. The units of b would be of $(mass)^2$. In the most general case there are four scales of time, energy, momentum and length that can be constructed from the three constants $b, c, \hbar$ as follows

$$\lambda_t \;=\; \sqrt{\frac{\hbar}{bc}}; \;\; \lambda_l \;=\; \sqrt{\frac{\hbar\ c}{b}}; \;\; \lambda_p \;=\; \sqrt{\frac{\hbar\ b}{c}}; \;\; \lambda_e \;=\; \sqrt{\hbar\ b\ c} \tag{1.3}$$

The gravitational constant can be written as $G = \alpha_G\ c^4/b$ where α_G is a dimensionless parameter to be determined experimentally. If $\alpha_G = 1$, then the four scales (2.3) coincide with the *Planck* time, length, momentum and energy, respectively.

The $U(1,1)$ group transformation laws of the phase-space coordinates X, T, P, E which leave the interval (2.2) invariant are [2]

$$T' = T \ cosh\xi \ + \ (\frac{\xi_v \ X}{c^2} \ + \ \frac{\xi_a \ P}{b^2}) \ \frac{sinh\xi}{\xi} \qquad (1.4a)$$

$$E' = E \ cosh\xi \ + \ (-\xi_a \ X \ + \ \xi_v P) \ \frac{sinh\xi}{\xi} \qquad (1.4b)$$

$$X' = X \ cosh\xi \ + \ (\xi_v \ T \ - \ \frac{\xi_a \ E}{b^2}) \ \frac{sinh\xi}{\xi} \qquad (1.4c)$$

$$P' = P \ cosh\xi \ + \ (\frac{\xi_v \ E}{c^2} \ + \ \xi_a \ T) \ \frac{sinh\xi}{\xi} \qquad (1.4d)$$

ξ_v is the velocity-boost rapidity parameter and the ξ_a is the force (acceleration) boost rapidity parameter of the primed-reference frame. These parameters are defined respectively in terms of the velocity $v = dX/dT$ and force $f = dP/dT$ (related to acceleration) as

$$tanh(\frac{\xi_v}{c}) = \frac{v}{c}; \quad tanh(\frac{\xi_a}{b}) \ = \ \frac{f}{F_{max}} \qquad (1.5)$$

It is straightforwad to verify that the transformations (1.4) leave invariant the phase space interval $c^2(dT)^2 - (dX)^2 + ((dE)^2 - c^2(dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = dT^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - c^2(dP)^2]$. Only the *combination*

$$(d\sigma)^2 \ = \ (d\tau)^2 \ \left(1 \ - \ \frac{F^2}{F^2_{max}} \right) \qquad (1.6)$$

is truly left invariant under force (acceleration) boosts (1.4).

We explored in [5] some novel consequences of Born's reciprocal Relativity theory in flat phase-space and generalized the theory to the curved spacetime scenario. We provided, in particular, *six* specific results resulting from Born's reciprocal Relativity and which are *not* present in Special Relativity. These are : momentum-dependent time delay in the emission and detection of photons; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations.

The purpose of this work is to analyze the *curved* $8D$ phase space (cotangent bundle) scenario within the context of the physics of a maximal proper force and Born's reciprocal relativity. The $8D$ tangent bundle of spacetime within the context of Finsler geometry and the physics of a limiting value of the proper acceleration in spacetime [4] has been studied by Brandt [3]. Generalized $8D$ gravitational equations reduce to ordinary Einstein-Riemannian gravitational equations in the $infinite$ acceleration limit. A pedagogical monograph on Finsler geometry can be found in [11] where, in particular, Clifford/spinor structures were defined with respect to nonlinear connections associated with certain nonholonomic modifications of Riemann–Cartan gravity.

The outline of this work is organized as follows. In section **2** the scalar curvature of the $8D$ cotangent bundle is explicitly evaluated and a generalized gravitational action in $8D$ is constructed that yields the observed value of the cosmological constant and the Brans-Dicke-Jordan Gravity action in $4D$ as two special cases. It is found that the geometry of the momentum space can be linked to the observed value of the cosmological constant when the curvature in *momentum* space is very large, namely the small size of P is of the order of $(1/R_{Hubble})$. More general $8D$ actions can be developed that involve sums of 5 distinct types of torsion squared terms and 3 distinct curvature scalars $\mathcal{R}, \mathcal{P}, \mathcal{S}$. Finally in section **3** we develop a Born's reciprocal complex gravitational theory as a local gauge theory in $8D$ of the $deformed$ Quaplectic group that is given by the semi-direct product of $U(1,3)$ with the $deformed$ (noncommutative) Weyl-Heisenberg group involving four *noncommutative* coordinates and momenta. The metric is complex and has symmetric real components and antisymmetric imaginary ones. An action in $8D$ involving 2 curvature scalars and torsion squared terms is presented.

2 Born's Reciprocal Gravity associated with the $8D$ Cotangent Bundle of Spacetime

2.1 Gravity as Gauge Theory of Diffeomorphisms

This introductory section is necessary to be able to construct the scalar curvature of the $8D$ Cotangent Bundle in the next section. The authors [6], [7], [8], [9] have shown that $m+n$-dim Einstein gravity can be identified with an m-dimensional generally invariant gauge theory of $Diffs\ \mathcal{N}$, where $\mathcal{N}$ is an n-dim manifold. This can be shown as follows.

Locally the $m+n$-dim space can be written as $\Sigma = \mathcal{M} \times \mathcal{N}$ and the metric G_{AB} can be decomposed as

$$G_{AB} = \begin{pmatrix} g_{\mu\nu}(x,y) \ + \ e^2 g_{ab}(x,y)\ A^a_\mu(x,y)\ A^b_\nu(x,y) & eA^a_\mu(x,y)\ g_{ab}(x,y) \\ eA^b_\nu(x,y)\ g_{ab}(x,y) & g_{ab}(x,y) \end{pmatrix}, \tag{2.1}$$

The connection $A^a_\mu(x,y)$ is an example of the *nonlinear* connection which appears in Lagrange-Finsler and Hamilton-Cartan spaces [10], [11]. The decomposition (2.1) must *not* be confused with the Kaluza-Klein reduction where one imposes an *isometry* restriction on the G_{AB} that turns A^a_μ into a gauge connection associated with the gauge group G generated by isometry. Dropping the isometry restrictions allows *all* the fields to depend on *all* the coordinates x, y. Nevertheless $A^a_\mu(x,y)$ can still be identified as a connection associated with the infinite-dim gauge group of $Diffs\ \mathcal{N}$.

The gauge transformations are now given in terms of the Lie derivatives w.r.t the *internal* space indices y^a

$$A_\mu \;\equiv\; A^a_\mu\,\partial_a,\; \xi \;\equiv\; \xi^a\,\partial_a \;\Rightarrow\; \mathcal{L}_{A_\mu}\,\xi \;=\; [\;A_\mu,\;\xi]^a \;=\; A^b_\mu\,\partial_b\,\xi^a \;-\; \xi^b\,\partial_b\,A^a_\mu. \qquad (2.2a)$$

as follows

$$\delta A^a_\mu \;=\; -\,\frac{1}{e}\,D_\mu\xi^a \;=\; -\,\frac{1}{e}\,(\;\partial_\mu\,\xi^a\;-e\;[\;A_\mu,\;\xi]^a\;). \qquad (2.2b)$$

$$\delta g_{ab} \;=\; \mathcal{L}_\xi\,g_{ab} \;=\; [\xi,\;g]_{ab} \;=\; \xi^c\,\partial_c\,g_{ab} \;+\; g_{ac}\,\partial_b\,\xi^c \;+\; g_{cb}\,\partial_a\,\xi^c. \qquad (2.2c)$$

$$\delta g_{\mu\nu} \;=\; \mathcal{L}_\xi\,g_{\mu\nu} \;=\; [\xi,\;g_{\mu\nu}] \;=\; \xi^a\,\partial_a\,g_{\mu\nu}. \qquad (2.2d)$$

In general, the Lie derivative $\mathcal{L}_X\mathbf{T}$ along the vector $X = X^a\,\partial_a$ of the mixed tensor $\mathbf{T}$ in the internal space is defined by [12]

$$\mathcal{L}_X\,T^{a_1a_2.....a_n}_{b_1b_2.......b_m} \;=\; (\;X^c\,\partial_c\,T^{a_1a_2.....a_n}_{b_1b_2.......b_m}\;) \;+\; \sum_{i=1}^{m}\,(\partial_{b_i}\,X^c)\,T^{a_1a_2......a_n}_{b_1b_2......c.....b_m} \;-$$

$$\sum_{i=1}^{n}\,(\partial_c\,X^{a_i})\,T^{a_1a_2......c.....a_n}_{b_1b_2...........b_m}. \qquad (2.2f)$$

there is a key *minus* sign in the last term of (2.2f) relative to the first two terms. Using eq-(2.1) the authors [6], [7], [8], [9] have shown that the curvature scalar $R^{(m+n)}$ in $m+n$-dim decomposes into the sum of several terms

$$R^{(m+n)} \;=\; g^{\mu\nu}\,R^{(m)}_{\mu\nu} \;+\; \frac{e^2}{4}\,g_{ab}\,F^a_{\mu\nu}\,F^b_{\rho\tau}\,g^{\mu\rho}\,g^{\nu\tau} \;+\; g^{ab}\,R^{(n)}_{ab} \;+$$

$$\frac{1}{4}g^{\mu\nu}\,g^{ab}\,g^{cd}\;[\;(D_\mu\,g_{ac})\,(D_\nu\,g_{bd}) \;-\; (D_\mu\,g_{ab})(D_\nu\,g_{cd})\;]\,+$$

$$\frac{1}{4}\,g^{ab}\,g^{\mu\nu}\,g^{\rho\tau}\;[\;\partial_a\,g_{\mu\rho}\,\partial_b g_{\nu\tau} \;-\; \partial_a\,g_{\mu\nu}\,\partial_b\,g_{\rho\tau}\;] \qquad (2.3)$$

plus total derivative terms given by $(\sqrt{|det\;g_{\mu\nu}|}\,\sqrt{|det\;g_{ab}|})^{-1}$ times

$$\partial_\mu(\;\sqrt{|det\;g_{\mu\nu}|}\,\sqrt{|det\;g_{ab}|}\;J^\mu\;) \;-\partial_a(\;\sqrt{|det\;g_{\mu\nu}|}\,\sqrt{|det\;g_{ab}|}\;eA^a_\mu J^\mu) \;+$$

$$\partial_a(\;\sqrt{|det\;g_{\mu\nu}|}\,\sqrt{|det\;g_{ab}|}\;\;J^a\;), \qquad (2.4)$$

with the currents:

$$J^\mu \;=\; g^{\mu\nu}\,g^{ab}\,D_\nu\,g_{ab}, \quad J^a \;=\; g^{ab}\,g^{\mu\nu}\,\partial_b\,g_{\mu\nu}. \qquad (2.5)$$

Finally, the Einstein-Hilbert action in $D = m+n$ dimensions can be rewritten as

$$S \;=\; \frac{1}{2\kappa^2}\,\int\,d^mx\;d^ny\;\sqrt{|det(g_{\mu\nu})|}\;\sqrt{|det(g_{ab})|}\;R^{(m+n)}(x,y). \qquad (2.6)$$

where the expression for $R^{(m+n)}(x,y)$ is given by (2.3) plus the total derivative terms (2.4). Therefore, Einstein gravity in $D = m+n$ dimensions describes an m-dim generally invariant field theory under the gauge transformations corresponding to the Diffs $\mathcal{N}$. Notice how A^a_μ couples to the graviton $g_{\mu\nu}$, meaning that the graviton is charged /gauged in this theory and also to the g_{ab} fields. The "metric" g_{ab} on $\mathcal{N}$ can be identified as a non-linear sigma field whose self interaction potential term is given by: $g^{ab}R^{(n)}_{ab}$.

The currents J^μ, J^a are functions of $g_{\mu\nu}, A_\mu, g_{ab}$. The "Ricci" tensor of the horizontal space is a *gauged* Ricci tensor meaning that it is constructed using the gauge covariant derivatives $\partial_\mu - eA^a_\mu \partial_a$. In the next section we shall display the explicit expression for the *gauged* Ricci tensor when the $8D$ space is the cotangent bundle of the D spacetime. The contribution of the currents to the action is essential when there are boundaries involved; i.e. the proyective/conformal boundary of AdS spaces which is relevant in the AdS/CFT correspondence.

When the internal manifold $\mathcal{N}$ is a homogeneous compact space one can perform a harmonic expansion of the fields w.r.t the internal y coordinates, and after integrating the action (2.6) w.r.t these y coordinates, one will generate an infinite-component field theory on the m-dimensional space represented by the x coordinates. A reduction of the Diffs $\mathcal{N}$, via the inner automorphims of a subgroup G of the Diffs $\mathcal{N}$, yields the usual Einstein-Yang-Mills theory interacting with a nonlinear sigma field. But in general, the theory described in (2.3) is by far *richer* than the latter theory. A crucial fact of the decomposition in (2.3) is that *each* single term is by itself independently invariant under Diffs $\mathcal{N}$.

In the special case when $g_{\mu\nu}(x)$ depends solely on x and $g_{ab}(y)$ depends on y then the spacetime gauged "Ricci scalar" coincides with the ordinary Ricci scalar $g^{\mu\nu}(x)\ R^{(m)}_{\mu\nu}(x)$ and the internal space "Ricci scalar" $g^{ab}(y)R^{(n)}_{ab}(y)$ becomes the true Ricci scalar of the internal space. However, the gauge field $A_\mu(x,y)$ still retains its full dependence on both variables x, y.

We have shown [13] that in this particular case the $D = m+n$ dimensional gravitational action restricted to $AdS_m \times S^n$ backgrounds admits a *holographic* reduction to a lower $d = m$-dimensional Yang-Mills-like gauge theory of diffs of S^n, interacting with a charged/gauged nonlinear sigma model plus boundary terms, by a simple tuning of the radius of S^n and the size of the throat of the AdS_m space. Namely, in the case of $AdS_5 \times S^5$, the holographic [13] reduction occurs if, and only if, the size of the AdS_5 throat *coincides* precisely with the radius of S^5 ensuring a *cancellation* of the scalar curvatures $g^{\mu\nu}R^{(m)}_{\mu\nu}$ and $g^{ab}R^{(n)}_{ab}$ in eq-(2.3) [13] such that the scalar curvature (Einstein-Hilbert Lagrangian) in $D = 10$ becomes

$$R^{(10)} \ = \ \frac{e^2}{4}\ g_{ab}(y)\ F^a_{\mu\nu}(x,y)\ F^b_{\rho\tau}(x,y)\ g^{\mu\rho}(x)\ g^{\nu\tau}(x)\ +$$

$$\frac{1}{4}\ g^{\mu\nu}(x)\ g^{ab}(y)\ g^{cd}(y)\ [\ (D_\mu g_{ac})\ (D_\nu g_{bd}) - \ (D_\mu g_{ab})\ (D_\nu g_{cd})\ ]. \qquad (2.7)$$

plus total derivative terms (boundary terms). The gauge covariant derivative

$$D_\mu \; g_{ab} \; = \; \partial_\mu g_{ab} \; + \; [\; A_\mu, \; g_{ab} \;]. \qquad (2.8a)$$

is defined in terms of the Lie-bracket above

$$[\; A_\mu \; , \; g_{ab} \;] \; = \; (\partial_a \; A^c_\mu(x^\mu, y^a)) \; g_{cb}(x^\mu, y^a) \; + \; (\partial_b \; A^c_\mu(x^\mu, y^a)) \; g_{ac}(x^\mu, y^a) \; +$$
$$A^c_\mu(x^\mu, y^a) \; \partial_c \; g_{ab}(x^\mu, y^a). \qquad (2.8b)$$

and the Yang-Mills like field strength is

$$F^a_{\mu\nu} \; = \; \partial_\mu \; A^a_\nu \; - \partial_\nu \; A^a_\mu \; - \; [\; A_\mu, \; A_\nu \;]^a \; =$$
$$\partial_\mu \; A^a_\nu \; - \partial_\nu \; A^a_\mu \; - \; A^c_\mu \; \partial_c \; A^a_\nu \; + \; A^c_\nu \; \partial_c \; A^a_\mu. \qquad (2.9)$$

Eq-(2.7) is nothing but the "holographic" like reduction of the $D = 10$-dim pure gravitational Einstein-Hilbert action to a 5-dim Yang-Mills-like action (of diffeomorphisms of the internal S^5 space) interacting with a charged nonlinear sigma model (involving the g_{ab} field) plus boundary terms. The previous argument can also be generalized to gravitational actions restricted to de Sitter spaces, like $dS_m \times H^n$ backgrounds as well, where H^n is an internal hyperbolic noncompact space of constant negative curvarture, and dS_m is a de Sitter space of positive constant scalar curvature.

2.2 Scalar curvature of the $8D$ Cotangent Bundle

The scalar curvature of the $8D$ tangent bundle manifold in the *anholonomic* frame adapted to the spacetime affine connection was given by [3] . The physics underlying such construction corresponded to a maximal proper *acceleration* principle in spacetime. In this section we shall evaluate, using a *different* method than the one provided by Brandt [3] and [10], the scalar curvature of the $8D$ *cotangent* bundle manifold based on the results of **2.1**. The physics in this case is that of a maximal proper *force* in spacetime which is associated with Born's reciprocal relativity principle.

The symplectic geometry of the cotangent bundle, the Poisson brackets of Hamiltonian systems, the nonlinear connection, the construction of torsion and curvature, the Bianchi and Ricci identities, the geodesic equations, etc ...... associated with Hamilton-Cartan spaces has been thoroughly studied by [10]. The geometry of the tangent bundle and Lagrange-Finsler spaces can also be found in [10], [11]. There is a duality (via the Legendre map) between Lagrange and Hamilton spaces and also between Finsler and Cartan spaces [10]. A recent analysis of a Lagrangian-Hamiltonian formalism for first and higher order field theories (higher order tangent spaces) has been provided by [15]. In this work we shall follow a *different* approach than the one presented in [10] to construct the scalar curvature of the $8D$ Cotangent Bundle.

In an $8D$ flat phase-space the infinitesimal interval is

$$(d\sigma)^2 \;=\; (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 + \frac{1}{b^2}\;\big((dE)^2 \;-\; (dp_x)^2 \;-\; (dp_y)^2 \;-\; (dp_z)^2\big) \;=$$

$$(d\tau)^2 \;\left(\; 1 \;+\; \frac{(dE/d\tau)^2 \;-\; (dp^i/d\tau)(dp_i/d\tau)}{b^2} \;\right) = (d\tau)^2 \;\left(\; 1 \;-\; \frac{F^2}{F^2_{max}} \;\right) \tag{2.10}$$

where one has factored out the proper time infinitesimal

$$g_{\mu\nu}\; dx^\mu\; dx^\nu \;=\; (d\tau)^2 \;=\; (dt)^2 \;-\; (dx)^2 \;-\; (dy)^2 \;-\; (dz)^2 \tag{2.11}$$

in (2.10) . The proper force square interval $(dE/d\tau)^2 - (dp^i/d\tau)(dp_i/d\tau) = -F^2 < 0$ is "spacelike" when the proper velocity interval $(dt/d\tau)^2 - (dx^i/d\tau)(dx_i/d\tau) > 0$ is timelike.

In the following we shall denote the momenta variables as $p^\alpha, \alpha = 1,2,3,4$. Rigorously speaking, the momenta variables in the $8D$ cotangent bundle should be denoted by p_μ and the coordinates by x^μ so that with respect to a symmetric nonlinear connection $N_{\nu\mu}(x,p)$ the canonical symplectic structure can be written as $\omega = (dp_\mu - N_{\nu\mu}(x,p)dx^\nu) \wedge dx^\mu$ [10]. Given the decomposition of the $8D$ cotangent space metric

$$G_{MN}(x,p) \;=\; \begin{pmatrix} g_{\mu\nu}(x,p) \;+\; \pi_{\alpha\beta}(x,p)\; A^\alpha_\mu(x,p)\; A^\beta_\nu(x,p) & A^\alpha_\mu(x,p)\; \pi_{\alpha\beta}(x,p) \\ A^\beta_\nu(x,p)\; \pi_{\alpha\beta}(x,p) & \pi_{\alpha\beta}(x,p) \end{pmatrix} \tag{2.12}$$

$A^\alpha_\mu(x,p)$ is the nonlinear connection in this case. The $8D$ cotangent space (curved phase-space) infinitesimal interval is given by

$$(d\sigma)^2 \;=\; g_{\mu\nu}\; dx^\mu\; dx^\nu \;+\; \frac{\pi_{\alpha\beta}}{\mathbf{b}^2}\; (dp^\alpha \;+\; \mathbf{b}A^\alpha_\mu\; dx^\mu)\; (dp^\beta \;+\; \mathbf{b}A^\beta_\nu\; dx^\nu) \;=$$

$$g_{\mu\nu}\; dx^\mu\; dx^\nu \;+\; \frac{\pi_{\alpha\beta}}{\mathbf{b}^2}\; (\frac{dp^\alpha}{d\tau} \;+\; \mathbf{b}A^\alpha_\mu\; \frac{dx^\mu}{d\tau})\; (\frac{dp^\beta}{d\tau} \;+\; \mathbf{b}A^\beta_\nu\; \frac{dx^\nu}{d\tau})\; (d\tau)^2 \;=$$

$$(d\tau)^2 \;\left(\; 1 \;-\; \frac{F^2}{F^2_{max}} \;\right);\quad F^2_{max} \;=\; \mathbf{b}^2 \tag{2.13}$$

after setting $g_{\mu\nu}dx^\mu dx^\nu = (d\tau)^2$ and

$$F^2 \;=\; F^\alpha F_\alpha \;=\; -\; \pi_{\alpha\beta}\; (\frac{dp^\alpha}{d\tau} \;+\; \mathbf{b}A^\alpha_\mu\; \frac{dx^\mu}{d\tau})\; (\frac{dp^\beta}{d\tau} \;+\; \mathbf{b}A^\beta_\nu\; \frac{dx^\nu}{d\tau}) \tag{2.14}$$

When all quantities do *not* have an explicit dependence on the momenta p^α, but they solely depend on the coordinates x^μ, the *correspondence* (the field A^α_μ is taken to be dimensionless)

$$\pi_{\alpha\beta} \leftrightarrow g_{\alpha\beta};\;\; \mathbf{b}\; A^\alpha_\mu \;\leftrightarrow\; p^\nu\; \Gamma^\alpha_{\mu\nu};\quad \frac{dx^\mu}{d\tau} \;\leftrightarrow\; \frac{p^\mu}{m};\quad F^\alpha \;\leftrightarrow\; \frac{dp^\alpha}{d\tau} \;+ m^{-1}\; \Gamma^\alpha_{\mu\nu}\; p^\mu\; p^\nu \tag{2.15}$$

gives

$$F^2 \ \leftrightarrow \ - \ g_{\alpha\beta} \ (\frac{dp^\alpha}{d\tau} \ + m^{-1} \ \Gamma^\alpha_{\mu\nu} \ p^\mu \ p^\nu) \ (\frac{dp^\beta}{d\tau} \ + m^{-1} \ \Gamma^\beta_{\mu\nu} \ p^\mu \ p^\nu) \ \rightarrow$$

$$- \ g_{\alpha\beta} \ (\frac{dp^\alpha}{d\tau} \ + m^{-1} \ \{^{\alpha}_{\mu\nu}\} \ p^\mu \ p^\nu) \ (\frac{dp^\beta}{d\tau} \ + m^{-1} \ \{^{\beta}_{\mu\nu}\} \ p^\mu \ p^\nu) \qquad (2.16)$$

Therefore, eq-(2.16) furnishes a *correspondence* from F^2 to the (spacelike) proper force squared experienced by a particle of mass m in ordinary curved Riemannian spacetime, with metric $g_{\alpha\beta}(\mathbf{x})$ and Levi-Civita connection $\{^{\alpha}_{\mu\nu}\}$, when $\Gamma^\alpha_{\mu\nu} \rightarrow \{^{\alpha}_{\mu\nu}\}$ (Christoffel symbols). Based on these findings, one may interpret the second/fourth terms in the right hand side of eq-(2.13) as the contributions to the curved phase-space interval due the effects of the proper force acting on the particle as indicated by the expression in the last term of eq-(2.13).

To simplify the calculations, it is expedient to choose an *anholonomic* (non-coordinate basis) adapted to the spacetime conection such that the bundle line element splits naturally into the sum of the spacetime line element and a fiber line element without cross terms. In the *anholonomic* basis the bundle metric has a simple block-diagonal form with entries $g_{\mu\nu}(\mathbf{x}, \mathbf{p}), g_{ab}(\mathbf{x}, \mathbf{p}); \mu, \nu = 1, 2, 3, 4; a, b = 1, 2, 3, 4$. $g_{\mu\nu}(\mathbf{x}, \mathbf{p})$ is now the metric of the base space of the $8D$ cotangent bundle (phase space) and its fiber space metric $g_{ab}(\mathbf{x}, \mathbf{p})$ becomes now the metric $\pi_{\alpha\beta}(\mathbf{x}, \mathbf{p})$ after a relabeling of the indices $a, b \rightarrow \alpha, \beta; \alpha, \beta = 1, 2, 3, 4$. Note that in general $\pi_{\alpha\beta} \neq g_{\alpha\beta}$ since the fiber space metric is not necessarily the same as the base space metric.

By recurring to the results of the prior section **2.1** one learns that the scalar curvature of the $8D$ cotangent bundle manifold can be decomposed as the sum

$$^{(8)}\mathbf{R} \ = \ ^{(8)}R_{MN} \ G^{MN} \ = \ ^{(h)}\mathcal{R} \ + \ ^{(v)}\mathcal{R} \ - \ \frac{1}{4} \ \pi_{\alpha\beta} \ F^\alpha_{\mu\nu} \ F^\beta_{\rho\tau} \ g^{\mu\rho} \ g^{\nu\tau} \ +$$

$$\frac{1}{4} g^{\mu\nu} \ \pi^{\alpha\beta} \ \pi^{\gamma\sigma} \ [\ (D_\mu \ \pi_{\alpha\gamma}) \ (D_\nu \ \pi_{\beta\sigma}) \ - \ (D_\mu \ \pi_{\alpha\beta}) \ (D_\nu \ \pi_{\gamma\sigma}) \ ] +$$

$$\frac{1}{4} \ \pi^{\alpha\beta} \ g^{\mu\nu} \ g^{\rho\tau} \ \mathbf{b^2} \ [\ (\partial_{p^\alpha} \ g_{\mu\rho}) \ (\partial_{p^\beta} \ g_{\nu\tau}) \ - \ (\partial_{p^\alpha} \ g_{\mu\nu}) \ (\partial_{p^\beta} \ g_{\rho\tau}) \ ] \qquad (2.17)$$

plus $(\sqrt{|det \ g_{\mu\nu}|} \ \sqrt{|det \ \pi_{\alpha\beta}|})^{-1}$ times the total derivative terms.

After *relabeling* indices, making no differentiation among $\mu, \nu, \rho...$ and α, β, γ, the ("horizontal") scalar curvature of the base space of the $8D$ *cotangent* bundle manifold $^{(h)}\mathcal{R}$ in eq-(2.17) is defined in terms of the horizontal symmetric connection coefficients $\Gamma^\mu_{\alpha\beta}(\mathbf{x}, \mathbf{p})$ given by

$$\Gamma^\mu_{\alpha\beta} \ = \ \frac{1}{2} \ g^{\mu\nu} \ [\ (\frac{\partial}{\partial x^\beta} \ - \ \mathbf{b} \ A^\lambda_\beta \ \frac{\partial}{\partial p^\lambda}) \ g_{\nu\alpha} \ + \ (\frac{\partial}{\partial x^\alpha} \ - \ \mathbf{b} \ A^\lambda_\alpha \ \frac{\partial}{\partial p^\lambda}) \ g_{\nu\beta} \ ] \ -$$

$$\frac{1}{2} \ g^{\mu\nu} \ [\ (\frac{\partial}{\partial x^\nu} \ - \ \mathbf{b} \ A^\lambda_\nu \ \frac{\partial}{\partial p^\lambda}) \ g_{\alpha\beta} \ ] \ \Rightarrow \qquad (2.18)$$

$$\Gamma^{\mu}_{\alpha\beta} \;=\; \{^{\mu}_{\alpha\beta}\} \;-\; \frac{1}{2}\;\mathbf{b}\; g^{\mu\nu} [\; A^{\lambda}_{\beta}\; \frac{\partial g_{\nu\alpha}}{\partial p^{\lambda}} \;+\; A^{\lambda}_{\alpha}\; \frac{\partial g_{\nu\beta}}{\partial p^{\lambda}} \;-\; A^{\lambda}_{\nu}\; \frac{\partial g_{\alpha\beta}}{\partial p^{\lambda}} \;] \tag{2.19}$$

$$\{^{\mu}_{\alpha\beta}\} \;=\; \frac{1}{2}\; g^{\mu\nu}\; [\; \frac{\partial}{\partial x^{\beta}}\; g_{\nu\alpha} \;+\; \frac{\partial}{\partial x^{\alpha}}\; g_{\nu\beta} \;-\; \frac{\partial}{\partial x^{\nu}}\; g_{\alpha\beta} \;] \tag{2.20}$$

The ("horizontal") scalar curvature of the base space of the $8D$ cotangent bundle becomes

$$^{(h)}\mathcal{R} \;=\; g^{\mu\nu} \left[(\frac{\partial}{\partial x^{\alpha}} \;-\; \mathbf{b}\; A^{\tau}_{\alpha}\; \frac{\partial}{\partial p^{\tau}})\; \Gamma^{\alpha}_{\mu\nu} \;-\; (\frac{\partial}{\partial x^{\nu}} \;-\; \mathbf{b}\; A^{\tau}_{\nu}\; \frac{\partial}{\partial p^{\tau}})\; \Gamma^{\alpha}_{\mu\alpha} \right] \;+$$

$$g^{\mu\nu} \left[\Gamma^{\alpha}_{\mu\nu}\; \Gamma^{\beta}_{\alpha\beta} \;-\; \Gamma^{\alpha}_{\mu\beta}\; \Gamma^{\beta}_{\alpha\nu} \right] \;=\; R \;+\; \Delta \tag{2.21}$$

where Δ are the *corrections* from the ordinary scalar Riemannian curvature.

The ("vertical") scalar curvature of the cotangent space (fiber space) of the $8D$ cotangent bundle is

$$^{(v)}\mathcal{R} \;=\; \pi^{\mu\nu} \left[\mathbf{b}\; (\frac{\partial}{\partial p^{[\alpha}}\; \Upsilon^{\alpha}_{\mu\nu]}) \;+\; \Upsilon^{\alpha}_{\mu[\nu}\; \Upsilon^{\beta}_{\alpha\beta]} \right] \tag{2.22}$$

where the anti-symmetrization of indices is indicated. The vertical symmetric connection coefficients $\Upsilon^{\mu}_{\alpha\beta}$ defined on the four-momentum cotangent space (fibers) are given by

$$\Upsilon^{\mu}_{\alpha\beta} \;=\; \frac{1}{2}\; \mathbf{b}\; \pi^{\mu\lambda}\; [\; \frac{\partial}{\partial p^{\alpha}}\; \pi_{\lambda\beta} \;+\; \frac{\partial}{\partial p^{\beta}}\; \pi_{\lambda\alpha} \;-\; \frac{\partial}{\partial p^{\lambda}}\; \pi_{\alpha\beta} \;] \tag{2.23}$$

The gauge field strength is

$$F^{\alpha}_{\mu\nu} \;=\; (\frac{\partial}{\partial x^{\mu}} \;-\; \mathbf{b}\; A^{\tau}_{\mu}\; \frac{\partial}{\partial p^{\tau}})\; A^{\alpha}_{\nu} \;-\; (\frac{\partial}{\partial x^{\nu}} \;-\; \mathbf{b}\; A^{\tau}_{\nu}\; \frac{\partial}{\partial p^{\tau}})\; A^{\alpha}_{\mu} \tag{2.24}$$

The gauge covariant derivative $D_{\mu}\pi_{\alpha\beta}$ is defined

$$D_{\mu}\; \pi_{\alpha\beta} \;=\; \frac{\partial}{\partial x^{\mu}} \pi_{\alpha\beta} \;-\; [\; A_{\mu},\; \pi_{\alpha\beta}\;]. \tag{2.25}$$

in terms of the Lie-bracket

$$[\; A_{\mu}\;,\; \pi_{\alpha\beta}\;] \;=\; \mathbf{b}\; (\frac{\partial}{\partial p^{\alpha}} A^{\lambda}_{\mu})\; \pi_{\lambda\beta} \;+ \mathbf{b}\; (\frac{\partial}{\partial p^{\beta}} A^{\lambda}_{\mu})\; \pi_{\alpha\lambda} \;+\; \mathbf{b}\; A^{\lambda}_{\mu}\; (\frac{\partial}{\partial p^{\lambda}} \pi_{\alpha\beta}). \tag{2.26}$$

By analogy to ordinary gravity, a candidate geometrical (gravitational) action defined in the $8D$ cotangent bundle is of the form

$$S \;=\; \frac{1}{2\kappa^{2}}\; \int_{\Omega}\; d^{4}x\; d^{4}p\; \sqrt{|det\; g_{\mu\nu}|}\; \sqrt{|det\; \pi_{\alpha\beta}|}\; ^{(8)}\mathbf{R}(\mathbf{x}, \mathbf{p}) \tag{2.27}$$

where ${}^{(8)}\mathbf{R}(\mathbf{x},\mathbf{p})$ is given by eqs-(2.17-2.26) and the domain of integration in phase-space is denoted by Ω. Using the natural units $\hbar = c = 1$ and after taking the curvature to have the canonical units of $(length)^{-2}$ leads to the units of $(length)^{-1}$ (mass) for the coupling κ. One could add explicit torsion squared terms to the action (2.27) as well, but for the moment we shall just focus on the scalar curvature term and set the torsion (and nonmetricity) terms to zero.

The action is invariant under gauge transformations (diffeomorphisms of the momentum space) given by

$$\delta A^\alpha_\mu \ = \ -\,(\ \partial_\mu\, \xi^\alpha \ - \ [\ A_\mu,\ \xi]^\alpha\ ) \qquad (2.28a)$$

$$[\xi,\ A_\mu]^\alpha \ = \ \mathbf{b}\, \xi^\sigma\ \partial_{p^\sigma}\ A^\alpha_\mu \ - \mathbf{b}\ A^\sigma_\mu\ \partial_{p^\sigma}\ \xi^\alpha \qquad (2.28b)$$

$$\delta\pi_{\alpha\beta} \ = \ [\xi,\ \pi]_{\alpha\beta} \ = \ \mathbf{b}\ \xi^\sigma\ \partial_{p^\sigma}\ \pi_{\alpha\beta} \ + \mathbf{b}\ \pi_{\alpha\sigma}\ \partial_{p^\beta}\ \xi^\sigma \ + \mathbf{b}\ \pi_{\sigma\beta}\ \partial_{p^\alpha}\ \xi^\sigma. \qquad (2.29)$$

$$\delta g_{\mu\nu} \ = \ [\xi,\ g_{\mu\nu}] \ = \ \mathbf{b}\ \xi^\alpha\ \partial_{p^\alpha}\ g_{\mu\nu} \qquad (2.30)$$

where $g_{\mu\nu}(\mathbf{x},\mathbf{p})$ and $\pi_{\alpha\beta}(\mathbf{x},\mathbf{p})$. The gauge field strength transforms homogeneously

$$\delta F^\alpha_{\mu\nu} \ = \ [\xi,\ F_{\mu\nu}]^\alpha \ = \ \mathbf{b}\ \xi^\sigma\ \partial_{p^\sigma}\ F^\alpha_{\mu\nu} \ - \mathbf{b}\ F^\sigma_{\mu\nu}\ \partial_{p^\sigma}\ \xi^\alpha \qquad (2.31)$$

The Lie bracket of a scalar Lagrangian *density* $\mathcal{L} = \sqrt{|detg|}\ \sqrt{|det\pi|}\ L$ of weight one and a vector field $\xi = \xi^\alpha \partial_{p^\alpha}$ is defined as

$$[\ \xi,\ \mathcal{L}\ ] \ = \ \xi^\alpha\ \partial_{p^\alpha}\ \mathcal{L} \ + \ \mathcal{L}\ \partial_{p^\alpha}\ \xi^\alpha \ = \ \partial_{p^\alpha}\ (\xi^\alpha\ \mathcal{L}) \qquad (2.32)$$

there is a second extra term in the r.h.s of (2.32), so that under the above infinitesimal gauge transformations the variation of the action S given by $\delta S = \int [\xi, \mathcal{L}] = \int \partial_{p^\alpha}(\xi^\alpha \mathcal{L})$ is a total derivative and it vanishes if $\mathcal{L}$ vanishes at $p^\alpha = \pm\infty$ and/or there are no boundaries in the integration domain of the p^α variables (the integration domain has *compact* support). Hence, the $8D$ action (2.27) is invariant under gauge transformations (diffeomorphisms of the momentum space).

In the most general case, one could have directly recurred to the local expressions for the distinguished tensors of torsion and curvature given in [10]. There are two different torsion 2-forms [10] in phase space involving 5 distinguished tensors $T^\mu_{\nu\rho}, R_{\mu\nu\rho}, C^{\mu\nu}_\rho, P^\mu_{\nu\rho}, S^{\mu\nu}_\rho$ and one curvature 2-form in phase space involving 3 distinguished tensors $R^\mu_{\nu\rho\tau}, P^{\mu\nu}_{\rho\tau}, S^{\mu\nu\rho}_\tau$. For further details we refer to [10]. The latter 3 curvature tensors correspond to the horizontal, mixed and vertical components of the curvature.

A natural action in the $8D$ phase space will involve sums of the 5 torsion squared terms and contractions of the above 3 curvature distinguished tensors with the metric leading to the curvature scalars $\mathcal{R}, \mathcal{P}, \mathcal{S}$. All torsion and curvature tensors are explicit functions of the nonlinear connection $N_{\mu\nu}(\mathbf{x},\mathbf{p})$, which is no longer purely symmetric since there is torsion, and the horizontal and vertical connection coefficients $\Gamma^{\mu}_{\nu\rho}(\mathbf{x},\mathbf{p}), \Upsilon^{\nu\rho}_{\mu}(\mathbf{x},\mathbf{p})$ and their derivatives [10]. Matter terms could also be introduced leading, if possible, to a generalized stress energy tensor on the $8D$ cotangent bundle and the extension of Einstein field equations with matter.

It is worth pointing out that the *horizontal* (geodesics) paths corresponding to a covariant derivative $\mathbf{D}$ associated with the nonlinear connection $N_{\mu\nu}(\mathbf{x},\mathbf{p})$ are characterized by the system of differential equations [10]

$$\frac{d^2x^\mu}{ds^2} \; + \; \Gamma^{\mu}_{\nu\rho}(\mathbf{x},\mathbf{p}) \; \frac{dx^\nu}{ds} \; \frac{dx^\rho}{ds} \; = \; 0; \quad \frac{dp_\mu}{ds} \; - \; N_{\nu\mu}(\mathbf{x},\mathbf{p}) \; \frac{dx^\nu}{ds} \; = \; 0 \qquad (2.33)$$

And the *vertical* (geodesic) paths at a point x^μ_0 are characterized by the system of differential equations

$$x^\mu \; = \; x^\mu_0; \quad \frac{d^2p_\mu}{ds^2} \; - \frac{1}{\mathbf{b}} \; \Upsilon^{\nu\rho}_{\mu}(\mathbf{x}_0,\mathbf{p}) \; \frac{dp_\nu}{ds} \; \frac{dp_\rho}{ds} \; = \; 0 \qquad (2.34)$$

s is a parameter associated with the parametrized curves and one has inserted the factor $1/\mathbf{b}$ to match units in (2.34). $\Gamma^{\mu}_{\nu\rho}(\mathbf{x},\mathbf{p}), \Upsilon^{\nu\rho}_{\mu}(\mathbf{x},\mathbf{p})$ are, respectively, the horizontal and vertical connection coefficients of the covariant derivative $\mathbf{D}$ associated with the nonlinear connection $N_{\mu\nu}(\mathbf{x},\mathbf{p})$ in the $8D$ cotangent bundle (phase-space).

2.3 Cosmological Constant, Brans-Dicke-Jordan Gravity and Field Equations

When the scalar curvature of the momentum space is *constant* and inversely proportional to the square of the characteristic momentum scale P^{-2}, after rescaling the momentum curvature by $\mathbf{b}^2$ in order to recover the same units as a scalar curvature in spacetime, one gets a contribution to the momentum-integral in eq-(2.27) defined on a compact momentum domain $\mathcal{D}_p$ region given by

$$\int_{\mathcal{D}_p} d^4p \; \sqrt{|det \; \pi_{\alpha\beta}|} \; {}^{(v)}\mathcal{R} \; = \; \frac{\Omega_p \; \mathbf{b}^2}{P^2} \; \sim \; \mathbf{b}^2 \; \frac{P^4}{P^2} \; = \; \mathbf{b}^2 \; P^2 \qquad (2.33)$$

one has taken into account that the volume Ω_p of a compact momentum domain region $\mathcal{D}_p$ is proportional to P^4. Inserting this contribution ${}^{(v)}\mathcal{R}$ inside the remaining spacetime integral of eq-(2.27) and equating it to the cosmological constant term in the action gives

$$\frac{1}{2\kappa^2} \int d^4x \; \sqrt{|det \; g_{\mu\nu}|} \; \mathbf{b}^2 \; P^2 \; = \; \frac{1}{16\pi G_N} \int d^4x \; \sqrt{|det \; g_{\mu\nu}|} \; 2 \; \Lambda \; \Rightarrow$$

$$4\pi \; \frac{L_P^2 \; \mathbf{b}^2 \; P^2}{\kappa^2} \; \sim \; \Lambda \; \sim \; \frac{1}{R^2_{Hubble}}; \quad G_N = L_P^2 = (m_P)^{-2} \qquad (2.34)$$

inserting into eq-(2.34) the following values (given in terms of the Planck mass m_P)

$$\mathbf{b} \sim m_P^2, \quad \kappa \ \sim \ m_P \ \Rightarrow \ P \ \sim \ \frac{1}{R_{Hubble}} \tag{2.35}$$

lead to the observed value of the extremely small cosmological constant when the size of the momentum space domain is extremely small and given by $P \sim \frac{1}{R_{Hubble}}$. If one assigns the Planck length scale L_P as the "minimal" scale, by Born's reciprocity one should have a corresponding "minimal" momentum scale that one may set to be $P \sim \frac{1}{R_{Hubble}}$. One may envision the compact momentum space region of integration as a bounded Cartan homogeneous domain like the ones studied by [14] in describing curved phase spaces. Noncompact domain regions, like a de Sitter hyperboloid in momentum space, with constant momentum curvature proportional to P^{-2} and a conformally flat metric in momentum space,

$$\pi_{\alpha\beta}(\mathbf{p}) \ \sim \frac{\eta_{\alpha\beta}}{1 - (p_\gamma p^\gamma / P^2)} \tag{2.36}$$

will yield an $infinite$ volume. For this reason it is desirable to choose compact regions in momentum space or choosing field configurations that vanish at infinity (compact support) such that the $8D$ action S in eq-(2.27) is $finite$. Despite that the size of $P \sim (1/R_{Hubble})$ is small does not mean that the rate of change of the momenta (the forces) have to be small. For example, electrons inside the atom are confined to a small region but have large velocities. Other range of values for κ and P are possible. If $m_P = \kappa = P$, the size of the momentum space region is now very large, all the way to the Planck momentum scale, but one gets a huge value for the cosmological constant in this case $\Lambda \sim (L_P)^{-2}$ instead of $(R_{Hubble})^{-2}$. This large value of the cosmological constant is compatible with a Planck size universe. A dynamical value of the cosmological constant over time is also an appealing possibility when ${}^{(v)}\mathcal{R}$ plays the role of a variable and effective cosmological "constant". As the universe expands, the value of the cosmological constant decreases from a very large initial value $(L_P)^{-2}$ to the present day one $(R_{Hubble})^{-2}$.

To sum up, we have seen how the geometry of the momentum space can be linked to the observed value of the cosmological constant when the curvature in *momentum* space is very large, namely the small size of P is of the order of $(1/R_{Hubble})$. Brans-Dicke-Jordan Gravity type of actions can also be recovered from the $8D$ action in the special case that

$$A_\mu = 0; \ \pi_{\alpha\beta}(\mathbf{x},\mathbf{p}) \ = \ \phi(\mathbf{x})\, \eta_{\alpha\beta}; \ \pi^{\alpha\beta} \ = \ \phi^{-1}(\mathbf{x})\, \eta_{\alpha\beta}; \ det(\pi_{\alpha\beta}) = \phi^4; \ g_{\mu\nu} = g_{\mu\nu}(\mathbf{x}) \tag{2.37}$$

inserting these values (2.37) into the terms of the action (2.27) gives ${}^{(v)}\mathcal{R} = 0; F^\alpha_{\mu\nu} = 0; ...$ and ${}^{(h)}\mathcal{R} = R$ (ordinary Ricci scalar curvature). After performing the momentum integral leads to a Brans-Dicke-Jordan Gravity-like action, up to numerical constants

$$\frac{V(\mathcal{D}_p)}{2\kappa^2} \int d^4x \sqrt{|det\ g_{\mu\nu}|}\ (\ \phi^2\ R\ -3\ g^{\mu\nu}\ \partial_\mu\phi\ \partial_\nu\phi\ ) \tag{2.38}$$

where $V(\mathcal{D}_p) = \int_{\mathcal{D}_p} d^4p$ is a measure of the momentum space region. The scalar ϕ does not have the correct canonical dimensions in $4D$. One may scale $\phi \to \kappa\phi = \Phi$ giving the following action in terms of the canonical scalar field Φ

$$\frac{V(\mathcal{D}_p)}{2\kappa^4} \int d^4x \sqrt{|det\ g_{\mu\nu}|}\ (\ \Phi^2\ R\ -3\ g^{\mu\nu}\ \partial_\mu\Phi\ \partial_\nu\Phi\ ) \tag{2.39}$$

hence, when the measure of the momentum space region obeys $(V(\mathcal{D}_p)/2\kappa^4) = (1/16\pi)$ one recovers the Brans-Dicke-Jordan action with the coupling parameter $\omega = \frac{3}{4}$ and after the change of variables $\varphi = \Phi^2$ is performed

$$\frac{1}{16\pi} \int d^4x \sqrt{|det\ g_{\mu\nu}|}\ (\ \varphi\ R\ -\ \omega\ \varphi^{-1}\ g^{\mu\nu}\ \partial_\mu\varphi\ \partial_\nu\varphi\ ) \tag{2.40}$$

The Einstein-Hilbert action for gravity is obtained when $\varphi = \frac{1}{G_N}$.

In the most general case, the equations of motion associated with the $8D$ action (2.27) associated with the geometry of the cotangent bundle are obtained by performing a variation

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0; \quad \frac{\delta S}{\delta \pi_{\alpha\beta}} = 0; \quad \frac{\delta S}{\delta A^\alpha_\mu} = 0 \tag{2.41}$$

with $g_{\mu\nu}(\mathbf{x},\mathbf{p}),\ \pi_{\alpha\beta}(\mathbf{x},\mathbf{p}),\ A^\alpha_\mu(\mathbf{x},\mathbf{p})$. These equations are quite cumbersome. In certain special cases, for example, one could calculate the corrections to the Schwarzschild solutions and the gravitational redshift found in the $8D$ tangent bundle case associated with a Finsler spacetime base manifold [3]. The universal upper limit on the attainable proper acceleration relative to the vacuum imposes certain constraints on the possible differential geometric structures in the $8D$ tangent bundle of the $4D$ spacetime. One is lead to a Finslerian structure for spacetime, in which the spacetime metric depends not only on the spacetime coordinates, but also on the four-velocity coordinates of the tangent-space. The redshift receives corrections proportional to the factor $(1 - a^2/a^2_{max})^{\frac{1}{2}}$ [3].

3 Born's Reciprocal Complex Gravity as a Gauge Theory of the Quaplectic Group

In this section we shall take a different approach than studying the geometry of the $8D$ cotangent bundle (phase-space) and instead construct an $8D$ local gauge theory of the Quaplectic group (group proposed by [2]) in phase space. For the sake of completeness, in this last section we extend our results [16] and construct a *deformed* Born reciprocal general relativity theory in curved phase space (without the need to introduce star products) as a local gauge theory of the *deformed* Quaplectic group that is given by the semi-direct product of $U(1,3)$ with the *deformed* (noncommutative) Weyl-Heisenberg group corresponding to *noncommutative* coordinates and momenta. The (deformed) Quaplectic group acts as the automorphism group along the internal fiber coordinates. Therefore we must *not* confuse the *deformed* complex gravity constructed here with the noncommutative gravity work in the literature [22] where the spacetime coordinates x^μ are not commuting.

The procedure that led to the construction of a Born reciprocal general relativity theory in a curved $4D$ spacetime [16] can be extended to an action in the $8D$ real-dimensional Phase Space associated with the cotangent bundle of $4D$ spacetime. The geometry of curved Phase spaces and bounded complex homogeneous domains has been studied by [14]. The presence of matter sources can be incorporated, for example, by recurring to the invariant action for a point-particle in Born's Reciprocal Relativity involving Casimir group invariant quantities associated with the world-line of the particle.

The deformed Weyl-Heisenberg algebra involves the generators

$$Z_a = \frac{1}{\sqrt{2}} \left(\frac{X_a}{\lambda_l} - i\frac{P_a}{\lambda_p} \right); \quad \bar{Z}_a = \frac{1}{\sqrt{2}} \left(\frac{X_a}{\lambda_l} + i\frac{P_a}{\lambda_p} \right); \quad a = 1,2,3,4. \tag{3.1}$$

Notice that we must *not* confuse the *generators* X_a, P_a (associated with the fiber coordinates of the internal space of the fiber bundle) with the ordinary base spacetime coordinates and momenta x_μ, p_μ. The gauge theory is constructed in the fiber bundle over the $8D$ base phase-space manifold which is an $8D$ curved space with *commuting* coordinates and momenta $x^\mu = x^0, x^1, x^2, x^3$; $p_\mu = p_0, p_1, p_2, p_3$. To properly match the dimensions we shall scale the momentum variables by the maximal proper-force $\mathbf{b}$ so that all coordinates have the dimensions of length. We denote the coordinates of the $8D$ phase-space collectively by $Y_M, M = 1,2,3,....,8$ where the first four coordinates correspond to x^μ and the last four coordinates correspond to $p_\mu/\mathbf{b}$.

The Hermitian generators $Z_{ab}, Z_a, \bar{Z}_a, I$ of the $U(1,3)$ algebra and the *deformed* Weyl-Heisenberg algebra obey the relations

$$(Z_{ab})^\dagger = Z_{ab}; \quad (Z_a)^\dagger = \bar{Z}_a; \quad I^\dagger = I; \quad a,b = 1,2,3,4. \tag{3.2}$$

The standard Quaplectic group [2] is given by the semi-direct product of the $U(1,3)$ group and the unmodified Weyl-Heisenberg $H(1,3)$ group : $\mathcal{Q}(1,3) \equiv U(1,3) \otimes_s H(1,3)$ and is defined in terms of the generators $Z_{ab}, Z_a, \bar{Z}_a, I$ with $a,b = 1,2,3,4$. A careful analysis reveals that the complex generators $Z_a, \bar{Z}_a$ (with Hermitian *and* anti-Hermitian pieces) of the *deformed* Weyl-Heisenberg algebra can be defined in terms of the Hermitian $U(1,4)$ algebra generators Z_{AB} , where $A,B = 1,2,3,4,5$; $a,b = 1,2,3,4$; $\eta_{AB} = diag\ (+,-,-,-,-)$, as follows

$$Z_a = (-i)^{1/2} (Z_{a5} - iZ_{5a}); \ \bar{Z}_a = (i)^{1/2} (Z_{a5} + iZ_{5a}); \ Z_{55} = \frac{\mathcal{I}}{2} \tag{3.3}$$

the Hermitian generators are $Z_{AB} \equiv \mathcal{E}_A^B$ and $Z_{BA} \equiv \mathcal{E}_B^A$; notice that the position of the indices is very relevant because $Z_{AB} \neq Z_{BA}$. The commutators are

$$[\mathcal{E}_a^b,\ \mathcal{E}_c^d] \ = \ -i\ \delta_c^b\ \mathcal{E}_a^d\ + i\ \delta_a^d\ \mathcal{E}_c^b; \quad [\mathcal{E}_c^d,\ \mathcal{E}_a^5] \ = \ -i\ \delta_a^d\ \mathcal{E}_c^5; \quad [\mathcal{E}_c^d,\ \mathcal{E}_5^a] \ = \ i\ \delta_c^a\ \mathcal{E}_5^d. \tag{3.4}$$

and $[\mathcal{E}_5^5,\ \mathcal{E}_5^a] = -i\ \delta_5^5\ \mathcal{E}_5^a$ .... such that now $\mathcal{I}(=2Z_{55})$ *no* longer commutes with $Z_a, \bar{Z}_a$. The generators Z_{ab} of the $U(1,3)$ algebra can be decomposed into the Lorentz-subalgebra generators $\mathcal{L}_{ab}$ and the "shear"-like generators $\mathcal{M}_{ab}$ as

$$Z_{ab} \equiv \frac{1}{2}\,(\mathcal{M}_{ab} - i\mathcal{L}_{ab}); \quad \mathcal{L}_{ab} = \mathcal{L}_{[ab]} \ = \ i\,(Z_{ab} - Z_{ba}); \quad \mathcal{M}_{ab} = M_{(ab)} \ = \ (Z_{ab} + Z_{ba}), \tag{3.5}$$

one can see that the "shear"-like generators $\mathcal{M}_{ab}$ are $Hermitian$ and the Lorentz generators $\mathcal{L}_{ab}$ are $anti-Hermitian$ with respect to the fiber internal space indices. The explicit commutation relations of the Hermitian generators Z_{ab} can be rewritten as

$$[\mathcal{L}_{ab},\ \mathcal{L}_{cd}] \ = \ (\eta_{bc}\mathcal{L}_{ad} - \eta_{ac}\mathcal{L}_{bd} - \eta_{bd}\mathcal{L}_{ac} + \eta_{ad}\mathcal{L}_{bc}). \tag{3.6a}$$

$$[\mathcal{M}_{ab},\ \mathcal{M}_{cd}] \ = \ -\,(\eta_{bc}\mathcal{L}_{ad} + \eta_{ac}\mathcal{L}_{bd} + \eta_{bd}\mathcal{L}_{ac} + \eta_{ad}\mathcal{L}_{bc}). \tag{3.6b}$$

$$[\mathcal{L}_{ab},\ \mathcal{M}_{cd}] \ = \ (\eta_{bc}\mathcal{M}_{ad} - \eta_{ac}\mathcal{M}_{bd} + \eta_{bd}\mathcal{M}_{ac} - \eta_{ad}\mathcal{M}_{bc}). \tag{3.6c}$$

Defining $Z_{ab} = \frac{1}{2}(\mathcal{M}_{ab} - i\mathcal{L}_{ab}),\ Z_{cd} = \frac{1}{2}(M_{cd} - iL_{cd})$ after straightforward algebra it leads to the $U(3,1)$ commutators

$$[\ Z_{ab},\ Z_{cd}\ ] \ = \ -\,i\ (\ \eta_{bc}\ Z_{ad}\ -\ \eta_{ad}\ Z_{cb}\ ). \tag{3.6d}$$

as expected, and which requires that the commutators $[M, M] \sim L$ otherwise one would not obtain the $U(3,1)$ commutation relations (3.9d) nor the Jacobi identities will be satisfied. The commutators of the (anti-Hermitian) Lorentz boosts generators L_{ab} with the $X_c,\ P_c$ generators are

$$[\mathcal{L}_{ab},\ X_c] \ = \ (\ \eta_{bc}\ X_a\ -\ \eta_{ac}\ X_b\ ); \quad [\mathcal{L}_{ab},\ P_c] \ = \ (\ \eta_{bc}\ P_a\ -\ \eta_{ac}\ P_b\ ). \tag{3.7a}$$

Since the Hermitian M_{ab} generators are the *reciprocal* boosts transformations which *exchange* X for P, in addition to boosting (rotating) those variables, one has in

$$[\mathcal{M}_{ab},\ \frac{X_c}{\lambda_l}] \ = \ -\ \frac{i}{\lambda_p}\ (\ \eta_{bc}\ P_a + \eta_{ac}\ P_b\ ); \quad [\mathcal{M}_{ab},\ \frac{P_c}{\lambda_p}] \ = \ -\frac{i}{\lambda_l}\ (\ \eta_{bc}\ X_a + \eta_{ac}\ X_b\ ) \tag{3.7b}$$

such that upon recurring to the above equations after lowering indices it leads to [1]

---

[1]These commutators $differ$ from those in [2] because he chose all generators X, P, M, L to be anti-Hermitian so there are no i terms in the commutators in the r.h.s of eq-(3.7b) and there are also sign changes

$$[\ Z_{ab},\ Z_c\ ]\ =\ -\frac{i}{2}\ \eta_{bc}\ Z_a\ +\ \frac{i}{2}\ \eta_{ac}\ Z_b\ -\frac{1}{2}\ \eta_{bc}\ \bar{Z}_a\ -\ \frac{1}{2}\ \eta_{ac}\ \bar{Z}_b$$

$$[\ Z_{ab},\ \bar{Z}_c\ ]\ =\ -\frac{i}{2}\ \eta_{bc}\ \bar{Z}_a\ +\ \frac{i}{2}\ \eta_{ac}\ \bar{Z}_b\ +\ \frac{1}{2}\ \eta_{bc}\ Z_a\ +\ \frac{1}{2}\ \eta_{ac}\ Z_b\ . \qquad (3.7c)$$

In the noncommutative Yang's phase-space algebra case [17], associated with a noncommutative phase space involving noncommuting spacetime coordinates and momentum x^μ, p^μ, the generator $\mathcal{N}$ which appears in the modified $[x^\mu, p^\nu] = i\hbar\eta^{\mu\nu}\mathcal{N}$ commutator is the *exchange* operator $x \leftrightarrow p$, $[p^\mu, \mathcal{N}] = i\hbar x^\mu / R_H^2$ and $[x^\mu, \mathcal{N}] = iL_P^2 p^\mu/\hbar$. L_P, R_H are taken to be the minimal Planck and maximal Hubble length scales, respectively. The Hubble upper scale R_H corresponds to a *minimal* momentum $\hbar/R_H$, because by "duality" if there is a minimal length there should be a minimal momentum also.

Yang's [17] noncommutative phase space algebra is isomorphic to the conformal algebra $so(4,2) \sim su(2,2)$ after the correspondence $x^\mu \leftrightarrow \mathcal{L}^{\mu 5}$, $p^\mu \leftrightarrow \mathcal{L}^{\mu 6}$, and $\mathcal{N} \leftrightarrow \mathcal{L}^{56}$. In the *deformed* Quaplectic algebra case, it is in addition to the $\mathcal{I}$ generator, the $\mathcal{M}_{ab}$ generator which plays the role of the exchange operator of X with P and which also appears in the *deformed* Weyl-Heisenberg algebra leading to a matrix-valued generalized Planck-constant, and noncommutative fiber coordinates, as follows

$$[\frac{X_a}{\lambda_l}, \frac{P_b}{\lambda_p}]\ =\ i\ \alpha_\hbar\ (\eta_{ab}\ \mathcal{I} + \mathcal{M}_{ab});\ [X_a, X_b]\ =\ -\ (\lambda_l)^2\ \mathcal{L}_{[ab]};\ [P_a, P_b]\ =\ (\lambda_p)^2\ \mathcal{L}_{[ab]}; \qquad (3.8)$$

One could interpret the term $\eta_{ab}\ \mathcal{I}\ +\ \mathcal{M}_{ab}$ as a matrix-valued Planck constant $\hbar_{ab}$ (in units of $\hbar$). The *deformed* (noncommutative) Weyl-Heisenberg algebra can also be rewritten as

$$[Z_a,\ \bar{Z}_b]\ =\ -\alpha_\hbar\ (\ \eta_{ab}\ \mathcal{I}\ +\ \mathcal{M}_{ab}\ );\quad [Z_a,\ Z_b]\ =\ [\bar{Z}_a,\ \bar{Z}_b]\ = -i\ Z_{[ab]}\ =\ -\mathcal{L}_{ab}.$$

$$[Z_a,\ \mathcal{I}]\ =\ 2\ \bar{Z}_a;\quad [\bar{Z}_a,\ \mathcal{I}]\ =\ -\ 2\ Z_a;\quad [Z_{ab},\ \mathcal{I}] = 0.\quad \mathcal{I}\ =\ 2\ Z_{55}. \qquad (3.9)$$

where $[\frac{X_a}{\lambda_l}, \mathcal{I}] = 2i\frac{P_a}{\lambda_p}$; $[\frac{P_a}{\lambda_p}, \mathcal{I}] = 2i\frac{X_a}{\lambda_l}$ and the metric $\eta_{ab} = (+1, -1, -1, -1)$ is used to raise and lower indices . The *deformed* Quaplectic algebra obeys the Jacobi identities. No longer $\mathcal{I}$ commutes with $Z_a, \bar{Z}_a$, it *exchanges* them, as one can see from eq-(3.9) since $Z_{55} = \mathcal{I}/2$.

The *complex* frame E^a_M which is *no* longer a square matrix and transforms under the fundamental representation of $U(1,3)$ is defined as

$$E^a_M\ =\ \frac{1}{\sqrt{2}}\ (\ e^a_M\ +\ if^a_M\ );\quad \bar{E}^a_M\ =\ \frac{1}{\sqrt{2}}\ (\ e^a_M\ -\ if^a_M\ ).\quad M = 1, 2, 3, ..., 8. \qquad (3.10)$$

The complex Hermitian metric is given by

$$G_{MN}\ =\ \bar{E}^a_M\ E^b_N\ \eta_{ab}\ =\ g_{(MN)}\ + ig_{[MN]}\ =\ g_{(MN)}\ + iB_{MN}. \qquad (3.11)$$

such that

$$(G_{MN})^\dagger \;=\; \bar G_{NM} \;=\; G_{MN}; \quad \bar G_{MN} \;=\; G_{NM}. \tag{3.12}$$

where the bar denotes complex conjugation. Despite that the metric is complex the infinitesimal line element is *real*

$$ds^2 \;=\; G_{MN}\, dY^M\, dY^N \;=\; g_{(MN)}\, dY^M\, dY^N, \;\; because \;\; i\, g_{[MN]}\, dY^M\, dY^N \;=\; 0. \tag{3.13}$$

The (deformed) Quaplectic-algebra-valued anti-Hermitian gauge field $(\mathbf{A}_M)^\dagger \;=\; -\mathbf{A}_M$ is given by

$$\mathbf{A}_M \;=\; \Omega^{ab}_M\; Z_{ab} \;+ \frac{i}{L_p}\;(\; E^a_M\; Z_a \;+\; \bar E^a_M\; \bar Z_a\;) \;+ i\; \Omega_M\; I\;. \tag{3.14}$$

where a length scale that we chose to coincide with the the Planck length scale L_P has been introduced in the second terms in the r.h.s since the connection $\mathbf{A}_M$ must have units of $(length)^{-1}$. In natural units of $\hbar = c = 1$ the gravitational coupling in $4D$ is $G = L_P^2$. Decomposing the anti-Hermitian components of the connection Ω^{ab}_M into anti-symmetric $[ab]$ and symmetric (ab) pieces with respect to the internal indices

$$\Omega^{ab}_M \;=\; \Omega^{[ab]}_M \;+ i\;\; \Omega^{(ab)}_M. \tag{3.15}$$

gives the anti-Hermitian $U(1,3)$-valued connection

$$\Omega^{ab}_M\; Z_{ab} \;=\; (\Omega^{[ab]}_M \;+ i\;\; \Omega^{(ab)}_M)\; \frac{1}{2}(\mathcal{M}_{ab} - i\; \mathcal{L}_{ab}) \;=$$

$$-\frac{i}{2}\; \Omega^{[ab]}_M\; \mathcal{L}_{ab} \;+\; \frac{i}{2}\; \Omega^{(ab)}_M\; \mathcal{M}_{ab} \;\Rightarrow\; (\Omega^{ab}_M\; Z_{ab})^\dagger \;=\; -\; \Omega^{ab}_M\; Z_{ab}. \tag{3.16}$$

since $(Z_{ab})^\dagger \;=\; Z_{ab}$

The *deformed* Quaplectic algebra-valued (anti-Hermitian) field strength is given by

$$\mathbf{F}_{MN} \;=\; \partial_M \mathbf{A}_N \;-\; \partial_N \mathbf{A}_M \;+\; [\mathbf{A}_M,\; \mathbf{A}_N] \;=$$

$$F^{ab}_{MN}\; Z_{ab} \;+ i\; (F^a_{MN}\; Z_a \;+\; \bar F^a_{MN}\; \bar Z_a) \;+\; F_{MN}\; I \;=$$

$$\frac{i}{2}\; F^{(ab)}_{MN}\; \mathcal{M}_{ab} \;-\frac{i}{2}\; F^{[ab]}_{MN}\; \mathcal{L}_{ab} \;+ i\; (F^a_{MN}\; Z_a \;+\; \bar F^a_{MN}\; \bar Z_a) \;+\; F_{MN}\; I \tag{3.17}$$

after decomposing $Z_{ab} \;=\; \frac{1}{2}(M_{ab} - iL_{ab})$. The components of the curvature two-form associated with the anti-Hermitian connection $\Omega^{ab}_M \;=\; \Omega^{[ab]}_M + i\Omega^{(ab)}_M$ are

$$-i\; F^{[ab]}_{MN} \;=\; \partial_M \Omega^{[ab]}_N \;-\; \partial_N \Omega^{[ab]}_M \;+\; \Omega^{[ac]}_{[M}\; \Omega^{[cb]}_{N]} \;-$$

$$\Omega^{(ac)}_{[M}\; \Omega^{(cb)}_{N]} \;+ \frac{1}{L_P^2}\; E^a_{[M}\; E^b_{N]} \;+ \frac{1}{L_P^2}\; \bar E^a_{[M}\; \bar E^b_{N]}\;. \tag{3.18}$$

$$i\, F^{(ab)}_{MN} \;=\; \partial_M \Omega^{(ab)}_N \;-\; \partial_N \Omega^{(ab)}_M \;+\; \Omega^{(ac)}_{[M}\; \Omega^{[cb]}_{N]} \;+\; \Omega^{(bc)}_{[M}\; \Omega^{[ca]}_{N]} \;+$$

$$\frac{1}{L_P^2}\; E^a_{[M}\; \bar E^b_{N]} \;+ \frac{1}{L_P^2}\; E^b_{[M}\; \bar E^a_{N]} \tag{3.19}$$

where a summation over the repeated c indices is implied and $[MN]$ denotes the anti-symmetrization of indices with weight one. Notice the presence of the *extra* terms EE in the above expressions for the *deformed* field strengths due to the *noncommutative* $[Z_a, Z_b] \neq 0$, and which in turn, *modifies* the Weyl-Heisenberg algebra due to the Jacobi identities. In the *undeformed* ordinary Quaplectic-algebra case these terms are *absent* because $[Z_a, Z_b] = 0, ...$ and, furthermore, there is *no* M_{ab} term in the ordinary Weyl-Heisenberg algebra. These extra terms $E^a \wedge E^b, ...$ in eqs-(3.18,3.19) are one of the hallmarks of the *deformed* Quaplectic gauge field theory formulation of the *deformed* Born's Reciprocal Complex Gravity.

The components of the torsion two-form are

$$F^a_{MN} \;=\; \partial_M E^a_N \;-\; \partial_N E^a_M \;-i\; \Omega^{[ac]}_{[M}\; E^c_{N]} \;+i\; \Omega^{(ac)}_{[M}\; \bar E^c_{N]} \;-2i\; \bar E^a_{[M}\; \Omega_{N]}. \tag{3.20a}$$

$$\bar F^a_{MN} \;=\; \partial_M \bar E^a_N \;-\; \partial_N \bar E^a_M \;+i\; \Omega^{[ac]}_{[M}\; \bar E^c_{N]} \;-i\; \Omega^{(ac)}_{[M}\; E^c_{N]} \;+2i\; E^a_{[M}\; \Omega_{N]}. \tag{3.20b}$$

The remaining field strength has roughly the same form as a $U(1)$ field strength in noncommutative spaces due to the additional contribution of $B_{\mu\nu}$ resulting from the nonabelian nature of the Weyl-Heisenberg algebra in the internal space (fibers) and which is reminiscent of the noncommutativity of the coordinates with the momentum :

$$F_{MN} \;=\; i\; \partial_M \Omega_N \;-\; i\; \partial_N \Omega_M \;+\; \frac{1}{L_P^2}\; E^a_M\; \bar E^b_N\; \eta_{ab} \;-\; \frac{1}{L_P^2}\; \bar E^a_M\; E^b_N\; \eta_{ab} \;=$$

$$i\; \partial_M \Omega_N \;-i\; \partial_N \Omega_M \;+\; \frac{1}{L_P^2}\; (\; G_{MN} \;-\; G_{NM}\;) \;=\; i\; \Omega_{[MN]} \;+i\; \frac{2}{L_P^2}\; G_{[MN]} \tag{3.21}$$

after recurring to the commutation relations (for $\alpha_h = 1$) in eqs-(3.8,3.9) and the Hermitian property of the metric

$$G_{MN} \;=\; \bar E^a_M\; E^b_N\; \eta_{ab} \;=\; [\; \eta_{ab}\; \bar E^b_N\; E^a_M\;]^* \;=\; (G_{NM})^* \;\Rightarrow\; (G_{MN})^* \;=\; G_{NM}. \tag{3.22}$$

where $*$ stands for (bar) complex conjugation.

The curvature tensor is defined in terms of the anti-Hermitian connection $\Omega^{[ab]}_M + i\; \Omega^{(ab)}_N$ as

$$\mathcal{R}^Q_{MNP} \;\equiv\; \frac{1}{4}\; (\; F^{[ab]}_{MN} \;+\; i\; F^{(ab)}_{MN}\;)\; (E^Q_a\; E_{bP} \;+\; \bar E^Q_a\; \bar E_{bP} \;+\; E^Q_a\; \bar E_{bP} \;+\; \bar E^Q_a\; E_{bP}). \tag{3.23}$$

where the explicit components $F^{[ab]}_{MN}$ and $F^{(ab)}_{MN}$ can be read from the defining relations (3.18, 3.19). Note that both values of values of $F^{[ab]}_{MN}$ and $F^{(ab)}_{MN}$ are purely *imaginary* such that one may rewrite the complex-valued F^{ab}_{MN} field strength as $(\mathcal{F}^{(ab)}_{MN} + i\mathcal{F}^{[ab]}_{MN})$ for real valued $\mathcal{F}^{(ab)}_{MN}$, $\mathcal{F}^{[ab]}_{MN}$ expressions. The contraction of indices yields *two* different complex-valued (Hermitian) Ricci tensors.

$$\mathcal{R}_{MP} \ = \ g^{KN}\, g_{QK}\, R^{Q}_{MNP} \ = \ \delta^{N}_{Q}\, R^{Q}_{MNP} \ = \ R_{(MP)} + i\, R_{[MP]}; \quad (\mathcal{R}_{MP})^* \ = \ \mathcal{R}_{PM} \tag{3.24}$$

and

$$\mathcal{S}_{M\lambda} \ = \ g^{KN}\, g_{KQ}\, R^{Q}_{MNP} \ = \ \mathcal{S}_{(MP)} \ + \ i\, \mathcal{S}_{[MP]}; \quad (\mathcal{S}_{MP})^* \ = \ \mathcal{S}_{PM} \tag{3.25}$$

due to the fact that

$$g^{KN}\, g_{QK} \ = \ \delta^{N}_{Q} \ \ and \ \ g^{KN}\, g_{KQ} \ \neq \ \delta^{N}_{Q}. \tag{3.26}$$

because $g_{KQ} \neq g_{QK}$. The position of the indices is crucial. There is a third Ricci tensor $Q_{[MN]} = \mathcal{R}^{Q}_{MNP}\delta^{P}_{Q}$ related to the curl of the *nonmetricity* Weyl vector Q_M [19] which one may set to zero. However, in the most general case one should include *nonmetricity*.

A further contraction yields the generalized (real-valued) Ricci scalars

$$\mathcal{R} \ = \ (g^{(MP)} \ + i\ g^{[MP]})\ (\ R_{(MP)} \ + \ i\ R_{[MP]}\ ) =$$

$$\mathcal{R} \ = \ g^{(MP)}\ R_{(MP)} \ - \ B^{MP}\ R_{[MP]}; \quad g^{[MP]} \equiv B^{MP}. \tag{3.27a}$$

$$\mathcal{S} \ = \ (g^{(MP)} \ + i\ g^{[MP]})\ (\ S_{(MP)} \ + \ i\ S_{[MP]}\ ) =$$

$$\mathcal{S} \ = \ g^{(MP)}\ \mathcal{S}_{(MP)} \ - \ B^{MP}\ \mathcal{S}_{[MP]}. \tag{3.27b}$$

The first term $g^{(MP)}\ R_{(MP)}$ corresponds to the usual scalar curvature of the ordinary Riemannian geometry. The presence of the extra terms $B^{MP}\ R_{[MP]}$ and $B^{MP}\ \mathcal{S}_{[MP]}$ due to the anti-symmetric components of the metric and the two different types of Ricci tensors are one of the hallmarks of the deformed Born complex gravity. We should notice that the inverse complex metric is

$$g^{(MP)} \ + \ ig^{[MP]} \ = \ [\ g_{(MP)} \ + \ ig_{[MP]}\ ]^{-1} \ \neq \ (g_{(MP)})^{-1} \ + \ (ig_{[MP]})^{-1}. \tag{3.28}$$

so $g^{(MP)}$ is now a complicated expression of both $g_{(MP)}$ *and* $g_{[MP]} = B_{MP}$. The same occurs with $g^{[MP]} = B^{MP}$. Rigorously we should have used a different notation for the inverse metric $\tilde{g}^{(MP)} \ + \ i\tilde{B}^{[MP]}$, but for notational simplicity we chose to drop the tilde symbol.

One could add an *extra* contribution to the complex-gravity real-valued action stemming from the terms $iB^{MP}F_{MP}$ which is very reminiscent of the BF terms in Schwarz Topological field theory and in Plebanksi's formulation of gravity. In the most general case, one must include both the contributions from the torsion and the $i\ B^{MP}F_{MP}$ terms. The contractions involving $G^{MP} = g^{(MP)} + iB^{MP}$ with the components F_{MP} (due to the antisymmetry property of $F_{MP} = -F_{PM}$) lead to

$$i\ B^{MP}\ F_{MP}\ =\ -\ B^{MP}\ (\ \partial_M\Omega_P\ -\ \partial_P\Omega_M\ )\ -\ 2\ B^{MP}\ B_{MP}\ =$$
$$-\ B^{MP}\ \Omega_{MP}\ -2\ B^{MP}\ B_{MP}\ . \tag{3.29}$$

where we have set the length scale $L_P = 1$ for convenience. These BF terms contain a mass-like term for the B_{MP} field. When the torsion is not constrained to vanish one must include those contributions as well. The real-valued torsion two-form is $(F^a_{MN}Z_a + \bar{F}^a_{MN}\bar{Z}_a)dY^M \wedge dY^N$ and the torsion tensor and torsion vector are

$$T^P_{MN}\ =\ F^a_{MN}\ E^P_a;\quad \bar{T}^P_{MN}\ =\ \bar{E}^P_a\ \bar{F}^a_{MN}\ ;\quad T_{MNP}\ =\ g_{PQ}\ T^Q_{MN};$$

$$\bar{T}_{MNP}\ =\ \bar{T}^Q_{MN}\ (g_{PQ})^*\ =\ \bar{T}^Q_{MN}\ g_{QP};\quad T_M\ =\ \delta^N_P\ T^P_{MN};\quad \bar{T}_M\ =\ \bar{T}^P_{MN}\ \delta^N_P. \tag{3.30}$$

The (real-valued) action, linear in the two (real-valued) Ricci curvature scalars and quadratic in the torsion is of the form

$$\frac{1}{2\kappa_2^2}\int_{\Omega_8} d^8Y\ \sqrt{|\ det\ (g_{(MN)} + iB_{MN})\ |}\ (\ a_1\ \mathcal{R} + a_2\ \mathcal{S} + a_3\ T_{MNQ}\ T^{MNQ} + a_4\ T_M\ T^M + c.c). \tag{3.31}$$

$$|\ det\ (g_{(MN)} + iB_{MN})\ |\ =\ \sqrt{\ det\ (g_{(MN)} + iB_{MN})\ det\ (g_{(MN)} - iB_{MN})} \tag{3.32}$$

where one must add the complex conjugate (cc) terms in order to have a real-valued action. κ_2^2 is a suitable coupling introduced to render the action dimensionless. The action (3.31) is invariant under infinitesimal $U(1,3)$ gauge transformations of the complex tetrad $\delta E^a_\mu = (\xi^a_{b(1)} + i\xi^a_{b(2)})E^b_\mu$ where the real $\xi^{(1)}_{[ab]}$ and imaginary $\xi^{(2)}_{(ab)}$ components of the complex parameter are anti-symmetric and symmetric, respectively, with respect to the indices a, b for anti-Hermitian infinitesimal $U(1,3)$ gauge transformations.

The a_1, a_2, a_3, a_4 are suitable numerical coefficients that will be constrained to have certain values if one wishes to avoid the presence of ghosts, tachyons and higher order poles in the propagator, not unlike it occurs in Moffat's nonsymmetric gravity theory [19]. The instabilities of Moffat's nonsymmetric gravity found by [20] are bypassed when one extends the theory to spacetimes with *complex* coordinates [21]. The $8D$ real-dim phase space can be realized as a $4D$ complex-dimensional space endowed with a symplectic and complex structure.

To conclude, the complex deformed Born Reciprocal Gravitational theory advanced here $differs$ from the modified gravitational theories in the literature [19], [21], [18], and it is mainly due to the fact that we have constructed a deformed complex Born's reciprocal gravitational theory in $4D$ as a gauge theory of the $deformed$ Quaplectic group given by the semidirect product of $U(1,3)$

with the *deformed* (noncommutative) Weyl-Heisenberg algebra of eqs-(3.8, 3.9). Finally, gravitational theories based on Born's reciprocal relativity principle involving a maximal speed limit and a maximal proper force, is a very promising avenue to quantize gravity that does *not* rely in *breaking* the Lorentz symmetry at the Planck scale, in contrast to other approaches based on deformations of the Poincare algebra, Hopf algebras, quantum groups, etc...

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