Horizons(Causally Disconnected Regions of Spacetime) and Infinite Doppler Blueshifts in both Alcubierre and Natario Warp Drive Spacetimes

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Abstract

Although both Alcubierre and Natario Spacetimes belongs to the same family of Einstein Field Equations of General Relativity and both have many resemblances between each other ,the Energy Density distribution in the Alcubierre Warp Drive is different than the one found in the Natario Warp Drive.The Horizons will arise in both Spacetimes when approaching Superluminal(Warp) speeds however due to a different distribution of Energy Density the Natario Warp Drive behaves slightly different when compared to the Alcubierre one. The major differences between the Natario and Alcubierre Warp Drive Spacetimes occurs when we study the Infinite Doppler Blueshifts that affect the Alcubierre Spacetime but not affect the Natario one because while in Alcubierre Spacetime the Negative Energy is distributed in a toroidal region above and below the ship perpendicular to the direction of the motion while in front of the ship.the space is empty having nothing to prevent a photon to reach the Horizon because in this case the Horizon lies on empty space,in the Natario Spacetime the Energy Density is distributed in a spherical shell that covers the entire ship and a photon sent to the front will be deflected by this shell of Negative Energy before reaching the Horizon because the Horizon also lies inside this shell and not on "empty" space.This shell avoids the occurrence of Infinite Doppler Blueshifts in the Natario Warp Drive Spacetime.We examine in this work the major differences between both Natario and Alcubierre Spacetimes outlining the repulsive character of the Negative Energy Density.The creation of a Warp Bubble in Alcubierre or Natario Spacetimes is beyond the scope of Classical General Relativity and will have to wait until the arrival of a real Quantum Gravity theory that must encompass Superluminal Non-Local Quantum Entanglement Effects in order to deal with the Horizon problem added to the Geometrical features of Classical General Relativity plus it must also provide a way to generate large outputs of Negative Energy Densities.Since this theory is ahead of our scientific capabilities,we discuss in the end of this work an approach that could be performed by our science in a short period of term.to increase our knowledge about the Warp Drive as a Dynamical Spacetime.

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1 Introduction-What exactly is a "Warp Drive"??

The "Warp Drive" as a solution of the Einstein Field Equations of General Relativity that would theoretically "allow" to travel faster than light was discovered by the Mexican mathematician Miguel Alcubierre from Universidad Nacional Autonoma de Mexico(UNAM)in 1994. He in that year published a paper called "The Warp Drive:Hyperfast Travel Within General Relativity". The "Warp Drive" as conceived by Alcubierre was a Bubble of Spacetime with a spaceship inside the Bubble at the rest with its local spacetime feeling no g-forces and no accelerations that otherwise would destroy the ship concerning faster than light velocities and the Bubble would be at the rest with respect to the rest of the Universe.Spacetime behind the Bubble would expand moving away the departure point and Spacetime would contract in the front of the Bubble bringing to the ship the destination point in a way the resembles the Big Bang(or the Big Crunch).A spaceship inside a "Warp Drive" would be able to attain large Superluminal velocities effectively travelling faster than light. Seven years later another paper on the "Warp Drive" appeared. In 2001 the Portuguese mathematician Jose Natario from Instituto Superior Tecnico(IST) conceived a "Warp Drive" that do not expands or contracts.The ship is still immersed in a "Warp Bubble" and this Bubble is carried out by the Spacetime "stream" at faster than light velocities with the ship at the rest with respect to its local neighborhoods inside the Bubble feeling no g-forces and no accelerations. Imagine an aquarium floating in the course of a river with a fish inside it...the walls of the aquarium are the walls of the Warp Bubble...Imagine that this river is a "rapid" and the aquarium is being carried out by the river stream...the aquarium walls do not expand or contract...an observer in the margin of the river would see the aquarium passing by him at an arbitrarily large speed but inside the aquarium the fish would be protected from g-forces or accelerations generated by the stream...because the fish would be at the rest with respect to its local spacetime inside the aquarium.Jose Natario in 2001 wrote the paper called "Warp Drive with Zero Expansion".The Natario Warp Drive is carried out by the Spacetime stream just like a fish in the stream of a river.The Natario Warp Drive is the main theme of this work. In this work we introduce first the Natario Generic Warp Drive Formalism to demonstrate that both Spacetimes are members of the same family of Einstein Field Equations of General Relativity(Section 2).Then we introduce the geometrical features of the Warp Drive With Zero Expansion(Section 3).After this we derive the Natario Continuous Shape Function $n(rs)$ using the Alcubierre Shape Function $f(rs)$ to reinforce the similarities between both Spacetimes(Section 4). We perform also a first study on the Negative Energy Densities in the Natario Warp Drive Spacetime(Section 5). The main scope of this work :The study of the differences between the Energy Densities distributions in both Alcubierre and Natario Spacetimes is given in Section 6.In this section we explore the Horizons and Infinite Doppler Blueshifts.An experienced reader can skip the previous Sections and go directly to Section 6 since the other Sections were written for readers not acquainted with the Natario Warp Drive Spacetime.In the end we present our conclusions with possible future developments.

2 Introducing the Generic Natario Warp Drive Formalism

The Warp Drive Spacetime according to Natario is defined by the following equation(pg 2 in [2])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{1}$$

where X^i is the so-called Shift Vector.This Shift Vector is the responsible for the Warp Drive behavior defined as follows(pg 2 in [2]):

$$X^i = X, Y, Z \curvearrowright i = 1, 2, 3 \tag{2}$$

The Warp Drive spacetime is completely generated by the Natario Vector nX(pg 2 in [2])

$$nX = X^i \frac{\partial}{\partial x^i} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}, \tag{3}$$

Note the difference between the Shift Vector X^i and the Natario Vector $nX = X^i \frac{\partial}{\partial x^i}$
Considering a Natario Warp Drive with motion over the x axis only we would have:

$$ds^2 = dt^2 - (dx^i - X^i dt)^2 \tag{4}$$

$$ds^2 = dt^2 - (dx - Xdt)^2 \tag{5}$$

- a)-coordinate

$$x^i = x \tag{6}$$

- b)-Shift Vector

$$X^i = X \tag{7}$$

- c)-Natario Vector

$$nX = X\frac{\partial}{\partial x} \tag{8}$$

Note that the Shift Vector X is used to define the Natario Vector nX.The relevance of the Natario work can be outlined in the following statement(pg 4 in [2]):

- 1)-Any Natario Vector nX generates a Warp Drive Spacetime if $nX = 0$ for a small value of $|x|$ defined by Natario as the interior of the Warp Bubble and $nX = -vs(t)$ or $nX = vs(t)$for a large value of $|x|$ defined by Natario as the exterior of the Warp Bubble with $vs(t)$ being the speed of the Warp Bubble.Again this encompasses many possible solutions.The Warp Drive is an entire family of solutions of the Einstein Field equations of General Relativity. There are at the present moment two solutions already discovered for the Warp Drive Spacetime:Alcubierre(1994) and Natario(2001)

The vector $|x|$ according to Natario is defined as(pg 4 in [2]):

$$|x| = x^i \frac{\partial}{\partial x^i} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \tag{9}$$

Again considering motion over the x axis only we have:

$$|x| = x^i \frac{\partial}{\partial x^i} = x\frac{\partial}{\partial x} \tag{10}$$

The Alcubierre solution of 1994 for the Warp Drive can be retrieved if we define the Shift Vector X as(pg 3 in [2])(pg 4 in [1])

$$X = vsf(rs) \tag{11}$$

$$Y = Z = 0 \tag{12}$$

$$vs = \frac{dxs}{dt} \tag{13}$$

$$rs = \sqrt{(x - xs)^2 + y^2 + z^2} \tag{14}$$

$$f(rs) = \frac{tanh[@(rs + R)] - tanh[@(rs - R)]}{2tanh(@R)} \tag{15}$$

The $f(rs)$ is the Alcubierre Shape Function with the values $f(rs) = 1$ in the interior of the Warp Bubble and $f(rs) = 0$ in the exterior of the Warp Bubble,xs is the center of the Warp Bubble,R is the radius of the Warp Bubble and @ is the thickness of the Warp Bubble.The Alcubierre Shape Function is very important.We will use it to define later the Natario Shape Function.The Natario Warp Drive with Zero Expansion we will present later is very different than the original Alcubierre Warp Drive with Expansion Contraction but the proof that both Warp Drive Spacetimes are members of the same family of solutions of the Einstein Field Equations of General Relativity is the possibility that the Shape Function defined for one of them can be used to construct the Shape Function for the other.

Considering the Natario Warp Drive Spacetime equation for the x-axis only:

$$ds^2 = dt^2 - (dx - Xdt)^2 \tag{16}$$

we would get the following expressions for the Shift Vector $X = vsf(rs)$:

$$ds^2 = dt^2 - (dx - vsf(rs)dt)^2 \tag{17}$$

Above we retrieved the original Alcubierre Warp Drive solution.The Natario Vector Field nX for the Alcubierre Warp Drive would then be defined as:

$$nX = X\frac{\partial}{\partial x} = vsf(rs)\frac{\partial}{\partial x} \tag{18}$$

The Expansion and Contraction of the Spacetime for the Alcubierre Warp Drive is a consequence of the choice made by Alcubierre in 1994 for the Shift Vector $X = vsf(rs)$ and is defined by the Expansion of the Normal Volume Elements as(pg 3 and 4 in [2])(pg 5 in [1]):

$$\theta = \partial_i X^i \tag{19}$$

$$\theta = \partial_x X = v_s f'(r_s)[\frac{x - x_s}{r_s}] \tag{20}$$

$$\theta = \partial_x X = v_s[\frac{df(rs)}{drs}][\frac{x - x_s}{r_s}] \tag{21}$$

It is easy to see that a different Shift Vector X or a different Shape Function $f(rs)$ would produce a different Expansion of the Normal Volume Elements.Natario in 2001 introduced a different Shift Vector that do not Expands or Contracts the Spacetime in the Warp Bubble.

The Warp Drive as a Dynamical Spacetime requires an amount of energy in order to be generated.The generic expression for the Energy Density for the Warp Drive and the same for the Alcubierre Warp Drive are given below(pg 4 in [2])(pg 8 in [1]):

$$\rho = \frac{1}{16\pi}\left[(\partial_x X)^2 - (\partial_x X)^2 - 2\left(\frac{1}{2}\partial_y X\right)^2 - 2\left(\frac{1}{2}\partial_z X\right)^2\right] \tag{22}$$

$$\rho = -\frac{1}{32\pi}v_s^2\left[f'(r_s)\right]^2[\frac{y^2 + z^2}{r_s^2}] \tag{23}$$

$$\rho = -\frac{1}{32\pi}v_s^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2 + z^2}{r_s^2}] \tag{24}$$

Again note the fact that a different Shift Vector or a different Shape Function would produce a different distribution of the Energy Density.This very important because the Warp Drive as a faster than light solution of the Einstein Field Equations requires enormous amounts of energy to be generated and this energy is negative.This is a critical issue that will perhaps await for a complete theory of Quantum Gravity in order to be solved since this is outside the scope of General Relativity.We will address this later in this work.The Warp Drive as a family of solutions of the Einstein Field Equations can have many Shift Vectors and many Shape Functions.The Warp Drive family can have many members.Some of these members of the family are perhaps impossible to be generated in a real fashion due to the distribution of the Energy Densities but other members of the same family look more feasible to be achieved from a realistic physical point of view.This will be exploited in Section 6.We will now examine the second member of the Warp Drive family introduced by Jose Natario in 2001 :"Warp Drive with Zero Expansion"

3 Warp Drive with Zero Expansion:

In 2001 the Portuguese mathematician Jose Natario from Instituto Superior Tecnico(IST) introduced a new Warp Drive spacetime defined using the Canonical Basis of the Hodge Star in spherical coordinates defined as follows(pg 4 in [2])[1]:

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r \sin\theta d\varphi) \tag{25}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim (r \sin\theta d\varphi) \wedge dr \tag{26}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \tag{27}$$

Applying the Natario equivalence between spherical and cartezian coordinates as shown below(pg 5 in [2])[2]:

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2 \sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{28}$$

Of course we consider here the following pedagogical approach

$$x = r\cos\theta \tag{29}$$

$$dx = d(r\cos\theta) \tag{30}$$

From the result written above we can review the Natario definition for the Warp Drive:

- 1)-A Natario Vector nX being $nX = 0$ for a small value of $|x|$(interior of the Warp Bubble)
- 2)-A Natario Vector $nX = -vs(t)$ or $nX = vs(t)$for a large value of $|x|$(exterior of the Warp Bubble)
- 3)-$vs(t)$-speed of the Warp Bubble seen by distant observers.

The concept of the Warp Drive with Zero Expansion introduced by Jose Natario in 2001 can be defined by the following Natario Vector nX as follows(pg 5 in [2]):

$$nX \sim -v_s(t)dx \tag{31}$$

or

$$nX \sim v_s(t)dx \tag{32}$$

The Natario Vector nX for the motion only in the x axis was defined originally as follows:

$$nX = X\frac{\partial}{\partial x} \tag{33}$$

[1]See Appendix on Hodge Stars and Differential 1-forms and 2-forms

[2]The Mathematical demonstration of this expression will be given in the Appendix on Hodge Stars and Differential Forms

but according to Natario we can use this approximation(pg 5 in [2]):

$$\frac{\partial}{\partial x} \sim dx \tag{34}$$

Hence the Natario Vector nX can be defined as follows:

$$nX = Xdx \tag{35}$$

The Shift Vector X for the Warp Drive with Zero Expansion is defined simply by:

$$X = -vs(t) \tag{36}$$

or by

$$X = vs(t) \tag{37}$$

And for dx we have(pg 5 in [2]) :

$$dx = d\left(\frac{1}{2}r^2 \sin^2\theta d\varphi\right) \tag{38}$$

Inserting all this stuff in the Natario Vector nX we would get the following expression(pg 5 in [2])

$$nX = Xdx = vs(t)d\left(\frac{1}{2}r^2 \sin^2\theta d\varphi\right) \tag{39}$$

Now a little bit of Warp Drive basics:An observer inside the Warp Bubble feel no accelerations and no g-forces if the Warp Bubble moves with a $vs = 200$(two hundred times faster than light) with respect to a distant observer because he is at the rest with respect to his local spacetime inside the Warp Bubble.In this case the Natario Vector is $nX = 0$.On the other hand an external observer faraway from the Warp Bubble sees the Warp Bubble passing by him at $vs = 200$.The Natario Vector in this case is $nX = -vs(t)$ or $nX = vs(t)$.Then we need a dx that is zero in the ship and 1 far from it.Note immediately the resemblances between this and the Alcubierre Shape Function $f(rs) = 1$ in the ship and $f(rs) = 0$ far from it.This is not a coincidence.Both Warp Drives belongs to the same family of solutions of the Einstein Field Equations.

Rewriting the Natario Vector as(pg 5 in [2])

$$nX = vs(t)d\left(\frac{1}{2}r^2 \sin^2\theta d\varphi\right) \tag{40}$$

Note that we replaced $\frac{1}{2}$ by $f(r)$ in order to obtain(pg 5 in [2]):

$$nX = vs(t)d\left(f(r)r^2 \sin^2\theta d\varphi\right) \tag{41}$$

$$nX = -v_s(t)d\left[f(r)r^2 \sin^2\theta d\varphi\right] \sim -2v_s f(r)\cos\theta \mathbf{e}_r + v_s(2f(r) + rf'(r))\sin\theta \mathbf{e}_\theta \tag{42}$$

From now on we will use this pedagogical approach that gives results practically similar the ones depicted in the original Natario Vector shown above[3]

$$nX = -v_s(t)d\left[f(r)r^2 \sin^2\theta d\varphi\right] \sim -2v_s f(r)\cos\theta dr + v_s(2f(r) + rf'(r))r\sin\theta d\theta \tag{43}$$

[3] Again see the Mathematical demonstration of the Natario Vector in the Appendix on Hodge Stars and Differential Forms

In order to make the Natario Warp Drive holds true we need for the Natario Vector nX a continuous Natario Shape Function being $f(r) = \frac{1}{2}$ for large r(outside the Warp Bubble) and $f(r) = 0$ for small r(inside the Warp Bubble) while being $0 <_f (r) < \frac{1}{2}$ in the walls of the Warp Bubble

In order to avoid contusion with the Alcubierre Shape Function $f(rs)$ we will redefine the Natario Shape Function as $n(r)$ and the Natario Vector as shown below

$$nX = -v_s(t)d\left[n(r)r^2\sin^2\theta d\varphi\right] \sim -2v_s n(r)\cos\theta dr + v_s(2n(r) + rn'(r))r\sin\theta d\theta \tag{44}$$

$$nX = -v_s(t)d\left[n(r)r^2\sin^2\theta d\varphi\right] \sim -2v_s n(r)\cos\theta dr + v_s(2n(r) + r[\frac{dn(r)}{dr}])r\sin\theta d\theta \tag{45}$$

Lets analyze the behavior of the Natario Vector nX for the Natario Shape Function $n(r)$ with a Shift Vector $X = -vs(t)$ and a Warp Bubble speed $vs(t) = 200$ with respect to a distant observer.Two hundred times faster than light.

- 1)-Inside the Warp Bubble $n(r) = 0$
- 2)-Outside the Warp Bubble $n(r) = \frac{1}{2}$
- 3)-In the Warp Bubble walls $0 < n(r) < \frac{1}{2}$
- A)-Inside the Warp Bubble $n(r) = 0$

  Inside the Warp Bubble $n(r) = 0$ as a constant value.Then the derivatives of $n(r)$ vanishes and we can rewrite the Natario Vector nX as follows:

$$nX = -2v_s n(r)\cos\theta dr + v_s(2n(r) + r[\frac{dn(r)}{dr}])r\sin\theta d\theta \tag{46}$$

$$nX = -2v_s[0]\cos\theta dr + v_s(2[0])r\sin\theta d\theta \tag{47}$$

$$nX = 0 \tag{48}$$

  No motion at all.The observer inside the Natario Warp Bubble is completely at the rest with respect to its Local Spacetime neighborhoods and this observer dont feel any acceleration or any g-forces..Then for this observer $\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta = 0$.Remember that the Natario Vector is still $nX = -vs(t)dx$.But the Shift Vector is still $X = -vs(t)$.The Natario Warp Bubble still moves with a speed $vs(t) = 200$ with respect to a distant observer.two hundred times faster than light but the internal observer inside the Warp Bubble is completely at the rest and completely in safe from the g-forces that would kill him moving ar such hyper-fast velocities.

- B)-Outside the Warp Bubble $n(r) = \frac{1}{2}$

  Outside the Warp Bubble $n(r) = \frac{1}{2}$ as a constant value.Then the derivatives of $n(r)$ vanishes and we can rewrite the Natario Vector nX as follows:

$$nX \simeq -2v_s n(r)\cos\theta dr + v_s(2n(r) + r[\frac{dn(r)}{dr}])r\sin\theta d\theta \tag{49}$$

$$nX \simeq -2v_s \frac{1}{2}\cos\theta dr + v_s(2\frac{1}{2})r\sin\theta d\theta \tag{50}$$

$$nX \simeq -v_s\cos\theta dr + v_s r\sin\theta d\theta \tag{51}$$

Remember that in this case we have the Natario Vector as still being $nX = -vs(t)dx$ with the Shift Vector defined as $X = -vs(t)$.But now we have $\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \neq 0$.The Natario Vector do not vanishes and an external observer would see the Natario Warp Bubble passing by him at a $vs(t) = 200$ two hundred times faster than light due to the Shift Vector $X = -vs(t)$

- C)-In the Warp Bubble walls $0 < n(r) < \frac{1}{2}$

  This is the region where the walls of the Natario Warp Bubble resides.It is not a good idea to place an observer here because the energy needed to distort the Spacetime generating the Warp Drive is placed in this region.The Natario Vector nX would then be:

$$nX = -v_s(t)d\left[n(r)r^2\sin^2\theta d\varphi\right] \sim -2v_s n(r)\cos\theta dr + v_s(2n(r) + r[\frac{dn(r)}{dr}])r\sin\theta d\theta \tag{52}$$

Now the reader can understand the point of view outlined by Jose Natario in 2001.The Natario Vector $nX = -vs(t)dx = 0$ vanishes inside the Warp Bubble because inside the Warp Bubble there are no motion at all because $dx = 0$ while being $nX = -vs(t)dx \neq 0$ not vanishing outside the Warp Bubble because an external observer sees the Warp Bubble passing by him with a speed defined by the Shift Vector $X = -vs(t)$ or $X = vs(t)$.

Applying the Extrinsic Curvature Tensor to the Shift Vector(pg 2 and 3 in [2]):

$$K_{ij} = \frac{1}{2}\left(\partial_i X^j + \partial_j X^i\right) \tag{53}$$

and equalizing both covariant scripts we get:

$$K_{ii} = \partial_i X^i \tag{54}$$

This is the Expansion of the Normal Volume Elements as defined in the previous Section(pg 3 in [2]):

$$\theta = \partial_i X^i \tag{55}$$

Redefining the Natario Vector nX as being the Rate-Of-Strain Tensor of Fluid Mechanics as shown below(pg 5 in [2]):

$$nX = X^r\mathbf{e}_r + X^\theta\mathbf{e}_\theta + X^\varphi\mathbf{e}_\varphi \tag{56}$$

Applying the Extrinsic Curvature for the Shift Vectors contained in the Natario Vector nX above we would get the following results(pg 5 in [2]):

$$K_{rr}=\frac{\partial X^r}{\partial r}=-2v_s n'(r)\cos\theta \tag{57}$$

$$K_{\theta\theta}=\frac{1}{r}\frac{\partial X^\theta}{\partial\theta}+\frac{X^r}{r}=v_s n'(r)\cos\theta; \tag{58}$$

$$K_{\varphi\varphi}=\frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi}+\frac{X^r}{r}+\frac{X^\theta\cot\theta}{r}=v_s n'(r)\cos\theta \tag{59}$$

$$K_{r\theta}=\frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{X^\theta}{r}\right)+\frac{1}{r}\frac{\partial X^r}{\partial\theta}\right]=v_s\sin\theta\left(n'(r)+\frac{r}{2}n''(r)\right) \tag{60}$$

$$K_{r\varphi}=\frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{X^\varphi}{r}\right)+\frac{1}{r\sin\theta}\frac{\partial X^r}{\partial\varphi}\right]=0 \tag{61}$$

$$K_{\theta\varphi}=\frac{1}{2}\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{X^\varphi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial X^\theta}{\partial\varphi}\right]=0 \tag{62}$$

Examining the first three results we can clearly see that(pg 5 in [2]):

$$\theta=K_{rr}+K_{\theta\theta}+K_{\varphi\varphi}=0 \tag{63}$$

The Expansion of the Normal Volume Elements in the Natario Warp Drive is Zero.

A Warp Drive With Zero Expansion.

The observer is still immersed in the Warp Bubble and this Bubble is carried out by the Spacetime "stream" at faster than light velocities with the observer at the rest with respect to its local neighborhoods inside the Bubble feeling no g-forces and no accelerations. Imagine an aquarium floating in the course of a river with a fish inside it...the walls of the aquarium are the walls of the Warp Bubble...Imagine that this river is a "rapid" and the aquarium is being carried out by the river stream...the aquarium walls do not expand or contract...an observer in the margin of the river would see the aquarium passing by him at an arbitrarily large speed but inside the aquarium the fish would be protected from g-forces or accelerations generated by the stream...because the fish would be at the rest with respect to its local Spacetime inside the aquarium.The Warp Drive is being carried out by the Spacetime "stream" like a fish in the stream of a river due to the resemblances between the Natario Vector nX and the Rate-Of-Strain Tensor of Fluid Mechanics

Spacetime Contraction in one direction(radial) is balanced by the Spacetime Expansion in the remaining direction(perpendicular)(pg 5 in [2]).

The Energy Density in the Natario Warp Drive is given by the following expression(pg 5 in [2]):

$$\rho=-\frac{1}{16\pi}K_{ij}K^{ij}=-\frac{v_s^2}{8\pi}\left[3(n'(r))^2\cos^2\theta+\left(n'(r)+\frac{r}{2}n''(r)\right)^2\sin^2\theta\right]. \tag{64}$$

$$\rho=-\frac{1}{16\pi}K_{ij}K^{ij}=-\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2\cos^2\theta+\left(\frac{dn(r)}{dr}+\frac{r}{2}\frac{d^2n(r)}{dr^2}\right)^2\sin^2\theta\right]. \tag{65}$$

This Energy Density is negative and depends of the Natario Shape Function $n(r)$ or its derivatives..In order to generate the Warp Drive as a Dynamical Spacetime large outputs of energy are needed and (again)this will remain a critical issue that will be solved perhaps by a real theory of Quantum Gravity but everything depends on the form or the behavior of the Natario Shape Function. We will address this later and we will introduce here the Natario Warp Drive Continuous Shape Function $n(r)$ in the next Section.

4 The Natario Warp Drive Continuous Shape Function $n(rs)$

We already know that the Natario Vector nX in the Warp Bubble walls where $0 <= n(r) <= \frac{1}{2}$ is given by:

$$nX = -v_s(t)d\left[n(r)r^2\sin^2\theta d\varphi\right] \sim -2v_s n(r)\cos\theta dr + v_s(2n(r) + r[\frac{dn(r)}{dr}])r\sin\theta d\theta \tag{66}$$

The Warp Bubble walls are the Natario Warped Region where the Negative Energy resides.This region also have a radius r and a thickness @ just like the Alcubierre Warped Region.

Now look again to the original form of the Natario Vector nX shown below:

$$nX = -v_s(t)dx \tag{67}$$

or

$$nX = v_s(t)dx \tag{68}$$

We already know that dx can be defined by:

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2\sin\theta\cos\theta d\theta\wedge d\varphi + r\sin^2\theta dr\wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{69}$$

But replacing r by the term rs we have:

$$\frac{\partial}{\partial x} \sim dx = d(rs\cos\theta) = \cos\theta drs - rs\sin\theta d\theta \sim rs^2\sin\theta\cos\theta d\theta\wedge d\varphi + rs\sin^2\theta drs\wedge d\varphi = d\left(\frac{1}{2}rs^2\sin^2\theta d\varphi\right) \tag{70}$$

Redefining the Natario Vector nX in function of this new quantity rs we have:

$$nX = -v_s(t)d\left[n(rs)rs^2\sin^2\theta d\varphi\right] \sim -2v_s n(rs)\cos\theta drs + v_s(2n(rs) + r[\frac{dn(rs)}{drs}])rs\sin\theta d\theta \tag{71}$$

Of course we need now to find a continuous expression for $n(rs)$ that is $n(rs) = 0$ inside the Warp Bubble and $n(rs) = \frac{1}{2}$ outside the Warp Bubble while being $0 <= n(rs) <= \frac{1}{2}$ in the Warp Bubble walls

The rs above cames from the rs and $f(rs)$ introduced by Miguel Alcubierre in 1994

$$rs = \sqrt{(x - xs)^2 + y^2 + z^2} \tag{72}$$

$$f(rs) = \frac{tanh[@(rs + R)] - tanh[@(rs - R)]}{2tanh(@R)} \tag{73}$$

We defined the Natario Shape Function $n(r)$ in function of a radius r small inside the Warp Bubble giving a $n(r) = 0$ and large outside the Warp Bubble giving a $n(r) = \frac{1}{2}$.Now we are using the Alcubierre rs because we need to know the thickness @ of the Warp Bubble:A Warp Bubble of small thickness implies that the behavior $0 <= n(rs) <= \frac{1}{2}$ is spanned over a small region and the derivatives of $n(rs)$ are high or have high values.Or in short the function $n(rs)$ falls rapidly from $\frac{1}{2}$ to 0 in the Natario Warped Region implying in large values for the derivatives affecting the Negative Energy requirements. If we span the

Natario Warp Bubble over a large region the derivatives will fall more smooth from $\frac{1}{2}$ to 0 and derivatives that falls slowly or have low values helps the Negative Energy requirements.Then we need a large thickness @ for the Natario Warp Bubble.

But we still need to find out the expression for the Natario Continuous Shape Function $n(rs)$ that is $n(rs) = 0$ inside the Warp Bubble and $n(rs) = \frac{1}{2}$ outside the Warp Bubble.

The Natario Continuous Shape Function $n(rs)$ for the Natario Warp Drive is defined by:

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{74}$$

$$n(rs) = \frac{1}{2}[1 - \frac{tanh[@(rs+R)] - tanh[@(rs-R)]}{2tanh(@R)}] \tag{75}$$

Lets examine the original Alcubierre Shape Function that is $f(rs) = 1$ inside the Warp Bubble for a small rs while being $f(rs) = 0$ outside the Warp Bubble for a large rs while being $0 <= f(rs) <= 1$ in the Alcubierre Warped Region and its implications for the behavior of the Natario Shape Function $n(rs)$:

- 1)-$f(rs) = 1$-Inside the Alcubierre Warp Bubble

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{76}$$

$$n(rs) = \frac{1}{2}[1 - 1] \tag{77}$$

$$n(rs) = 0 \tag{78}$$

This matches the requirements of the Natario Warp Drive inside the Natario Warp Bubble

- 2)-$f(rs) = 0$-Outside the Alcubierre Warp Bubble

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{79}$$

$$n(rs) = \frac{1}{2}[1 - 0] \tag{80}$$

$$n(rs) = \frac{1}{2} \tag{81}$$

This matches the requirements of the Natario Warp Drive outside the Natario Warp Bubble

- 3)-0 <= $f(rs)$ <= 1-In the Alcubierre Warped Region

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{82}$$

If:

$$f(rs) = \frac{1}{4} \tag{83}$$

Then:

$$n(rs) = \frac{1}{2}[1 - \frac{1}{4}] = \frac{1}{2}[\frac{3}{4}] = \frac{3}{8} = 0,375 \tag{84}$$

$$0 <= n(rs) <= \frac{1}{2} \curvearrowright 0 <= n(rs) <= 0,5 \tag{85}$$

This matches the requirements of the Natario Warp Drive in the Natario Warped Region

According to our definition of the Natario Shape Function $n(rs)$ and considering the original Alcubierre Shape Function $f(rs)$ when we have $f(rs) = 1$ we will have a $n(rs) = 0$ and when we have a $f(rs) = 0$ we will have a $n(rs) = \frac{1}{2}$

We derived the requirements of the Natario Shape Function $n(rs)$ for the Natario Warp Drive in function of the original Alcubierre 1994 Shape Function $f(rs)$

We pointed out before that the Warp Drive was an entire family of solutions of the Einstein Field Equations of General Relativity. The Warp Drive is an entire class of Dynamical Spacetimes like the Wormholes or Black Holes.The Natario Warp Drive although many different from the Alcubierre Warp Drive and in Section 6 we will the the differences is a member of the same family.This is the reason why we were able to use the Continuous Shape Function of Alcubierre $f(rs)$ to create the Natario Continuous Shape Function $n/rs)$.

We can now proceed with the study of the cartography of the Natario Warp Drive

5 Energy Density Requirements and the Continuous Shape Function $n(rs)$ for the Natario Warp Drive

We know that the Warp Drive requires an enormous amount of energy in order to be generated and to make the things worst this energy is negative.Large amounts of Negative Energy Densities are outside the scope of General Relativity.

The Energy Density for the Natario Warp Drive is given by:

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3(n'(rs))^2 \cos^2\theta + \left(n'(rs) + \frac{rs}{2} n''(rs) \right)^2 \sin^2\theta \right]. \tag{86}$$

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3(\frac{dn(rs)}{drs})^2 \cos^2\theta + \left(\frac{dn(rs)}{drs} + \frac{rs}{2} \frac{d^2n(rs)}{drs^2} \right)^2 \sin^2\theta \right]. \tag{87}$$

With the Natario Continuous Shape Function $n(rs)$ being given by:

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{88}$$

$$n(rs) = \frac{1}{2}[1 - \frac{tanh[@(rs+R)] - tanh[@(rs-R)]}{2tanh(@R)}] \tag{89}$$

Since the square of the Warp Bubble speed vs appears in the Energy Density and since this makes the things even worst(we will see this in Section 6)we need very low derivatives in order to keep it at physical attainable levels if we want to have a Warp Bubble speed $vs = 200$.two hundred times faster than light.Assuming a constant Warp Bubble radius R and thickness @ we have:

$$n'(rs) = \frac{dn(rs)}{drs} \tag{90}$$

$$n'(rs) = -\frac{1}{4tanh(@R)} \frac{d[tanh[@(rs+R)] - tanh[@(rs-R)]]}{drs} \tag{91}$$

$$n''(rs) = \frac{d^2n(rs)}{drs^2} \tag{92}$$

$$n''(rs) = -\frac{1}{4tanh(@R)} \frac{d^2[tanh[@(rs+R)] - tanh[@(rs-R)]]}{drs^2} \tag{93}$$

Lets examine first the term $-\frac{1}{4tanh(@R)}$

$$tanh(@R) = \frac{sinh(@R)}{cosh(@R)} = \frac{\varepsilon^{@R} - \varepsilon^{-@R}}{\varepsilon^{@R} + \varepsilon^{-@R}} \cong 1 \tag{94}$$

$$coth(@R) = \frac{cosh(@R)}{sinh(@R)} = \frac{\varepsilon^{@R} + \varepsilon^{-@R}}{\varepsilon^{@R} - \varepsilon^{-@R}} \cong 1 \tag{95}$$

The above assumptions are valid for a Warp Bubble radius $R = 100$ meters because $\varepsilon^{-@R} = \frac{1}{\varepsilon^{@R}} \cong 0$ due to the large value of $\varepsilon^{@R}$

Now lets examine the derivatives:

$$\frac{d[tanh[@(rs+R)]}{drs} = \frac{@}{cosh^2[@(rs+R)]} \tag{96}$$

Making:

$$U = rs + R \tag{97}$$

But since:

$$cosh(@U) = \frac{\varepsilon^{@U} + \varepsilon^{-@U}}{2} \tag{98}$$

We can make the following approximation again taking in mind that $\varepsilon^{-@U} = \frac{1}{\varepsilon^{@U}} \cong 0$ due to the large value of $\varepsilon^{@U}$ for a Warp Bubble radius of $R = 100$

$$cosh(@U) = \frac{\varepsilon^{@U}}{2} \tag{99}$$

$$cosh^2(@U) = \frac{\varepsilon^{2@U}}{4} \tag{100}$$

Look again to the derivative:

$$\frac{d[tanh[@(rs+R)]}{drs} = \frac{@}{cosh^2[@(rs+R)]} \tag{101}$$

The term $\frac{\varepsilon^{2@U}}{4}$ is so big because we are raising $2,718281^{2000}$ for a Warp Bubble Radius of $R = 100$ and thickness $@ = 10$ regardless of a small rs inside the Warp Bubble or a large rs outside the Warp Bubble.This number is enormous.Dividing a Warp Bubble thickness @ of a size of $@ = 10$ by such an enormous number will make this derivative close to zero.And since we are concerned in the energy of the Warp Bubble walls we focuses the region where rs approaches R rasing of course the power factor.

Lets examine now the second derivative:

$$\frac{d[tanh[@(rs-R)]}{drs} = \frac{@}{cosh^2[@(rs-R)]} \tag{102}$$

Making:

$$V = rs - R \tag{103}$$

But since:

$$cosh(@V) = \frac{\varepsilon^{@V} + \varepsilon^{-@V}}{2} \tag{104}$$

Here we have a curious situation:Inside the Warp Bubble $rs < R$ and V is negative while outside the Warp Bubble $rs > R$ and V is positive.Note that the change of the signs of the power factor reverse the rules of the exponentials but the result is almost similar to the previous case except that in the point where $rs = R$ the derivative is equal to the Warp Bubble thickness @.
Then for the first order derivative $n'(rs)$ of the Natario Shaoe Function $n(rs)$ the term that counts for the integral is:

$$n'(rs) = +\frac{1}{4tanh(@R)}\frac{d[tanh[@(rs-R)]]}{drs} \tag{105}$$

Because we dropped the first hyperbolic tangent derivative keeping the one that also possesses a negative sign.
But we already know that the term $tanh(@R) \cong 1$ .Then we have

$$n'(rs) = +\frac{1}{4}\frac{@}{cosh^2[@(rs-R)]} \tag{106}$$

Note again that this expression have low values when $rs \neq R$ achieving the value of the Warp Bubble thickness @ as a maximum value when $rs = R$.
Remember also that the Energy Density in the Natario Warp Drive uses the square of the derivative of the Natario Shape Function.
Then we must integrate the following expression:

$$(n'(rs))^2 = +\frac{1}{16}\frac{@^2}{cosh^4[@(rs-R)]} \tag{107}$$

The integrals of this function are known from tables of integrals of hyperbolic functions and are given by[4]:

$$\int \frac{dx}{cosh^n(ax)} = \frac{sinh(ax)}{a(n-1)cosh^{n-1}(ax)} + \frac{n-2}{n-1}\int \frac{dx}{cosh^{n-2}(ax)} \tag{108}$$

Making $a = @$ and $x = V$ being $V = rs - R$ we have:

$$\int \frac{dV}{cosh^4(@V)} = \frac{sinh(@V)}{3@cosh^3(@V)} + \frac{2}{3}\int \frac{dV}{cosh^2(@V)} \tag{109}$$

$$\int \frac{dV}{cosh^4(@V)} = \frac{8(\varepsilon^{@V} - \varepsilon^{-@V})}{3@(\varepsilon^{@V} + \varepsilon^{-@V})^3} + \frac{2}{3}\int \frac{dV}{cosh^2(@V)} \tag{110}$$

$$\int \frac{dV}{cosh^2(@V)} = \frac{sinh(@V)}{@cosh(@V)} + \int \frac{dV}{cosh(@V)} \tag{111}$$

From the previous approximations we can write the integral as follows with $dV = drs$ since R is constant:

$$\int \frac{dV}{cosh^2(@V)} = \frac{\varepsilon^{@V} - \varepsilon^{-@V}}{@(\varepsilon^{@V} + \varepsilon^{-@V})} + \int \frac{dV}{cosh(@V)} \tag{112}$$

$$\int \frac{dV}{cosh(@V)} = \frac{2}{@}\arctan(\varepsilon^{@V}) \tag{113}$$

[4]Integral Table taken from Wikipedia

Note that the following term

$$\frac{8(\varepsilon^{@V} - \varepsilon^{-@V})}{3@(\varepsilon^{@V} + \varepsilon^{-@V})^3} \tag{114}$$

Have very low values even in the region where $rs < R$ due to the power of 3 vanishing in the region where $rs = R$.
Note also that the following term

$$\frac{\varepsilon^{@V} - \varepsilon^{-@V}}{@(\varepsilon^{@V} + \varepsilon^{-@V})} \tag{115}$$

Also have low values because we are dividing the exponentials by the thickness of the Bubble @ and also vanished in the region where $rs = R$
Note that a hyperbolic sinh in the upper part of the fractions of the terms that are not integrated helps ourselves because in the regions where $rs < R$ or $rs > R$ these terms have extremely low values and these terms vanished in the region where $rs = R$.
So we are left with only this integral

$$\int \frac{dV}{cosh(@V)} = \frac{2}{@} \arctan(\varepsilon^{@V}) \tag{116}$$

We lowered the terms in the integral of the Natario Shape Function first order derivative
But we must concern ourselves with the square of the Warp Bubble speed vs that appears in the Natario Warp Drive Energy Density and for a Warp Bubble speed $vs = 200$ two hundred times faster than light this means a $vs = 6 \times 10^{10}$.Its square would then be:

$$vs^2 = 3,6 \times 10^{21} \tag{117}$$

This is a number of 21 degrees of magnitude.More on this in Section 6.

Examining the derivative of second order of the Natario Shape Function:

$$n''(rs) = -\frac{1}{4tanh(@R)} \frac{d^2[tanh[@(rs+R)] - tanh[@(rs-R)]]}{drs^2} \tag{118}$$

We know that we can make the following approach:

$$n''(rs) = -\frac{1}{4} \frac{d^2[tanh[@(rs+R)] - tanh[@(rs-R)]]}{drs^2} \tag{119}$$

Computing the derivatives using $U = rs + R$ and $V = rs - R$ we would get the following result:

$$n''(rs) = -\frac{1}{4}[-2@^2 \frac{sinh(@U)}{cosh^3(@U)} + 2@^2 \frac{sinh(@V)}{cosh^3(@V)}] \tag{120}$$

From the approximations we developed before we can see that this term is so low that almost vanishes due to the powers of $\varepsilon^{3@U}$ or $\varepsilon^{3@V}$.Remember that our study is a first order approximation and remember also that $R = 100$,$@ = 10$ so regardless of the value of rs that will make the term $@V$ vanish completely when $rs = R$ we are raising ε to a power of 3000.So by now we will neglect the derivative of second order of the Natario Shape Function.

From the Natario Warp Drive Energy Density we can make this approach considering only the first derivative of the Natario Shape Function $n(rs)$:

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs)\right)^2\sin^2\theta\right]. \tag{121}$$

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(rs)}{drs})^2\cos^2\theta + \left(\frac{dn(rs)}{drs}\right)^2\sin^2\theta\right]. \tag{122}$$

$$(n'(rs))^2 = +\frac{1}{16}\frac{@^2}{cosh^4[@(rs-R)]} \tag{123}$$

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[\frac{3}{16}\frac{@^2}{cosh^4[@(rs-R)]}\cos^2\theta + \frac{1}{16}\frac{@^2}{cosh^4[@(rs-R)]}\sin^2\theta\right]. \tag{124}$$

Note that in the direction parallel to the motion(the front of the ship) $\cos\theta = 1$ and $\sin\theta = 0$ so in front of the ship the energy is approximately

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[\frac{3}{16}\frac{@^2}{cosh^4[@(rs-R)]}\cos^2\theta\right]. \tag{125}$$

This is very important we have Negative Energy Density in front of the ship.More on this in the next section.
On the other hand in the direction perpendicular to the motion(above the ship) $\cos\theta = 0$ and $\sin\theta = 1$ so above the ship the energy is approximately

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[\frac{1}{16}\frac{@^2}{cosh^4[@(rs-R)]}\sin^2\theta\right]. \tag{126}$$

So we can see that the ship is completely covered by the Negatrive Energy of Warp Bubble in the Natario Warp Drive[5] that is being carried out by the Spacetime Stream.Just like a fish inside an aquarium and the aquarium is floating and being carried away by the stream of a river

[5]See the Artistic Presentation of the Natario Warp Drive Spacetime

6 Horizon and Doppler Blueshifts in the Natario and Alcubierre Warp Drive Spacetimes

We will now examine the problem of the Horizon and Infinite Doppler Blueshifts that occurs naturally in the Natario and Alcubierre Warp Drive Spacetimes due to the Geometry of the Spacetime itself .While the Horizon exists in both Warp Drive Spacetimes and behaves similar from one Warp Drive to the another the Infinite Doppler Blueshifts behaves very different from a Warp Drive(Natario) to the other/Alcubierre).

The Horizon problem is being discussed in pg 6 in [2] and also in pg 3 in [16]

We will provide here a step by step[6] procedure to demonstrate the existence of the Horizon.

Starting with the Natario Warp Drive with motion only in the $x - axis$ as defined by pg 6 in [2] we have the following expressions given below:

$$ds^2 = dt^2 - (dx - X^i dt)^2 \tag{127}$$

$$ds^2 = dt^2 - (dx - X dt)^2 \tag{128}$$

with the Shift Vector X being defined as:

$$X = vs(t) \tag{129}$$

We have:

$$ds^2 = dt^2 - (dx - vsdt)^2 \tag{130}$$

Expanding the square powers we should expect for:

$$ds^2 = dt^2 - (dx^2 - 2vsdxdt + vs^2 dt^2) \tag{131}$$

$$ds^2 = dt^2 - vs^2 dt^2 + 2vsdtdx - dx^2 \tag{132}$$

$$ds^2 = (1 - vs^2)dt^2 + 2vsdtdx - dx^2 \tag{133}$$

$$ds^2 = (1 - X^2)dt^2 + 2X dtdx - dx^2 \tag{134}$$

But also from pg 6 in [2] we know that a Null-Like geodesics must satisfy the following condition:

$$ds^2 = 0 \tag{135}$$

Then we arrive at the following second-degree equation

$$0 = (1 - X^2)dt^2 + 2X dtdx - dx^2 \tag{136}$$

[6]For experienced readers this may be somewhat tedious but we are concerned on the novice or newcomer readers to the Warp Drive Science

And we know that we can divide both side by dt to obtain the following result

$$0 = (1 - X^2) + 2X\frac{dx}{dt} - \frac{dx^2}{dt^2} \tag{137}$$

Defining U as below:

$$U = \frac{dx}{dt} \tag{138}$$

Our second-degree equation becomes now:

$$0 = (1 - X^2) + 2XU - U^2 \tag{139}$$

Defining C as below:

$$C = (1 - X^2) \tag{140}$$

Our second-degree equation becomes after the multiplication of the terms in both sides by -1 :

$$U^2 - 2XU - (1 - X^2) = 0 \tag{141}$$

$$U^2 - 2XU - C = 0 \tag{142}$$

The solutions of the second-degree equation are given by:

$$U = \frac{2X \pm \sqrt{4X^2 + 4C}}{2} = \frac{2X \pm \sqrt{4X^2 + 4(1 - X^2)}}{2} = \frac{2X \pm \sqrt{4}}{2} = \frac{2X \pm 2}{2} = X \pm 1 \tag{143}$$

From above we will get two roots shown below:

$$U = X \pm 1 = vs \pm 1 \tag{144}$$

We must examine carefully the meaning of each one of these roots.

Initially and according to fig 2 pg 8 in [2] $X = vs = 0$ inside the Natario Warp Bubble and $X = vs$ outside the Natario Warp Bubble and then assuming a continuous growth of X from 0 to vs we have the region where $0 < X < vs$ and this region is the Natario Warped Region.We are expecting that a photon sent to the direction of the Warp Bubble from inside the Bubble will cross the Natario Warped Region in order to emerge outside the Warp Bubble

Then we are left ourselves with 3 possible scenarios:

- 1)-inside the Warp Bubble $X = 0$
- 2)-outside the Warp Bubble $X = vs$ and $vs > 1$
- 3)-in the Natario Warped Region $0 < X < vs$ and $vs > 1$

We found two roots for our second-degree equation.Each root depicts a possible direction for a photon sent from inside the Warp Bubble to the Warp Bubble Walls in the motion over the $x - axis$.

- 1)- photon sent towards the front of the Warp Bubble $U = X - 1 = vs - 1$
- 2)- photon sent towards the rear of the Warp Bubble $U = X + 1 = vs + 1$

But remember that

$$U = \frac{dx}{dt} \tag{145}$$

Then we should expect for:

- 1)- photon sent towards the front of the Warp Bubble $\frac{dx}{dt} = X - 1 = vs - 1$
- 2)- photon sent towards the rear of the Warp Bubble $\frac{dx}{dt} = X + 1 = vs + 1$

Or even better:

- 1)- photon sent towards the front of the Warp Bubble $\frac{dx}{dt} - X = -1$
- 2)- photon sent towards the rear of the Warp Bubble $\frac{dx}{dt} - X = +1$

Then we can easily see that(pg 6 in [2]).:

$$\| \frac{dx}{dt} - X \| = 1 \tag{146}$$

But if $X = 0$ inside the Warp Bubble and $X = vs$ outside the Warp Bubble with $vs > 1$ then in order for X to pass from 0(inside) to vs(outside) with a continuous growth then X must enter in the region where $0 < X < vs$ which means to say that X enters in the Natario Warped Region and in a given region of the Natario Warped Region $X = 1$ and for the photon sent towards the front of the Warp Bubble we have:

$$\frac{dx}{dt} = X - 1 = 1 - 1 = 0 \tag{147}$$

The photon speed becomes zero!.Note that although we are working with the Natario Warp Drive according to pg 3 in [16] this also occurs for the Alcubierre Warp Drive but due to the next topic of Doppler Blueshifts we keep ourselves with the Natario Warp Drive.

The photon stops inside the Natario Warped Region where $X = 1$ never reaching the region outside the Warp Bubble where $X = vs$

Of course we are aware of the result above when $\frac{dx}{dt} = 0$

$$\| \frac{dx}{dt} - X \| = 1 = \| 0 - X \| = 1 = \| X \| = 1 \tag{148}$$

According to Natario assuming cylindrical symmetry about the x-axis, there exists a point on the positive x-axis where $\|\mathbf{X}\| = 1$ as seen before by ourselves and the cylindrically symmetric surface in this point whose angle α with $\mathbf{X}$ is given by(pg 6 in [2]).

$$\sin\alpha = \frac{1}{\|\mathbf{X}\|} \tag{149}$$

$$\sin\alpha = 1 \dashrightarrow \alpha = \frac{\pi}{2} \dashrightarrow \|\mathbf{X}\| = 1 \tag{150}$$

This angle α is known as the Natario angle.

The point where the photon stops is the Horizon.

In the Horizon, events inside the Warp Bubble cannot causally influence events on the outer side of the Warp Bubble because the photon stops in this point inside the Natario Warped Region.

The photon enters the Natario Warped Region and cross the portion of the Natario Warped Region where $0 < X < 1$ stopping effectively where $X = 1$.This will be very important when examining the next topic in this section:Infinite Doppler Blueshifts.

Of course we consider here only Classical General Relativity where light speed ($c = 1$ or $vs = 1$ or $X = 1$) is the maximum speed allowed to send information from one point to another.But see the Superluminal Effects in [6] to [10].

See also pg 3 in [16] where is mentioned that communication with the front part of the Warp Bubble needs Superluminal signs.

We still do not have a Quantum Gravity theory that encompasses the Non-Local Entanglements of Quantum Mechanics[7]with the Spacetime Geometry described by General Relativity.

This issue of the Horizon is still an open question in Warp Drive Science and we will use here a pedagogical example to illustrate this:Imagine that we have two supersonic jet planes one in front of the another but with the radios turned off due malfunction and the only thing both planes have to communicate between each other are phonon[8] machines.Initially both planes are at subsonic speeds and a phonon sent by the rear plane can reach the plane in the front so both planes are "causally connected".They have synchronized clocks and in a given time both planes passes the speed of the sound and both enters in Supersonic(Mach) speeds .When this happens a phonon from the rear plane can no longer reach the plane in the front because the phonon sent by the rear plane to reach the plane in the front is outrunned by the speed of the rear plane itself.Then from the point of view of the rear plane the front plane is "causally disconnected".But a phonon sent by the front plane will reach the plane in the rear.Anyone will figure out that this is a similar situation between our pairs of photons sent to the front or the rear part of the Warp Bubble.So when the Warp Bubble achieves Superluminal(Warp) Speed it happens something similar to the Mach shockwave for supersonics speeds(see pg 15 in [5]).

Notice that outside the Warp Bubble $X = vs$ with $vs > 1$ and we have (pg 6 in [2]).See also fig 1 pg 7 in [2].

[7]eg Einstein-Podolsky-Rosen Paradox or the Instantaneous Communication between pairs of polarized particles like the Alain Aspect or Raymond Chiao experiments as described in [6] to [10]

[8]equivalent to the photon but for sound waves

$$\sin\alpha = \frac{1}{v_s} \tag{151}$$

According to Natario this is the familiar expression for the Mach cone angle[9].For Superluminal(Warp) Speed scenario we prefer to redefine this as the Natario cone angle.See fig 3 pg 9 in [2]

But if $vs > 1$ then $\sin\alpha < 1$ and $\alpha < \frac{\pi}{2}$ and as higher are the values of vs then $\sin\alpha \simeq 0$ and also $\alpha \simeq 0$

As higher are the values of vs the Natario cone angle becomes zero so we can see that vs defines the inclination of the Natario cone angle. For "Low" Superluminal(Warp) Speeds the inclination is close to $\frac{\pi}{2}$ because although $vs > 1$ vs stands close to 1 while for "High" Superluminal(Warp) Speed the Natario cone angle gets closer to zero because $vs >> 1$.[10].Then according to pg 6 in [2]) we are left with 3 possible scenarios for the inclination of the Natario cone angle:

- 1)-$\|\mathbf{X}\| = 1$ and $vs = 1$.

$$\sin\alpha = \frac{1}{\|\mathbf{X}\|} \tag{152}$$

$$\sin\alpha = 1 \dashrightarrow \alpha = \frac{\pi}{2} \dashrightarrow \|\mathbf{X}\| = 1 \tag{153}$$

  This situation happens when $X = vs$ and $vs = 1$.The Warp Drive reaches Luminal Speed.This means to say that the the Warp Drive reaches the Horizon.The Natario cone angle similar to the Mach cone angle appears(see pg 15 in [5]) with an inclination of $\frac{\pi}{2}$ because $sin\alpha = 1$ which means to say that the Natario cone angle appears perpendicular to the direction of motion

- 2)- $X = vs$ and $vs > 1$.

$$\sin\alpha = \frac{1}{v_s} \tag{154}$$

$$\sin\alpha < 1 \dashrightarrow \alpha < \frac{\pi}{2} \dashrightarrow \|\mathbf{X}\| > 1 \dashrightarrow \sin\alpha \simeq 1 \dashrightarrow \alpha \simeq \frac{\pi}{2} \tag{155}$$

  This situation happens when $X = vs$ and $vs > 1$ and the Warp Drive reaches "Low" Superluminal(Warp) Speed.The Natario cone angle similar to the Mach cone angle appears(see pg 15 in [5]) now with an inclination of less than $\frac{\pi}{2}$ but still close to it because vs is still close to 1.Example for a $vs = 2$(two times light speed) then $\sin\alpha = \frac{1}{2}$ and $\alpha = \frac{\pi}{6}$

- 3)-$X = vs$ and $vs >> 1$

$$\sin\alpha = \frac{1}{v_s} \tag{156}$$

[9]See [17] or the Graphical Presentation of the Mach cone angle.Compare this with fig 1 pg 7 in [2].Although we consider $\frac{1}{v_s}$ because we make $c = 1$ the expression really means $\frac{c}{v_s}$

[10]see the Artistical Presentations in the end of this work

$$\sin\alpha << 1 \dashrightarrow \alpha << \frac{\pi}{2} \dashrightarrow \|\mathbf{X}\| >> 1 \dashrightarrow \sin\alpha \simeq 0 \dashrightarrow \alpha \simeq 0 \tag{157}$$

This situation happens when $X = vs$ and $vs >> 1$ and the Warp Drive reaches "High" Superluminal(Warp) Speed.The Natario cone angle similar to the Mach cone angle appears(see pg 15 in [5]) now with an inclination of much less than $\frac{\pi}{2}$ and highly inclinated almost parallel to the direction of motion because $\sin\alpha \simeq 0$ so $\alpha \simeq 0$ too.Example for a $vs = 200$(two hundred times light speed) then $\sin\alpha = \frac{1}{200} \simeq 0$.

Now we will examine the problem of the Doppler Blueshifts in the Natario Warp Drive.We have two kinds of Doppler Blueshifts:

- 1)-Incoming photons coming to the front of the ship with the frequency Blueshifted
- 2)-Infinite Doppler Blueshifts in the Horizon for a photon sent from inside the Warp Bubble to the front .

According to pg 9 in [15] and pg 8 in [2] photons coming to the front of the Warp Bubble appears with the frequency hight Doppler Blueshifted. Computing the Blueshift using the classical Doppler-Fizeau formula for a photon approaching the ship from the front we have:[11]

$$f = f_0 \frac{c + va}{c - vb} \tag{158}$$

The terms above are:

- 1)-f is the photon frequency seen by an observer
- 2)-f_0 is the original frequency of the emitted photon
- 3)-c is the light speed.in our case $c = 1$
- 4)-va is the speed of the light source approaching the observer.In our case is vs or X(Natario)
- 5)-vb in the speed of the light source moving away from the observer. In our case because the photon is coming to the observer $vb = 0$

Rewriting the Doppler-Fizeau expression for an incoming photon approaching the Warp Bubble from the front we should expect for(see eq 26 pg 9 in [15],see also pg 8 in [2]):

$$f = f_0(1 + vs) \tag{159}$$

$$f = f_0(1 + X) \tag{160}$$

Energy E is Planck Constsnt $\hbar$ multiplied by frequency so for the energy we would have:

$$E = E_0(1 + vs) \tag{161}$$

[11]Remember that the Warp Drive do not obey Lorentz transformations so the classical formula can be applied to get the results of [2] and [15]

$$E = E_0(1 + X) \tag{162}$$

Note that as larger is $vs > 1$ as large is the Blueshift and this is a serious obstacle that compromises the physical feasibility of the Warp Drive.See pg 11 in [15].Incoming Blueshifted photons are hazardous for any crew inside the Warp Bubble. We used previously a Shift Vector $X = vs = 200$ but for this and still according to pg 11 in [15] COBE photons are Blueshifted in the front of the Warp Bubble to energies equivalent to a Solar Photosphere.

Look also to fig 2 pg 8 in [2].Outside the Natario Warp Bubble $X = vs$ while in the interior $X = 0$ then a photon coming from outside arrives at the Natario Warped Region with an energy E_0 according to the Doppler-Fizeau formula multiplied by $1 + X$ being the incoming energy $E = E_0(1 + X)$ then the photon crosses the Natario Warped Region where $0 < X < vs$ and finally arrives at the region inside the Warp Bubble where $X = 0$ so $E = E_0$.Note that $E_0(1 + X) >> E_0$ which means to say that the incoming photon passed from a state of high energy $E_0(1 + X)$ to a state of low energy E_0.

At this point one would ask:what happened with the remaining energy??

This excess of energy is "released" when the photon crosses the Warped Region rendering this region unstable and generating the Hawking temperatures that are another big problem that compromises the physical stability of the Warp Drive(see pgs 6 and 7 in [11])).

At first sight it seems that both Alcubierre and Natario Warp Drive behaves similar and are affected by the same physical constraints in this Doppler Blueshift scenario but actually Natario Warp Drive behaves different than Alcubierre Warp Drive because it possesses a different distribution of Energy Density.

The Energy Density for the Alcubierre Warp Drive is given by the following expressions(pg 4 in [2])(pg 8 in [1])(eq 140 pg 28 in [12])(eq 5.8 pg 75 in [14])(eq 8 pg 6 in [13]):

$$\rho = -\frac{1}{32\pi} v_s^2 \left[f'(r_s)\right]^2 [\frac{y^2 + z^2}{r_s^2}] \tag{163}$$

$$\rho = -\frac{1}{32\pi} v_s^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2 + z^2}{r_s^2}] \tag{164}$$

The Energy Density in the Natario Warp Drive is given by the following expressions(pg 5 in [2]):

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3(n'(r))^2 \cos^2\theta + \left(n'(r) + \frac{r}{2} n''(r)\right)^2 \sin^2\theta\right]. \tag{165}$$

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3(\frac{dn(r)}{dr})^2 \cos^2\theta + \left(\frac{dn(r)}{dr} + \frac{r}{2}\frac{d^2 n(r)}{dr^2}\right)^2 \sin^2\theta\right]. \tag{166}$$

We will start with the Alcubierre Energy Density:see fig 2 pg 4 in [11],fig 3 pg 28 in [12],fig 5.3 pg 76 in [14],fig 3 pg 7 in [13]
The Energy Density in the Alcubierre Warp Drive is located in a toroidal region above and below the ship perpendicular to the direction of motion.[12]

[12]see also our Artistic Presentation of the Energy Density in the Alcubierre Warp Drive

Note that in front of the ship we have no Energy Density displacements.In front of the ship we have a region of "empty" or "hollow" space.This is the region where Spacetime contracts in the front of the Alcubierre Warp Bubble.
Considering a photon motion only over the $x - axis$ coming towards the Alcubierre Warp Bubble and highly Doppler Blueshifted.
In this case $y^2 + z^2 = 0$ and hence the Energy Density is also zero in the point where the photon arrives at the Warp Bubble neighborhoods.
It is easy to see that only when $|y^2 + z^2| > 0$ the Energy Density becomes not null.Now its easy to figure out why the the Energy Density in the Alcubierre Warp Drive is located above and below the ship while in the front we have "hollow" space.Also we know that a negative Energy Density have repulsive gravitational behavior[13]that could in principle "deflect" photons but we have no Energy Density in the front of the Alcubierre Warp Bubble.This means to say that there is nothing left to stop the photon.

When $y^2 + z^2 = 0$ we have for the Alcubierre Energy Density:

$$\rho = -\frac{1}{32\pi} v_s^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2 + z^2}{r_s^2}] = 0 \tag{167}$$

On the other hand lets examine the Natario Energy Density when a highly Blueshifted photon approaches the front of the Natario Warp Bubble with a motion over the $x - axis$ only:

In this case $\sin\theta = 0$ and $\cos\theta = 1$ because the photon is placed right on the direction of motion.Then we have:

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2 \cos^2\theta + \left(\frac{dn(r)}{dr} + \frac{r}{2}\frac{d^2n(r)}{dr^2}\right)^2 \sin^2\theta\right]. \tag{168}$$

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2 \cos^2\theta\right] \tag{169}$$

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2\right] \tag{170}$$

Note that in the Natario Warp Drive the Energy Density in the $x - axis$ do not vanish.This is the reason why we presented in this work a section entirely devoted to Energy Density calculations and another section related to the Natario Shape Function before this section.This negative Energy Density have "repulsive"[14] behavior and can in principle "deflect" incoming photons

From what we presented above we can conclude that if the Negative Energy "deflect" photons then we cannot send signals using photons to the regions where the Negative Energy is located neither in Alcubierre nor in Natario Warp Drives.We already outlined before that Warp Bubbles are ahead of the scope of General Relativity and must wait for a Quantum Gravity theory that encompasses Non-Local Quantum Entanglements.Plus we already outlined before that the process to generate large outputs of negative Energy Density remains unknown.

[13]for example the negative energy of the Cosmological Vacuum eg the Einstein Cosmological Constant that is accelerating the Expansion of the Universe.We will not pursue this so far here but see the Appendix on the Einstein Cosmological Constant
[14]Appendix on the Einstein Cosmological Constant

But note also that since there are no negative Energy Density in front of the Alcubierre Bubble because it is located above and below the ship perpendicular to the direction of motion then the photon sent to the front of the Warp Bubble will not be stopped and will reach the Horizon.

On the other hand we know that a photon sent to the front of the Natario Warp Bubble initially travels in the region where $X = 0$ (inside the Warp Bubble) and will arrive at the Natario Warped Region where $0 < X < vs$ being supposed to stop in the part inside the Natario Warped Region where $X = 1$(the Horizon).

But we also know that $n(rs) = 0$ inside the Natario Warp Bubble and $0 < n(rs) < \frac{1}{2}$ in the Natario Warped Region while being $n(rs) = \frac{1}{2}$ outside the Warp Bubble.The derivatives of $n(rs)$ vanishes inside the Warp Bubble and outside the Warp Bubble but do not vanishes in the Natario Warped Region keeping there a non null negative Energy Density.Thene we have for the motion in the $x - axis$ the following conditions:

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3 (\frac{dn(r)}{dr})^2 \right] \tag{171}$$

$$0 < n(rs) < \frac{1}{2} \tag{172}$$

$$0 < X < vs \tag{173}$$

When the photon enters the region where $0 < X < vs$ it also enters the region where the derivatives of $n(rs)$ do not vanish.The negative Energy Density "deflect" the photon before reaching the Horizon.The photon is deflected before reaching $X = 1$ because in front of the ship the space in the Natario Warp Drive is not empty. Remember that for a Warp Bubble speed $vs = 200$ two hundred times faster than light this means a $vs = 6 \times 10^{10}$.Its square would then be:

$$vs^2 = 3,6 \times 10^{21} \tag{174}$$

Giving an Energy Density in the front of the Natario Warp Bubble of:

$$\rho = -\frac{v_s^2}{8\pi} \left[3 (\frac{dn(r)}{dr})^2 \right] = -\frac{3,6 \times 10^{21}}{8\pi} \left[3 (\frac{dn(r)}{dr})^2 \right] \tag{175}$$

Note that an Energy Density with magnitudes of powers of 10^{21} when integrating over the volume of the Warp Bubble will give a total Energy requirement to maintain the Warp Bubble extremely high.This is a major concern[15].

Then while in the Alcubierre Warp Drive the incoming photon towards the Warp Bubble is not stopped and will reach the Bubble and the photon sent towards the front of the Warp Bubble from inside will reach the Horizon because Alcubierre Warp Drive have empty space in front of the ship,in the Natario Warp Drive both photons are stopped by the negative Energy Density in front of the ship and the Horizon cannot be reached from inside:

[15]In our Appendix on the Einstein Cosmological Constant we will discuss magnitudes of Energy Densities

We will now examine the last topic in this work:Infinite Doppler Blueshifts in the Horizon suffered by photons sent towards the front of the Warp Bubble.See pg 6 and 8 in [2].

$$n = \frac{dx}{dx} - X \tag{176}$$

$$E_0 = E(1 + n.X) \tag{177}$$

Look that the expression of n is familiar to ourselves from the solutions of the second-degree equation when computing Horizons. In fact we have :

- 1)- photon sent towards the front of the Warp Bubble $\frac{dx}{dt} - X = -1$

When $X = 0$ inside the Warp Bubble then $n = -1$ and $E_0 = E$ as would be expected for an energy measured by the observer inside the Warp Bubble but when $X = 1$ in the Horizon we are left with:

$$\frac{dx}{dt} = X - 1 = 1 - 1 = 0 \tag{178}$$

$$n = \frac{dx}{dx} - X = 0 - 1 = -1 \tag{179}$$

So when a photon reaches the Horizon we have the following conditions: $n = -1$ and $X = 1$.Inserting these values in the equation of the energy(pg 8 in [2]) we have:

$$E = \frac{E_0}{(1 + n.X)} = \frac{E_0}{(1 + -1.1)} = \frac{E_0}{0} \tag{180}$$

And then we have the Infinite Doppler Blueshift in the Horizon as mentioned by Natario in pg 8 in [2].

Note that this occurs in the Alcubierre Warp Drive but not in the Natario Warp Drive because as we already knows the photon is stopped in the Natario Warp Drive before reaching the Horizon due to a different distribution of Energy Density.

We can demonstrate the same effect in a way different than the one used by Natario(geometric projections for the Eulerian observer) using the Classical Doppler-Fizeau formula

$$f = f_0 \frac{c + va}{c - vb} \tag{181}$$

The terms above are:

- 1)-f is the photon frequency seen by an observer
- 2)-f_0 is the original frequency of the emitted photon
- 3)-c is the light speed.in our case $c = 1$
- 4)-va is the speed of the light source approaching the observer.Since the photon is moving away from the observer then $va = 0$

- 5)-vb in the speed of the light source moving away from the observer.In our case $vb = X$

In this case we are concerned with a photon sends towards the front of the Bubble then the photon is moving away from the observer.In this case we have $va = 0$ and $vb = X$.Making $c = 1$ we have:

$$f = f_0 \frac{1}{1 - X} \tag{182}$$

When $X = 0$ inside the Warp Bubble then $f = f_0$ but when $X = 1$ in the Horizon we are left with

$$f = f_0 \frac{1}{1 - X} = f_0 \frac{1}{1 - 1} = f_0 \frac{1}{0} \tag{183}$$

And again we have the Infinite Doppler Blueshift in the Horizon as mentioned by Natario in pg 8 in [2].

7 Conclusion-What Next For The Warp Drive??

The Warp Drive as a valid solution of the Einstein Field Equations of General Relativity faces some physical problems:

- 1)-The Causally Disconnected portions of spacetime also known as The Horizon Problem:It is true that the observer inside the Warp Bubble wether in an Alcubierre or Natario Warp Drive cannot send signals to the front of the Warp Bubble eg photons with the light speed.The Horizons Problem must be postponed to a more advanced theory that encompasses both General Relativity and Quantum Mechanics specially the Non-Local effects of the Quantum Entanglements eg Einstein-Podolsky-Rosen Paradox or the experiments of Raymond Chiao or Alain Aspect with Superluminal Instantaneous Communication between pairs of polarized photons at a great distance from each other.More Superluminal Effects in [6] to [10].
- 2)-The Negative Energy Problem:It is true that we can create small amounts of Negative Energy by the Casimir Effect as noted by Alcubierre himself while the Negative Energy Requirements for a Warp Drive are large specially considering the square of the Warp Bubble Speed that appears in the equations.We need to discover new ways to create Negative Energy again perhaps with a new theory of Quantum Gravity.We demonstrated that a Shape Function with low derivatives will help in this context.[16]
- 2)-The Doppler Blueshift Problem:Space is not empty.It is fulfilled with the photons of Cosmic Background Radiation(COBE) and a Warp Bubble at two hundred times faster than light would impact some of these Doppler Blueshifted photons to the wavelengths of Synchrotron Radiation which is lethal and we have too many of these photons per cubic centimeter of space. Note that Natario Warp Drive have Negative Energy covering all the ship and Negative Energy means a repulsive Spacetime Curvature of General Relativity that could deflect the photons.

To terminate:Warp Drives are Dynamical Spacetimes and members of the same family of solutions of the Einstein Field Equations of General Relativity. The first member was discovered by Alcubierre in 1994 while the second was discovered by Natario in 2001. But Natario was inspired by Alcubierre. Due to a different distribution of Energy Density the Natario Warp Drive seems to perform better in the case of the Doppler Blueshifts and since we have no third candidate or third member[17] while we wait the arrival of the so-called Quantum Gravity to solve all the problems related to the Warp Drive at least we could make the complete study of the Cartography of the Natario Warp Drive Spacetime.We know that Natario used a constant Shift Vector $X = vs$ to arrive at this distribution of Energy.But a "real" Warp Drive is expected to accelerate or de-accelerate or is expected to make curves.A "real" Warp Drive is expected to change velocites:is expected to change the Shift Vector.What would happen to the Hodge Star if the Shift Vector varies with time?.What would happen to the Natario Vector if the Shift Vector changes?What would happen to the Riemmann, Ricci and Einstein Tensors if the Shift Vector varies? The complete Cartography of the Natario Warp Drive with variable Shift Vectors is something that although mathematically extremely difficult may well be achieved with our Classical General Relativity if we devote ourselves to this.We are confident that one day the desired Quantum Gravity will arrive and then at least we would have the Natario Warp Drive ready and prepared to evolve to a more advanced Quantum Form.

[16] See the Appendix on the Einstein Cosmological Constant

[17] As far as we can tell

The Descendant of the Natario Warp Drive may well be the one that will lead the Human Race to the Stars:To Our Future:To the Undiscovered Country.Live Long And Prosper[18]

[18]from the movie Star Trek VI:The Undiscovered Country:A Journey Into the Future

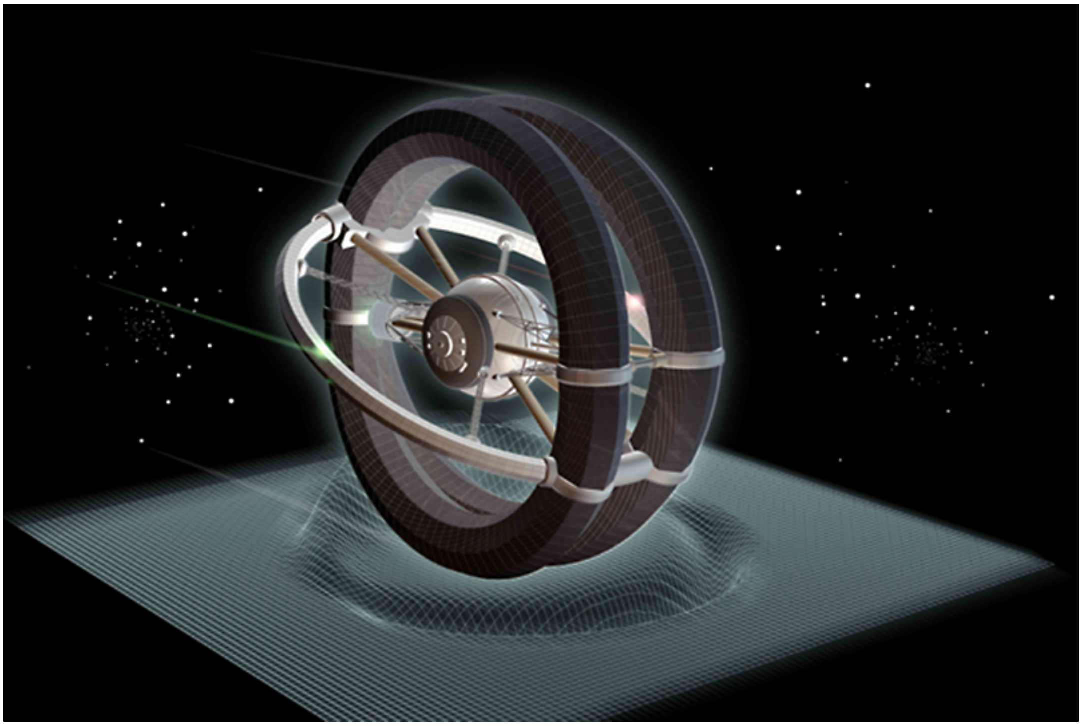

Figure 1: Artistic representation of the Natario Warp Drive Bubble .Note the Alcubierre Expansion of the Normal Volume Elements below.(Source:Internet)

8 Artistic Graphical Presentation of the Natario Warp Drive

Note that according to the geometry of the Natario Warp Drive the Spacetime Contraction in one direction(radial) is balanced by the Spacetime Expansion in the remaining direction(perpendicular)(pg 5 in [2]). Remember also that the Expansion of the Normal Volume Elements in the Natario Warp Drive is given by the following expression(pg 5 in [2]). :

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{184}$$

If we expand the radial direction the perpendicular direction contracts to keep the Expansion of the Normal Volume Elements equal to zero.
This figure is a pedagogical example of the graphical presentarion of the Natario Warp Drive.The "metal bars"[19] in the figure actually do not exists but were included to illustrate how the expansion in one direction can be counter-balanced by the contraction in the other directions.These "metal bars" keeps the Expansion of the Normal Volume Elements in the Natario Warp Drive equal to zero.
Note also that the graphical presentation of the Alcubierre Warp Drive Expansion of the Normal Volume Elements according to fig 1 pg 10 in [1] is also included

[19]These metal bars could be also regarded as Artistic Presentations of the Negative Energy Density in the Natario Warp Drive

Note also that the Energy Density in the Natario Warp Drive being given by the following expressions(pg 5 in [2]):

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(r))^2\cos^2\theta + \left(n'(r) + \frac{r}{2}n''(r)\right)^2\sin^2\theta\right]. \quad (185)$$

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2\cos^2\theta + \left(\frac{dn(r)}{dr} + \frac{r}{2}\frac{d^2n(r)}{dr^2}\right)^2\sin^2\theta\right]. \quad (186)$$

Is being distributed around all the space involving the ship(above the ship $\sin\theta = 1$ and $\cos\theta = 0$ while in front of the ship $\sin\theta = 0$ and $\cos\theta = 1$).The Negative Energy in front of the ship "deflect"[20]photons so these will not reach the Horizon and will not suffer from Infinite Doppler Blueshifts.

-)-Above the ship

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[\left(\frac{dn(r)}{dr} + \frac{r}{2}\frac{d^2n(r)}{dr^2}\right)^2\sin^2\theta\right]. \quad (187)$$

-)-In front of the ship

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2\cos^2\theta\right]. \quad (188)$$

Note that although we can have low values for the derivatives of the Natario Shape Function the term vs^2 will not ameliorate the needs of large outputs of Negative Energy.This is beyond the scope of Classical General Relativity

[20]See the Appendix on the Einstein Cosmological Constant and the repulsive properties of the Negative Energies

Figure 2: Artistic representation of the Natario Warp Drive Bubble . (Source:Internet)

9 Artistic Graphical Presentation of the Natario Warp Drive entering in the Warp "Shockwave" $X = 1$, Natario angle=$\alpha = \frac{\pi}{2}$

Although the figure above was "borrowed" from science fiction it depicts correctly what would happen for a Natario Warp Drive when it reaches Luminal speeds($X = 1$)[21]. We already know that when $\|\mathbf{X}\| = 1$ and $vs = 1$ according to pg 6 in [2],(see also pg 15 in [5])

$$\sin\alpha = \frac{1}{\|\mathbf{X}\|} = 1 \tag{189}$$

$$\sin\alpha = 1 \dashrightarrow \alpha = \frac{\pi}{2} \dashrightarrow \|\mathbf{X}\| = 1 \tag{190}$$

From above we see that the white light "disk" appears in front of the ship perpendicular to the direction of motion because the Natario angle in this case is $\frac{\pi}{2}$.Compare this with the Graphical Presentation of the Mach Cone angle

[21]We know that the photon cannot be used to send signals to the front of the Natario Warp Bubble because photons will be deflected by the Negative Energy Density that exists in the front of the ship before reaching the Horizon.However we assume that a Quantum Graviry theory that encompasses the Non-Local Quantum Entanglements and the Superluminal Quantum Effects of [6] to [10] coupled to the Classical General Relativity will be able to do the task

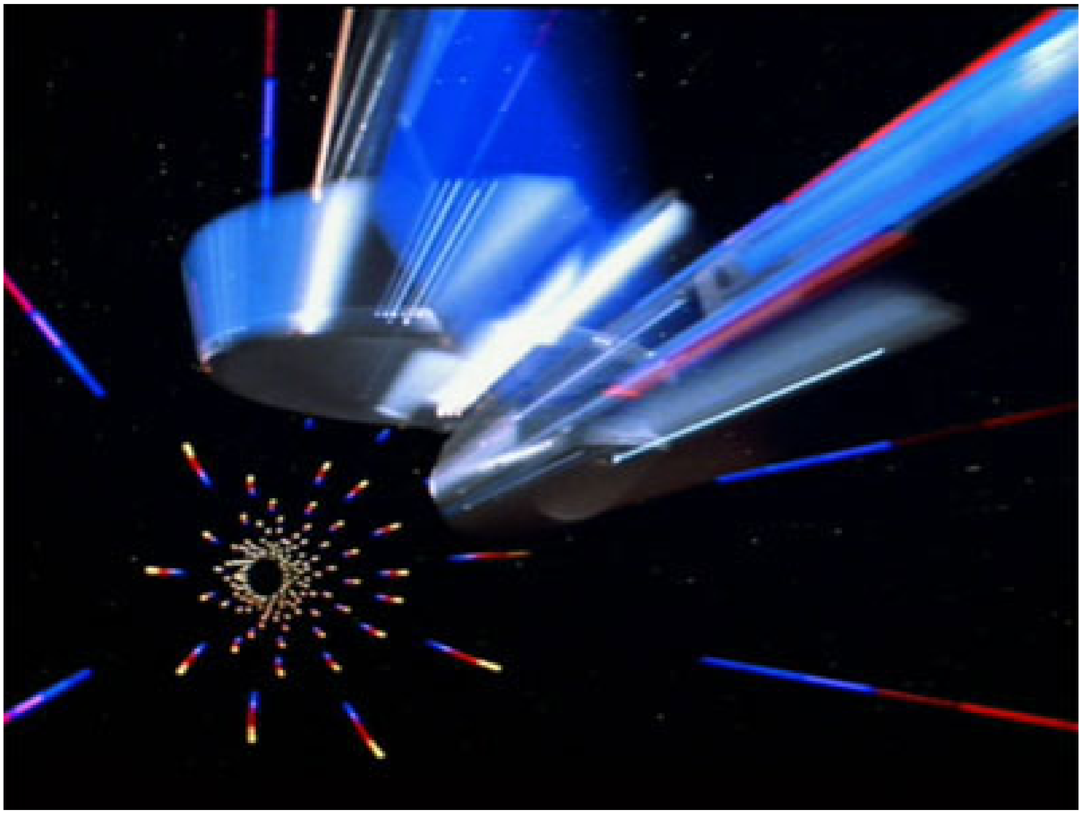

Figure 3: Artistic representation of the Natario Warp Drive Bubble at "Low" Superluminal(Warp) Speed. Note the Doppler lines . (Source:Internet)

10 Artistic Graphical Presentation of the Natario Warp Drive at "Low" Superluminal(Warp) Speed $X = vs$ and $vs > 1$. Natario angle $\alpha < \frac{\pi}{2}$ $\alpha \simeq \frac{\pi}{2}$

Like in the previous Artistic Presentation the figure above was also "borrowed" from science fiction and also depicts correctly what would happen for a Natario Warp Drive when it reaches "Low" Superluminal(Warp) speeds($X > 1$)[22]. We already know that when $X = vs$ and $vs > 1$ (pg 6 in [2]).

$$\sin\alpha = \frac{1}{v_s} \tag{191}$$

$$\sin\alpha < 1 \dashrightarrow \alpha < \frac{\pi}{2} \dashrightarrow \|\mathbf{X}\| > 1 \dashrightarrow \sin\alpha \simeq 1 \dashrightarrow \alpha \simeq \frac{\pi}{2} \tag{192}$$

From above we can see that $\sin\alpha < 1$ and $\alpha < \frac{\pi}{2}$.This means that the Natario angle is no longer perpendicular to the direction of motion and now have a small inclination given by the Natario angle itself.Compare this with the Graphical Presentation of the Mach Cone angle.

[22]See the footnote on Luminal speeds in the previous presentation.

Figure 4: Artistic representation of the Natario Warp Drive Bubble at High Superluminal(Warp) Speed. Note the Doppler lines . (Source:Internet)

11 Artistic Graphical Presentation of the Natario Warp Drive at "High" Superluminal(Warp) Speed $X = vs$ and $vs >> 1$. Natario angle $\alpha << \frac{\pi}{2}$ $\alpha \simeq 0$

Like in the two previous Artistic Presentation the figure above was also "borrowed" from science fiction and also depicts correctly what would happen for a Natario Warp Drive when it reaches "High" Superluminal(Warp) speeds($X >> 1$)[23]. We already know that when $X = vs$ and $vs >> 1$ (pg 6 in [2]).

$$\sin\alpha = \frac{1}{v_s} \tag{193}$$

$$\sin\alpha << 1 \dashrightarrow \alpha << \frac{\pi}{2} \dashrightarrow \|\mathbf{X}\| >> 1 \dashrightarrow \sin\alpha \simeq 0 \dashrightarrow \alpha \simeq 0 \tag{194}$$

In this case since $X >> vs$ $\sin\alpha << 1$ and $\alpha << \frac{\pi}{2}$.The Natario angle now appears almost parallel to the direction of motion and the inclination of the Natario angle is so hight that approaches zero.Compare this with the Graphical Presentation of the Mach cone angle.

[23]See the footnote on Luminal speeds in the second previous presentation.

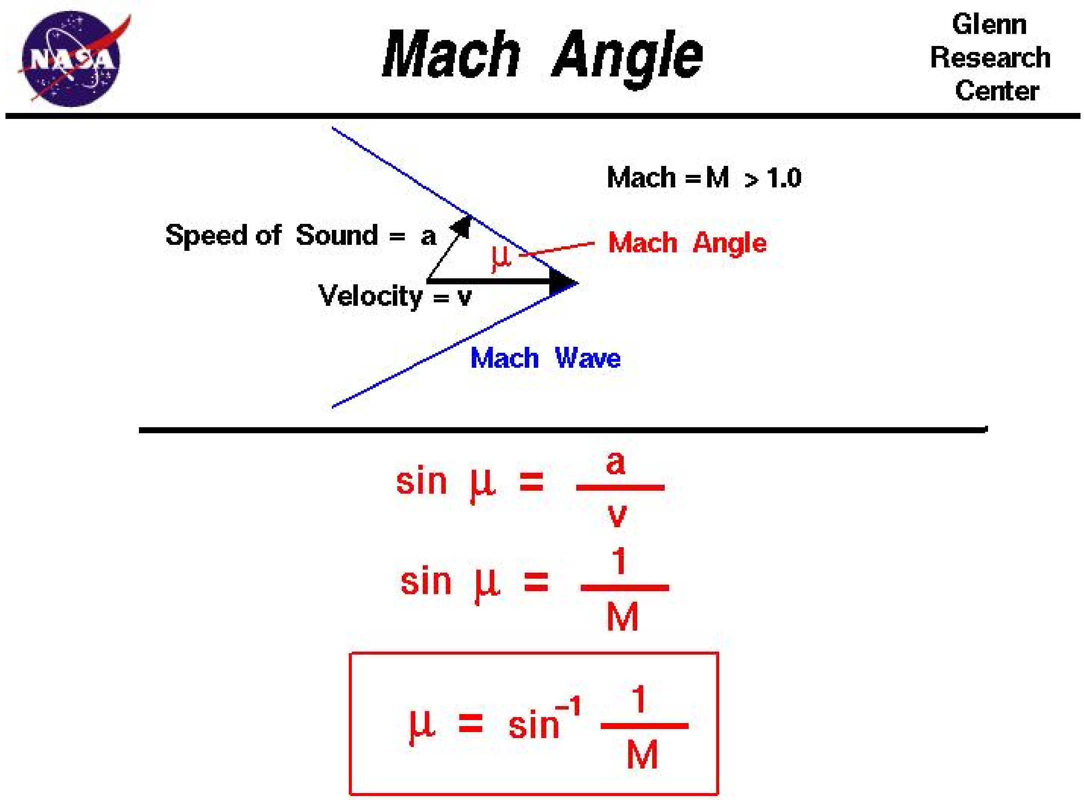

Figure 5: Graphical Presentation of the Mach Cone angle . (Source:NASA [17])

12 Graphical Presentation of the Mach Cone angle

Above is being presented the graphical presentation of the Mach Cone angle.We included this presentation here[24] to demonstrate the similarities that exists between the Supersonic(Mach) speeds and the Mach Cone angle with the Superluminal(Warp) speeds and the Natario Cone angle.

Note that when a plane achieves the speed of the sound its velocity v equals the speed of the sound a and then

$$\sin\mu = \frac{a}{v} = 1 \tag{195}$$

In the above case the angle $\mu = \frac{\pi}{2}$ and the Mach "shockwave" is perpendicular to the direction of motion of the plane.

But when the plane accelerates to Supersonic(Mach) speeds then $v >> a$ and $\sin\mu = \frac{a}{v} << 1$ with the angle $\mu \cong 0$.As fast the plane accelerates the Mach angle approaches zero and the Mach shockwave inclination tend to be parallel to the speed of the plane itself.

[24]See the Remarks Section on [17]

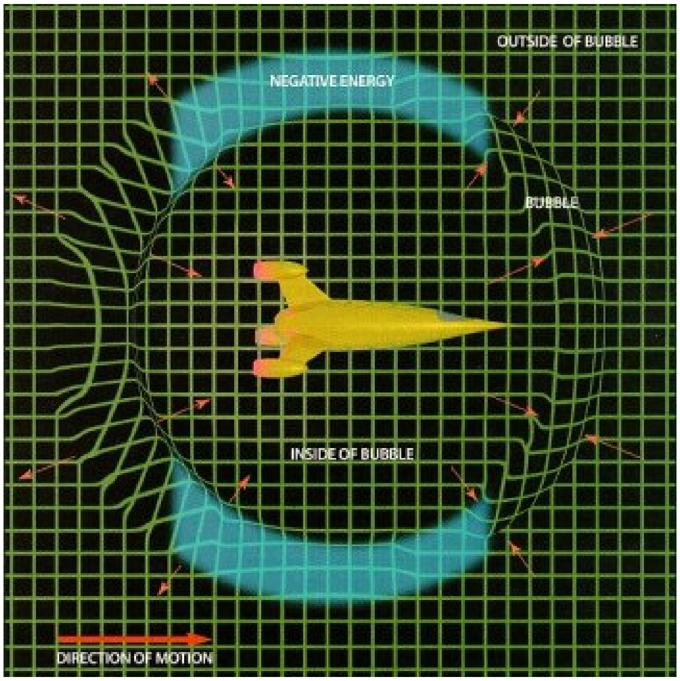

Figure 6: Artistic representation of the Energy Density distribution in the Alcubierre Warp Drive . (Source:fig 2 pg 4 in [11])

13 Artistic Graphical Presentation of the Energy Density distribution in the Alcubierre Warp Drive

Above is being presented the Artistic Graphical Presentation of the Energy Density for the Alcubierre Warp Drive given by the following expressions(pg 4 in [2])(pg 8 in [1])(eq 140 pg 28 in [12])(eq 5.8 pg 75 in [14])(eq 8 pg 6 in [13]):

$$\rho = -\frac{1}{32\pi} v_s^2 \left[f'(r_s)\right]^2 [\frac{y^2+z^2}{r_s^2}] \tag{196}$$

$$\rho = -\frac{1}{32\pi} v_s^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2+z^2}{r_s^2}] \tag{197}$$

Note that the Negative Energy Density is located on a toroidal region above and below the ship and perpendicular to the direction of motion due to the term $y^2 + z^2 \neq 0$.The front of the ship is "empty" space where the contraction of Spacetime in front occurs. There are no Negative Energy Densities in front of the ship in the Alcubierre Warp Drive

In front of the ship in the $x - axis$ only we have $y^2 + z^2 = 0$ and then the Alcubierre Energy Density will have the following value:

$$\rho = -\frac{1}{32\pi} v_s^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2+z^2}{r_s^2}] = 0 \tag{198}$$

Note that an observer stationary inside an Alcubierre Warp Drive that moves with a Superluminal(Warp) speed $vs > 1$ with respect to the rest of the Universe[25] will suffer from the Horizons and Doppler Blueshift problems because there are no Negative Energies in front of the ship to "deflect" photons.This is the reason why the Hawking temperatures that compromises the physical stability of the Alcubierre Warp Drive(see pgs 6 and 7 in [11]) cannot be avoided . Although Natario and Alcubierre Warp Drive Spacetimes are members of the same family of the Einstein Field Equations of General Relativity and although the Shape Function used for Alcubierre Warp Drive can be used to construct the Shape Function for the Natario Warp Drive due to the different geometrical nature of the Negative Energy Densities distributions one(Natario) will assume a different behavior when compared to the other(Alcubierre).This was the main idea outlined in this work

[25] Observer at rest inside the Bubble $X = 0$ while outside the Bubble $X = vs$

14 Appendix:Einstein Cosmological Constant,Vacuum Negative Energy Density and the Acceleration of the Expansion Rate of the Universe

We know that the Warp Drive wether Alcubierre or Natario requires lots of Negative Energy and to make the things worst this energy is directly proportional to the square of the Warp Bubble speed vs^2.If the speed achieves Superluminal(Warp) values for example $X = vs = 200$(two hundred times light speed in order to cross reasonable interstellar distances in our local neighborhoods in reasonable amounts of time) this implies in an Energy Density in the front of the Natario Warp Bubble of a magnitude of 10^{21} integrated by the volume of the Warp Bubble itself[26]. For large Bubbles(capable to contains a ship example a Bubble with 100 meters of radius) the demand of Negative Energy to sustain the Warp Bubble would be enormous independent of the values of the derivatives of the Natario Shape Function :

$$\rho = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2\right] = -\frac{3,6\times 10^{21}}{8\pi}\left[3(\frac{dn(r)}{dr})^2\right] \tag{199}$$

The generation of a Warp Bubble as we stated before is beyond the scope of our science because we need lots of Negative Energy and Casimir Effect can only "manufacture" microscopical amounts of it.

Rather surprisingly it was the Natario in pg 10 and 11 in [5] that gave to us an idea about how large outputs of Negative Energy can be "discovered" in the Universe.We know that the Universe is expanding and galaxies are moving away from each other at Superluminal(Warp) speeds.(pg 10).
Since positive Energy Density means also positive mass and positive gravitational field that attracts matter then the mechanism responsible for the Galaxies receding at Superluminal(Warp) speeds must be of repulsive gravity and with an enormous energy enough to send billions of stars at these velocities.

Recent discoveries in Cosmology[27]confirmed that the Universe as a matter of fact is accelerating the expansion rate and this mechanism seems to be of "exotic" nature. see eq 9 pg 4 in [18]

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{200}$$

Above is the equation of the acceleration of the Universe.$\frac{\ddot{a}}{a}$ is the acceleration measured by astronomers in terms of the Hubble Parameter and its derivatives ,$-\frac{4\pi G}{3}(\rho + 3p)$ is the term due to the positive matter,pressure and Energy Densities that due to the attractive behavior of gravity would tend to de-accelerate the expansion rate and now look to the term $\frac{\Lambda}{3}$:if the Universe is accelerating then $\frac{\ddot{a}}{a} >> 0$ and this means to say that $\frac{\Lambda}{3} >> -\frac{4\pi G}{3}(\rho + 3p)$.The term $\frac{\Lambda}{3}$ is the exotic matter that have repulsive gravity Negative Energy Density(stress energy momemntum tensor of the vacuum)and is accelerating the Universe also known as the Einstein Cosmological Constant.

see eq 12 pg 5 in [18] and eq 1.3 pg 2 in [19]

$$T_{\mu\nu}^{vacuum} = -\rho_{vacuum} g_{\mu\nu} \tag{201}$$

see eq 14 pg 5 in [18] and eq 1.5 pg 3 in [19]

$$\rho_{vacuum} = \rho_{\Lambda} = \frac{\Lambda}{8\pi G} \tag{202}$$

[26] $2\times 10^2 \times 3\times 10^8 = 6\times 10^{10} m/s$ the square gives 10^{21}

[27] This section is only a theoretical approach.We will not pursue this subject any further here.

The value of the Einstein Cosmological Constant is subject of major controversies in Physics and we will not extend this matter here but could "theoretically" generate large outputs of Negative Energy Densities.

see eq 19 pg 7 in [18] and eq 1.9 pg 3 in [19]

$$\rho_{vacuum} = \rho_{\bigwedge} = 2 \times 10^{110} \frac{ergs}{cm^3} \tag{203}$$

$$\rho_{vacuum} = \rho_{\bigwedge} = 2 \times 10^{93} \frac{joules}{cm^3} \tag{204}$$

Note that these enormous values appears in the Energy Density of the vacuum with magnitudes much larger than 10^{21} and this Energy Density is negative.Perhaps the Universe already possesses large amounts of Negative Energy and we need to learn how to "collect" this energy for the Warp Drive.

This subject is well beyond the current capabilities of our science and will have to wait for a real Quantum Gravity theory.

15 Appendix:Differential Forms,Hodge Star and the Natario Vector nX

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{205}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{206}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{207}$$

From above we get the following results

$$dr \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{208}$$

$$rd\theta \sim r\sin\theta(d\varphi \wedge dr) \tag{209}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{210}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(pg 8 in [4]):

$$*dr = r^2 \sin\theta(d\theta \wedge d\varphi) \tag{211}$$

$$*rd\theta = r\sin\theta(d\varphi \wedge dr) \tag{212}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{213}$$

Back again to the Natario equivalence between spherical and cartezian coordinates(pg 5 in [2]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2\sin\theta\cos\theta d\theta\wedge d\varphi + r\sin^2\theta dr\wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{214}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{215}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{216}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{217}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi\wedge dr)] \tag{218}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] - [r\sin^2\theta(d\varphi\wedge dr)] \tag{219}$$

We know that the following expression holds true(see pg 9 in [3]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{220}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] + [r\sin^2\theta(dr\wedge d\varphi)] \tag{221}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [2]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{222}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{223}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{224}$$

According to pg 10 in [3] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{225}$$

Because and according to pg 10 in [3]:

$$d(\alpha+\beta) = d\alpha + d\beta \tag{226}$$

$$d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{227}$$

$$d(dx) = d(dy) = d(dz) = 0 \tag{228}$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta \wedge dd\varphi = 2sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{229}$$

$$*[d(r^2)d\varphi] = 2rdr\wedge d\varphi + r^2\wedge dd\varphi = 2r(dr\wedge d\varphi) \tag{230}$$

And then we derived again the Natario result of pg 5 in [2]

$$r^2\sin\theta\cos\theta(d\theta\wedge d\varphi) + r\sin^2\theta(dr\wedge d\varphi) \tag{231}$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [2] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \tag{232}$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \tag{233}$$

$$f(r)r^2 2sin\theta\cos\theta(d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{234}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{235}$$

Comparing the above expressions with the Natario definitions of pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta)\wedge(r\sin\theta d\varphi) \sim r^2\sin\theta(d\theta\wedge d\varphi) \tag{236}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi)\wedge dr \sim r\sin\theta(d\varphi\wedge dr) \sim -r\sin\theta(dr\wedge d\varphi) \tag{237}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr\wedge(rd\theta) \sim r(dr\wedge d\theta) \tag{238}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(d\theta\wedge d\varphi)] + 2f(r)\ sin\theta[r\sin\theta(dr\wedge d\varphi)] + f'(r)r\sin\theta[r\sin\theta(dr\wedge d\varphi)] \tag{239}$$

$$2f(r)\ cos\theta e_r - 2f(r)\ sin\theta e_\theta - rf'(r)\sin\theta e_\theta \tag{240}$$

$$*d[f(r)r^2\sin^2\theta d\varphi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{241}$$

Defining the Natario Vector as in pg 5 in [2] with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{242}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{243}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [2]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{244}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{245}$$

With our pedagogical approaches

$$nX = 2vs(t)f(r)\ cos\theta dr - vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{246}$$

$$nX = -2vs(t)f(r)\ cos\theta dr + vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{247}$$

16 Epilogue

- "The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C.Clarke[28]
- "The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein[29][30]

[28] special thanks to Maria Matreno from Residencia de Estudantes Universitas Lisboa Portugal for providing the Second Law Of Arthur C.Clarke

[29] "Ideas And Opinions" Einstein compilation, ISBN $0-517-88440-2$, on page 226."Principles of Research" ([Ideas and Opinions],pp.224-227), described as "Address delivered in celebration of Max Planck's sixtieth birthday (1918) before the Physical Society in Berlin"

[30] appears also in the Eric Baird book Relativity in Curved Spacetime ISBN $978-0-9557068-0-6$

17 Remarks

References 3 and 4 were taken from Internet although not from a regular available site like the one of references 1 and 2(arXiv.org).We can provide the Adobe PDF Acrobat Reader of these references for those interested.Reference 5 is available to the public from the home page of Jose Natario at IST and in principle it will remain available although we can also provide copies of reference 5 for those interested.Reference 17 is available to the public from the NASA Glenn Research Center for high school students $K-12$

18 Legacy

This work is dedicated to the $10th$ anniversary of the Natario Warp Drive Spacetime.The first version appeared in the arXiv.org as gr-qc/0110086 in 19 October 2001

References

[1] *Alcubierre M., (*1994*). Class.Quant.Grav. 11 L73-L77,gr-qc/0009013*

[2] *Natario J.,(*2002*). Class.Quant.Grav. 19 1157-1166,gr-qc/0110086*

[3] *Introduction to Differential Forms,Arapura D.,*2010

[4] *Teaching Electromagnetic Field Theory Using Differential Forms,Warnick K.F. Selfridge R. H.,Arnold D. V.,IEEE Transactions On Education Vol* 40 *Num* 1 *Feb* 1997

[5] *Natario J.,(*2003*). Warp Drive.Lecture Seminar given by Natario J.at IST Lisbon Portugal*

[6] *Geroch R.(*2010*). arXiv:gr-qc/1005.1614v1*

[7] *Chiao Y.R,Merced.C.U.(*2010*). arXiv:gr-qc/1011.1325v1*

[8] *Chiao Y.R.(*2010*). arXiv:gr-qc/1011.1326v1*

[9] *Chiao Y.R.(*2010*). arXiv:gr-qc/1011.3185v1*

[10] *Chiao Y.R.(*2010*). arXiv:gr-qc/1012.0797v1*

[11] *Barcelo C.,Finazzi S.,Liberati S.(*2010*). arXiv:gr-qc/1001.4960v1*

[12] *Lobo F.S.N.(*2007*). arXiv:gr-qc/0710.4474v1*

[13] *Ford L.H. ,Pfenning M.J., (*1997*). Class.Quant.Grav. 14 1743-1751,gr-qc/9702026*

[14] *Ford L.H. ,Pfenning M.J., (*1998*). Post-Doctoral Dissertation of Pfenning M.J.gr-qc/9805037*

[15] *Clark C.,Hiscock W.,Larson S.,(*1999*). Class.Quant.Grav. 16 3965-3972,gr-qc/9907019*

[16] *Everett A.E.,Roman T.A,(*1997*) ,Phys.Rev. D56 024023,gr-qc/9702049*

[17] *http://www.grc.nasa.gov/WWW/K-12/airplane/machang.html*

[18] *Carroll S.(*2000*). arXiv:astro-ph/0004075v1*

[19] *Carroll S.(*2003*). arXiv:astro-ph/0310342v1*

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