# Why the Colombeau Algebras Cannot Handle Arbitrary Lie Groups ?

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*Dedicated to Marie-Louise Nykamp*

**Abstract**

It is briefly shown that, due to the growth conditions in their definition, the Colombeau algebras cannot handle arbitrary Lie groups, and in particular, cannot allow the formulation, let alone, solution of Hilbert's Fifth Problem.

"History is written with the feet ..."

Ex-Chairman Mao, of the Long March fame ...

Science is not done scientifically, since it is mostly done by non-scientists ...

Anonymous

A "mathematical problem" ?
For sometime by now, American mathematicians

have decided to hide their date of birth
and not to mention it in their academic CV-s.
Why ?
Amusingly, Hollywood actors and actresses have their
birth date easily available on Wikipedia.
Can one, therefore, trust American
mathematicians ?
Why are they so blatantly against transparency ?
By the way, Hollywood movies have also for long
been hiding the date of their production ...

A bemused non-American mathematician

## 1. Global Lie Group Actions on PDEs : the Parametric Method

The advantages of being able to define *global* actions for arbitrary Lie groups are well known for at least six decades by now, as presented systematically in the celebrated text of Chevalley, [2], for instance.
Yet, even in the case of Lie groups acting on Euclidean spaces, and not on manifolds in general, the customary approach has not been able to go beyond a mere local definition, when it comes to actions on functions by arbitrary Lie groups, see for instance [1,6,11-13].

Rather surprisingly, this failure to define globally the action on functions of arbitrary Lie groups is due to an elementary difficulty, which can easily be overcome by a *parametric* definition of functions, as shown for the first time in [21], see also [22,24,25].

This parametric approach proves to have in fact two important advantages, namely, one of *calculus*, and the other of *functorial* nature. The calculus advantage relates to the simple and well known fact that the partial derivatives of any order of a parametrically given function can be computed from it, without first having to bring the function to the usual, nonparametric form. The functorial advantage, relating perhaps even to a simpler fact, is the one which will actually allow the most easy, direct and natural global definition of arbitrary Lie group actions on functions. In fact, as shown in [21,22,24,25] and mentioned in the sequel, it allows as well for the equally easy global definition of a far larger class of Lie *semigroup* actions.

As a general remark about the parametric approach to the global definition of arbitrary Lie group actions on functions, it is rather ironic to note that, in an embryonic, partial and *local* manner, this approach has in fact been in use for a long time by now.

Indeed, suppose given a smooth function $f : \Omega \longrightarrow \mathbb{R}$, with $\Omega \subseteq \mathbb{R}^n$ nonvoid, open. Further, suppose given an arbitrary Lie group $G$ acting on $M = \Omega \times \mathbb{R}$ according to

$$G \times M \longrightarrow M$$

Then the usual way this Lie group action on $M$ is extended to such functions $f$, and thus to $\mathcal{C}^\infty(\Omega, \mathbb{R})$, is as follows. We consider the graph of $f$, that is, the set

$$\gamma_f = \{ \ (x, f(x)) \mid x \in \Omega \ \} \subseteq M$$

Therefore, for any $g \in G$, we can define point-wise the action $g\gamma_f$ and obtain again a subset of $M$.

Unfortunately however, in general, it will *not* be true that

$$g\gamma_f = \gamma_h$$

for a certain smooth function $h : \Omega \longrightarrow \mathbb{R}$, which function $h$ if it existed, it would obviously correspond to the *global* action of $g$ on $f$, that is, we would have

$$gf = h$$

And then, the usual way to define arbitrary Lie group actions on functions overcomes this difficulty at the cost of no less than a *double localization*, [11-13], namely

- $g$ is *restricted* to a neighbourhood of the identity $e \in G$, and in addition

- $f$ is *restricted* to suitable nonvoid, open subsets $\triangle$ of $\Omega$.

It is clear, however, that the consideration of the graph $\gamma_f$ of $f$ amounts to replacing $f : \Omega \longrightarrow \mathbb{R}$ by the following special *parametric* form of it, see (3.3), (3.4) in the sequel, namely $f_* : \Omega \longrightarrow M$, where $\Omega \ni x \longmapsto f_*(x) = (x, f(x)) \in M$. Furthermore, in this case $g\gamma_f$ is nothing else but $gf_*$, that is, the action of $g$ on $f_*$, which can *always* be defined globally, irrespective of the function $f$, or of the Lie group action $G$ on $M$.

Thus it becomes clear that the *only* difficulty we have ever faced when trying to define globally arbitrary Lie group actions on functions is *not* at all related to Lie groups or functions, but solely to our rather unformulated, and yet quite implacable *intent* to have $gf_*$ retranslated into a usual, *nonparametric* function $h : \Omega \longrightarrow \mathbb{R}$.

On the other hand, the parametric approach to Lie group actions introduced in [21,22,24,25], is adopted and pursued in its *full* extent, that is, without any sort of localization, this being the simple and fundamental reason for the fact that *arbitrary* Lie group actions can be defined *globally* on smooth functions.
Furthermore, as shown in [21,22,24,25], this possibility to define globally arbitrary Lie group actions on smooth functions can easily be extended to actions on large classes of generalized functions, and in particular, distributions, one of the effects of such an extension being the first general solution of Hilbert's Fifth Problem, [21,22].
Also as mentioned and shown briefly in the sequel, one can define globally on functions the action of far larger classes of Lie semigroups. This comes as a rather unexpected *bonus*, and the effect of the mentioned functorial nature of the parametric approach to Lie group actions which allows the definition of arbitrary smooth - thus typically *noninvertible* - actions. Such noninvertible actions can, of course, no longer belong to Lie group actions, but only to Lie *semigroup* actions, [21,22,24,25].

Let us mention here in passing that the interest in such Lie semigroups of actions comes from the fact that they range over a significantly

*larger* class of actions than those corresponding to Lie groups. Therefore, when applied to the study of solutions of PDEs - this time as *semisymmetries* - they can offer new additional insights.
Furthermore, as pointed out by P J Olver, semigroups of actions appear quite naturally in several aspects of the classical Lie theory, see for details [21, chap. 13], [???].

Let us briefly illustrate the essence of difficulties with Lie Group actions on usual functions.

**Classical Lie Group Actions**. For convenience, let us consider the familiar and important setup when Lie group actions are used in the study of PDEs. In such cases, we are given a linear or nonlinear PDEs of the general form

$$(1.1) \quad T(x, D)U(x) = 0, \quad x \in \Omega \subseteq \mathbb{R}^n$$

where $\Omega$ is nonvoid open, $U : \Omega \longrightarrow \mathbb{R}$ is the unknown function, while $T(x, D)$ is a $\mathcal{C}^\infty$-smooth linear or nonlinear partial differential operator. The relevant Lie groups $G$ act on the open subset $M = \Omega \times \mathbb{R} \subseteq \mathbb{R}^{n+1}$, according to

$$(1.2) \quad G \times M \ni (g, (x, u)) \longmapsto g(x, u) = (g_1(x, u), g_2(x, u)) \in M$$

where $x \in \Omega, \ u \in \mathbb{R}$ are the independent and dependent variables, respectively, and

$$(1.3) \quad \begin{array}{l} G \times M \ni \ (g, (x, u)) \longmapsto \ g_1(x, u) \in \Omega \\ G \times M \ni \ (g, (x, u)) \longmapsto \ g_2(x, u) \in \mathbb{R} \end{array}$$

with $g_1$ and $g_2$ being $\mathcal{C}^\infty$-smooth.

We note that, given $g \in G$, in view of the Lie group axioms, it follows that the mapping

$$(1.4) \quad M \ni (x, u) \stackrel{g}{\longmapsto} g(x, u) \in M$$

is a $\mathcal{C}^\infty$-smooth diffeomorphism.

A first basic problem in Lie group theory, when applied to PDEs, is how to *extend* the action in (1.2), (1.3) of the Lie group $G$ on the open subset $M$, to an action of $G$ on the $\mathcal{C}^\infty$-smooth functions

$$(1.5) \quad U : \Omega \longrightarrow \mathbb{R}$$

or more generally, on $\mathcal{C}^\infty$-smooth functions

$$(1.6) \quad U : \triangle \longrightarrow \mathbb{R}$$

where $\triangle \subseteq \Omega$ is nonvoid, open. And unless one solves this problem, one simply cannot speak about the Lie group invariance of classical solutions of PDEs.

From this point of view, the Lie group actions (1.2), (1.3) are divided in two types, [11,12].

The simpler ones, called *projectable*, or *fibre preserving*, satisfy the condition, see (1.3)

$$(1.7) \quad g_1(x,u) \;=\; g_1(x), \quad g \in G,\ (x,u) \in M$$

The special interest in Lie group actions (1.7) comes from the fact that they allow an easy *global* extension to action on $\mathcal{C}^\infty$-smooth functions. Indeed, in this case, in view of (1.4), it follows that for $g \in G$, we obtain the $\mathcal{C}^\infty$-smooth diffeomorphism

$$(1.8) \quad \Omega \ni x \stackrel{g_1}{\longmapsto} g_1(x) \in \Omega$$

Now, given $g \in G$ and $U$ in (1.6), it is easy to define the respective global Lie group action

$$(1.9) \quad g\,U \;=\; \widetilde{U} : \widetilde{\triangle} \;=\; g_1(\triangle) \longrightarrow \mathbb{R}$$

by

$$(1.10) \quad \widetilde{U}(g_1(x)) \;=\; g_2(x, U(x)), \quad x \in \triangle$$

Indeed, (1.4) implies that in (1.9), we have $\widetilde{\triangle} \subseteq \Omega$ nonvoid, open, while (1.10) is equivalent with

$$(1.11) \quad \widetilde{U}(\widetilde{x}) \; = \; g_2(g_1^{-1}(\widetilde{x}), U(g_1^{-1}(\widetilde{x}))), \quad \widetilde{x} \in \widetilde{\triangle}$$

However, an arbitrary Lie group action (1.2), (1.3) need *not* be projectable. And in such a case the global extension of the Lie group action (1.2), (1.3) to $\mathcal{C}^\infty$-smooth functions (1.5), or in general (1.6), will typically *fail*. In this way, we are obliged, [11,12], to limit ourselves to *local* Lie group actions on functions, and thus return to the pre-Chevalley stage of Lie group theory.

Indeed, in the case of general, nonprojectable Lie group actions (1.2), (1.3), we may immediately run into the problem of possible *noninvertibility*. Namely, certain $\mathcal{C}^\infty$-smooth mappings involved in the definition of the group action $g\ U = \widetilde{U} : \widetilde{\triangle} \longrightarrow \mathbb{R}$ may fail to have inverses, let alone, $\mathcal{C}^\infty$-smooth ones. Let us illustrate this phenomenon in more detail. Given $g \in G$, let us write (1.3) in the form

$$(1.12) \quad \begin{array}{rcl} \widetilde{x} & = & g_1(x, u) \\ \widetilde{u} & = & g_2(x, u) \end{array}$$

where $(x, u)$, $(\widetilde{x}, \widetilde{u}) \in M$. Given now $U : \triangle \longrightarrow \mathbb{R}$ as in (1.6), the natural way to define the group action $gU = \widetilde{U} : \widetilde{\triangle} \longrightarrow \mathbb{R}$ would be by the relation, see (1.12)

$$(1.13) \quad \widetilde{U}(g_1(x, U(x))) \; = \; g_2(x, U(x)), \quad x \in \triangle$$

which means that $\widetilde{U}(\widetilde{x}) = \widetilde{u}$. However, in order that (1.13) be a correct definition, we have to be able to obtain $x \in \triangle$ as a $\mathcal{C}^\infty$-smooth function of $\widetilde{x} \in \widetilde{\triangle}$, by using the first equation in

$$(1.14) \quad \begin{array}{rcl} \widetilde{x} & = & g_1(x, U(x)) \\ \widetilde{u} & = & g_2(x, U(x)) \end{array}$$

and thus by replacing $x \in \triangle$ in the second equation above, in order to obtain $\widetilde{u}$ as a function of $\widetilde{x}$, that is, the relation (1.13). Furthermore, one also has to obtain $\widetilde{\triangle} \subseteq \Omega$ as being nonvoid, open. The crucial issue here is, therefore, the $\mathcal{C}^\infty$-*smooth invertibility* of the mapping

$$(1.15) \quad \triangle \ni x \stackrel{\alpha}{\longmapsto} g_1(x, U(x)) \in \Omega$$

which obviously depends on $g$ and $U$. And as seen in the very simple example next, this in general is not possible.

**Example 2.1.**

Let us consider the following *nonprojectable* case of the Lie group action (1.2), (1.3), where $\Omega = \mathbb{R}$, $M = \Omega \times \mathbb{R} = \mathbb{R}^2$, $G = (\mathbb{R}, +)$, and for $\epsilon = g \in G = \mathbb{R}$, $(x, u) \in M$, we have

$$\begin{aligned} \widetilde{x} &= x + \epsilon u^2 \\ \widetilde{u} &= u \end{aligned}$$

Let us take $\triangle = \Omega = \mathbb{R}$ and the simple function $U : \triangle \longrightarrow \mathbb{R}$ defined by $U(x) = x$, with $x \in \triangle$. Then (1.15) becomes

$$\mathbb{R} \ni x \stackrel{\alpha}{\longmapsto} x + \epsilon x^2 \in \mathbb{R}$$

which is *not* invertible as a function, let alone as a $\mathcal{C}^\infty$-smooth function, except for the trivial group action corresponding to $\epsilon = 0$, that is, to the identical group transformation.

□

The usual way to deal with this situation, [11,12], is to consider the group action (1.2), (1.3) as well as the mapping $\alpha$ in (1.15), and therefore the function to be acted upon $U : \triangle \longrightarrow \mathbb{R}$, *only locally*, that is, to restrict all of them to such suitable neighbourhoods of the neutral element $e \in G$, as well as of points $x \in \triangle$, on which $\alpha$ is $\mathcal{C}^\infty$-smooth invertible.

It is useful to note however that, depending also on the function $U$ in (1.6), the mapping $\alpha$ in (1.15) can sometime happen to have a global, and not only local $\mathcal{C}^\infty$-smooth inverse, even in the case of a nonprojectable Lie group action. For instance, this happens if in the above Example 2.1., we consider $\widetilde{x} = x + \epsilon u$.

Let us mention what happens when the mapping $\alpha$ in (1.15) is invertible, regardless of the Lie group action being projectable or not, and when its inverse $\alpha^{-1}$ is also a $\mathcal{C}^\infty$-smooth mapping. Then we can indeed turn to (1.13) in order to define the group action $g\, U = \widetilde{U}$ by

$$(1.16) \quad \widetilde{U}\,(\widetilde{x}) \;=\; g_2(\alpha^{-1}(\widetilde{x}), U(\alpha^{-1}(\widetilde{x}))), \quad \widetilde{x} \in \widetilde{\triangle}$$

where

$$(1.17) \quad \widetilde{\triangle} \;=\; \alpha(\triangle) \;\text{ is open}$$

Obviously, the case of *projectable* Lie group actions in (1.7) - (1.11) is included in (1.16), (1.17).

As mentioned in the Introduction, here, following [21,22], we take a *new route*, when dealing with the difficulties in (1.12) - (1.15), which we face in the case of general, *nonprojectable* Lie group actions (1.2), (1.3). This new route will *not* require the above mentioned traditional localisation of $g \in G$, $\alpha$ or $U$. In other words, we are able to perform *globally* arbitrary Lie group actions on functions $U$ defined on the whole of their unrestricted, original domains, as for instance in (1.5) and (1.6). Fortunately, this construction is particularly simple and applicable without any undue restrictions.

**A Simple, Basic Observation**. To summarize. The basis upon which we can delvelop this global approach is the following rather simple observation :

- The usual impediment which prevents us from extending arbitrary Lie group actions (1.2), (1.3) to global actions on functions (1.5) or (1.6) is *not* at all related to Lie groups, but to the usual way of representing functions, by discriminating between independent and dependent variables. Once one does away with such a discrimination, by using a parametric representation of functions, the way to a natural and easy global Lie group action on functions is open.

**Parametrisation** in its essence amounts to the following *embedding* of the usual definition of a function into a larger concept. Namely, a usual function

$$(1.18) \quad A \ni x \overset{f}{\longmapsto} y \;=\; f(x) \in B$$

is actually *constrained* to be a correspondence from the set $A$ of its independent variable $x$, to the set $B$ of its dependent variable $y$.

On the other hand, a parametric representation of $f$ can be given by any *pull-back* type mapping

$$(1.19) \quad P \ni p \overset{h}{\longmapsto} h(p) \;=\; (x(p), y(p)) \in A \times B$$

which maps any suitably given parameter domain $P$ into the graph of $f$, under the following two conditions :

$$(1.20) \quad y(p) \;=\; f(x(p)), \quad p \in P$$

and

$$(1.21) \quad P \ni p \longmapsto x(p) \in A \;\text{ is surjective}$$

With respect to $P$, this, in general, only implies that its cardinal is not smaller than that of $A$.

However, when dealing with Lie group actions, the parameter domain $P$ is required to be a suitable open subset in an Euclidean space, while the parametrisation $h$ is assumed $\mathcal{C}^\infty$-smooth.

It follows that, in general, a parametric representation will introduce an *additional* variable $p$, ranging over $P$, which this time is mapped into the pair $(x(p), y(p))$ of the original independent and dependent variables, pair which is an element in the cartesian product $A \times B$.

This kind of embedding, obtained by introducing an additional variable, and thus going beyond the constraint of only dealing with the usual independent and dependent variables, proves to have an important and naturally built in advantage. Namely, it allows for the first time - and in a straightforward manner - the *global* definition of arbitrary Lie group actions on functions.

In the usual, that is, nonparametric approach, however, when one wanst to define the Lie group action on a function, and obtain again a function, one cannot in general do so, unless at the end one is able to *separate* the independent and dependent variables, by expressing the latter as a function of the former. And in the nonprojectable case of Lie group actions, this typically is not possible, except locally in the independent variable, and also, near to the trivial, identical Lie group transformation.

On the other hand, if one starts, and ends, with parametrically given functions, then as shown in [21,22] and seen in the sequel, one has no difficulties at all.

## 2. A Solution of Hilbert's Fifth Problem

In [21] a complete solution to Hilbert's Fifth Problem was obtained for the first time in the literature.

## 3. The Growth Conditions in Colombeau Algebras Cannot Handle Arbitrary Lie Groups

Let us briefly recall the way *growth conditions* are essential in defining the Colomebau algebras [3,4,14]. For simplicity, we shall consider the case of the domains $\Omega = \mathbb{R}^n$, and on them, of the general Colombeau algebras first introduced in [3]. Their construction starts with the auxiliary family of smooth functions

$$(3.1) \quad \Phi_m(\Omega) = \left\{ \phi \in \mathcal{D}(\Omega) \;\middle|\; \begin{array}{ll} (i) & \int_\Omega \phi(x)dx = 1 \\ \\ (ii) & \int_\Omega x^p \phi(x)dx = 0, \qquad p \in \mathbb{N}^n, 1 \leq |p| \leq m \end{array} \right\}$$

Further, for $\epsilon > 0$ and $\phi \in \mathcal{D}(\Omega)$, we define $\phi_\epsilon \in \mathcal{D}(\Omega)$ by

$$(3.2) \qquad \phi_\epsilon(x) = \phi(x/\epsilon)/\epsilon^n, \qquad x \in \Omega$$

Now, our basic space of function will be

$$(3.2) \qquad \mathcal{E}(\Omega) = (\mathcal{C}^\infty(\Omega))^{\Phi(\Omega)}$$

which is obviously a *differential algebra* with the term-wise operations.

The general Colombeau algebra on $\Omega = \mathbb{R}^n$ is constructed in three steps.

First, we consider the *differential subalgebra* $\mathcal{A}(\Omega)$ in $\mathcal{E}(\Omega)$, given by all the functions $f \in \mathcal{E}(\Omega)$ which satisfy the *growth condition*

$$(3.3) \qquad \begin{array}{l} \forall \ \text{compact } K \subseteq \Omega, \ p \in \mathbb{N}^n \ : \\ \exists \ m \in \mathbb{N}, \ m \geq 1 \ : \\ \forall \ \phi \in \Phi_m(\Omega) \ : \\ \exists \ \eta, c > 0 \ : \\ \forall \ x \in K, \ \epsilon \in (0, \eta) \ : \\ \qquad |\, D^p f(\phi_\epsilon, x) \,| \leq c/\epsilon^m \end{array}$$

Second, we consider in the algebra $\mathcal{A}(\Omega)$ the *ideal* $\mathcal{I}(\Omega)$ given by by all the functions $f \in \mathcal{A}(\Omega)$ which satisfy the *growth condition*

$$
(3.4)\qquad
\begin{aligned}
&\forall \ \text{compact } K \subseteq \Omega, \ p \in \mathbb{N}^n \ : \\
&\exists \ k \in \mathbb{N}, \ k \geq 1, \ \beta \in B \ : \\
&\forall \ m \in \mathbb{N}, \ m \geq k, \ \phi \in \Phi_m(\Omega) \ : \\
&\exists \ \eta, c > 0 \ : \\
&\forall \ x \in K, \ \epsilon \in (0, \eta) \ : \\
&\qquad | \, D^p f(\phi_\epsilon, x) \, | \leq c \, \epsilon^{\beta(m) - k}
\end{aligned}
$$

where

$$
(3.5)\qquad B = \left\{ \beta \in (0, \infty)^{\mathbb{N}} \;\middle|\; \begin{array}{ll} (i) & \beta \text{ is non-decreasing} \\ & \\ (ii) & \lim_{m \to \infty} \beta(m) = \infty \end{array} \right\}
$$

Third, the general Colombeau algebra of generalized functions on $\Omega = \mathbb{R}^n$ is the *quotient algebra*

$$
(3.6)\qquad \mathcal{G}(\Omega) = \mathcal{A}(\Omega) / \mathcal{I}(\Omega)
$$

The reason for the failure of the Colombeau algebras (1.6) in dealing with arbitrary Lie groups which can appear in the study of PDEs becomes now easily obvious. Namely, the algebras $\mathcal{A}(\Omega)$ in (3.3) which are used in the definition (3.6) of the Colombeau algebras do *not* allow arbitrary smooth, and not even arbitrary analytic operations on Colombeau generalized functions. And this is obviously due to the specific *growth conditions* in the definition (3.3) of these algebras $\mathcal{A}(\Omega)$. Indeed, let us consider the following set of *slowly increasing* smooth functions

$$
(3.7)\qquad \mathcal{O}(\mathbb{R}^r) = \left\{ \alpha \in \mathcal{C}^\infty(\mathbb{R}^r) \;\middle|\; \begin{array}{l} \forall \ p \in \mathbb{N}^r \ : \\ \\ \qquad D^p \alpha \ \text{ is slowly increasing} \end{array} \right\}
$$

where a function $\beta \in \mathcal{C}^\infty(\mathbb{R}^r)$ is called *slowly increasing*, if and only if there exist $K, c > 0$, such that

$$(3.8) \qquad |\,\beta(\xi)\,| \leq K(\,1 + |\,\xi\,|\,)^c, \qquad \xi \in \mathbb{R}^r$$

The mentioned limitation regarding smooth operations on Colombeau generalized functions is described by the following result, [3,19], :

Given Colombeau generalized functions $T_1, \ldots, T_m \in \mathcal{G}(\Omega)$ and $\alpha \in \mathcal{O}(\mathbb{R}^{2n})$, then there exists a Colombeau generalized function $\alpha(T_1, \ldots, T_m) \in \mathcal{G}(\Omega)$, and it is defined by

$$(3.9) \qquad \alpha(T_1, \ldots, T_m) = \alpha(f_1, \ldots, f_m) + \mathcal{I}(\Omega) \in \mathcal{G}(\Omega)$$

where

$$(3.10) \qquad T_i = f_i + \mathcal{I}(\Omega) \in \mathcal{G}(\Omega), \qquad 1 \leq i \leq m$$

The problem here clearly is in the fact that, as soon as a smooth nonlinear operation $\alpha$ is no longer in $\mathcal{O}(\mathbb{R}^{2n})$, one cannot in general obtain the growth condition (3.3) being satisfied by $\alpha(f_1, \ldots, f_m)$ in (3.9) for all Colombeau generalized functions $T_1, \ldots, T_m \in \mathcal{G}(\Omega)$.

In view of the above it is obvious that the Colombeau algebras cannot deal with Lie groups which are associated with a variety of large classes of PDEs.

## 4. The Inevitable Infinite Branching in the Multiplication of Singularities

The rather amusing fact, after decades of studies in the nonlinear algebraic theory of generalized functions, see subject 46F30 in the AMS classification, is what appears to be the inability on the part of not a few specialists involved to realize and understand that multiplication of generalized functions does quite *inevitably branches* when faced with dealing with singularities, [18-20,28-30]. And as seen easily, this branching has most simple algebraic, more precisely, ring theoretic reasons.

As it happens, however, realizing the presence and importance of that branching seems not to be so easy, since it has so far eluded several notable mathematicians, as mentioned for instance in [30].

The immediate and most obvious consequence of the mentioned inevitable *infinite* branching is that a *variety* of differential algebras of generalized functions should be considered when, for instance, solving nonlinear PDEs, or studying Lie group actions on solutions of such equations. After all, such an approach is in no way a novelty in the solution of PDEs, as for more than seven decades by now a large variety of Sobolev spaces have been used for such a purpose.

As for the Colombeau algebras, they obviously have a number of convenient properties. Moreover, as stressed in [19,20], their construction has a rather important *natural* feature which, however, is seldom mentioned, let alone used in the literature.
However, as with all mathematical constructs, so with the Colombeau algebras, they manifest clear limitations in certain important situations.

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