The fundaments of quantum physics are still not well established.

This report tries to find the cracks in these fundaments and suggests repair procedures

This leads to unconventional solutions.

On the origin of physical dynamics

Report of a research project

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On the origin of physical dynamics

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Abstract

When physics must be based on an axiomatic foundation then the law set of traditional quantum logic is a valid candidate. However, at first sight, these axioms do not treat physical fields and they do not treat dynamics. It only prescribes the static relations that exist between quantum logical propositions that treat static subjects. These subjects are considered to be physical subjects or their properties. Amongst these propositions statements exist that describe everything that can be said about the static condition of a given physical item. Such propositions represent that item.

Traditional quantum logic is lattice isomorphic to the set of closed subspaces of an infinite dimensional separable Hilbert space **H**. That is why quantum mechanics is usually done with the aid of Hilbert space features.

The representation of a physical field does not fit in a Hilbert subspace. Physical fields have a universe wide range and their presentation would cover all of a whole Hilbert space.

Piron has shown that a candidate Hilbert space can be defined over one of three division rings. The choice comprises the real numbers, the complex numbers and the quaternions. The choice for the quaternions means that manipulations of the Hilbert space, such as Fourier transforms, in general use such multi-dimensional numbers.

According to Helmholtz decomposition theorem, the quaternionic Fourier transform can be divided in a complex longitudinal Fourier transform and a transverse Fourier transform. For quaternionic functions this means that they can be *locally* split into a one-dimensional rotation free part and a two-dimensional divergence free part.

This e-paper indicates that traditional quantum logic can be expanded to extended quantum logic, which includes the influences of physical fields in the form of potential propositions that concern virtual items. Extended quantum logic is lattice isomorphic with the set of subspaces of *a set of* Hilbert spaces. The fields take care of the coherence between these Hilbert spaces.

In this complicated way the axioms of traditional quantum logic form the constraints of the dynamics of quantum physics. When the dynamics of the universe would be put to a hold, then the axioms of extended quantum logic would describe all static constraints that are put to that universe. Dynamics means that universe steps from one static status quo to the next.

After the step the conditions are changed and the static constraints are reestablished. If we find the laws that control the steps, then we have found a complete axiomatic foundation of physics. Classical physics forms another constraint of dynamical quantum physics. This e-paper studies what happens during the step.

In the process a classification of skew Hilbert fields will be generated that corresponds closely to the Maxwell fields. Further, this e-paper investigates what happens in the infinitesimal steps that nature takes in order to arrive at the next static status quo. In this way the origin of dynamics and the origin of special relativity may be revealed.

Contents

It is undeniable that a creator exists.
It is easy to give a creator a place in physics
It is our fate to interpret who or what this creator is.
Fourier transforms are abundantly present in physics and have the habit to convert something that is strongly localized into something that is very widespread and vice versa.
Consider what the Fourier transform of the universe will be!

Introduction

The aim of this paper is to build upon a fundament consisting of a minimal set of axioms and then derive as much as is possible from fundamental physics by using only purely mathematical methods.

Observations and measurements at low dose rates learn that ALL information that we receive comes to us in small packages that we call quanta. The distribution of the probability of these information quanta determines the picture that we get from reality.

The development of quantum physics in its early days went violently. The consequence is that many of the fundaments of this theory are not constructed carefully. Fundamental repair is required.

The fundament

This e-paper builds on the following postulates:

1. The rock fundament of physics is an ordered sequence of instances of traditional quantum logic.
2. All physical information is transferred in the form of clouds of information carrying quanta.
3. The shape of this cloud is determined by a probability amplitude distribution that generates a tendency to keep these quanta together.

Equations of motion

All equations of motion are in fact continuity equations that treat the local information generation, annihilation and transfer.

Total change within V = flow into V + production inside V

The logic of the model

This e-paper is not about reality. It is about a model that could be a reflection of part of reality.

When reasoning about physical reality, it is safe to follow the rules of classical logic. If one starts with a true statement and these rules are followed then the path of reasoning stays with truth. Classical logic is based on about 25 axioms. A significant part of these axioms defines the structure of the logic as a half-ordered set and some other axioms expand this to define the set as a lattice. The other axioms have more to do with the rules that must be

followed in order to reason logically. May be it is a good starting point to use logic itself as a fundament of physics.

In the first decades of the last century it was discovered that nature itself cheats with classical logic. Numerous observations of the behavior of small particles revealed that some of the interrelations between these observations are in conflict with classical logic. Birkhoff and von Neumann interpreted this conflict and came to the conclusion that nature obeys its own kind of logic. They named this logic quantum logic.

The model builds its foundation on traditional quantum logic. This e-paper is not about quantum logic. It uses quantum logic because traditional quantum logic defines the **static** framework in which quantum dynamics takes place. Traditional quantum logic prescribes the potential relations that may exist between quantum logical propositions. Amongst these propositions statements exist that describe everything that can be said about the static condition of a given physical item. Such propositions represent that item. These propositions form the top of a hierarchy of propositions that treat the current values of the properties of the considered item.

Traditional quantum logic is lattice isomorphic to the set of closed subspaces of an infinite dimensional separable Hilbert space **H**.That is why quantum mechanics is usually done with the aid of Hilbert space features. The representation of a physical field does not fit in a Hilbert subspace. Physical fields have a universe wide territory and their presentation would cover all of a complete Hilbert space.

Piron has shown that a candidate Hilbert space can be defined over one of three division rings. The choice comprises the real numbers, the complex numbers and the quaternions. The choice for the quaternions means that manipulations of the Hilbert space, such as the Fourier transforms, in general operate on these multi-dimensional numbers. In the model the representations of physical fields are Hilbert fields. Hilbert fields are blurred Hilbert distributions. Hilbert distributions are sets of Hilbert vectors. The blur is a local excitation that is attached to a Hilbert vector. The blur is characterized by a continuous spread function. The values of this function may reach all Hilbert vectors. In this way these fields not only cover the whole Hilbert space, but they also become differentiable.

Apart from Hilbert fields the much simpler Hilbert functions exist. Hilbert functions can be defined with the help of a normal operator. Using the eigenvalues and the inner products of the eigenvectors with a selected Hilbert vector that vector can be converted in a hyper complex function. Hilbert functions can be interpreted as sampled versions of continuous functions. They are not differentiable.

As a consequence, the theory that is derived here is largely based on the properties of these multidimensional transforms and on the properties of Hilbert fields and Hilbert functions. Any Hilbert field can be split in a rotation free longitudinal part and a divergence free transverse part. The direction in which a field is rotation free may change with the values of the (current) coordinates. As long as the direction stays stationary, the corresponding coordinates can be considered as belonging to a complex plane that is embedded in a quaternionic space.

The blurs that constitute the Hilbert fields do not fit inside the realm of an infinite dimensional separable Hilbert space, but their values can be attached to Hilbert vectors. On the other hand the separable Hilbert space $\mathbf{H}$, can be embedded in a rigged Hilbert space $\mathbb{H}$ in which the blurs of a Hilbert field get a supported place.

Fourier transforms can be defined in a separable Hilbert space, but there they expose sampling characteristics that do not occur in a corresponding rigged Hilbert space $\mathbb{H}$. Similarly the notions of differentiation and integration are easily implemented in a rigged Hilbert space $\mathbb{H}$. Without the blurring trick, differentiation is impossible in a separable Hilbert space.

However, the set of closed subspaces of a rigged Hilbert space $\mathbb{H}$ is no longer lattice isomorphic with the set of propositions in a traditional quantum logic system. We do not want to offer the isomorphism with quantum logic in order to achieve differentiability of functions. This differentiability is already introduced by the blurs that are attached to the Hilbert vectors. This approach delivers a cleaner model that becomes even better comprehensible when later on we interpret the blur as a probability amplitude distribution.

In a three dimensional vector space a Fourier transform of a vector field can locally be divided in a one-dimensional longitudinal, (locally) complex Fourier transform and a two-dimensional transverse Fourier transform. The longitudinal transform works only on the longitudinal part of the field that is being transformed. The transverse transform works only on the transverse part of the field that is being transformed. This also applies to the case where this vector space is formed by the imaginary quaternions and the fields have quaternionic values.

The division in a longitudinal part and a transverse part of a function or a field *has only a local validity*. It holds as long as the direction in which the considered function of field is sufficiently (= within accepted inaccuracy) stationary. The split is the subject of the Helmholtz decomposition theorem. Multi-dimensional Dirac delta functions show the same decomposition as the multi-dimensional Fourier transform. The splits lose their significance when the field gets too wild.

The fact that this field categorization has only local validity and that it is related to an imaginary direction causes that the quaternionic Fourier transform must be considered to operate in a curved coordinate space. The differentiability of quaternionic functions and Hilbert fields also offers this categorization. Fourier transformation converts differentiation into multiplication with the canonical coordinate.

For a given field this situation can be solved by using two coordinate systems. One in which the coordinates show the curvature of the field and one in which the field has no curvature. This concept can be extended to a covering field, which is the superposition of all Hilbert fields that exist in the Hilbert space. Using the coordinate system for which the covering field has no curvature the universe wide Fourier transform can be taken.

For a given field and a given coordinate system it is possible to define a decomposition related local curvature. That curvature can be used to define a derived field. We will call this partner field the curvature field of that combination.

In a given coordinate system the fields can be categorized according to their symmetry properties. These categorizations must also cope with the curvature of that coordinate space.

In the described way, traditional quantum logic and the Helmholtz decomposition theorem together form a set of laws that define the static relational status quo that would exist in nature when dynamics could be put to a hold. This paper points out that traditional quantum logic can be expanded such that it includes the representation of *static* physical fields.

Dynamics can be included by representing nature by a sequence of such extended quantum logics. The fields regulate the coherence between subsequent quantum logics. This also means that the model can include dynamics by representing nature by a sequence of Hilbert spaces. The blurs in the Hilbert fields regulate the coherence between subsequent Hilbert spaces.

As a consequence the subsequent Hilbert spaces do not differ much. The same holds for the configuration of the fields in sub-sequent stages of the static status quo. In fact the fields can be seen as a storage place for the conditions that determine the relation between the past, the current and the future status quo.

In this view the fields represent relations between sets of potential propositions. These propositions are statements that say everything that can be said about the static condition of virtual items. The set of propositions in dynamic extended quantum logic is no longer isomorph with the set of closed subspaces of a single Hilbert space. It is isomorphic with the closed subspaces of a series of Hilbert spaces. One member of this set of Hilbert spaces is the currently actual Hilbert space. It contains the representatives of actual physical items. The other members are *virtual* Hilbert spaces. They contain the representations of virtual physical items.

Each virtual Hilbert space corresponds to an instance of a virtual (traditional) quantum logic. These virtual logics represent potential replacements of the actual traditional quantum logic. A virtual logic differs from other virtual logics in the fact that their propositions have a different configuration in terms of their atomic predicates or in terms of their sub-ordered propositions. In a similar sense they will differ from the actual logic.

The replacement will be made between logics that have a great resemblance with each other and the values of the predicates within equivalent propositions will be the same or close to each other. At each replacement the existing items are annihilated and replacing items are created. It is possible that annihilation is not followed by creation or that a created item is not the replacement of an annihilated item.

A redefiner, which steps from the actual Hilbert space to a virtual one, implements dynamics. The redefinition step exchanges the actual Hilbert space against a virtual Hilbert space whose selection is derived from the representation of the fields.

In order to be able to control the coherence between subsequent Hilbert spaces, the blurs that constitute the Hilbert fields act as probability density distributions. These distributions have a form that minimizes change during the step from the current Hilbert space to its successor. As a consequence physical quantities do not become observable as continuous objects. Observables become available in the form of information quanta that form the outcome of stochastic processes. The form of the clouds of information quanta is described by the probability amplitude distributions that together form the Hilbert fields.

Dynamics can be interpreted as a sequence of steps in which each step leads nature from the conditions of one status quo to the conditions of the next status quo. The laws that define the status quo are clear. During these steps several things happen. The laws that govern the dynamics are still obscure. The steps couple the static ingredients into a dynamic mixture. The steps are taken universe wide. A redefiner with a universe wide domain controls these steps. The step counter presents a universe-wide progression parameter. This parameter must not be confused with our common notion of time, but it cannot be denied that it has some relation with it.

During the step particles move forced by their own momentum and by the surrounding covering field to their new position. As a consequence their private fields get redistributed in space. Thus the covering field and its derived partner the curvature field will change. This delivers the preconditions for the next step. These activities are all infinitesimal.

Inertia shows that fields do not interfere with uniform movement. However, acceleration goes together with field action. The words "goes together with" mean that no causal relation exists. Inertia represents the influence of the whole universe on the condition of a local physical item. In fact it is a bilateral relation.

Thus, the changes of the covering field go together with acceleration of the particles.
This is all that happens during the infinitesimal progression step.
We must now analyze what acceleration does during an infinitesimal progression step.
Inertia can guide the way. Roughly, the driving force comes from the difference $\Delta \boldsymbol{E}(\boldsymbol{r}, t)$ between the current curvature field and the previous curvature field. It is contained in an equivalent of the stripped version of one of Maxwell's equations.

$$\Delta \boldsymbol{E}(\boldsymbol{r}, t) = G \cdot \frac{\partial \mathbf{v}}{\partial \mathrm{t}} \tag{1}$$

G is the gravitational constant.
This happens during the progression step to all particles.
Please notice the switch from covering field to curvature field. It is essential!

Recapitulation and extrapolation

In summary: Traditional quantum logic is usually defined via its structure as an orthomodular lattice. This logic only defines part of the *static* skeleton of the frame in which quantum physics operates. It does not state anything about physical fields. The Helmholtz/Hodge decomposition theorem defines the structure of static physical fields. In that way this theorem plays a similar role as traditional quantum logic. However, the decomposition has only local validity, where quantum logic has global validity. Extended quantum logic encompasses both law sets. These law sets do not specify or even touch the source of *dynamics*. Dynamics couples the static fields. The coupling not only applies to parts of the same field. It also concerns different fields. For example dynamics couples electrostatic fields with magnetostatic fields into dynamical electromagnetic fields. The gravitation field describes the curvature of observable space that is caused by the decomposition properties of the other fields. Thus, instead of a separate field the gravitation field can be considered as the result of the rotation properties of the other fields. Inertia reveals the importance of the gravitation field.

Both the propositions about a quantum physical system and physical fields are closely related. However, this relation only gets relevant when dynamics comes into play. Dynamics causes a continuing redefinition of the propositions. This converts the current static status quo into the next one. When one proposition is changed it interchanges its constituting atomic predicates with other predicates. The change can even involve the exchange of atomic predicates against atomic predicates that are of another type. It is also possible that the configuration of a complex system that consists of simpler components is altered.

The static physical fields can be interpreted as storage of the preconditions for the next step. The physical fields are the representatives of the influences that go together with the sticky resistance of the set of propositions against the changes that occur due to the redefinitions of the propositions that describe physical items. This sticky resistance also occurs in propositions that are sub-ordered to other propositions. Inertia is a feature that shows this resistance explicitly.

The propositions about quantum physical items can be represented by closed subspaces of a Hilbert space. The presence of dynamics means that the relations between these subspaces are not stationary. They change between subsequent Hilbert spaces. It is also possible to give the physical fields a representation by attaching them to the Hilbert spaces. Their values can be attached to Hilbert vectors. However, it must be clear that quantum physical items and physical fields are not the same stuff. Physical fields cannot be represented by closed Hilbert subspaces. They cover the whole universe and as a consequence they cover the whole Hilbert space. However, the strength of individual fields may be concentrated around separate excited places that are represented by single Hilbert vectors or a small set of Hilbert vectors. Such private fields may diminish with distance. Together the private fields form a covering field. For a given coordinate system that covering field has a partner curvature field that can be interpreted as gravitational field.

The actual physical items are distributed in space and are surrounded by potentials that act as a kind of blur. This is why quantum physics has much in common with optics. The superposition of the separate blurs characterizes the information transfer quality of the

corresponding field. For each particle a separate blur characterizes the quantum generation properties of that particle. At not too short distances the electromagnetic fields have the same shape as gravitational field. Locally, the EM fields and the gravitation field are based on the same Hilbert distributions. As is indicated above, the gravitation field is a derived field. The main difference lays in the fact that the charges of electromagnetic fields have the same size but may have different sign such that they may partly compensate each other's influence. The charge (mass) of the gravitational field is always positive, but it may differ in size. Another difference is that the gravitation field is the consequence of the decomposition properties of the other fields. Mass appears to be an expression of space curvature and on its turn this curvature is an expression of the rotation properties of the non-gravitational fields. The curvature fields that correspond with private fields do not compensate each other's influence. The masses of all physical items work together in order to create the immense potential that causes inertia.

GPS coordinates

One of the most intriguing facts is that a GPS operator that resides in the separable Hilbert space $\mathbf{H}$, cannot be used to define the position of particles. Due to the granularity of its eigenspace it would immediately introduce unnatural preferred directions. In contrast its equivalent, the GPS operator that resides in rigged Hilbert space $\mathbb{H}$ is very useful as a coordinate system for determining the location of field values. In this way it can be used to locate the field excitations that go together with particles. This continuous GPS operator cannot be used to locate the Hilbert vectors that represent particles. Thus there exists no GPS like operator that can be used to locate particles in Hilbert space. An alternative is formed by the strand operator. The strand operator uses the continuous GPS operator as a background coordinate system. Its eigenspace depends on the configuration of the covering field.

Test proposition

It is an elucidating experience to try to implement a complicated quantum logical proposition in the representation of quantum logic in Hilbert space. In that way we may discover how dynamics emerges in this static skeleton. For that reason, we choose as an example a predicate with quantifiers rather than a clean proposition.

The selected example proposition (♠) is

> "*All items in universe influence each other's position*".

We already gave the final conclusion of this experiment here: A well-ordered replacement of atomic predicates in an enveloping proposition appears to occur without extra field activity, but any deviation of a well ordered replacement causes an extra field activity in the form of influence of the complete set of all propositions. This explains the interaction between fields and physical items. A local deviation of the uniformity of the distribution of physical items can still cause a slight influence of neighboring items. At small distances the influences can be large. The influence of fields can be implemented in the Hilbert space. Via an action = reaction game the interaction between fields and Hilbert subspaces form the source of dynamics.

What further happens during the implementation of our example proposition (♠) is completely governed by mathematics. Thus, *for our example* no further extension of quantum logic is needed to transform it into a useful version of dynamic quantum logic. However, nothing is said yet of what occurs during the infinitesimal progression steps. During this step one static status quo is converted to the next static status quo. This will be the main subject of this e-paper.

Numbers

As number spaces we use the 2^n-ons of Warren Smith rather than the hyper complex numbers based on the Cayley-Dickson construction. Up to the octonions the corresponding number spaces are similar. (See http://www.math.temple.edu/~wds/homepage/nce2.pdf or the appendix). For higher n the 2^n-ons behave in a nicer way. They keep more of their number characteristics. We use the quaternions (n=2) as the number space that is used to define the inner product of the Hilbert space. However, we tolerate operators to have eigenvalues that are higher dimensional 2^n-ons. We also use 2^n-ons in order to set the values of physical fields.

When we use these numbers as eigenvalues or as field values, then we apply their number characteristics as well as their storage capacity. A 2^n-on contains 2^n real numbers. We also tolerate that eigenvalues of operators and values of fields support multiple sign selections, such as the inversion of the real axis and the handedness (chirality) of external vector products for their eigenvalues. 2^n-ons offer n sign selections and contain n independent imaginary base numbers. Each new independent base number introduces a new sign selection. The sign selections translate into an n-fold hyper complex conjugation and 2^n different skew fields.

With $n > m$, the 2^n-ons act like 2^m-ons in their lower m dimensions. Further, the 2^n-ons contain several subspaces of 2^m-ons. We may use smoothly curved manifolds that are crossed by curves which form trails of 2^n-on numbers and that are locally touched by tangent spaces that can be interpreted as 2^n-on number spaces.

When the members of a set 2^n-ons approach zero, then in their mutual arithmetic actions they are getting more and more the characters of lower dimensional 2^m-ons. In the same sense, when two 2^n-ons approach each other, their mutual arithmetic actions are getting more and more the characteristics the arithmetic of lower dimensional 2^m-ons.

The implementation of the proposition (♠) leads to a story of manipulators and manipulated observables. The number waltz feature ($c=ab/a$) of the 2^n-ons that becomes a noticeable effect for n>1 seems to play a significant role in our model. If this model applies to quantum physics, then it may reveal why special relativity exists and brings clearness in the different notions of time that exist in quantum physics. The curvature introduced by the spatial variance of what the longitudinal direction is reveals how the mentioned influences can be implemented as component fields which are defined on a curved coordinate system. This holds for gravitational fields as well as for the other fields such as electromagnetic fields.

Implementing quantum physics in a complex Hilbert space hides these interesting features and diminishes the insight that higher dimensional 2^n-ons can reveal.

Prospect

The article shows that there is a need to extend traditional quantum logic such that it not only includes the representations of fields but also includes axioms, which specify the dynamic underpinning of quantum physics.

In the course of this project several fundamental aspects of physics get uncovered.

Comments

Version 2

Version 2 builds on the content of version 1. This new version stands on itself, but it reorganizes and extends the contents of version 1. One reason is that the text in some paragraphs relies on the text of a series of other paragraphs, so that it is impossible to configure the paragraphs in a streaming order without repeating much of the content. Here we take the solution to refer to future paragraphs and add a "back to XXX" reference after the target text.

Project

This project is far from finished. Most parts I have rewritten several times. Some ingredients are already included before they are finalized and before they are put at the proper position in the context. I try to make the whole paper consistent with its parts and I try to keep my goal to include nothing that did not follow directly or indirectly from the axioms of traditional quantum logic. I only tolerate mathematics as a valid tool and ingredient. I will not use or accept intuition as a reason to include a subject in the text. However, I will use indications retrieved from previous experiences. I will also not tolerate the usefulness of a concept or its acceptance in the physical community as a valid argument to include that concept. If you encounter places where I did not succeed in that goal, then you may conclude that I still have to work on that section. When the paper gets its final version, then no deviations of my goal should result. However, partly due to my progressed age, I might never reach that condition. Then, *you* reader might take over and finish the job. But first think of the possibility that we will succeed. *What does it mean that all of fundamental physics is based on mathematics and on the ~25 axioms of traditional quantum logic?!*

References

This e-paper contains no lengthy reference list. References to other documents are usually presented inline and are mostly put in the form of hyperlinks. A sometimes referenced toolkit contains a collection of stuff that otherwise must be grasped from internet. As is done in this article, much of the contents of the toolkit are directly or indirectly obtained from Wikipedia. In that case the text is adapted to the requirements of the papers that use this toolkit. Most texts on internet are based on complex Hilbert spaces, so where necessary I have converted these texts into quaternionic versions.

Equation editor

This paper is prepared with MS Word 2010. This word processor version contains a rather capable equation editor and a large series of fonts including Cambria Math. However, the equation editor does not cooperate with the paragraph indexing in order to automatically enumerate the out of line equations. For that reason equations are enumerated manually and relative to the current paragraph header. References inside that paragraph just use the equation number. References from outside of the paragraph are hyperlinks that refer to the

paragraph header. The hyperlink text will then include the equation number. So, you must move manually to the equation inside the target paragraph.

EM fields

This paper draws significantly from the book on electromagnetic field theory of Bo Thidé. That book has a different goal and uses different premises. The book does not use the quaternionic field approach as is done here, but its contents easily translate to quaternions. Further its formulation is very precise, it links formulas to physical concepts and most of all it is online: http://www.plasma.uu.se/CED/Book/EMFT_Book.pdf.

Strands

I took many ideas from the research of Christoph Schiller as it is presented in his online book http://www.motionmountain.net/research.html .
If strand theory is a valid approach to a model of physical reality and if the theory presented here is also a valid approach, then strands must be representable in this model. For that reason examples of possible embedded definitions are presented. First strands are specified as item type definitions. There a single strand closely equals a photon type. Next a normal operator is defined that has eigenvalue sets in the form of strands.

Notation note

This paper uses {} in order to indicate a set or a function.
Depending on the context $\{|f_s\rangle\}_s$ means an ordered set of vectors $|f_s\rangle$ where s is the ordering index. In other contexts$\{|f_s\rangle\}_s$ means a vector function $|f(s)\rangle$ where s is the (discrete or quaternionic) parameter. Continuous functions are presented in the normal way.
$f(\{q_j\}_j)$ is a function $f(q_1, q_2, q_3, q_4,\ldots q_n,)$ of the set of parameters $(\{q_j\}_j$, where $j = 1, 2, \ldots, n$. The index constraint n might be infinity.

The appendix and the toolkit contain information about other notation and naming conventions that are used in this paper.

Acquired indications

Several indications stimulated the development of the theory that is presented in this paper. They are listed in this section.

Studying physics

When I was studying physics I was triggered by two facts that have significant influence on my current insights.

Classical versus quantum physics

After finishing the semesters that treated classical physics I started taking lessons in quantum mechanics and I was immediately amazed by the large difference in the way that classical mechanics was handled and the way that quantum mechanics was done. Questioning the teachers did not bring much relief. Their explanation was that the difference is due to the superposition principle. Investigating this reply reveals quickly that this is an alternative description of the different way of working, but no explanation. So, I dived into the library and into scientific bookshops until I finally found a booklet from P. Mittelstaedt: (*Philosophische Probleme der modernen Physik*, BI Hochschultaschenbücher, Band 50, 1963) that contained a chapter on quantum logic. I concluded that this produced the answer that I was looking for. Small particles obey a kind of logic that differs from classical logic. As a result their dynamic behavior differs from the behavior of larger objects. I searched further and encountered papers from Garret Birkhoff and John von Neumann that explained the correspondence between quantum logic and Hilbert spaces. In those years C. Piron wrote his papers that finalized my insight in this subject, but first I must explain the other fact that triggered me.

The rediscovery of quaternions

Quantum physics appeared to be done in the realm of Hilbert spaces. Operators in those spaces delivered the eigenvalues that played the role of values of observable quantities.

I had problems with the fact that according to the in those days commonly accepted theory the operators, which deliver observable values as their eigenvalues or as their expectation values, had to be self-adjoint and as a consequence these operators could only deliver real valued eigenvalues. Nature has a clear 3+1 dimensional structure and there was no logical indication in the quantum theory that was lectured in those days that explained why four observable values must cling together. I started searching for a number system that could deliver this extra connectivity and I quickly discovered a number system with 3+1 dimensions that supported addition, multiplication and division. It took me more time to discover that this number system was already discovered more than a century before by William Rowan Hamilton when he was walking with his wife over a bridge in Dublin. He was so glad about his discovery that he carved the corresponding formula into the sidewall of the bridge. The inscription has faded away, but it is now molded in bronze and fixed to the same wall. When an assistant professor told me the story I started to read papers on quaternions and discovered the work of Constantin Piron.

Birkhoff and von Neumann already discovered that the set of propositions in a traditional quantum logic system is lattice isomorphic with the set of closed subspaces of an infinite dimensional separable Hilbert space $\mathbf{H}$.Piron proved that the inner product of this Hilbert space must be specified with members of a division ring. There are only three division rings:

the real numbers, the complex numbers and the quaternions. I went for the widest choice and started studying quaternionic Hilbert spaces.

Representation restriction

After discovering that traditional quantum logic can be represented inside an infinite dimensional separable Hilbert space, it is a disappointment to discover that this presentation does not cover physical fields and does not cover dynamics.

Extended quantum logic

However, it appears that it is possible to expand traditional quantum logic in a way that corresponds to adding blurs to a subset of the Hilbert vectors. The blurred subsets may represent particles. In this way the superposition of the blurs may represent the physical fields. This leads directly to the existence of Maxwell-like fields in the realm of such an extended Hilbert space. This also leads to an extended quantum logic that covers physical fields.

Dynamic quantum logic

Dynamics can be implemented by representing dynamic quantum logic as a sequence of extended traditional quantum logics that each represents a static status quo. This dynamic step can be detailed further. See Progression step details. With respect to the Hilbert space the dynamic model uses a sequence of extended Hilbert spaces. It can be compared to a book, where each page represents a static status quo.

Curved space

When quaternions are taken as the division ring, then Fourier transforms become quaternionic Fourier transforms. The ideal Euclidean formulation of the multi-dimensional Fourier transform cannot cope with a variable direction in which the Fourier analyzed function or field is rotation free. This can be circumvented by converting the field by a coordinate transformation such that the resulting field has a stationary direction in which it is rotation free. This corresponds with accepting the existence of a curved coordinate space. This curved space is subject of general relativity. With other words, extended quantum logic supports general relativity.

Intensified imaging

After finishing my study I started my career in the development laboratory for high-tech electronic appliances of a big electronics company. My task consisted of the analysis and measurement of the visual trajectory, starting from the radiation source and ending after interpretation of the image in the brain of the observer. At those times (~1975) this was fundamental research, because both the measuring methods and the modeling methodology in this area were still in their childhood. The target products for the laboratory were night vision devices and X-ray image intensifiers.

Intensified imaging is required at low dose rates and in situations where the radiation detection capability of the human eye is unsuitable. This occurs with starlight scene imaging and with X-ray shadow imaging. The low dose rate is necessary due to the fact that no active scene lightning can be supplied or due to the fact that hazardous gamma ray effects must be avoided.

When the snowing image produced by image intensifying equipment is observed, then it becomes immediately clear that this image is built up from a large number of separate spots that together form a rather noisy picture of the object.

The research not only concerned perception experiments and measurement. We also had to devise the standards for the measurements as well. So we took part in the establishment of develop worldwide standards for the specification and measurement of the optical transfer function (OTF) and its modulus the MTF. We also took part in the committees that created the standards for the detective quantum efficiency (DQE).

The fact that these standards were not only required but were also successful is in itself very astonishing. We needed these standards because we could model the visual trajectory as a chain of which the first elements consisted of a set of Poisson generators. The generators are characterized with their efficiency and a spatial, angular and chromatic distribution.

Next in the chain are attenuating binomial processes. Statistically a blur can also be considered as a binomial process. The information is spread over a larger area. A Poisson process can be combined with a subsequent binomial process into a generalized Poisson process that has a lower efficiency.

The chain also contains light lenses and particle lenses. Further, the equipment aided chain contains detection surfaces that convert radiation quanta into electrons or electrons into radiation. This chain might also contain scintillation layers that convert high energy X-ray quanta in large series of low energy light quanta. It might contain fiber optic plates that just transport images, usually from a curved to a flat surface. It might contain channel plates that convert single electrons into clouds that contain about hundred thousand electrons. It might contain image receivers that convert the image into an electric signal or into a photographic plate.

The equipment aided chain may and the unaided chain will also contain the eye of a human observer. Intensified images are detected by the cones in the fovea. At very low light levels the adapted eye detects the images via the rods in the fovea. Rods have a much lower acuity than cones. Therefore they have a much longer integration time. In general, measures that reduce noise have both positive and negative effects on the information content of the signal. There is an optimum condition. My task was to find that condition.

After the detection in the fovea the received signal is handled by a large series of preprocessors that act in parallel as well as in sequence. The preprocessors associate the signals that are received by receptors that lay in each other's neighborhood. The association tests a detail pattern that is typical for the considered preprocessor. The associated signal is only passed further when its signal to noise ratio surpasses a given boundary level. In this way the higher regions of the information processing are not disturbed by unnecessary noise.

All preprocessors work in this way as noise filtering decision centers. The association results in a categorization of the encoded image. The signal that reaches the folded fourth layer of the visual cortex represents the completely coded version of the received image. In the

human brain, a folded surface of about four square millimeters is devoted to each image receptor in the fovea. This code is interpreted further in the brain. As early as possible the filtering process stops noise and details of the image that do not carry useful information from proceeding further in the chain.

Due to this design, already the unaided brain-eye combination is well suited to perceive and interpret images in a very large dynamic range of circumstances. Apart from the fact that the visual channel can adapt from somewhat above starlight conditions until bright daylight conditions, the visual trajectory appears to be optimized for handling signals that enter the eye in the form of clouds of quanta that are generated by Poisson processes.

All vertebrate visual trajectories work according to the sketched principles. Over billions of years evolution has exploited the fact that information that comes to living species is generated by Poisson processes. The visual trajectory of vertebrates is optimized for handling this information for the benefit of the survival of the owner of this channel. See: http://www.crypts-of-physics.eu/Howthebrainworks.pdf .

This fact is a strong indication that all visual information comes to us in the form of clouds of quanta. When looking at low dose rates through an intensified viewer, it becomes clear that this assumption is valid. The perceived noisy image is built from separate dots that represent the detected quanta. No radiation wave is visible. What you see is just a streaming cloud of quanta.

The fact that visual information is generated by Poisson processes indicates a more general feature of physics. ALL information that is transferred by electromagnetic radiation can be considered to be generated by Poisson processes. This becomes apparent when measurements are done at very low dose rates. In the static model the Poisson processes represent a lateral distribution. In addition, taken over a sequence of Hilbert spaces the Poisson processes represent a temporal distribution.

Granular GPS

In separable Hilbert space $\mathbf{H}$, the granularity of the eigenspace of a GPS-like coordinate operator presents problems with the fact that a dense packaging of the granules generates unrealistic preferred directions. Its equivalent in the corresponding rigged Hilbert space $\mathbf{\mathbb{H}}$ does not suffer this restriction. It can be used as coordinate system for fields, but it cannot be used to locate particles inside the separable Hilbert space $\mathbf{H}$.The fields are attached to a subset of the Hilbert vectors and all Hilbert vectors touch their values.

Progression step details

After the former indications the theory reaches the stage that it becomes sensible that the model of nature, which takes its foundation on quantum logic, steps from one static status quo to the next. It dawns that this is the way that dynamics is implemented. What happens during these steps is still mysterious. The Hilbert space itself only suggests a Euclidean signature of observable space time. However, Einstein proved that observable spacetime has a Minkowski signature. This discrepancy must have its origin in what occurs inside the dynamic step. An early conclusion is that coordinate time does not play the role of the fourth dimension in the quaternionic eigenspace of a spacetime-like operator. Both inertia and

Feynman's approach of the path integral may guide what happens during an infinitesimal dynamical step.

Release and removal of quanta

During the step interactions take place and particles are emitted or absorbed. The information is carried by clouds of quanta. The quanta carry the information that they collect GPS and GMS related data in the dynamical step.
An indication for this fact houses in the structure of the creation and annihilation operators. These operators consist of a part that relates directly to the GPS operator and a part that directly relates to the GMS operator.

Fields and probability amplitude distributions

Some subsets of Hilbert vectors represent elementary particles. It means that they are blurred. The blur is a probability amplitude distribution whose form is typical for the elementary particle type. Elementary particles combine to form more complex particles.
The superposition of all probability amplitude distributions that correspond to the separate particles forms the covering field.
A repositioning of a particle means a reconfiguration of the covering field and vice versa.

A detailed list of indications and considerations

1. All information comes to us in the form of clouds of quanta.
2. These clouds get their shape via a combination of probability amplitude distributions.
3. Each type of elementary particle is characterized by a set of Hilbert vectors and a particular kind of probability amplitude distribution.
4. The information contained in the quanta and in the cloud is the only information that becomes observable.
5. This information consists of the information that is carried by the separate quanta and by the probability distribution that describes the cloud.
6. The probability amplitude distribution that characterizes a particle becomes part of the field that exists in the surroundings of the particle.
7. Physical fields consist from the superposition of the probability amplitude distributions of the separate particles.
8. Each quantum in the cloud carries a set of information data.
9. This set contains a 3D position, a 3D momentum and a binary chirality.
10. Curvature and torsion of the path of the particle are secondary characteristics, which are introduced via the probability distributions that make up the field that exists in the direct environment of the particle.
11. Curvature in the path of a particle is caused by the local rotation that exists in the surrounding field(s).
12. The rotation properties of the field determine the local decomposition of the static field.
13. This local decomposition determines a curvature of observable space.
14. On its turn this curvature specifies a curvature field.
15. The curvature field has all the characteristics of the gravitation field.

16. The generation of a given kind of quantum has a typical probability.
17. There exist anti-quanta. The generation of an anti-quantum is equivalent to the annihilation of the corresponding quantum.
18. Creation and annihilation operators have probability amplitude distributions as their eigenfunctions.
19. In their simplest form these probability distributions are Poisson distributions.
20. The generation of shot noise is characterized by Poisson distributions.
21. At high dose rates the Poisson distributions become Gaussian (normal) distributions.
22. For more complicated configurations the probability amplitude distribution must be considered rather than its squared modulus: the probability density distribution.
23. Bosons are characterized by probability amplitude distributions that remain invariant under a rotation of 2π.
24. The probability amplitude distribution of a two boson system is invariant under perturbation of the bosons.
25. The creation and annihilation operators of bosons are characterized by a non-zero commutator.
26. Photons form the simplest boson type.
27. The probability distribution of the corresponding quanta resembles a Poisson distribution.
28. Fermions are characterized by probability amplitude distributions that change sign under a rotation of 2π.
29. The probability amplitude distribution of a two fermion system changes sign under perturbation of the fermions.
30. With each fermion type an anti-type exists.
31. A quaternionic probability amplitude distribution can also contain chirality information.
32. When chirality is taken into account then a probability amplitude distribution must be used rather than a probability density distribution.
33. The creation and annihilation operators of fermions are characterized by a non-zero anti-commutator.
34. Creation and annihilation operators can be split in a part that resides in configuration space and a part that resides in Fourier space.
35. A quant can be emitted (created), absorbed (annihilated) and it can be virtual.
36. Virtual quanta belong to previous or future events.
37. Only actual quanta deliver observable information.
38. Emitted and absorbed actual quanta belong to the current version of events.
39. During each dynamical step information is collected both from configuration space related sources and from momentum space related sources.
40. The part of the collected information that resides in configuration space delivers the 3D position information to the emitted/absorbed/virtual quant.

41. The part of the collected information that resides in Fourier space delivers the 3D momentum information to the emitted/absorbed/virtual quant.
42. The sum of an even function and its Fourier transform is invariant under Fourier transformation.
43. The difference between an odd function and its Fourier transform is invariant under Fourier transformation.
44. Apart from a scale factor, the functions that characterize linear and spherical harmonics are invariant under Fourier transformation.
45. The scale factor is 1, i, -1 or –i.
46. The harmonic functions are also related by creation and annihilation operators.
47. The harmonic functions contain a factor that equals a Gaussian probability distribution.
48. Any knot can be represented topologically by equations in Cartesian coordinates x, y, z of the form: $x = f(t), y = g(t), z = h(t)$, where $f(t)$, $g(t)$ and $h(t)$ are Fourier series with finitely many terms.
49. Only in 3D space knots cannot all be unknotted.
50. There exist three basic types of elementary particles that can be distinguished via the number of strands / Hilbert vectors involved. These basic types are the bosons, the quarks and the leptons.
51. The quarks and the leptons can be distinguished in three generations.
52. The particles are distinguished via the probability amplitude distribution of the corresponding quanta and the kind of information that is carried by these quanta.
53. There exist three basic forms of interaction that are distinguished via the number of strands/Hilbert vectors that are involved in the interaction event.
54. These basic forms of interaction can be related to Reidemeister moves.
55. Particles become observable via their interactions, thus via the quanta that are generated due to these interactions.
56. All motion observed in nature minimizes action.
57. Uniform motion preferably occurs via a geodesic and obeys the geodesic equation.
58. The visual trajectory of vertebrates is devised in order to cope with a huge dynamical range of light conditions ranging from starlight conditions up to bright daylight conditions
59. Over billions of years, evolution has exploited the fact that information that comes to living species is generated by Poisson processes. The visual trajectory of vertebrates is optimized for handling this information for the survival of the owner of this channel.
60. The Banach–Tarski theorem states that a spherical surface can be split in five peaches that can form two spheres of the same volume. The statement does not hold in the eigenspace of a Hilbert coordinate operator.
61. At least one coordinate operator has lattice sampling properties. Its eigenspace shows preferred directions.
62. At the lowest scale it is not clear how the granules of an eigenspace of a Hilbert position operator are geometrically arranged.

63. The geometric sampling of normal operators between subsequent Hilbert spaces may differ.
64. Particles can be considered as sources and drains of information carrying quanta.
65. These sources and drains play their role in a continuity equation that treats information carried by quanta.
66. The concept of measurement has no significance at Planck scales.

The indications and considerations that are treated in this chapter will steer the development of the theory that is subject of this e-paper.

First conclusion

The standard model can be retrieved via categorization of the particle types and their interactions. This comes down to categorizing probability amplitude distributions and categorization of information packages that are carried by generated quanta.

Logic

Logics

Quantum logic differs from classical logic in one of its axioms. The propositions in classical logic is isomorphic with the set of Venn diagrams. The set of propositions of traditional quantum logic is lattice isomorphic with the set of closed subspaces of an infinite dimensional separable Hilbert space **H**. This isomorphism means that quantum logical propositions can be represented by closed subspaces of a Hilbert space. The inner products of that Hilbert space can be defined by using numbers of a 2^n-on number space. Taking $n>2$ for that purpose raises numeric problems with the closure of the subspaces. Traditional quantum logic does not include any axioms that treat dynamics and it does not treat the influences of physical fields. It only specifies stationary relations that are possible between physical items and their properties.

Example proposition

In order to discover the emergence of dynamics we will implement a quantum logical proposition in Hilbert space and test its truthfulness. We will introduce in this example proposition physical fields as well as dynamics.

The example proposition(♠) is:

All items in universe influence each other's position. (♠)

It can be answered with either yes or no. And, if we succeed, it can be implemented in Hilbert space. So, in that case it is a valid quantum logical proposition.
Proving 'yes' is cumbersome, but the 'no' is hardly less difficult. It requires finding an item of which the position is not influenced by at least one of the other items. For this purpose it is necessary to implement notions of *items*, the *universe*, *influences* and *position* in Hilbert space.

The statement includes quantifiers (*position*) and operational elements (*influence*). The set of axioms of traditional quantum logic does not treat these subjects. At least it does not do that in a realistic way. As we will see, the *influence* of the *universe* of propositions (*items*) will put particular restrictions to the extension of quantum logic into the realm of an extended dynamic logic. This restriction is manifested in the occurrence of physical fields and inertia.

Translated in physical terms inertia means that in contrast to a *uniform movement*, the *acceleration* of an *item* will go together with the *action* of a *physical field*. Notice that we use the words "goes together with" instead of "generates" or "causes".

Translated in logical terms a conclusion of the analysis of inertia runs:
"During a redefinition of a proposition the exchange of atomic predicates in that proposition must be done in well-ordered and controlled steps. Otherwise the *community of propositions* will *influence* the considered proposition."

Again it must be noticed that there is no causal relation between the event of being well-ordered and the event of influencing. The interaction is instantaneous.

When nature's logic is put in axioms, then influences that correspond to physical fields must follow from the axioms. Together with the specification of the origin of dynamics this will then result in a dynamic version of quantum logic.

I assume that this category of logic does not yet exist in mathematics. There exists a version of dynamic operational quantum logic, but it does not cover or mention the effects of the representation of physical fields in logic and it does not specify the *origin of dynamics*.

Atomic predicates

Atomic propositions are statements that are either true or false and which cannot be broken down into other simpler propositions. When an atomic proposition concerns a property, then it may contain the value of that property. We will call that kind of atomic propositions atomic predicates. For example "The speed is 5." The identity or the category of the property is "speed". The value of the property is 5 with a certain inaccuracy. The inaccuracy is typical for the category of the atomic statement. The dimension of the value is "meter per second", but that is another atomic statement and it is a fixed statement. Both the dimension and the inaccuracy form extra information that is part of the type definition of the atomic predicate category "speed".

The atomic predicates form a set with a particular lattice structure. In this set we only consider atomic predicates that are independent of all other atomic predicates. Several choices of such sets exist. A subset consisting of members of a chosen set may be canonical conjugates of members of another set.

In Hilbert space the type definitions of atomic predicates that concern numeric variables are represented by **operators**. The values of the properties in the atomic predicates correspond to the eigenvalues of the operators or they are expectation values. Expectation values are statistically determined via a probability characteristic that characterizes both the operator and a physical item. See Wave function. In separable Hilbert space **H**, the eigenspaces of all normal operators are granular.

Type definitions

Type definitions are propositions that describe and categorize subjects without specifying their variable values. An **atomic predicate type** is the type definition of a category of atomic predicates and specifies the type of property that these propositions treat. The definition also contains the *physical dimension* (unit) of the property, the inaccuracy and the allowed *range* of the potential values of this property. For example, if that category is "speed", then the definition contains the *physical dimension* meters per second. The minimum of the absolute value is zero and the maximum of the absolute value is c. Speed is an imaginary quaternion.

When the type definition concerns a more complex object that can act as an individual the definition will be called an **item type** definition. Item type definitions use **atomic predicate types**.

When that item cannot be broken into simpler objects that still can act as an individual, then the type definition is an **elementary type** definition. Elementary type definitions are constructed of type definitions of atomic predicates.

The elementary types form (a rather small) subset of the whole set of type definitions. Elementary types appear to divide into two categories**: bosons** and **fermions**. The fermions can be divided in leptons and quarks

If the item is not an elementary type, then its type specification is a **system or sub-system type** definition. A (sub)-system type definition is constructed of elementary item type definitions and atomic predicate types.

The type definitions form a set with a different lattice structure. Its structure is isomorph with the structure of classical logic.

In Hilbert space no representation for item type definitions exists. However, in Hilbert space atomic predicate types are represented by operators.

Strands as type definitions

The notion of strand is introduced by Christoph Schiller. Apart from its crossing switch events, a strand is not observable. In strand theory the boson type are represented by a single strand. Reversely a strand is equivalent to the type definition of the simplest boson, which is the photon.
This e-paper defines a strand operator that has an eigenspace in which chains of granules reside. These chains come close to Schiller's strands. In each chain only the current granule and its immediate neighborhood deliver observable values.

Items

The first problem that is raised by constructing the representation of proposition (♠) is to determine what in this representation stands for an item. The simplest solution is to attach a subspace of the Hilbert space to the item. The corresponding proposition can be phrased as: "*This is the item*". Something either belongs to the subspace or it is outside that subspace. Everything that can be attributed to the item can also be attributed to this subspace. Each of these propositions belongs to a hierarchy for which the mentioned proposition forms the top. All sub-ordered propositions correspond to subspaces of the item's subspace. In this way the universe of items can be represented by a set of mutual orthogonal subspaces of the Hilbert space. Rays that are spanned by a single Hilbert vector and that are connected with a numeric value can be considered as atomic predicates. Subspaces spanned by such rays that are related to the same type of value can be considered as statements with a wider scope. The rays can be subspace of an items subspace. The subspace that corresponds to a conglomerate of elementary items also represents that conglomerate as an item. The configuration of the subspace that represents an item will change as a function of the parameter that measures the progression of the dynamic behavior of the item. It is possible that not only the values of the atomic predicates change. The types of these atomic predicates may change as well. This happens for example with atomic types that are each other's canonical conjugate. It is also possible that the configuration of the subspace changes more drastically.

In a set of subsequent Hilbert spaces the subspace that represents the item can be moved around with respect to a selected base consisting of eigenvectors of a normal operator. In this way it may be possible to implement the dynamics of items. This moving around does not mean that the vectors are moved around. It means that at each step of the move the set of

vectors that span the considered subspace is **redefined**. The redefinition corresponds to a redefinition of the corresponding proposition. Alternatively, it is also possible to redefine the selected normal operator. Thus, redefinition and the laws that govern redefinition convert the static quantum logic into a dynamic version of quantum logic. It will be shown that physical fields play a significant role in this redefinition.

With his bra-ket notation Dirac has provided us with a marvelous symbolism for vectors and even for operators. He did not provide us with symbols for subspaces. However, it is easy to extend his symbolism and indicate a subspace with a set of vectors that spans that subspace. For example $\{|f_s\rangle\}_s$ indicates a set of element vectors $|f_s\rangle$ with enumerator s that span a closed subspace. This set identifies the subspace. Different sets may identify the same closed subspace.

It is sensible to have one vector inside the item's subspace that is considered as characteristic for the location of the representation of the item in Hilbert space. We reserve the name **locator** for this vector. When the item is redefined, that vector may be redefined as well. This characteristic vector can be used to obtain a precise location of the subspace in Hilbert space. The process via which the locator is determined depends on the requirements. The requirements may be set in relation to an operator. For example the vector that corresponds with the expectation value of the operator for that subspace can be chosen as the locator. In that case the state vector that corresponds with that operator may play the role of the locator. Two or more bosons can share the same locator. Fermions that possess the same property values cannot share the same vector as a locator.

Atomic predicates are not considered to be statements that fully describe a physical item. The statement "This is the item" forms the top of a hierarchy of statements that all say something about the item. The hierarchy contains statements that define the item's type. Other members of the hierarchy specify the items constituents. Still other statements concern the item's atomic variables that together with the type definition specify the item's identity. For atoms the variables of the subsystems are hidden from the outside of the atom. This means that atoms can be considered as modules.

Representation of items

A subspace in a single separable Hilbert space $\mathbf{H}$ cannot represent all properties of a physical item. The fact that the position of the item is known means that an eigenvector of the position operator resides within the subspace that represents the item. Say that this subspace covers position values in a certain region. Heisenberg's uncertainty principle now states that the value of the momentum of the item is uncertain. Any values of this property must correspond with eigenvectors of the momentum operator that also reside in this subspace. For elementary particles the subspace will be too small in order to guarantee sufficiently sure property values. Sufficient information could be collected when the Hilbert space also contains past and future data, such that the momentum can be derived/estimated from those data. The physical fields contain such preconditions. For a free elementary particle the momentum can be derived from the Fourier transform of the probability amplitude distribution that controls the position of the particle. This probability amplitude distribution is the wave function of the particle. Together with the subspace that represents the particle, the wave function represents all information that can be retrieved from the particle. Since all

particles have such probability amplitude distributions these private fields get intermixed. Thus in the neighborhood of other particles the superposition of the private fields must be reckoned rather than a single private field.

Vacuum

Multidimensional subspaces exist that do not represent a dynamical item. They can be considered as **vacuum**. It is still possible that the subspace represents a ground state. We will assume that on the average the 'filled' and the vacuum subspaces are evenly distributed over a connected part of the Hilbert space. The phrase "evenly distributed" means that the distance between the representations of items makes sense. Here we do not mean the distance related to the norm of Hilbert vectors, but the coordinate related distance that will be introduced later.

"Vacuum" does not say that these subspaces are empty. It is rather an indication that the subspace does not represent a dynamical object. Instead the subspace may represent a ground state.

Vacuum does not generate observable information quanta. In vacuum the clouds of quanta are empty.

Hilbert spaces

Dual views of a Hilbert space

In Hilbert space normal operators exist whose eigenvectors form an orthonormal base of the Hilbert space. The canonical conjugate of that normal operator has a set of eigenvectors that is completely disjoint of the former orthonormal base. This fact defines pairs of views of the same Hilbert space that are related via canonical conjugation.

The corresponding orthogonal bases do not touch. Every base vector is a linear combination with non-zero coefficients of all members of "the other base". All coefficients have the same norm.

Position

The original proposition (♠) speaks about the position of the item. The position must be related to something that is available in the Hilbert space. The Hilbert space is defined over a number space. Thus we might attach a number of this number space (or a higher 2n-on) to the Hilbert subspace that represents the item. That number must represent position. The natural way of attaching numbers to subspaces of a Hilbert space is via the concept of eigenvalues of normal operators. Any symmetry transform of a selected normal GPS coordinate operator meets the requirements.

There exists one significant drawback. The eigenspaces of all normal operators that reside in a separable Hilbert space $\mathbf{H}$ are granular. A dense packaging of the granules generates preferred directions. If we want to avoid this, then we must use the equivalent operator that resides in the corresponding rigged Hilbert space $\mathbf{\mathbb{H}}$. That operator has a continuum as its eigenspace. Apart from the real axis of the hyper complex number system it shows no preferred directions. So for position observations we must take rescue in rigged Hilbert space $\mathbf{\mathbb{H}}$. We can only use the corresponding GPS operator for providing a background coordinate system. The GPS operator is not a part of the separable Hilbert space $\mathbf{H}$.It cannot be used to locate the vectors of the separable Hilbert space!!! However, we can use it to give field values a location.

For a given field we may choose a position operator Q such that for that field we can work with the ideal form of the quaternionic Fourier transform. That means that by using these coordinates as parameters, the field that will be analyzed has decompositions that run along straight lines in the eigenspace of Q. Q introduces a new coordinate system that is curved with respect to the original GPS coordinate system.

The new coordinates are characterized by the fact that the considered field when formulated using these coordinates shows a decomposition into static parts that runs along straight coordinate lines. A Fourier transform taken in these coordinates has universe wide validity. The canonical conjugate P of operator Q also shows a similar behavior for the Fourier transform of the analyzed field that was first stated in Q coordinates and after transformation is specified in P coordinates.

However, due to the fact that the Hilbert space is separable, the situation is far more complicated than is sketched in this section.

Generating a Hilbert space GPS

The first step is the introduction of a suitable GPS system in Hilbert space. This can be done by taking an orthonormal base of Hilbert vectors and add quaternion values to them. The number set must be countable. Let us take the rational quaternions as an example. This construction defines a normal operator Q with countable infinite number of eigenvectors |q> and corresponding eigenvalues q. We will use the name **coordinate space** for the eigenspace of the **coordinate operator** Q. As indicated earlier the eigenspace of Q shows preferred directions.

The quaternions clearly have an origin. In contrast, the unit sphere of the Hilbert space, which contains all eigenvectors of Q is an affine space. The eigenvectors of Q form an orthonormal base. This singles out the eigenvector that belongs to the origin of the eigenspace. It indicates that Q must only be used for relative locations.

When we speak about the (Q) **coordinate distance** between two vectors |f> and |g> in Hilbert space, then we mean the numerical distance between the values of <f|Q f>/<f|f> and <g|Q g>/<g|g>.

Q has an infinite but countable number of eigenvalues. A location in coordinate space represents a location on the unit sphere of Hilbert space.

The fact that Q must be bounded means that Q has a **boundary** ∋ at a finite distance from its origin.

Every location in the eigenspace of Q has a unique representation in the boundary ∋ of Q and vice versa.

Take the polar decomposition of the normal coordinate operator Q in a unitary part Ų and a positive operator Ṇ. The eigenspace of Ų is the **uni-coordinate space**. Like the unit sphere of the Hilbert space, the uni-coordinate space *is an affine space*. Besides of that also no preferred direction should exist in this unit sphere. But that is not the case!

The eigenspace of Q consists of all eigenvalues of Q. The eigenspace is not a closed set and it does not include infinity. The eigenspace of Q is granular. In order to be able to act as a kind of GPS the granules must have a fixed size. A dense packing of the granules will create preferred directions. Thus, Q is not isotropic. In contrast, the unit sphere of the Hilbert space is isotropic. This sphere contains all eigenvectors of Q. With granularity spread in a regulated order, the granularity decouples three (preferred) imaginary directions. As a consequence the size of an infinitesimal step will depend on direction. This does not generally correspond with physical reality. Only in condensed matter such conditions may occur.

With artificial means the eigenspace of the coordinate operator may be closed by adding all limits of converging rows of eigenvalues. In this away a closed set of quaternions results. However, most members of this closed set are not eigenvalues of the coordinate operator Q. The set is eigenspace of a corresponding coordinate operator Ǫ̆ in a rigged Hilbert space **Ħ**. Still, the use of the separable Hilbert space **Ħ**, coordinate operator Q will always introduce preferred directions. Thus, for realistic physical conditions an alternative for this coordinate operator Q must be sought. The coordinate operator Ǫ̆that has its residence in the rigged Hilbert space **Ħ** does not suffer from preferred imaginary directions. For that reason we can use it as a **background coordinate operator**. In the future we will indicate the background operator Ǫ̆ as **the GPS operator**. We will use the name **GPS like operator** for any

operator that has an eigenspace that can be obtained via an invertible continuous transform or a reflection from the eigenspace of operator $\mathfrak{Q}$.

Canonical conjugate

The four dimensions of the quaternions enable the split of Q into one Hermitian and three anti-Hermitian components. Via the inner product of the Hilbert space, each of these components gets a canonical conjugate. This creates a GMS-like operator.

$$< \mathfrak{q}_\mu | \mathfrak{p}_\mu > = \tilde{f}_\mu \left(\mathfrak{p}_\mu \right) = < \mathfrak{p}_\mu | \mathfrak{q}_\mu >^* = f_\mu{}^* \left(\mathfrak{q}_\mu \right) = exp\left(\boldsymbol{n_\mu} \cdot \mathfrak{p}_\mu \cdot \mathfrak{q}_\mu / \hbar \right) \tag{1}$$

$\mu = 0, 1, 2, 3$ is the index of the dimension.

$\boldsymbol{n}_\mu = i,\ \boldsymbol{i},\ \boldsymbol{j},\ \boldsymbol{k}$ are imaginary base numbers.

$$\boldsymbol{n}_\mu^2 = -1 \tag{2}$$

The Hilbert vector $|\mathfrak{q}_\mu >$ is eigenvector of operator $\mathfrak{Q}_\mu$ and corresponds with eigenvalue $\mathfrak{q}_\mu$.
The Hilbert vector $|\mathfrak{p}_\mu >$ is eigenvector of operator $\mathcal{P}_\mu$ and corresponds with eigenvalue $\mathfrak{p}_\mu$.

The constant $\hbar$ relates to the size of the granules.
For each dimension index μ holds:

$$< \mathfrak{q}_\mu \left| \mathcal{P}_\mu f \right. > \cdot \Delta \mathfrak{q}_\mu = \boldsymbol{n}_\mu \cdot \hbar \cdot \Delta < \mathfrak{q}_\mu \left| f \right. > = \boldsymbol{n}_\mu \cdot \hbar \cdot \Delta f(\mathfrak{q}_\mu) \tag{3}$$

$$[\mathcal{P}_\mu, \mathfrak{Q}_\mu] = \mathcal{P}_\mu Q - \mathfrak{Q}_\mu \mathcal{P}_\mu = \boldsymbol{n}_\mu \cdot \hbar \tag{4}$$

The definitions of the four canonical conjugates also define four (decoupled) complex Fourier transforms. The granularity decouples the Fourier transforms.

The Hilbert space GMS

The GMS operator $\check{\mathcal{P}}$ of the Hilbert space is the canonical conjugate of the Hilbert GPS operator $\check{\mathfrak{Q}}$. Both operators reside in the corresponding rigged Hilbert space $\mathbf{H}$. The canonical conjugate $\mathcal{P}$ of $\mathfrak{Q}$ is formed from the combination of the four $\mathcal{P}_\mu$ operators. The same reasoning that is used for the $\mathfrak{Q}$ operator also holds for the $\mathcal{P}$ operator. It means that also the $\mathcal{P}$ operator has a granular eigenspace and it has a boundary $\mathfrak{C}$. Both boundaries have a one to one correspondence with the unit sphere $\mathfrak{O}$ of the Hilbert space, but none of the eigenvectors of the $\mathfrak{Q}$ operator coincides with an eigenvector of the $\mathcal{P}$ operator.

GPS stands for Global Positioning System.
GMS stands for Global Momentum System.

The fourth dimension

Often time is perceived as the fourth dimension. However, Piron and Einstein prove that our common notion of time is unfit to act as the fourth dimension of the hyper complex number space. Einstein's special relativity indicates that a Minkowski signature characterizes the common spacetime concept. It means that a rectangular triangle relation exists between the spacetime step, the space step and the coordinate time step, where the time steps acts as the hypotenuse. Thus the coordinate time step is not perpendicular to the space step as the

fourth dimension would be. In combination with Garret Birkhoff and John von Neumann, Constantin Piron proved that the values of inner products of Hilbert vectors must belong to a division ring. When observables must stay expressible in such numbers, then they can maximally be quaternions. In that case, the real part of the quaternion of which the imaginary part represents a space coordinate, cannot be coordinate time.

Another argument is the following. Neither traditional quantum logic nor the corresponding separable Hilbert space $\mathbf{H}$ can represent dynamics. Thus, time is not an observable that fits in this separable Hilbert space $\mathbf{H}$.However, both traditional quantum logic and the corresponding Hilbert space may contain items that represent the precondition of change.

Due to the fact that the Hilbert space is separable, the observable quantities must be granular. For example the granularity of space is characterized by the Planck length l_{Pl}. The fourth dimension must be granular as well.

Two quantities, other than space and time that are known to be granular are action and entropy. The granularity of action is characterized by the Planck constant. The granularity of entropy is characterized by the Boltzmann constant. Both are valid candidates for the fourth dimension. Energy is not a valid candidate, because it represents action per unit of time. Thus, it would introduce a notion of time via this backdoor. Action represents change. Entropy represents potential change. Field values represent preconditions of change.

Another possibility is to use the spacetime step as the fourth dimension. This step is perpendicular to the space step. This interpretation immediately poses the question what then the physical significance is of this spacetime step. In contrast to the action step it does not show up in equations of motion.

Next lures the interpretation of the spacetime step as something that directly relates to the action step. This would give the spacetime step an acceptable physical interpretation.

The relation must be hard and global. Otherwise the Minkowski signature loses the general validity that it appears to have in physical reality. If it is true, then the action step is always perpendicular to the space step. If it is not true, then it is still possible that part of the action step is parallel to the spacetime step. For that part the mentioned relation can be in correspondence with the Minkowski signature. Then, still the interpretation of the other part of the action step results.

Until we encounter the requirement to fill it, the gap of the fourth dimension can be left open. One thing is sure; coordinate time does not fit in that gap.

Time and dynamics

Dynamics and its parameter time do not fit in a Hilbert space that can only represent a static status quo. That means that this Hilbert space does not support the corresponding operators. However, the static representation of the preconditions of change are represented in this Hilbert space. Its interpretation is then as the precondition for the change that will follow in the next dynamical step.

For example potential displacement is characterized by momentum, which is the canonical conjugate of space. An external time step is required in order to determine the actual displacement. The time step occurs between subsequent Hilbert spaces. As a consequence the displacement gets its significance by comparing subsequent Hilbert spaces.

Action is change. In this respect its role is similar to the role of displacement. Also the action step gets its significance by comparing subsequent Hilbert spaces. In this respect action can be compared with space. Fields represent the preconditions for the next action step.

When the smallest possible space step $l_{Pl} = \sqrt{\hbar G/c^3}$ and the smallest possible coordinate time step $t_{Pl} = \sqrt{\hbar G/c^5}$ are put into the Minkowski signature, $\boldsymbol{\Delta t}^2 = \boldsymbol{\Delta \tau}^2 - \boldsymbol{\Delta q}^2/c^2$ then the corresponding spacetime step $\boldsymbol{\Delta \tau}$ is zero.

Hilbert functions

Coordinate operators enable the definition of a special type of functions. Take a coordinate operator Q. Next take an arbitrary Hilbert vector $|f>$. Construct the inner products of this vector with all eigenvectors $\{|\varrho>\}$. of Q. Use the eigenvalues $\{\varrho\}$ of Q as variable. Now we have defined **Hilbert function** $f(\varrho)$ as

$$f(\varrho) = <f|\varrho>; for\ all\ \varrho\ in\ \{\varrho\} \tag{1}$$

Now the Fourier transform $\tilde{f}(\wp)$ of $f(\varrho)$ is defined using the canonical conjugate $\mathcal{P}$ of Q via:

$$\tilde{f}(\wp) = <\tilde{f}|\wp>; for\ all\ \wp\ in\ \{\wp\} \tag{2}$$

This is again a Hilbert function, but it uses a different coordinate operator ($\mathcal{P}$).
Hilbert functions are sampled functions. They are not differentiable. They can be approximated by a corresponding continuous function, which may be differentiable.

The continuous approximation of $f(\varrho)$ is indicated as $\tilde{\tilde{f}}(\tilde{\tilde{\varrho}})$. Both the function and its parameters are smooth.
The sampled version of continuous function $g(q)$ is indicated as $\ddot{g}(\ddot{q})$.

The components of Hilbert functions are always decoupled. The same holds for their Fourier transforms. *For Hilbert functions no divergence and no curl exists.*

The components of (quaternionic) continuous functions are always coupled. The same holds for their Fourier transforms. For continuous functions divergence and curl may exist. However, inside a separable Hilbert space **H**, continuous functions only can act as Hilbert vectors. This is the case in ℓ^2. (See: http://en.wikipedia.org/wiki/Lp_space#Hilbert_spaces .)

Due to the granularity of their parameter space, Hilbert functions show different behavior in different directions due to the difference in sampling in those directions.

Limitedness

Soft granularity

The separable Hilbert space **H** has a countable dimension. It means that the eigenvalues of normal operators may offer a dense coverage of a connected part of the number space, but it is not a closed coverage. It does not include all limits of all convergent rows. At least the

whole number space is densely covered by the set of eigenvectors. An eigenvector represents an atomic predicate that represents the corresponding attribute of the considered item. The eigenvector lies inside the subspace that represents the item. The corresponding atomic predicate states that the corresponding attribute of the item lies inside the environment that is represented by the eigenvector.

Hard granularity

The fact that the separable Hilbert space $\mathbf{H}$ has a countable number of orthonormal base vectors renders the eigenspace of every normal operator granular. The way this granularity is distributed may cause particular features. For example dense packing causes preferred directions. Preferred directions do not commonly occur in nature. Such directions occur in condensed matter. Thus, dense packing or any other kind of organized packing does not generally occur in nature. This means that the use of a Q type coordinate operator is restricted to specific situations. However, from the Q operator a corresponding background GPS operator $\breve{Q}$ can be derived that resides in the corresponding rigged Hilbert space $\mathbf{H}$. The set of closed subspaces of this rigged Hilbert space $\mathbf{H}$ is no longer lattice isomorph with traditional quantum logic. Thus, it is not a proper model of that logic. This means that we must find a possible replacement for the Q type operator that resides in the separable Hilbert space.

Investigating an alternative operator

The alternative operator cannot support a multidimensional eigenspace otherwise it would also generate preferred directions. Still it must deliver positions as eigenvalues. Thus it must relate to a background GPS coordinate system.

A corresponding continuous GPS coordinate operator that can deliver such a GPS background coordinate system can only reside in the rigged Hilbert space $\mathbf{H}$ that corresponds to the considered separable Hilbert space.

The eigenspace of the target operator may consist of

- a set of separate points
- a set of curves

These elements are located with respect to the mentioned background coordinate system. It must be possible to locate the current position of ALL physical particles with this eigenvalue set.

On the other hand, the operator must exhibit the granularity of the position attribute. At the same time, the position must not be related to a fixed lattice. As a consequence: Any position difference must be equal or larger than the Planck length.

Between subsequent Hilbert spaces the position may stay stationary. However, when a difference occurs, it must again be equal or larger than the Planck length. On its turn this means that between subsequent Hilbert spaces the eigenspaces of the target operator must be related.

The solution may be given by a set of chains of granules. Each chain has a sub-chain of past granules, a current granule and a sub-chain of future granules. During the step to the next

Hilbert space, the first granule in the future chain of the current Hilbert space comes to be the current granule in the subsequent Hilbert space. The current granule turns into the last granule in the past chain. Another possibility is that during the step the position of the current granule does not change.

In each chain only the current granules will deliver observable values.
Fields take care that in each chain sufficient smoothness exists around the current granules. For that reason the field in the surrounds of the current granules acts like a probability amplitude distribution that regulates the position of that granule. The result of this investigation is a strand operator.

Schiller's approach takes another route. In his strand theory the fluctuations of strands determine the field that surrounds the strand. Here we let the field determine the fluctuation of the chain. However, this relation plays its major role in the neighborhood of the current granule.

Schiller does not distinguish strongly between different fields. Here we consider all fields to be constituted from the same stuff. However, the curvature (=gravitation) field is derived from the superposition of the other fields.

Strand operator

Strands do not suffer the anomalies of the eigenspace of the Q operator. The strand operator does not cause preferred directions. The elements in its eigenspace have a direction, but that direction is related to local physical conditions.

A strand operator $\mathbb{Q}$ can be defined along the following steps:

- Take a chain of granules.
- All granules have the same size.
- Each granule in this chain can be given an integer ordering number.
- The background coordinate GPS operator can be used to give each granule in a strand a unique position.
- Each chain consists of a past sub-chain, a current granule and a future sub-chain. The ordering number of the current granule is zero.
- If the set of Hilbert spaces steps to the subsequent Hilbert space, then the position of the current granule stays stationary or it becomes the position of the last granule in the past sub-chain.
- A probability amplitude distribution takes care that in each chain sufficient smoothness exists around the current granules.
- Define a set of such chains. We will call them **strands**.
- These measures leave a freedom that corresponds to a fluctuation of the strands over subsequent Hilbert spaces.
- Taken over a small set of subsequent Hilbert spaces, the movement of the current granule reflects the influence of the probability amplitude distribution that controls the smoothness of the chain in the surround of the current granule.
- This distribution describes a moving, rotating and diffusing cloud of information carrying quanta.
- The squared modulus of the probability amplitude distribution describes the probability density of the information carried by the quanta.

Further:

1. The eigenspace of the strand operator can only house a finite number of strands.
2. The eigenspace of the strand operator does not house surfaces.
3. The eigenspace of the strand operator does not house volumes.

Because the normal operator is bounded, a boundary surface exists at a finite distance from the origin of the background GPS coordinate system. We will call this boundary the **outer horizon**.

The covered space has an outer horizon, but it may also contain closed **inner horizons**. Outside the outer horizon and inside the inner horizons no strands exist.

A strand may be a closed chain. Closed chains reflect tangentially at the border of horizons. An open chain connects horizons. This may be one and the same horizon. Thus, the eigenspace does not contain lose ends of strands.

The probability amplitude distribution that guides the current granule becomes part of the surrounding fields. It forms the basic constituent of the field. Its introduction extends the concept of separable Hilbert space. In a similar way it extends the concept of quantum logic.

These rules only define the immediate neighborhood of the current granule of the strand. If required, the local direction of a strand near other granules in the chain is also guided by the local properties of the surrounding field(s). The covering field represents the combination of these fields.

Strand interpretation

In a single Hilbert space a strand may represent a piece of a potential past, present and future path of a particle. The present part of the path is formed by the direct surround of a single granule that acts as the current granule. In this single Hilbert space the granule corresponds to a Hilbert vector which is an eigenvector of the strand operator. In subsequent Hilbert spaces this potential path may differ. The path is determined by the current configuration of the field(s) that influence(s) the path. As a consequence, when taken over a sequence of Hilbert spaces, the strands fluctuate. This gives strands a place in Hilbert space.

- A boson is represented by a single strand. Thus, a boson corresponds with a single Hilbert vector in a Hilbert space.
- A quark is represented by a pair of strands. Thus, a quark corresponds with a Hilbert vector pair in a Hilbert space.
- A lepton is represented by a triple of strands. Thus, a lepton corresponds with a triple of Hilbert vectors in a Hilbert space.

Taken over a sequence of Hilbert spaces the granules that represent the actual state of the strand represent the actual paths of the corresponding particle.

Only the current state of the strand becomes observable. An event marks the observation. The event is characterized by a set of basic values:

- The space step, measured in Planck lengths.
- The progression step, measured in Planck times.
- The change (action), measured in units of the size of the Planck constant.

- The potential change measured in units of the size of the Boltzmann constant.

This current state gets its values from the granules in the direct surround of the current granule.

Secondary information can be derived from the field in the immediate surrounds of the current granule(s).

Fundamental measures and units

Events are instants of creation or annihilation of quanta. After creation the quantum becomes observable. After annihilation the quantum is no longer observable.
A ***change*** is the stepwise variation of the information carried by a quantum.
The ***information*** carried by a quantum is its position, its displacement and its chirality.

The ***distance*** between two items equals the number of granules that fit between them.
The ***progression time*** between two events equals the number of progression steps between them.
The ***action*** in a progression interval equals the number of steps in that interval during which a change took place.
The ***entropy*** of a system equals the number of steps during which a change can take place in that system.

In relation to the covering field, a probability amplitude distribution provides secondary information.

The basic measures of physics are:

$$\text{The Planck length, } l_{Pl} = \sqrt{\hbar G/c^3} = 1.6 \cdot 10^{-35}\ \text{m} \quad (1)$$

$$\text{The Planck (coordinate) time, } t_{Pl} = \sqrt{\hbar G/c^5} = 5.4 \cdot 10^{-44}\ s \quad (2)$$

$$\text{The unit of action is the constant of Planck, } \hbar = 1.055 \cdot 10^{-34} \quad (3)$$

$$\text{The unit of entropy is the constant of Boltzmann, } k = 1.38 \times 10^{-23} \quad (4)$$

Numbers

Sign selections

Four possibilities exist due to the sign selections of the quaternions. One sign selection is covered by the conjugation $a \rightarrow a^*$. The other is caused by the handedness (chirality) $a \rightarrow a^{\otimes}$. When both combine then the superscript $a \rightarrow a^{\oplus}$ is used. It is also possible to use the extended **quaternionic conjugation**:

$$a^{*} = a^{①} \tag{1}$$

$$a^{\otimes} = a^{②} \tag{2}$$

$$a^{\oplus} = a^{③} \tag{3}$$

$$a^{****} = a^{\otimes\otimes} = a^{④} = a^{⓪} = a \tag{4}$$

This differs from the complex conjugation:

$$a^{**} = \left(a^{①}\right)^{①} = a \tag{5}$$

The effects of the quaternionic conjugation are visible in the base numbers 1, **i**, **j**, **k**:

$$1^{*} = 1 \tag{6}$$

$$\mathbf{i}^{①} = -\mathbf{i} \tag{7}$$

$$\mathbf{i}^{②} = \mathbf{j} \tag{8}$$

$$\mathbf{i}^{③} = -\mathbf{j} \tag{9}$$

$$\mathbf{i}^{④} = \mathbf{i} \tag{10}$$

$$\mathbf{j}^{①} = -\mathbf{j} \tag{11}$$

$$\mathbf{j}^{②} = \mathbf{i} \tag{12}$$

$$\mathbf{j}^{③} = -\mathbf{i} \tag{13}$$

$$\mathbf{j}^{④} = \mathbf{j} \tag{14}$$

$$\mathbf{k}^{①} = -\mathbf{k} \tag{15}$$

$$\mathbf{k}^{②} = \mathbf{k} \tag{16}$$

$$\mathbf{k}^{③} = -\mathbf{k} \tag{17}$$

$$\mathbf{k}^{④} = \mathbf{k} \tag{18}$$

Thus **k** follows the rules of complex conjugation. This renders its direction to a special direction. This direction is called the **longitudinal** direction. The directions of **i** and **j** are transverse directions. Apart from that they are mutual perpendicular and perpendicular to **k**, they have no preferred direction.

These properties hold locally, they are related to other properties of the operators or the fields that carry these properties.

Product rule

We use the quaternionic product rule. It has eight (16-8) versions. When either a or b is fixed, then the product has four versions:

$$ab = a^{⓪}b^{⓪} = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} \tag{1}$$

$$a^{⓪}b^{①} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} + \boldsymbol{a} b_0 - \boldsymbol{a} \times \boldsymbol{b} \tag{2}$$

$$a^{⓪}b^{②} = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} + \boldsymbol{a} b_0 - \boldsymbol{a} \times \boldsymbol{b} \tag{3}$$

$$a^{⓪}b^{③} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} \tag{4}$$

$$a^{①}b^{⓪} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} + \boldsymbol{a} b_0 - \boldsymbol{a} \times \boldsymbol{b} = a^{⓪}b^{①} \tag{5}$$

$$a^{①}b^{①} = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} - \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} \tag{6}$$

$$a^{①}b^{②} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} - \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} \tag{7}$$

$$a^{①}b^{③} = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} - \boldsymbol{a} b_0 - \boldsymbol{a} \times \boldsymbol{b} \tag{8}$$

$$a^{②}b^{⓪} = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} + \boldsymbol{a} b_0 - \boldsymbol{a} \times \boldsymbol{b} = a^{⓪}b^{②} \tag{9}$$

$$a^{②}b^{①} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} = a^{⓪}b^{③} \tag{10}$$

$$a^{②}b^{②} = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} = a^{⓪}b^{⓪} \tag{11}$$

$$a^{②}b^{③} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} = a^{⓪}b^{③} \tag{12}$$

$$a^{③}b^{⓪} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} - \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} = a^{①}b^{②} \tag{13}$$

$$a^{③}b^{①} = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} - \boldsymbol{a} b_0 - \boldsymbol{a} \times \boldsymbol{b} = a^{①}b^{③} \tag{14}$$

$$a^{③}b^{②} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} - \boldsymbol{a} b_0 - \boldsymbol{a} \times \boldsymbol{b} \tag{15}$$

(16)

$$a^{③}b^{③} = a_0b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0\boldsymbol{b} - \boldsymbol{a}b_0 + \boldsymbol{a} \times \boldsymbol{b} = a^{①}b^{①}$$

Products of the form *aa* have two versions.

$$aa = a^{⓪}b^{⓪} = a^{⓪}a^{②} = a^{②}a^{⓪} \tag{17}$$

$$= a^{①}a^{③} = a^{③}a^{①} = a^{①}a^{①} = a^{②}a^{②} = a^{③}a^{③}$$

$$= a_0a_0 - \langle \boldsymbol{a}, \boldsymbol{a} \rangle$$

$$a^{⓪}a^{①} = a^{①}a^{⓪} = a^{⓪}a^{③} = a^{③}a^{⓪} = a^{②}a^{③} \tag{18}$$

$$= a^{③}a^{②} = a^{①}a^{②} = a^{②}a^{①}$$

$$= a_0a_0 + \langle \boldsymbol{a}, \boldsymbol{a} \rangle$$

Product sub-terms

The product *ab* contains two particular sub-terms:

$$a_0b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0\boldsymbol{b} + \boldsymbol{a}b_0 + \boldsymbol{a} \times \boldsymbol{b} \tag{1}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = a_1b_1 + a_2b_2 + a_3b_3 \tag{2}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{i}(a_2b_3 - a_3b_2) + \boldsymbol{j}(a_3b_1 - a_1b_3) + \boldsymbol{k}(a_1b_2 - a_2b_1) \tag{3}$$

The products $a_0\boldsymbol{b}$ and $\boldsymbol{a}b_0$ have two versions.
The product $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ has two versions.
The product $\boldsymbol{a} \times \boldsymbol{b}$ has two versions.
The sum $a_0\boldsymbol{b} + \boldsymbol{a}b_0$ has four versions. If either a or b is fixed it has two versions.

Operators

The sign selections of operator $\nabla = (\nabla_0, \boldsymbol{\nabla})$ depend on the sign selections of position operator Q, which determines the sign selections for its eigenvalues $q = (q_0, \boldsymbol{q})$.

Normally we consider the sign selection for operators Q and ∇fixed to operators $Q^{⓪}$ and $\nabla^{⓪}$. Sometimes we chose instead operator $\nabla^{①}$.

Quaternionic conjugation is directly connected with the concepts of **parity** and **spin**.

For quaternionic functions symmetry reduces the differences that are produced by conjugation and anti-symmetry stresses the differences. The same holds for operators.

Matrices

Another possibility is to present sign selections by matrices.

$$\alpha_1 = \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{1}$$

$$\alpha_2 = \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{2}$$

$$\alpha_3 = \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{3}$$

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4}$$

Construction

The Cayley-Dickson construction formula enables the generation of a quaternion from two complex numbers:

$$p = a_0 + a_1\mathbf{k} + \mathbf{i}(b_0 + b_1\mathbf{k}) \tag{1}$$

$$q = c_0 + c_1\mathbf{k} + \mathbf{i}(d_0 + d_1\mathbf{k}) \tag{2}$$

$$(a, b)\,(c, d) = (ac - db^*;\ a^*d + cb) \tag{3}$$

$$r = pq \tag{4}$$

$$r_0 = a_0c_0 - a_1c_1 - b_0d_0 - b_1d_1 \tag{5}$$

$$r_k = a_0c_1 - a_1c_0 - b_0d_1 + b_1d_0 \tag{6}$$

$$r_i = a_0d_0 + a_1d_1 + b_0c_0 - b_1c_1 \tag{7}$$

$$r_j = -a_1d_0 + a_0d_1 + b_0c_1 + b_1c_0 \tag{8}$$

Quaternionic conjugation ② means the exchange of r_i and r_j.

Colors

In the following text, the consequences for the product of the sign choices of the conjugate ① is indicated by blue color ±. The consequence for the product of the choice of the handedness ② of the cross product is indicated by red color ±. The mixed conjugate ③ acts accordingly on both colors.

The sign selections split the ring of quaternions in four different realizations.

Path characteristics

The Frenet-Serret frame is devised for describing curved paths of particles

Let $\{\boldsymbol{\alpha}_{qt}\}_t = \boldsymbol{\alpha}(q,t)$ describe a curved path consisting of infinitesimal steps through a landscape $\{\boldsymbol{\alpha}_q\}_q = \boldsymbol{\alpha}(q)$ of imaginary quaternions $\boldsymbol{\alpha}_{qt}$, such that $||\dot{\boldsymbol{\alpha}}(q(t))|| = 1$ for all t.

The 3D Frenet-Serret frame for the above path is given by:

$$\boldsymbol{T}\big(q(t)\big) \coloneqq \frac{\partial \boldsymbol{\alpha}\big(q(t)\big)}{\partial t} = \boldsymbol{T}(t) = \dot{\boldsymbol{\alpha}}(t) \tag{1}$$

$$\kappa(t) \coloneqq ||\dot{\boldsymbol{T}}(t)|| \tag{2}$$

$$\kappa(t) \cdot \boldsymbol{N}(t) \coloneqq \dot{\boldsymbol{T}}(t) \tag{3}$$

$$\boldsymbol{B}(t) \coloneqq \boldsymbol{T}(t) \times \boldsymbol{N}(t) \tag{4}$$

$$||\boldsymbol{T}(t)|| = ||\boldsymbol{N}(t)|| = ||\boldsymbol{B}(t)|| = 1 \tag{5}$$

$\boldsymbol{T}(t)$ is the **tantrix** of curve $\boldsymbol{\alpha}(q(t))$ at instance t.
$\boldsymbol{N}(t)$ is the **principal normal** of curve $\boldsymbol{\alpha}(q(t))$ at instance t. It is only defined when $\kappa(t) \neq 0$.
$\boldsymbol{B}(t)$ is the **binormal** of curve $\boldsymbol{\alpha}(q(t))$ at instance t.
$\boldsymbol{T}(t)$, $\boldsymbol{N}(t)$ and $\boldsymbol{B}(t)$ are imaginary quaternions.
$\kappa(t)$ is the curvature of curve at $\boldsymbol{\alpha}(q(t))$ at instance t.
$r(t) = 1/\kappa(t)$ is the radius of curvature at instance t.
$\tau(t)$ is the torsion of curve $\boldsymbol{\alpha}(q(t))$ at instance t.

$$\begin{bmatrix} \dot{\boldsymbol{T}}(t) \\ \dot{\boldsymbol{N}}(t) \\ \dot{\boldsymbol{B}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}(t) \\ \boldsymbol{N}(t) \\ \boldsymbol{B}(t) \end{bmatrix} \tag{6}$$

The Frenet-Serret curves have particular characteristics. The path may be curved and curled. The path is completely determined by its tantrix, curvature and torsion given by functions of t. Each coordinate of the quaternionic function $\boldsymbol{\alpha}(q(t))$ has its own set of characteristics. This means that for a given quaternionic function these characteristics are quaternions rather than real numbers and they are all functions of parameter t.

Path equations

The path equations are given by

$$\dot{\boldsymbol{T}}(t) = \kappa(t) \cdot \boldsymbol{N}(t) \tag{1}$$

$$\dot{\boldsymbol{N}}(t) = -\kappa(t) \cdot \boldsymbol{T}(t) + \tau(t) \cdot \boldsymbol{B}(t) = -\kappa(t) \cdot \boldsymbol{T}(t) + \tau(t) \cdot \boldsymbol{T}(t) \times \boldsymbol{N}(t) \tag{2}$$

$$\dot{\boldsymbol{B}}(t) = -\tau(t) \cdot \boldsymbol{N}(t) = \boldsymbol{T}(t) \times \dot{\boldsymbol{N}}(t) + \dot{\boldsymbol{T}}(t) \times \boldsymbol{N}(t) \tag{3}$$

$$= \tau(t) \cdot \boldsymbol{T}(t) \times \boldsymbol{B}(t)$$

Curve length

The curve length $l(a, b)$ is defined by:

$$l(a,b) = \int_{x=a}^{x=b} |\dot{\boldsymbol{\alpha}}(q(x))|\, dx \tag{1}$$

The integration over the square of the modulus delivers the **action** S of the curve.

$$S(a,b) = \int_{x=a}^{x=b} |\dot{\boldsymbol{\alpha}}(q(x))|^2\, dx \tag{2}$$

Reparameterization

The path characteristics κ(t) and τ(t) together with the curve length and the curve action are independent of any reparameterization $s(t)$ of the progression parameter t.
A natural reparameterization is given by $s(t) = l(t_0, t)$.
This turns the curve $\boldsymbol{\alpha}(q(t))$ into a **natural** curve $\boldsymbol{\gamma}\big(q(s)\big)$:

$$\boldsymbol{\gamma}\big(q(s)\big) = \boldsymbol{\alpha}(q(t)) \tag{1}$$

Curves on a surface which minimize length between the endpoints are called geodesics.
The natural curve corresponds to a geodesic.
The consequence is that in three-dimensional space the corresponding movement obeys the geodesic equation. The Lagrangian is an equivalent of this equation.

Functions and fields

Distributions in quaternionic Hilbert space

Using a compact normal operator $\mathcal{Q}$ and a second distribution operator ρ with the same eigenvectors $\{|q_i>\}_q$ but with eigenvalues $\{\rho_i\}$ we can generate a Hilbert distribution.

$$\rho(q_i) = < q_i|\rho\ q_i > \tag{1}$$

Operator ρ need not be a compact normal operator. Its spectrum of eigenvalues may be confined to a discrete set of points. Its eigenvectors are used. Its eigenvalues need not be used. . If they are used, the eigenvalues may consist of any kind of hyper complex number.

A Hilbert distribution is not differentiable. It can be seen as a combination of a set of Dirac delta functions that are multiplied with hyper complex numbers. If all numbers are quaternions, then it is a linear combination of Dirac delta functions that each represents a Hilbert vector.

The Hilbert space is separable. This means that the set of eigenvalues of an operator is countable. Thus a Hilbert distribution $\rho(q_i)$ is always discrete:

$$\rho(q_i) = \sum_{i=1}^{N} q_{E\,i} \cdot \delta(q_i - q_i) \tag{1}$$

The factors $q_{E\,i}$ are hyper complex 2^n-ons.

Convolution of a Hilbert distribution with a blurring spread function can render the result (mostly) differentiable. Depending on the blur, the result may still be singular for example on the definition points of the Hilbert distribution. The blur may represent a probability distribution.

With other words, every Hilbert function is also a Hilbert distribution. (The reverse is not true).

Another form of Hilbert distribution is the representation of a probability amplitude distribution as a Hilbert function.

A special kind of Hilbert distributions is formed by the **elementary Hilbert distributions**. These distributions contain a single or only a few Hilbert vectors. They form the fundament under the private fields, which represent elementary particles. Private fields are special kinds of Hilbert fields.

As stated before, every Hilbert function can be split into four decoupled components. And every Hilbert function has a Fourier transform that consists of four decoupled Fourier transforms.

As in the case of a Hilbert function, a Hilbert distribution can represent a very dense coverage. In that case the distribution may become quasi differentiable.

Hilbert field

By blurring the Hilbert distribution with a suitable spread function, the distribution can be transformed into a mostly continuous function. When the blur is the same for every element of the Hilbert distribution, then this converts the Hilbert distribution $\rho(q)$ into a skew Hilbert field $\phi(q)$ via the convolution:

$$\phi(q) = f(q) \circ \rho(q) \quad (1)$$

With a given Hilbert distribution $\rho(q)$, each blurring function $f(q)$ causes in this way a corresponding Hilbert field that is characterized by the blurring function $f(q)$.

Mathematically this convolution is impossible in a separable Hilbert space, but the corresponding rigged Hilbert space $\mathbf{H}$ is a suitable alternative. The vectors in an orthonormal base consisting of eigenvectors of the normal operator $\mathcal{Q}$ that resides in the separable Hilbert space $\mathbf{H}_1$ are represented in the rigged Hilbert space $\mathbf{H}$ by corresponding Dirac delta functions. We only use the vectors that belong to the Hilbert distribution $\rho(q)$. The values of the result of the convolution can be attached to the same orthonormal base vectors of the separable Hilbert space $\mathbf{H}_1$.This procedure attaches the field onto the separable Hilbert space $\mathbf{H}_1$.In principle the field covers all unit sized vectors of the separable Hilbert field.

When the blurs differ per element of the Hilbert distribution, then the Hilbert field can still be interpreted as the superposition of the contributing blurs, but it can no longer be considered as a convolution. Like with the convolution, the location of the blur must be reckoned in this superposition.

When there are only a few types of blurs, then each type constitutes via convolution with a corresponding Hilbert distribution a type specific Hilbert field. The covering Hilbert field can then be interpreted as the superposition of the (blur) type specific Hilbert fields.

The blurs are not hanging as a lose substance in the separable Hilbert space $\mathbf{H}_1$.The blurs are spread over the Hilbert vectors. Each Hilbert vector in the domain of a blur touches this blur and carries the local value of that blur.

Hilbert fields that correspond to the same Hilbert distribution form a Hilbert field set.

A Hilbert field or type specific subfield can be categorized according to its:

- Symmetries
- Conjugation
- Corresponding blur function
- Corresponding Hilbert distribution

Hilbert fields are differentiable. The dimension related components of a Hilbert field are coupled. The differential of a symmetric field or field part is anti-symmetric.
The differential of an anti- symmetric field or field part is symmetric.

Sampled Hilbert field

A **sampled Hilbert field** consists of its values attached to the eigenvectors of a normal operator, whose eigenspace acts as a coordinate system. A sampled Hilbert field is NOT differentiable. Its dimension related components are decoupled. It closely approximates a corresponding Hilbert field. The granular eigenspace of the normal operator closely matches the corresponding continuous eigenspace of a coordinate operator that resides in rigged Hilbert space $\mathbf{H}$.

It can be interpreted as the distributed superposition of a number of Hilbert functions.

Blur function

The blur is a spread function. It is the reason of the significant similarity between optics and quantum physics. On the other hand, the blur is a probability distribution. This is the source of quantum noise. The probability distribution can be a probability amplitude distribution or its squared modulus, which is a probability density distribution. A quaternionic probability amplitude distribution has the advantage that its squared modulus can specify the probability

and the parameter can specify the full location, while the resulting factor represents related data in the form of a unitary quaternion. This quaternion can also carry its sign selection data, which includes its chirality and its parity. Compared to a complex amplitude distribution, this is a wealth of extra information. The shape of the blur contains secondary information. For example the Fourier transform of the blur offers momentum related data and the rotation of the blur represents angular momentum related data.

The simplest kind of blur that belongs to a particle relates to its ground state.
Hermite functions, which are eigenfunctions of the Fourier transformation, have a Gaussian blur.
Coherent states, which are eigenfunctions of creation or annihilation operators have a Poissonian blur.

Quantum shot noise produces a Poisson distribution. When large numbers of quanta are produced the distribution approaches a Gaussian distribution. A binomial process that follow a noise generating Poisson process can be combined with that binomial process into a generalized Poisson process with a lower efficiency. The binomial process represents a weakening effect. Spatial blur can be interpreted as a binomial process. This is because it represents a spatial diffusion effect. In the static model the Poisson processes only represent a lateral distribution. Taken over a sequence of Hilbert spaces the Poisson processes represent an additional temporal distribution. The efficiency of the detection of quanta is characterized by the detective quantum efficiency (DQE) of the detector. Together with the Fourier transform of the spatial spread function this determines the signal to noise ratio in the information stream. The spread has an integrating (smoothing) effect. A sharper spread improves the signal, but also increases the noise. Any temporal integration reduces the noise. The effect of the lateral spread can be characterized by the optical transfer function (OTF).

When the quanta are given a direction, then the blur becomes the equivalent of a probability amplitude distribution. In strand theory the observable values of crossing switches of strands form probability amplitude distributions. See: http://www.motionmountain.net/research.html .

The blur plays a role when canonical conjugate operators occur together or in sequence. An extra blur is caused by the inaccuracy of the combination of these operators.
The blur has many functions and interpretations:

- Convolution with a smooth spread function makes a Hilbert distribution differentiable.
- The spread ensures the compactness of corresponding operators. It also reduces the frequency range that is covered by its Fourier transform.
- The spread function represents a probability density distribution or more in detail a quaternionic probability amplitude distribution.
- Each elementary blur has a Fourier transform
- The probability distribution is characteristic for the inaccuracy of the expectation value of a category of operators, such as the GPS operator, the GMS operator, the Fourier transform, the creation/annihilation operators, the ladder operators and the number and ladder operators.
- The ground state is characterized by a typical spread function.
- The spread represents the probability that virtual items exchange roles with actual items.
- The virtual items represent subspaces of virtual Hilbert spaces that are ready to exchange roles with the currently valid Hilbert space.
- The virtual items represent virtual quantum logical propositions that may exchange roles with currently actual propositions.
- The virtual quantum logical propositions are elements of a virtual traditional quantum logic that is ready to exchange roles with the currently actual traditional quantum logic.

- The blur can be interpreted as a spatial quantum noise distribution.
- The blur can be interpreted as a spatial distribution of crossing switches of strands.
- The blur can be interpreted as a spatial distribution of generations or annihilations of quanta. The annihilation of a quant is equivalent to the generation of the corresponding anti-quant.
- The blur works as storage of past, present and future conditions.
- The blur can be squeezed in order to reflect the importance of momentum versus position.
- A basic (ground) blur has in each direction a symmetric cut. An odd-times differentiated blur has in one direction an asymmetric cut. An even-times differentiated blur has in each direction a symmetric cut.
- The blur represents the sticky resistance of the universe against unordered changes (= changes of uniform movement in a geodesic). This is proved by the existence of inertia.
- The blur represents the sticky resistance of the collection of all propositions against unordered redefinitions.
- Blurs can be categorized according to the corresponding particle type.
- The superposition of blurs forms a field.
- A particle can be interpreted as the local excitation of this field.
- During a progression step the blur may get distorted.

In short: Without blur (quantum) physics is impossible!

Decomposition

The imaginary part of a Hilbert field can be decomposed in a rotation free part and one or two divergence free parts. This decomposition depends on the choice of the selected coordinate system. In general such decomposition runs along curved lines. However, it is possible to select a coordinate system for which the decomposition runs along straight lines. For this coordinate system it is possible to define a globally valid multidimensional Fourier transform. With respect to this coordinate system, other coordinate systems possess a locally defined curvature.

Bypassing granularity

The fact that the Hilbert space is separable means that normal operators have a countable number of eigenvalues. That may still be an infinite number, but it means that the eigenspace of these operators are granular. It is possible to define a procedure that attaches a natural number to each eigenvector and to each eigenvalue. It means that functions that are defined using such eigenvalues as parameters or as function values cannot be differentiated.

This does not mean that differentiable functions cannot exist in Hilbert space. For example, $\ell^2$is isomorphic with a separable Hilbert space $\mathbf{H}$, and consists of integrable and differentiable functions, but, as with any separable Hilbert space, the eigenvalues of operators in ℓ^2 do not form a closed set. The mentioned functions act as Hilbert vectors. They are NOT Hilbert functions.

It is possible to use a trick that enables differentiation of fields that are defined as functions with eigenvalues of a normal operator as their parameter values. The trick consists of blurring all or a subset of the corresponding eigenvectors. When the blur is differentiable, then the field becomes differentiable as well. Still, if the blur extends far enough, all Hilbert vectors touch a value of this field.

The blur does not fit IN the considered separable Hilbert space **H**.It is ***fixed*** to a vector of this separable Hilbert space **H**.In addition, values of the blur touch all other Hilbert vectors.

The fact that differentiable quaternionic functions have an isotropic multi-dimensional parameter space (in the imaginary part of the quaternions) means that in contrast to the eigenspaces of coordinate operators in separable Hilbert space **H**, this parameter space is coherent. All its dimension related components are coupled. Instead a decoupling exists along curved coordinates that decompose rotation free and divergence free parts of the quaternionic functions.

In our model the real part of quaternions that are applied as parameters appears to play a rather minor or at least a quite different role. For that reason, in most cases the results of differential geometry are more applicable than the theory of regular quaternionic functions.

Differential geometry also decomposes local space into three independent coordinate directions. These dimensions are selected according to the divergence and rotation properties of the analyzed functionality. This is similar to the approach in the Helmholtz or Hodge decomposition theorem. For example, the Frenet-Serret frame features three mutually perpendicular directions.

The basic constituent and private field

There is only one basic type of constituent to Hilbert fields. That constituent type is a probability amplitude distribution. A small subset of Hilbert vectors forms an elementary Hilbert distribution. A corresponding **private field** is formed from the convolution of a basic constituent with this elementary Hilbert distribution. It is also possible that the basic constituents differ. In that case the private field is the superposition of the basic constituents. In this superposition the separate locations of the basic constituents must be reckoned.
All other Hilbert vectors are covered by values of this private field. Via the attach points and via the touching values the private fields are embedded in separable Hilbert space **H**.The private field represents an elementary particle and the physical fields that are emitted by that particle. Different types of elementary particles exist. The private field is a continuous function with parameters that are taken from a selected coordinate system. The field itself is independent of the selection of this coordinate system. Only the field, taken as a function depends on that selection. We assume that in the context of this e-paper all private fields are differentiable. Due to the fact that the basic constituent has divergence and curl, its dimension related components are **coupled**. The field can be decomposed in rotation free part and one or two divergence free parts. As a consequence it has coupled Fourier transforms. There exists a coordinate system for which the decomposition runs along straight coordinate lines. An appropriate reorientation of this coordinate system puts the decomposition along the coordinate axes. In this coordinate system the ideal form of the Fourier transform can be applied.

When subsequent Hilbert spaces are considered, the private fields move together with the corresponding elementary Hilbert distribution. Apart from a linear movement the private fields may rotate. When a given Hilbert distribution contains just one Hilbert vector, then the constituent can rotate free around that point. If it contains two vectors, then one axis is fixed with respect to the anchor points. If it contains three independent vectors, then the basic

constituent can only rotate together with these anchor points. The movements are stochastic and have average characteristics such as position, speed, rotation axis, rotation phase and chirality. At each position within the private field these data may differ. Also the relative position of the carrying Hilbert vectors with respect to each other may change. The basic constituent can be interpreted as the probability amplitude distribution that describes the average location of the carrying Hilbert vectors.
The granularity of the eigenspace of the position operator determines the minimal distance that can exist between the carrying vectors.

It is sensible to select the coordinate system such that the members of the elementary Hilbert distribution are eigenvectors of the corresponding position operator. When a Fourier transform is taken, then this can no longer be valid. In that case the members of the elementary Hilbert distribution must be linear combinations of the eigenvectors of the canonical conjugate of the original coordinate operator.

The Hilbert vectors that are member of a given elementary Hilbert distribution can be interpreted as eigenvectors of a strand operator. The corresponding eigenvalue is a granule that is a characteristic part of the strand.

Covering field

Physical fields are not physical items. In our model, physical fields are represented by Hilbert fields. For each Hilbert field, every member of an orthonormal base of the Hilbert space corresponds to a value of the field. Each elementary particle corresponds to a **private field**. A **covering field** is formed by the superposition of these private fields. Each private field that belongs to an elementary particle is characterized by a **blur function** that is typical for the particle type. That blur function can be interpreted as a *probability amplitude distribution*. Each elementary particle is **based on a small number of Hilbert vectors** that together span a Hilbert subspace. These vectors form an elementary Hilbert distribution. Types of elementary particles can be characterized by the number of Hilbert vectors on which they are based.

The blurs of elementary particles are smooth and fade out at long distances. As a consequence the covering field is smooth as well and can be considered as a single - very wide spread - probability distribution. The simplest form of blur is a Poisson distribution.

Depending on the type of the constituting particles the covering field can be divided in subfields. Each type has its own subfield.

Depending on the selected coordinate operator, the parameters of the probability distributions are GPS related or GMS related. However, also other coordinate systems can be used.

Covering field summary

A **covering field** can be defined as the superposition of all private fields. For each Hilbert vector the value of the covering field is formed by the sum of the private field values that touch this vector. The covering field is differentiable. Like the components of the private fields the components of the covering field are coupled. Its divergence and curl exist

everywhere. On its turn, this means that the covering field can be **decomposed** in a rotation free part and one or two divergence free parts. This decomposition may run along curved surfaces. This decomposition replaces the decomposition along imaginary quaternionic dimensions that characterizes Hilbert functions.

Curvature field

The effects of field decomposition depend on the selection of the coordinate operator. For the coordinate operator $\breve{Q}$ that is derived from normal operator Q the decomposition runs along curved lines. For the coordinate operator $\check{Q}$ that is derived from normal operator Q the decomposition runs along the straight coordinate axes. Through the curvature of the eigenspace of $\check{Q}$ with respect to the eigenspace of $\breve{Q}$ the multidimensional quaternionic Fourier transform of the covering field in $\check{Q}$ coordinates becomes a universe wide validity range.

The decomposition properties of the covering field determine the curvature of a secondary coordinate system with respect to the original GPS coordinate system. That curvature on its turn defines a secondary field which we will call **curvature field**.

In this view the curvature field is a tertiary field, which is derived from the covering field, which is a secondary field that is built via superposition from the private fields of the separate particles that in this sense can be considered as primary fields.

The curvature that is caused by the blur of an elementary distribution represents a **private curvature field** and is independent of any electric charge (or color charge) of the elementary distribution. The "charge" of the private curvature field is non-negative. It can be thought of being distributed over the private curvature field or being located at a center point. This "charge" is called mass and the center point is the center of mass.

Curvature field summary

The decomposition depends on the selected coordinate system. This coordinate system is defined by a corresponding coordinate operator that resides in rigged Hilbert space $\mathbb{H}$. Selecting an appropriate coordinate operator may cause a new decomposition that runs along straight lines. This new coordinate operator only does that for a given covering field configuration. When that configuration changes, the straightening coordinate operator must change as well. The straightening coordinate operator can be used together with the original coordinate operator in order to derive a curvature field. The curvature field assigns a curvature characteristic to each Hilbert vector. It is possible to average these values by using the probability densities that are determined by the private fields. In this way, each elementary particle gets is own curvature characteristic in the form of a private curvature field. This private curvature field can be used to determine the mass of the particle.

At each location the local curvature can also have a function as a guide for the local direction of strands in that environment.

Functions in quaternionic Hilbert space

Due to their definition the Hilbert functions are only defined for an infinite but countable number of parameter values that lay dense in quaternion space. The Hilbert functions are close to a corresponding differentiable function that resides in the corresponding rigged

Hilbert space **Ħ**. In contrast to what holds for the Hilbert function, the dimension related components of the differentiable function are coupled.

A locatable probability distribution can be described by the convolution of a Dirac delta function that corresponds to the Hilbert vector, which represents the location of the weighted center of that distribution and a function $f(q)$ that describes the distribution relative to that location. In this way a blurred Hilbert vector is defined. This means that a blurred Hilbert vector can be closely approximated by a Hilbert function that is defined by the combination of a sharp *locator* Hilbert vector and a sharp *shape* Hilbert vector. We will use the addition "Hilbert" to the name of a continuous function for the Hilbert function that closely approximates that continuous function.
Thus, in Hilbert space the representative of the blurred locator Hilbert vector by a Hilbert function is a **Hilbert blur** or more specifically a **Hilbert probability amplitude distribution**.

It is also possible to use an elementary Hilbert distribution as the anchor of the continuous probability amplitude distribution. This construct may represent an elementary particle. It is closely approximated by a private Hilbert field that is formed by the superposition of the Hilbert functions that are formed by a small set of locator Hilbert vectors and a single shape Hilbert vector.

Pure states are characterized by blurred elementary Hilbert distributions.

Elementary Hilbert distribution

An elementary Hilbert distribution is a discrete distribution in which a single or a small number of Hilbert vectors participate. Together these Hilbert vectors span a closed subspace that represents an un-blurred elementary item.

A unary distribution uses only one Hilbert vector that is eigenvector of the position operator.
A binary distribution uses two Hilbert vectors that are different eigenvectors of the position operator.
A ternary distribution uses three Hilbert vectors that are different eigenvectors of the position operator.

If the eigenvectors are selected such that they belong to mutually perpendicular imaginary (base) eigenvalues, then the elementary distributions are restricted to the mentioned three classes.

When the Hilbert vectors in an elementary distribution are blurred with the same spread function then the resulting subfield gets a typical elementary spread function.

Characteristic functions

Now the position is connected to eigenvectors of the strand operator. The physical item is connected to a subspace rather than to a single vector. This subspace is spanned by the eigenvectors. So we can use a localizer that represents the (weighted) average position as a more precise indicator of the position of the physical item. On the other hand physical items are characterized by a state.

A state is either a wave function or a probability density operator. Both use background coordinate position as their parameter. The wave function is a probability amplitude distribution. Each wave function can be approximated by a Hilbert function. The squared modulus of the wave function indicates the probability of finding the position of the localizer.

The probability density operator is a weighted projection operator that is related both to the subspace that represents the item and to the position operator. It represents the probability that after measuring the position the parameter of the density distribution is found as the result.

Differentiation

Let $\check{Q}$ be the selected coordinate operator.
The operator ∇ is directly related to operator $\check{Q}$. Thus, the sign selections for $\check{Q}$ transfer to the sign selections for operator ∇. Due to sign selection four nabla operations exist when the field is held fixed. With a fixed nabla operator there may exist four results.

$$\mathrm{g}(q) = \nabla f(q) = \nabla_0 f_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) \pm (\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q)) \tag{1}$$

$$\bar{\mathrm{g}}(q) = \nabla^{①} f(q) \tag{2}$$

$$= \nabla_0 f_0(q) \pm \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \pm \nabla_0 \boldsymbol{f}(q) - \boldsymbol{\nabla} f_0(q) \pm (\mp \boldsymbol{\nabla} \times \boldsymbol{f}(q))$$

∇ turns a symmetric field $f(q)$ into an anti-symmetric field $\nabla f(q)$ and an anti-symmetric field into a symmetric field.

The fact that $\nabla f(q) = 0$ means that $f(q)$ is constant or that at location q function $f(q)$ is in a maximum, a minimum, a saddle point or an asymmetric plateau. The consequence of this restriction is:

$$\nabla_0 f_0(q) = \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \tag{3}$$

$$\pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) = \mp (\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q)) \tag{4}$$

The fact that $\nabla^{①} f(q) = 0$ leads to different equations.

$$\nabla_0 f_0(q) = \pm \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \tag{4}$$

$$\pm \nabla_0 \boldsymbol{f}(q) - \boldsymbol{\nabla} f_0(q) = \pm (\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q)) \tag{5}$$

The quaternionic Laplace operator Δ is defined by

$$h(q) = \Delta f(q) = \nabla^{①} \nabla f(q) = \nabla \nabla^{①} f(q) \tag{5}$$

$$= {\nabla_0}^2 f(q) + \boldsymbol{\nabla}^2 f(q)$$

A quaternionic function that fulfills $\Delta f(q) = 0$ is a harmonic function.
A quaternionic function that fulfills $\boldsymbol{\nabla}^2 f(q) = 0$ is a spatial harmonic function.

Harmonic functions

Spatial Harmonic functions are suitable spread functions.
For a harmonic function $f(q)$ holds:

$$\Delta f(q) = \nabla\nabla^{①} f(q) = 0 \tag{1}$$

See for example the section Potential.

Conservation of charge

When $\rho_0(q)$ is interpreted as a charge density distribution, then the conservation of charge is given by the continuity equation:

Total change within V = flow into V + production inside V (1)

$$\frac{d}{dt}\int_V \rho_0 \, dV = \oint_S \hat{\boldsymbol{n}}\rho_0 \boldsymbol{v} \, dS + \int_V s_0 \, dV \tag{2}$$

Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface S, $\boldsymbol{v}(q)$ is the velocity at which the charge density $\rho_0(q)$ enters volume V and s_0 is the source density inside V
With the help of Gauss theorem this converts into:

$$s_0(q) = \nabla_0\rho_0(q) \pm \langle \boldsymbol{\nabla}, (\rho_0(q)\boldsymbol{v}(q) + \boldsymbol{\nabla} \times \boldsymbol{a}(q)) \rangle \tag{3}$$

$$= \nabla_0\rho_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle$$

This equation represents a balance equation for charge (or mass) density.
Here $\boldsymbol{\rho}(q)$is the current density and $\boldsymbol{a}(q)$ is an arbitrary differentiable vector function.
The blue colored ± indicates quaternionic sign selection through conjugation ① or ③.
The quaternionic charge density is given by:

$$\rho(q) = \rho_0(q) + \boldsymbol{\rho}(q) = \rho_0(q) - \rho_0(q)\boldsymbol{v}(q) \tag{4}$$

The quaternionic source density $s(q)$ is given by:

$$s(q) = s_0(q) + \boldsymbol{s}(q) = \nabla\rho(q) \tag{5}$$

$$\boldsymbol{\nabla} s_0(q) = \nabla_0\boldsymbol{\nabla}\rho_0(q) \mp \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle \tag{6}$$

Probability amplitude, probability density and probability current

The probability amplitude distribution ψ(q) can be used to define a probability density and probability current. See http://www.vttoth.com/qt.htm.

$$\rho_0(q) = |\psi(\mathrm{q})|^2 = (\psi^*(\mathrm{q})\psi(\mathrm{q})) \tag{1}$$

$$\boldsymbol{\rho}(q) = \psi^*(\mathrm{q})\, \boldsymbol{\alpha}\, \psi(\mathrm{q}) \tag{2}$$

By using in $\boldsymbol{\alpha}$ the matrix form of the sign selections we get the Dirac equation:

$$\alpha_1 = \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{3}$$

$$\alpha_2 = \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{4}$$

$$\alpha_3 = \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{5}$$

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{6}$$

Compare Dirac:

$$\frac{\partial \psi}{\partial t} = (-\boldsymbol{\alpha} i \nabla \pm \beta m)\psi \tag{7}$$

There exist also a relation between $\alpha_1, \alpha_2, \alpha_3$ and the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$:

$$\boldsymbol{\sigma} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \tag{8}$$

$$1 \mapsto I, \qquad \boldsymbol{i} \mapsto i\,\sigma_1, \qquad \boldsymbol{j} \mapsto i\,\sigma_2, \qquad \boldsymbol{k} \mapsto i\,\sigma_3 \tag{9}$$

Canonical conjugate

Remember that the operator $\check{Q}$ that resides in rigged Hilbert space $\mathbf{H}$ is defined such that the decomposition of the covering field runs along straight lines.

The canonical conjugate of the operator $\check{Q}$ is the operator $\check{P}$. It is defined by using a complex subspace of a quaternionic number space that is used to specify inner products. It is defined by specifying the function that defines the inner products of the eigenvectors |q> of $\check{Q}$ and |p> of $\check{P}$ with real eigenvalues q and p.

$$< q|p > = \tilde{f}(p) = < p|q >^* = f^*(q) = exp\,(\boldsymbol{k} \cdot p \cdot q/\hbar) \tag{1}$$

The constant $\boldsymbol{\hbar}$ is Planck's constant and relates to the granularity of the eigenspaces. The imaginary base number $\boldsymbol{k}$ belongs to a complex subspace of the quaternionic number space.

Due to its specification, the canonical conjugate operator $\check{P}$ can be interpreted as a generator of displacement of the eigenvalues of $\check{Q}$. For this purpose the considered function $f(q)$ must be differentiable.

$$\check{P} = \boldsymbol{k} \cdot \hbar \cdot \frac{\partial}{\partial q} \tag{2}$$

$$< q|\check{P}\, f >= \boldsymbol{k} \cdot \hbar \cdot \frac{\partial}{\partial q} f(q) \tag{3}$$

This interpretation of the operator $\check{\boldsymbol{P}}$ shows that the complex canonical conjugate shown here corresponds with the imaginary direction in which the differentiated function $f(q)$ is rotation free.

The definition leads to the commutator:

$$[\check{P}, \check{Q}] \;=\; \check{P}\check{Q} - \check{Q}\check{P} \;=\; \boldsymbol{k} \cdot \hbar \tag{4}$$

The sign selections of $\check{P}$ depend on the sign selections of $\check{Q}$.

Complex Fourier transform

The specification of the canonical conjugate also defines a **complex** Fourier transform.

Also here the imaginary direction must correspond to the direction in which the analyzed function is rotation free.

Let |f> be the generator of a quaternionic function that is generated with the help of the eigenvectors and eigenfunctions of operator $\check{Q}$ with canonical conjugate $\check{P}$.

The Fourier transform $\mathcal{F}_{\mathbf{i}} = U_{qp}$ converts the base $\{|q>\}_q$ into the base $\{|p>\}_p$. The inverse Fourier transform U_{pq} does the reverse. These transforms reside in rigged Hilbert space $\mathbb{H}$.

$$< q|f >= < f|q >^* = \; f^*(q) \;=\; \sum_p (< q|p > \cdot < p|f >) \tag{1}$$

$$= \sum_p < p\, (< p|q >)|f >$$

$$= \; \sum_p < p\, U_{pq}|f >$$

$$= \sum_p < p|U_{qp}\, f >$$

$$< p|f >= \sum_q (< p|q > \cdot < q|f >) \tag{2}$$

$$= \sum_q < q|U_{pq}\, f >$$

The complex Fourier transform of a symmetric (complex) function is a cosine transform. It is a real function.

The complex Fourier transform of an anti-symmetric (complex) function is a sine transform. It is an imaginary function.

Through complex Fourier transformation the operators $\check{P}$ and $\check{Q}$ exchange roles.

The Hilbert function $\tilde{f}(p) =< f|p >$ denotes the Fourier transform of the Hilbert function $f(q) =< f|q >$

Heisenberg's uncertainty

The Heisenberg's uncertainty principle is a consequence of the definition of the combination of the canonical conjugate and the definition of the Hilbert field. It means that a small spread of q values goes together with a large spread of p values and vice versa.

$$\Delta q \cdot \Delta p \geq \hbar/2 \qquad (1)$$

A **squeezed** coherent state is any state such that the uncertainty principle is saturated. That is:

$$\Delta q \cdot \Delta p = \hbar/2$$

See: http://en.wikipedia.org/wiki/Squeezed_coherent_state.
For animations: http://gerdbreitenbach.de/gallery/.

Preferred direction

The longitudinal direction is a preferred direction. The complex Fourier transform specified above is a **longitudinal quaternionic** Fourier transform $F_{\mathbf{k}}$ that works on the complex part of the function that is defined via the imaginary base number **k**. The longitudinal Fourier transform concerns the rotation free part of the function or field. When arbitrary GPS coordinates ($\check{\Psi}$) are used the longitudinal direction varies with position. In that case the longitudinal direction concerns a local property, so in these coordinates the definition of the longitudinal part of a field or function has only local validity. A non-stationary longitudinal direction also means that the Fourier transform has only local validity and a diminished accuracy.

However, the position operator $\check{Q}$ is defined such that when the analyzed function or field is specified with $\check{Q}$ coordinates the longitudinal direction is stationary. $\check{Q}$ varies with the analyzed field. To the current covering field belongs a corresponding operator $\check{Q}$. This relation must not hold for subfields and it need not hold for fields that are private for particles. In $\check{Q}$ coordinates the longitudinal direction is a globally preferred direction. The corresponding longitudinal Fourier transform leaves the other two dimensions untouched. It behaves as if the analyzed function is constant in these other dimensions. It is well-known that the Fourier transform of a constant delivers a Dirac delta function. Thus, the longitudinal Fourier transform equals a cut through the quaternionic Fourier transform of the full 3D imaginary quaternionic function or field.

This is similar to the cut through the 2D optical transfer function that is obtained when the Fourier analysis of the imaging device is confined to the image of a thin slit.

We confine Fourier analysis in this paper to the ideal quaternionic Fourier transform. Thus, when nothing is indicated we presume $\check{Q}$ coordinates and analysis of the covering field.

Affine space

The eigenvectors of a normal operator form an orthonormal base of the separable Hilbert space **H**.This orthonormal base defines an affine unit sphere. Apparently, the correspondence with a $\breve{Q}$ type GPS operator that is equipped with an origin in its eigenspace is not natural thing for this affine Hilbert unit sphere. On the other hand, like this Hilbert unit sphere, the imaginary eigenspace of the $\breve{Q}$ type GPS operator has no preferred direction.

If a field covers all vectors of an orthonormal base, then it covers all of Hilbert space. The orientation along the longitudinal direction of the (covering) field is not natural for the Hilbert space, but it is natural for the field. Apart from a shift of the origin, the position operator $\breve{Q}$ is fully determined by the properties of the field.

The origin of the eigenspace of the $\breve{Q}$ operator may be interpreted as the position of the observer. That selection consumes the last freedom for this operator.

There exists a point to point relation between an arbitrary $\breve{Q}$ type GPS operator and the $\breve{Q}$ operator. This point to point relation defines the curvature field.

Complex Fourier transform invariance properties

Each even function $f(q) \Leftrightarrow \tilde{f}(p)$ induces a Fourier invariant:

$$h(q) = \sqrt{2\pi}\, f(q) + \tilde{f}(q).$$

$$\tilde{h}(q) = \sqrt{2\pi}\, h(q)$$

Each odd function $f(q) \Leftrightarrow \tilde{f}(p)$ induces a Fourier invariant:

$$h(q) = \sqrt{2\pi}\, f(q) - \tilde{f}(q).$$

A function $f(q)$ is invariant under Fourier transformation *if and only if* the function f satisfies the differential equation

$$\frac{\partial^2 f(q)}{\partial q^2} - t^2 f(q) = \alpha\, f(q), \text{ for some scalar } \alpha \in C.$$

See the quantum harmonic oscillator.
The Fourier transform invariant functions are fixed apart from a scale factor. That scale factor can be 1, $\boldsymbol{k}$, -1 or $-\boldsymbol{k}$. $\boldsymbol{k}$ is an imaginary base number in the longitudinal direction.

Fourier-invariant functions show iso-resolution, that is, $\Delta_p = \Delta_q$ in the Heisenberg's uncertainty relation.

For proves see: http://www2.ee.ufpe.br/codec/isoresolution_vf.pdf.

Quaternionic Fourier transform split

The longitudinal Fourier transform represents only part of the full quaternionic Fourier transform.

$$\mathcal{F}_{\mathbf{k}}\big(f(q)\big) = \mathcal{F}\big((f((q,\mathbf{k})),\mathbf{k})\big) \tag{1}$$

Or

$$\mathcal{F}_{\parallel}\big(f(q)\big) \equiv \mathcal{F}\left(f_{\parallel}(q)\right) \tag{2}$$

It relates to the full quaternionic Fourier transform $\mathcal{F}$

$$\mathcal{F}\big(f(q)\big) = \tilde{f}(p) \tag{3}$$

The inverse Fourier transform runs:

$$\mathcal{F}^{-1}\big(\tilde{f}(p)\big) = f(q) \tag{4}$$

The split in longitudinal and transverse Fourier transforms corresponds to a corresponding split in the multi-dimensional Dirac delta function.

The analysis of path characteristics showed that the changes occur in three perpendicular directions rather than two. First we tackle the situation that $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular rather than parallel. Thus instead $\langle \boldsymbol{p}, \boldsymbol{q} \rangle$ the inner product $< p|q >$ is constructed using $\boldsymbol{p} \times \boldsymbol{q}$. With respect to the path analysis this transform treats the curvature part. The third dimension will treat the torque part.

The transverse Fourier transforms

Apart from the longitudinal quaternionic Fourier transform two **transverse quaternionic** Fourier transform exist. Like the longitudinal part, the definition of the transverse part of a function or field has only local validity.
The longitudinal transform is:

$$\mathcal{F}_{\parallel}\big(f(q)\big) \equiv \mathcal{F}\left(f_{\parallel}(q)\right) \tag{1}$$

$$< q|p_{\parallel} > = \tilde{f}_{p_{\perp}}(p_{\parallel}) = < p_{\parallel}|q >^{*} = f_{\parallel}{}^{*}(q) = \exp\left(-\frac{\boldsymbol{p}\langle \boldsymbol{p}, \boldsymbol{q} \rangle}{|\boldsymbol{p}|\,\hbar}\right) \tag{2}$$

The first transverse transform is

$$\mathcal{F}_{\perp}\big(f(q)\big) \equiv \mathcal{F}\big(f_{\perp}(q)\big) \tag{3}$$

$$< q|p_{\perp} > = \tilde{f}_{p_{\perp}}(p_{\perp}) = < p_{\perp}|q >^{*} = f_{\perp}{}^{*}(q) = \exp\left(\frac{\boldsymbol{p} \times \boldsymbol{q}}{\hbar}\right) \tag{4}$$

The second transverse transform:

$$\mathcal{F}_{\vdash}\big(f(q)\big) \equiv \mathcal{F}\big(f_{\vdash}(q)\big) \tag{5}$$

$$< q|p_{\vdash} > = \tilde{f}_{\vdash}(p_{\vdash}) = < p_{\vdash}|q >^{*} = f_{\vdash}{}^{*}(q) = \exp\left(\frac{\boldsymbol{p} \times (\boldsymbol{p} \times \boldsymbol{q})}{|\boldsymbol{p}|\,\hbar}\right) \tag{6}$$

$$= \exp\left(\frac{|\mathbf{p}|\boldsymbol{q}}{\hbar}\right)$$

Transverse plane

The Cayley-Dickson construction, as well as Warren Smith's construction formula shows that the transverse part can be considered as a complex number space multiplied with a fixed imaginary quaternionic base number. The selection of the imaginary base number $\boldsymbol{i}$ is arbitrary as long as it is perpendicular to $\boldsymbol{k}$. The resulting plane is spanned by axes $\boldsymbol{i}$ and $\boldsymbol{ik}$. When base number $\boldsymbol{i}$ is divided away, then a normal complex number space results.
Also here a complex Fourier transform can be defined in a way that is similar to the longitudinal Fourier transform. It must be reckoned that the sign selections for these directions differ.

Alternative approach

The following draws from the work of S. Thangavelu.

Let us take the non-abelian group $\mathbb{H}_1$ which is $\mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R}$ with the group law

$$(x, y, t)(x_0, y_0, t_0) = (x + x_0;\ y + y_0;\ t + t_0 + x\, y_0) \tag{1}$$

Then it is clear that $\mathbb{H}_1$ is non-abelian and the Lebesgue measure dx dy dt is both left and right invariant Haar measure on $\mathbb{H}_1$. With this measure we can form the Hilbert space $L^2(\mathbb{H}_1)$. Let $\Gamma = \mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z}$. Then it is easy to check that Γ is a subgroup of $\mathbb{H}_1$ so that we can form the quotient $M = \Gamma/\mathbb{H}_1$ consisting of all right cosets of Γ. Functions on M are naturally identified with left Γ-invariant functions on $\mathbb{H}_1$. As the Lebesgue measure dx dy dt is left Γ-invariant we can form $L_2(M)$ using the Lebesgue measure restricted to M. As a set we can identify M with $[0, 1)^3$ and we just think of $L^2(M)$ as $L^2([0, 1)^3)$.

Fourier expansion in the last variable allows us to decompose $L^2(M)$ into a direct sum of orthogonal subspaces. Simply define $\mathcal{H}_k$ to be the set of all $f \in L^2(M)$ which satisfy the condition

$$f(x, y, t + s) = \exp(2\,\pi\, i\, k\, s)\, f(x, y, t) \tag{2}$$

Then $\mathcal{H}_k$ is orthogonal to $\mathcal{H}_j$ whenever $k \neq j$ and any $f \in L^2(M)$ has the unique expansion

$$f = \sum_{k=-\infty}^{\infty} f_k\,;\ f_k \in \mathcal{H}_k \tag{3}$$

In quaternionic terms, the split sees *ik* as imaginary quaternion ***k*** and the quaternionic Hilbert space is split in components according to the imaginary direction of ***k***, where the choice is between three mutually perpendicular directions.

For the moment, we are mainly interested in $\mathcal{H}_1$ which is a Hilbert space in its own right. It is interesting to note that functions in $\mathcal{H}_1$ are also invariant under the left action of Γ.

Our next example of a unitary operator is the following. Consider the map $J : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ given by

$$J(x,y,t) = (-x\,,y,t - xy) \tag{4}$$

$$J^{\dagger}(x,y,t) = (x,-y,t - xy) \tag{5}$$

$$J^{\dagger} = J^{-1} \tag{6}$$

$$J^2(x,y,t) = J(-x,\cdot\, y;\; t - xy) = (-x,-\,y;t) \tag{7}$$

$$J^4 = I \tag{8}$$

$$J(0,0,t) = (0,0,t) \tag{9}$$

$$J\,f(x,y,t) = f(J(x,y;\;t)) = f(-x,\cdot\, y,t - xy) \tag{10}$$

Weil-Brezin transform

Next consider the Weil-Brezin transform V:

$$V\,f(x,y,t) = exp(2\,\pi\,\boldsymbol{k}\,t)\sum_{n} f(x\;+\;n)\,\exp(2\,\pi\,\boldsymbol{k}\,n\,y) \tag{11}$$

$$\int_{y=0}^{1} |V\,f(x,y,t)|^2 dy = \int_{x=0}^{1}\sum_{n=-\infty}^{n=\infty} |f(x\;+\;n)|^2\,dx \tag{12}$$

$$\iiint_{0}^{1} |V\,f(x,y,t)|^2 dx\,dy\,dt = \int_{0}^{1} |\,f(x)|^2 dx \tag{13}$$

V is unitary.
See also Zak transform

Fourier transform

We define the Fourier transform $\mathcal{F}$ by:

$$\mathcal{F} = V^{\dagger}\,J\,V \tag{14}$$

- $\mathcal{F}^4\,f = f$; for every $f \in L^2(\mathbb{R})$
- $\mathcal{F}^2\,f(x) = f(-x)$; for almost every $x \in \mathbb{R}$
- $\|\mathcal{F}\,f\|^2 = \|f\|^2$

For $f \in L^1(\mathbb{R}) \setminus L^2(\mathbb{R})$ the Fourier transform is given by

$$\mathcal{F}\mathrm{f}(\xi) = \int_{x \in \mathbb{R}} \mathrm{f(x)} \cdot \exp(2\,\pi\, \boldsymbol{k}\, x) dx \tag{15}$$

If we further assume that $\mathcal{F}f \in L^1(\mathbb{R})$ then for almost every x we have

$$\mathrm{f}(\xi) = \int_{\xi \in \mathbb{R}} \mathcal{F}\mathrm{f}(\xi) \exp(2\,\pi\, \boldsymbol{k}\, x\, \xi) dx \tag{16}$$

Functions invariant under Fourier transform

In this section we confine to a complex part of the Hilbert space.
See http://en.wikipedia.org/wiki/Hermite_polynomials.
There exist two types of Hermite polynomials: (1, 2)

1. The probalist's Hermite polynomials:

$$H_n^{prob}(z) = (-1)^n \exp(½z^2)\, \frac{d^n}{dz^n} \exp(-½z^2).$$

2. The physicist's Hermite polynomials

$$H_n^{phys}(z) = (-1)^n \exp(z^2) \frac{d^n}{dx^n} \exp(-z^2) = \exp(½z^2) \left(z - \frac{d}{dz}\right) \exp(-½z^2)$$

These two definitions are *not* exactly equivalent; either is a rescaling of the other:

$$H_n^{phys}(z) = 2^{n/2}\, H_n^{prob}\left(z\sqrt{2}\right) \tag{3}$$

In the following we focus on the physicist's Hermite polynomials.

The Gaussian function ϕ(z) defined by

$$\varphi(x) = exp(-\pi\, z^2) \tag{4}$$

is an eigenfunction of $\mathcal{F}$. It means that its Fourier transform has the same form.

As $\mathcal{F}^4 = \mathrm{I}$ any λ in its spectrum $\sigma(\mathcal{F})$ satisfies $\lambda^4 = 1$: Hence,

$$\sigma(\mathcal{F}) = \{1; -1;\, i; -i\}. \tag{5}$$

We take the Fourier transform of the expansion:

$$exp(-½\, z^2 + 2\, z\, c - c^2) = \sum_{n=0}^{\infty} exp(-½\, z^2)\, H_n(z)\, c^n/n! \tag{6}$$

First we take the Fourier transform of the left hand side:

$$\frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\infty} exp(-\boldsymbol{k}\, z\, p_z)\, exp(-½\, z^2 \;+\; 2\, z\, c - c^2)\; dz \tag{7}$$

$$= \; exp(-½\, p_z^2 \;-\; 2\, \boldsymbol{k}\, p_z\, c \;+\; c^2)$$

$$= \; \sum_{n=0}^{\infty} exp(-½\, p_z^2)\; H_n(p_z)\; (-\boldsymbol{k}\, c)^n / n!$$

The Fourier transform of the right hand side is given by

$$\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \int_{z=-\infty}^{\infty} exp(-\boldsymbol{k}\, z\, p_z) \cdot exp(-½\, z^2)\; H_n(z)\; c^n / n!\; dz \tag{8}$$

Equating like powers of c in the transformed versions of the left- and right-hand sides gives

$$\tfrac{1}{\sqrt{2\pi}} \textstyle\int_{z=-\infty}^{\infty} \exp(-\boldsymbol{k}\, z\, p_z) \cdot exp(-½\, z^2)\; H_n(z)\; c^n / n!\; dz \tag{9}$$

$$= \; (-\boldsymbol{k})^n \cdot \exp(-½\, p_z^2)\; H_n(p_z)\; \frac{c^n}{n!}$$

Let us define the Hermite functions $\psi_n(z)$

$$\psi_n(z) \;\equiv\; \; < z|\psi_n > = \mathrm{c}_n \exp(-½\, z^2)\; H_n(z) \tag{10}$$

$$|\mathcal{F}\, \psi_n > = \; |\psi_n > \; (-\boldsymbol{k})^n \tag{11}$$

with suitably chosen c_n so as to make

$$\|\psi_n\|^2 \;=\; 1 \tag{12}$$

$$\mathrm{c}_n = \frac{1}{\sqrt{2^n n!\, \sqrt{\pi}}} \tag{13}$$

The importance of the Hermite functions lie in the following theorem.

"The Hermite functions ψ_n; $n \in \mathbb{N}$ form an orthonormal basis for $L^2(\mathbb{R})$"

Consider the operator

$$H \;=\; -½ \frac{d^2}{dz^2} + \; ½\, z^2 \tag{14}$$

Apply this to $\psi_n(z)$:

$$H \cdot \psi_n(z) = \; (½ \;+\; n)\, \psi_n(z) \tag{15}$$

Thus, ψ_n is an eigenfunction of *H*.

Let $f = \psi_{4k+j}$ be any of the Hermite functions. Then we have

$$\sum_{n=-\infty}^{\infty} f(y + n) \cdot \exp\left(-2\pi \boldsymbol{k}\, x\, (y+n)\right) \tag{16}$$

$$= (-\boldsymbol{k})^j \sum_{n=-\infty}^{\infty} f(x + n)\, exp(2\pi \boldsymbol{k}\, n\, y)$$

Proof: As

$$\mathcal{F} = V^{\dagger} J V \tag{17}$$

the equation

$$\mathcal{F} f = (-\boldsymbol{k})^j f \tag{18}$$

translates into

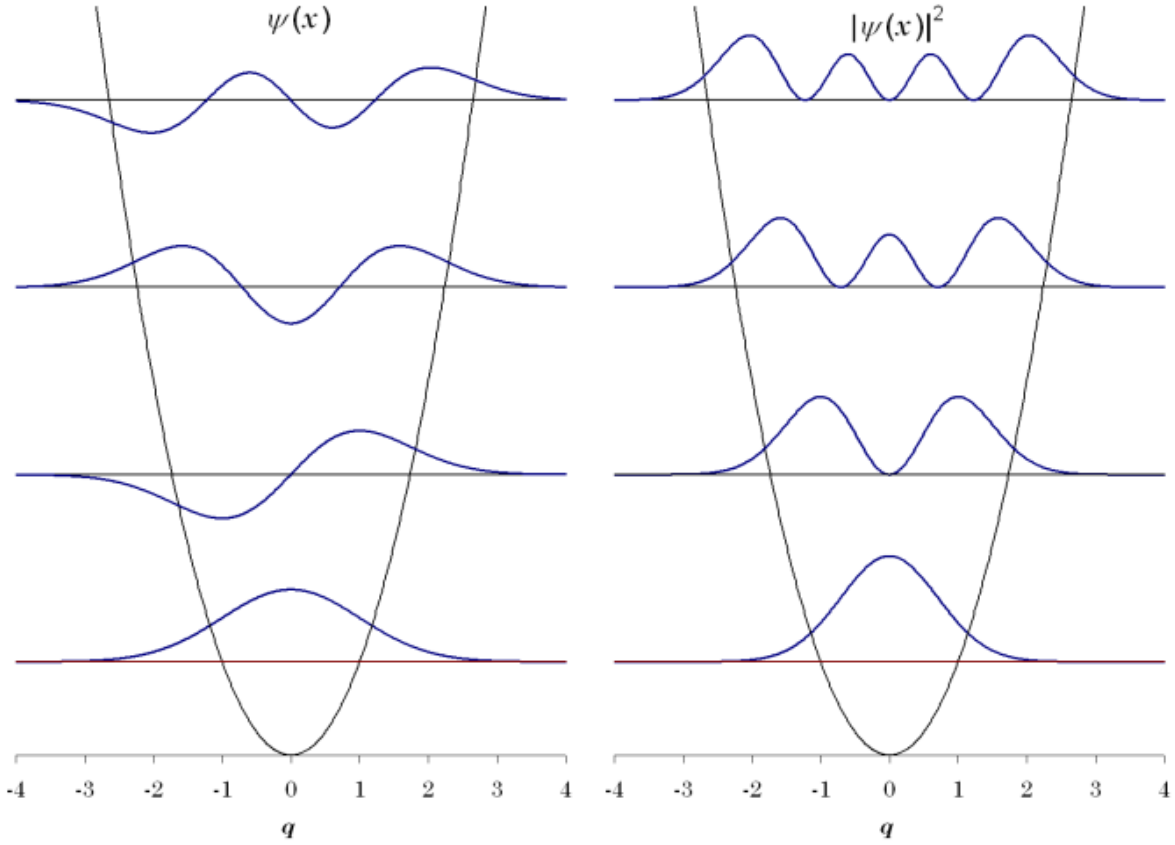
$$JVf(x;\, y;\, t) = (-\boldsymbol{k})^j\, Vf(x;\, y;\, t) \tag{19}$$

With the definition of *V* and $t = xy$:

$$V\, f(x,y,t) = exp(2\pi \boldsymbol{k}\, t) \sum_n f(x + n)\, exp(2\pi \boldsymbol{k}\, n\, y) \tag{20}$$

QED.

The vectors $|\psi_n\rangle$ are eigenvectors of the Fourier transform operator with eigenvalues $(-\boldsymbol{k})^n$.
The eigenfunctions $\psi_n(x)$ represent eigenvectors $|\psi_n\rangle$ that span the complex Hilbert space $\mathbf{H}_k$.
For higher *n* the central parts of $\psi_n(x)$ and $|\psi_n(x)|^2$ become a sinusoidal form.



A **coherent state** is a specific kind of state of the quantum harmonic oscillator whose dynamics most closely resemble the oscillating behavior of a classical harmonic oscillator system. The ground state is a squeezed coherent state.

Ladder operator

The Hermite functions ψ_n represent Fock states.

Boson ladder operators are characterized by

$$\mathcal{A}|\psi_n> = \sqrt{n}\,|\psi_{n-1}> \tag{1}$$

$$\mathcal{A}^\dagger|\psi_n> = \sqrt{n+1}\,|\psi_{n+1}> \tag{2}$$

$$\mathcal{A} = \frac{1}{\sqrt{2}}\left(c_1\frac{d}{dq} + c_2 q\right) = -\boldsymbol{k}\,\check{P}\sqrt{\frac{1}{2\,\hbar\,m\,\omega}} + \check{Q}\sqrt{\frac{m\,\omega}{2\,\hbar}}$$

$$\mathcal{A}^\dagger = \frac{1}{\sqrt{2}}\left(-c_1\frac{d}{dq} + c_2 q\right) = \boldsymbol{k}\,\check{P}\sqrt{\frac{1}{2\,\hbar\,m\,\omega}} + \check{Q}\sqrt{\frac{m\,\omega}{2\,\hbar}}$$

In the Heisenberg picture, the operators have the following time dependence:

$$\mathcal{A}(t) = \mathcal{A}(t_0)\exp\left(-\boldsymbol{k}\,\omega\,(\mathrm{t} - t_0)\right) \tag{3}$$

$$\mathcal{A}^\dagger(t) = \mathcal{A}^\dagger(t_0)\exp\left(\boldsymbol{k}\,\omega\,(\mathrm{t} - t_0)\right) \tag{4}$$

We can also define an enumeration operator N which has the following property:

$$N = \mathcal{A}^\dagger \mathcal{A} \tag{5}$$

$$N|\psi_n> = |\psi_n> n \tag{6}$$

In deriving the form of $\mathcal{A}^\dagger$, we have used the fact that the operators X and P_x, which represent observables, are Hermitian. These observable operators can be expressed as a linear combination of the ladder operators as

$$\check{Q}(t) = \sqrt{\frac{\hbar}{2\, m\, \omega}}\ (\mathcal{A}^\dagger(t) + \mathcal{A}(t)) \tag{7}$$

$$\check{P}(t) = \boldsymbol{k}\sqrt{½\,\hbar\, m\, \omega}\ (\mathcal{A}^\dagger(t) - \mathcal{A}(t)) \tag{8}$$

The $\check{\boldsymbol{Q}}$ and $\check{\boldsymbol{P}}$ operators obey the following identity, known as the canonical commutation relation:

$$[\check{Q}, \check{P}] = \boldsymbol{k}\,\hbar \tag{9}$$

Using the above, we can prove the identities

$$H = \hbar\,\omega\left(\mathcal{A}^\dagger\,\mathcal{A} + ½\right) = \hbar\,\omega\,(N + ½) \tag{10}$$

$$[\mathcal{A}^\dagger, \mathcal{A}] = 1 \tag{11}$$

Now, let $|f_E>$denote an energy eigenstate with energy E. The inner product of any ket with itself must be non-negative, so

$$< \psi_E\, \mathcal{A}|\mathcal{A}\,\psi_E > = < \psi_E E|\, \mathcal{A}^\dagger\, \mathcal{A}\, \psi_E > \geq 0 \tag{12}$$

Expressing $\mathcal{A}^\dagger\mathcal{A}$ in terms of the Hamiltonian H:

$$< \psi_E|(H/(\hbar\,\omega) - ½)\,\psi_E > = (E/(\hbar\,\omega) - ½) \geq 0 \tag{13}$$

so that

$$E \geq ½\,\hbar\,\omega. \tag{14}$$

Note that when $|\mathcal{A}\,\psi_E > = |0>$ (is the zero ket i.e. a ket with length zero), the inequality is saturated, so that

$$E = ½\,\hbar\,\omega \tag{15}$$

It is straightforward to check that there exists a state satisfying this condition; it is the ground state

$$|\psi_{ground} >= |\psi_{E_n} >;\ (n = 0) \tag{16}$$

Using the above identities, we can now show that the commutation relations of $\mathcal{A}$ and $\mathcal{A}^\dagger$ with H are:

$$[H, \mathcal{A}] = -\hbar\,\omega\,\mathcal{A} \tag{17}$$

$$\left[H, \mathcal{A}^\dagger\right] = \hbar\,\omega\,\mathcal{A}^\dagger \tag{18}$$

Thus, provided $|\mathcal{A}\ \psi_E >$ is not the zero ket,

$$\begin{aligned} |H\,\mathcal{A}\,\psi_E >= &\ |[H, \mathcal{A}] + \mathcal{A}\,H\,\psi_E > \\ = &\ |-\hbar\,\omega\,\mathcal{A} + \mathcal{A}\,E\,\psi_E > \\ = &\ |-\hbar\,\omega\,\mathcal{A} + \mathcal{A}\,E\,\psi_E > \\ = &\ (E - \hbar\,\omega)\,|\mathcal{A}\,\psi_E > \end{aligned} \tag{19}$$

Similarly, we can show that

$$|H\,\mathcal{A}^\dagger\ \psi_E >= (E + \hbar\,\omega)\,|\mathcal{A}^\dagger\,\psi_E > \tag{20}$$

In other words, $\mathcal{A}$ acts on an eigenstate of energy E to produce, up to a multiplicative constant, another eigenstate of energy $E - \hbar\,\omega$, and $\mathcal{A}^\dagger$ acts on an eigenstate of energy E to produce an eigenstate of energy $E + \hbar\,\omega$. For this reason, a is called a "lowering operator", and $\mathcal{A}^\dagger\mathcal{A}$ "raising operator". The two operators together are called ladder operators. In quantum field theory, $\mathcal{A}$ and $\mathcal{A}^\dagger$ are alternatively called "annihilation" and "creation" operators because they destroy and create particles, which correspond to our quanta of energy.

Given any energy eigenstate, we can act on it with the lowering operator $\mathcal{A}$, to produce another eigenstate with $\hbar\,\omega$-less energy. By repeated application of the lowering operator, it seems that we can produce energy eigenstates down to $E = -\infty$. However, this would contradict our earlier requirement that $E \geq \hbar\,\omega/2$.

Ground state

Therefore, there must be a ground-state energy eigenstate, which we label $|f_{ground}>$, such that

$$|\mathcal{A}\,\psi_{ground} >= |0 >; \text{(zero ket)}. \tag{1}$$

In this case, subsequent applications of the lowering operator will just produce zero kets, instead of additional energy eigenstates. Furthermore, we have shown above that

$$|H\,\psi_{ground} >= (\tfrac{1}{2}\,\hbar\,\omega)\,|\psi_{ground} > \tag{2}$$

Finally, by acting on $|\psi_{ground}>$ with the raising operator and multiplying by suitable normalization factors, we can produce an infinite set of energy eigenstates

$$\{|\psi_{ground}>, |\psi_{E_1}>, |\psi_{E_2}>, \ldots., |\psi_{E_n}>\}, \tag{3}$$

such that

$$|H\,\psi_{E_n}> = \hbar\,\omega\,(n + ½)\,|\psi_{E_n}> \tag{4}$$

which matches the energy spectrum.
This method can also be used to quickly find the ground state wave function of the quantum harmonic oscillator.
Indeed

$$|\mathcal{A}\,\psi_{ground}> = |0> \tag{5}$$

becomes

$$\psi_{ground}(x) = \psi_0(x) \equiv < x|\,\psi_{ground}> = -\frac{\hbar}{m\,\omega}\frac{d}{dx}\,\psi_0(x) \tag{6}$$

so that

$$d\,\psi_0(x) = \psi_0(x)\,\frac{\hbar}{m\,\omega}\,x\,d\,x \Rightarrow \ln\big(\psi_0(x)\big) = \frac{m\,\omega}{2\hbar}\,x^2 + const \tag{7}$$

After normalization this leads to the following position space representation of the ground state wave function.

$$\psi_0(x) = \sqrt[4]{\frac{m\,\omega}{\pi\hbar}}\;e^{-\frac{m\,\omega}{2\hbar}x^2} \tag{8}$$

Coherent state

A **coherent state** is a specific kind of state of the quantum harmonic oscillator whose dynamics most closely resemble the oscillating behavior of a classical harmonic oscillator system.

The coherent state |α> is defined to be the 'right' eigenstate of the annihilation operator $\mathcal{A}$. Formally, this reads:

$$|\mathcal{A}\,\alpha> = \alpha|\alpha> \tag{1}$$

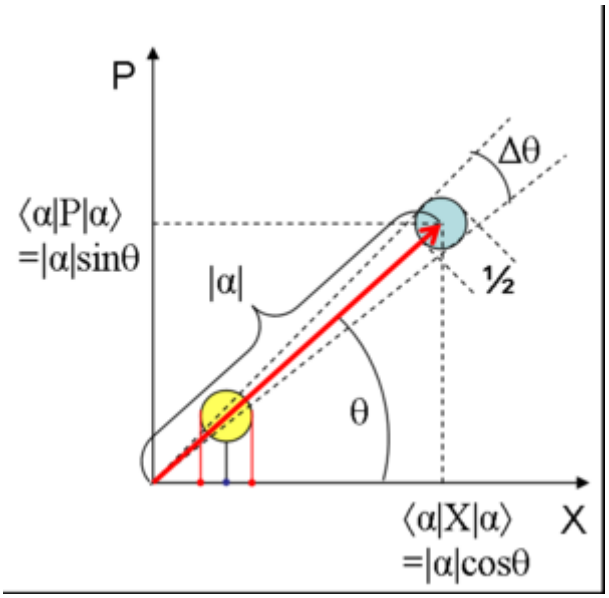
Since $\mathcal{A}$ is not Hermitian, α is a hyper complex number that is not necessarily real, and can be represented as

$$\alpha = |\alpha|\,\exp(\boldsymbol{k}\,\theta) \tag{2}$$

where θ is a real number. $|\alpha|$ is the amplitude and θ is the phase of state |α>.

This formula means that a coherent state is left unchanged by the annihilation or the creation of a particle. The eigenstate of the annihilation operator has a Poissonian number distribution A Poisson distribution is a necessary and sufficient condition that all annihilations are statistically independent.

The coherent state's location in the complex plane (phase space) is centered at the position and momentum of a classical oscillator of the same phase θ and amplitude. As the phase increases the coherent state circles the origin and the corresponding disk neither distorts nor spreads. The disc represents Heisenberg's uncertainty. This is the most similar a quantum state can be to a single point in phase space.



Phase space plot of a coherent state. This shows that the uncertainty (blur) in a coherent state is equally distributed in all directions. The horizontal and vertical axes are the X and P quadratures of the field, respectively. Oscillations that are said to be *in quadrature*, if they are separated in phase by π/2 radians. The red dots on the x-axis trace out the boundaries of the quantum noise. Further from the origin the relative contribution of the quantum noise becomes less important.

The representation of the coherent state in the basis of Fock states is:

$$|\alpha> = \exp(-½|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n> = \exp(-½|\alpha|^2) \exp\left(\alpha \mathcal{A}^\dagger\right) |0> \quad (3)$$

where |n> are Hermite functions (eigenvectors of the Hamiltonian). This is a Poissonian distribution. The probability of detecting n photons is:

$$\mathcal{P}(n) = \exp(-\langle n \rangle) \frac{\langle n \rangle^n}{n!} \quad (4)$$

Similarly, the average photon number in a coherent state is

$$\langle n \rangle = \langle \mathcal{A}^\dagger \mathcal{A} \rangle = |\alpha|^2 \tag{5}$$

and the variance is

$$(\Delta n)^2 = Var\left(\mathcal{A}^\dagger \mathcal{A}\right) = |\alpha|^2 \tag{6}$$

Squeezing

The squeezing operator can squeeze a state more or less in the direction of either P or Q. The operator is defined as:

$$Sq(z) = \exp\left(½\left(z^* \mathcal{A} + z\mathcal{A}^\dagger\right)\right) \tag{1}$$

$$z = r\exp(\boldsymbol{k}\,\theta) \tag{2}$$

The ground state is a saturated squeezed coherent state where

$$\Delta p = \Delta q \text{ and } \Delta q \cdot \Delta p = \hbar/2 \tag{3}$$

Base transforms

Now we have discovered the following base transforms:
Position⇔momentum:

$$< q|p > = \frac{1}{\sqrt{2\pi\hbar}} \exp(\frac{\boldsymbol{k}\, q\, p}{\hbar}) \tag{1}$$

Position⇔Fock state:

$$< q|n > = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2^n\, n!}} \exp(-\frac{m\omega}{2\hbar}\, q^2) H_n(q\sqrt{\frac{m\omega}{\hbar}})$$

Fock state⇔coherent state:

$$< n|z > = \frac{1}{\sqrt{n!}}\, z^n \exp(-½|z|^2)$$

Rotational symmetry

In case of rotational symmetry in the imaginary part of quaternion space, the exponential function must be replaced by a Bessel function. The corresponding Fourier transform then becomes a Hankel transform.

The spherical harmonics are eigenfunctions of the square of the orbital angular momentum operator $-i\hbar \boldsymbol{r} \times \boldsymbol{\nabla}$ and therefore they represent the different quantized configurations of atomic orbitals.

Spherical harmonics

The following draws from the work of S. Thangavelu.

In this subsection we look for eigenfunctions of the Fourier transform which have spherical symmetry. As in the one dimensional case we consider functions of the form

$$f(x) = p(x)\exp(-\pi|x|^2) \tag{1}$$

This will be an eigenfunction of $\mathcal{F}$ if and only p satisfies

$$\int_{\mathbb{R}^n} p(x - iy)\exp(-\pi\cdot|x|^2)\,dx = \lambda\, p(y) \tag{2}$$

Here in quaternion terms x and iy represent two mutually perpendicular imaginary numbers while x and y are parallel. Thangavelu uses complex numbers. We keep as close as is possible to his text.

If (2) is true for all $y \in \mathbb{R}^n$ then we should also have

$$\int_{\mathbb{R}^n} p(x+y)\exp(-\pi\cdot|x|^2)\,dx = \lambda\, p(iy) \tag{3}$$

Integrating in polar coordinates the integral on the left is

$$\int_{r=0}^{\infty} |\mathbb{S}^{n-1}|\left(\int_{\mathbb{S}^{n-1}} p(y + r\,\omega)\,d\sigma(\omega)\right) exp(-\pi\cdot r^2)\cdot r^{n-1}dr \tag{4}$$

where $d\sigma(\omega)$ is the normalised surface measure on the unit sphere $\mathbb{S}^{n-1}$.

If p is homogeneous of degree m then

$$p(i\,y) = i^m\, p(y) \tag{5}$$

and hence for such polynomials the equation

$$\int_{\mathbb{R}^n} p(x+y)\exp(-\pi\cdot|x|^2)\,dx = \lambda\, i^m\, p(iy) \tag{6}$$

will be satisfied for

$$\lambda = (-i)^m \tag{7}$$

if p has the mean value property

$$\int_{\mathbb{S}^{n-1}} p(y + r\,\omega)d\sigma(\omega) = p(y) \tag{8}$$

Such functions are precisely the harmonic functions satisfying

$$\Delta u = 0 \tag{9}$$

Thus we have proved:

Let

$$f(x) = p(x)\, exp(-\pi |x|^2) \tag{10}$$

where p is homogeneous of degree m and harmonic. Then

$$\mathcal{F} f = (-i)^m f \tag{11}$$

Let $\mathcal{P}^m$ stand for the finite dimensional space of homogeneous harmonic polynomials of degree m:

The above theorem says that the finite dimensional subspace of $L^2(\mathbb{R}^n)$ consisting of functions of the form

$$p(x) \exp(-\pi |x|^2);\ p \in \mathcal{P}^m \tag{12}$$

is invariant under the Fourier transform.
We claim that the following extension is true.
Let

$$f \in L^2(\mathbb{R}^n \tag{13}$$

be of the form

$$f(x) = p(x)g(|x|);\ p \in \mathcal{P}^m \tag{14}$$

Then

$$\mathcal{F} f(\xi) = p(\xi)g(|\xi|) \tag{15}$$

Thus the subspace of functions of the form

$$f(x) = p(x)g(|x|);\ p \in \mathcal{P}^m \tag{16}$$

is invariant under the Fourier transform.

Let

$$f \in L^2(\mathbb{R}^n) \tag{17}$$

be of the form

$$f(x) = p(x)g(|x|);\ p \in \mathcal{P}^m \tag{18}$$

Then

$$\mathcal{F}_n\,(\mathrm{f}) = (-\mathrm{i})^m\, p\, \mathcal{F}_{n+2m}\, g \tag{19}$$

The above result is known as the Hecke-Bochner formula for the Fourier transform.

We conclude our discussion on invariant subspaces with the following result which shows that the Fourier transform of a radial function reduces to an integral transform whose kernel is a Bessel function. This relates to the Hankel transform.

Let J_α stand for the Bessel function of type $\alpha > -1$
If

$$f(x) = g(|x|) \tag{20}$$

is radial and integrable then

$$\mathcal{F}_n\,(\mathrm{f})(\xi) = \mathrm{c}_n \int_0^\infty \mathrm{g(r)} \cdot \frac{\mathrm{J}_{\frac{n}{2}-1}(2\,\pi\,\mathrm{r}\,|\xi|)}{(2\,\pi\,\mathrm{r}\,|\xi|)^{\frac{\mathrm{n}}{2}-1}} \cdot \mathrm{r}^{\mathrm{n}-1} dr \tag{21}$$

Spherical harmonics eigenvalues

See: http://en.wikipedia.org/wiki/Spherical_harmonics for more details.
Spherical harmonics are best presented in polar coordinates. There exists a corresponding polar Fourier transform. This Fourier transform also has invariant functions. Like in the rectangular case, they form the basis for spherical harmonics.

Laplace's equation in spherical coordinates is:

$$\nabla^2 f = r^{-2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta}\left(\sin(\theta) \frac{df}{d\theta}\right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 f}{\partial \varphi^2} = 0 \tag{1}$$

Try to find solutions in the form of the eigenfunctions of the Fourier transform.
By separation of variables, two differential equations result by imposing Laplace's equation:

$$\mathrm{f}(r, \theta, \varphi) = R(r) \cdot Y(\theta, \varphi) \tag{2}$$

$$R^{-1} \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \lambda \tag{3}$$

$$\frac{1}{Y \sin(\theta)} \frac{\partial}{\partial \theta}\left(\sin(\theta) \frac{\partial Y}{\partial \theta}\right) + \frac{1}{Y \sin^2(\theta)} \frac{\partial^2 Y}{\partial \varphi^2} = -\lambda \tag{4}$$

The second equation can be simplified under the assumption that Y has the form

$$Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi) \tag{5}$$

Applying separation of variables again to the second equation gives way to the pair of differential equations

$$\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2} = -m^2 \tag{6}$$

$$\lambda \sin^2(\theta) + \frac{\sin(\theta)}{\Theta(\theta)}\frac{d}{d\theta}\left[\sin(\theta)\frac{d\Theta}{d\theta}\right] = m^2 \tag{7}$$

for some number m. A priori, m is a complex constant, but because Φ must be a periodic function whose period evenly divides 2π, m is necessarily an integer and Φ is a linear combination of the complex exponentials $\exp(\pm i\, m\, \theta)$. The solution function $Y(\theta, \varphi)$ is regular at the poles of the sphere, where $\theta = 0, \pi$. Imposing this regularity in the solution Θ of the second equation at the boundary points of the domain is a Sturm–Liouville problem that forces the parameter λ to be of the form $\lambda = \ell(\ell + 1)$ for some non-negative integer with $\ell \geq |m|$; this is also explained below in terms of the orbital angular momentum.
Furthermore, a change of variables $t = cos\theta$ transforms this equation into the Legendre equation, whose solution is a multiple of the associated Legendre function. $P_l^m(cos(\theta))$.
Finally, the equation for R has solutions of the form $R(r) = A\, r^{\ell} + B\, r^{-\ell-1}$; requiring the solution to be regular throughout $\mathbb{R}^3$ forces $B = 0$.
Here the solution was assumed to have the special form

$$Y(\theta, \varphi) = \Theta(\theta)\, \Phi(\varphi) \tag{8}$$

For a given value of ℓ, there are $2\ell + 1$ independent solutions of this form, one for each integer m with $-\ell \leq m \leq \ell$. These angular solutions are a product of trigonometric functions, here represented as a complex exponential, and associated Legendre functions:

$$Y_l^m(\theta, \varphi) = N \exp(i\, m\, \varphi)\, P_l^m(cos(\theta)) \tag{9}$$

which fulfill

$$r^2\, \nabla^2 Y_l^m(\theta, \varphi) = -l\, (l + 1)\, Y_l^m(\theta, \varphi) \tag{10}$$

Here Y_l^m is called a spherical harmonic function of degree ℓ and order m, P_l^m is an associated Legendre function, N is a normalization constant, θ represents the colatitude and ϕ represents the longitude. In particular, the colatitude θ, or polar angle, ranges from 0 at the North Pole to π at the South Pole, assuming the value of π/2 at the Equator, and the longitude φ, or azimuth, may assume all values with $0 \leq \varphi < 2\pi$. For a fixed integer ℓ, every solution $Y(\theta, \varphi)$ of the eigenvalue problem

$$r^2\, \nabla^2 Y = -l\, (l + 1)\, Y \tag{11}$$

is a linear combination of Y_l^m. In fact, for any such solution, $r^{\ell}\, Y(\theta, \varphi)$ is the expression in spherical coordinates of a homogeneous polynomial that is harmonic, and so counting dimensions shows that there are $2\ell + 1$ linearly independent of such polynomials.
The general solution to Laplace's equation in a ball centered at the origin is a linear combination of the spherical harmonic functions multiplied by the appropriate scale factor r^l,

$$f(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m r^l Y_l^m(\theta,\varphi) \tag{12}$$

where the f_l^m are constants and the factors $r^l Y_l^m$ are known as solid harmonics. Such an expansion is valid in the ball

$$r < R = 1/\lim_{l\Rightarrow\infty} sup \left|f_l^m\right|^{1/l} \tag{13}$$

Orbital angular momentum

In quantum mechanics, Laplace's spherical harmonics are understood in terms of the orbital angular momentum

$$\boldsymbol{L} = -i\,\hbar\,\boldsymbol{x}\times\boldsymbol{\nabla} = L_x\,\boldsymbol{i} + L_y\boldsymbol{j} + L_z\,\boldsymbol{k} \tag{1}$$

The spherical harmonics are eigenfunctions of the square of the orbital angular momentum

$$\boldsymbol{L}^2 = -r^2\nabla^2 + \left(r\frac{\partial}{\partial r} + 1\right)r\frac{\partial}{\partial r} \tag{2}$$

$$= \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta} - \frac{1}{\sin^2(\theta)}\cdot\frac{\partial^2}{\partial\varphi^2}$$

Laplace's spherical harmonics are the joint eigenfunctions of the square of the orbital angular momentum and the generator of rotations about the azimuthal axis:

$$L_z = -i\cdot\hbar\cdot\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) = i\,\hbar\,\frac{\partial}{\partial\varphi} \tag{3}$$

These operators commute, and are densely defined self-adjoint operators on the Hilbert space of functions f square-integrable with respect to the normal distribution on $\mathbb{R}^3$:

$$(2\,\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^3}|f(x)|^2\exp(-|x|^2/2) < \infty \tag{4}$$

Furthermore, $\boldsymbol{L}^2$ is a positive operator.
If Y is a joint eigenfunction of L^2 and L_z, then by definition

$$\boldsymbol{L}^2\,Y = \lambda\,Y \tag{5}$$

$$L_z\,Y = m\,Y \tag{6}$$

for some real numbers m and λ. Here m must in fact be an integer, for Y must be periodic in the coordinate φ with period a number that evenly divides $2\,\pi$. Furthermore, since

$$\boldsymbol{L}^2 = L_x^2 + L_y^2 + L_z^2 \tag{7}$$

and each of L_x , L_y, L_z are self-adjoint, it follows that $\lambda \geq m^2$.

Denote this joint eigenspace by $E_{\lambda,m}$, and define the raising and lowering operators by

$$L_+ = L_x + i\,L_y \tag{8}$$

$$L_- = L_x - i\,L_y \tag{9}$$

Then L_+ and L_- commute with $\boldsymbol{L}^2$, and the Lie algebra generated by L_+, L_-, L_z is the special linear Lie algebra, with commutation relations

$$[L_z, L_+] = L_+ \tag{10}$$

$$[L_z, L_-] = -L_- \tag{11}$$

$$[L_+, L_-] = 2\,L_z \tag{12}$$

Thus $L_+ : E_{\lambda,m} \to E_{\lambda,m+1}$ (it is a "raising operator") and $L_- : E_{\lambda,m} \to E_{\lambda,m-1}$ (it is a "lowering operator"). In particular, $L_+^k : E_{\lambda,m} \to E_{\lambda,m+k}$ must be zero for k sufficiently large, because the inequality $\lambda \geq m^2$ must hold in each of the nontrivial joint eigenspaces. Let $Y \in E_{\lambda,m}$ be a nonzero joint eigenfunction, and let k be the least integer such that

$$L_+^k\, Y = 0 \tag{13}$$

Then, since

$$L_-\, L_+ = \boldsymbol{L}^2 - L_z^2 - L_z \tag{14}$$

it follows that

$$0 = L_-\, L_+^k\, Y = \left(\lambda - (m+k)^2 - (m+k)\right) Y \tag{15}$$

Thus $\lambda = \ell(\ell+1)$ for the positive integer $\ell = m + k$. (16)

Spherical harmonics expansion

The Laplace spherical harmonics form a complete set of orthonormal functions and thus form an orthonormal basis of the Hilbert space of square-integrable functions. On the unit sphere, any square-integrable function can thus be expanded as a linear combination of these:

$$f(\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m Y_l^m(\theta,\varphi) \tag{1}$$

This expansion holds in the sense of mean-square convergence — convergence in L^2 of the sphere — which is to say that

$$\lim_{N\to\infty} \int_0^{2\pi} \int_0^{\pi} \left| f(\theta,\varphi) - \sum_{l=0}^{N} \sum_{m=-l}^{l} f_l^m Y_l^m(\theta,\varphi) \right|^2 \sin(\theta)\, d\theta\, d\varphi = 0 \tag{2}$$

The expansion coefficients are the analogs of Fourier coefficients, and can be obtained by multiplying the above equation by the complex conjugate of a spherical harmonic, integrating over the solid angle Ω, and utilizing the above orthogonality relationships. This is justified rigorously by basic Hilbert space theory. For the case of orthonormalized harmonics, this gives:

$$f_l^m = \int_\Omega f(\theta,\varphi)\, Y_l^{m*}(\theta,\varphi) d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin(\theta)\ f(\theta,\varphi)\, Y_l^{m*}(\theta,\varphi) \tag{3}$$

If the coefficients decay in ℓ sufficiently rapidly — for instance, exponentially — then the series also converges uniformly to f.
A real square-integrable function f can be expanded in terms of the real harmonics $Y_{\ell m}$ above as a sum

$$f(\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m Y_l^m(\theta,\varphi) \tag{4}$$

Convergence of the series holds again in the same sense.

Spin weighted spherical harmonics

Regard the sphere $\mathbb{S}^2$ as embedded into the three-dimensional imaginary part of the quaternionic number field. At a point **x** on the sphere, a positively oriented orthonormal basis of tangent vectors at **x** is a pair **a**, **b** of vectors such that

$$(\boldsymbol{x},\boldsymbol{a}) = (\boldsymbol{x},\boldsymbol{b}) = (\boldsymbol{a},\boldsymbol{b}) = 0 \tag{1}$$

$$(\boldsymbol{a},\boldsymbol{a}) = (\boldsymbol{b},\boldsymbol{b}) = 1 \tag{2}$$

$$\langle \boldsymbol{x},\boldsymbol{a} \times \boldsymbol{b}\rangle > 0 \tag{3}$$

where the first pair of equations states that **a** and **b** are tangent at **x**, the second pair states that **a** and **b** are unit vectors, **a** and **b** are orthogonal, and the $\{\boldsymbol{x},\boldsymbol{a},\boldsymbol{b}\}$ is a right-handed basis of $\mathbb{R}^3$.

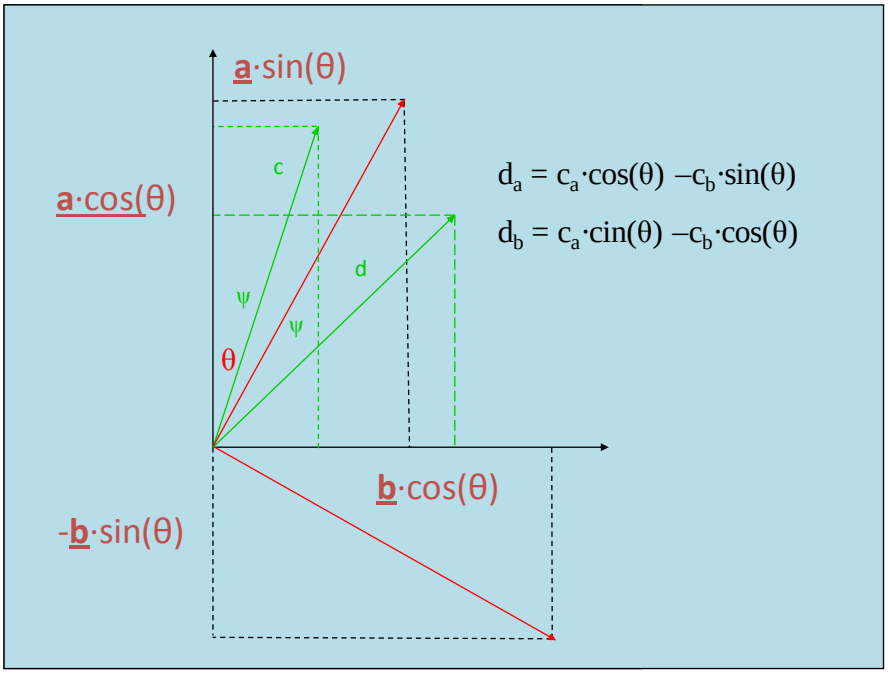


Figure 3: θ and the parameters a and b of the spin-weight function f.

A spin-weight s function f is a function accepting as input a point $\mathbf{x}$ of S^2 and a positively oriented orthonormal basis of tangent vectors at $\mathbf{x}$, such that

$$f\big(\boldsymbol{x}, \boldsymbol{a}\, cos(\theta) - \boldsymbol{b}\, sin(\theta), \boldsymbol{a}\, sin(\theta) + \boldsymbol{b}\, cos(\theta)\big) = \exp(\boldsymbol{i}\, s\, \theta)\, f(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) \tag{4}$$

for every rotation angle θ.

Following Eastwood & Tod (1982), denote the collection of all spin-weight s functions by $\mathbf{B}(s)$. Concretely, these are understood as functions f on $\boldsymbol{C^2}\backslash\{0\}$ satisfying the following homogeneity law under complex scaling

$$f(\lambda\, \boldsymbol{z}, \bar{\lambda}\, \bar{\boldsymbol{z}}) = \left(\frac{\bar{\lambda}}{\lambda}\right)^{s} f(\boldsymbol{z}, \bar{\boldsymbol{z}}) \tag{5}$$

This makes sense provided s is a half-integer.
Abstractly, $\mathbf{B}(s)$ is isomorphic to the smooth vector bundle underlying the antiholomorphic vector bundle $O^*(2\cdot s)$ of the Serre twist on the complex projective line $\boldsymbol{CP^1}$. A section of the latter bundle is a function g on $\boldsymbol{C^2}\backslash\{0\}$ satisfying

$$g\big(\lambda\, \boldsymbol{z}, \bar{\lambda}\, \bar{\boldsymbol{z}}\big) = \big(\bar{\lambda}\big)^{2s}\, g(\boldsymbol{z}, \bar{\boldsymbol{z}}) \tag{6}$$

Given such a g, we may produce a spin-weight s function by multiplying by a suitable power of the Hermitian form

$$P(\boldsymbol{z}, \bar{\boldsymbol{z}}) = \boldsymbol{z}\, \bar{\boldsymbol{z}} \tag{7}$$

Specifically, $f = P^{-s} g$ is a spin-weight s function. The association of a spin-weighted function to an ordinary homogeneous function is an isomorphism.

Eth

The spin weight bundles **B**(s) are equipped with a differential operator ð (eth). This operator is essentially the Dolbeault operator,

$$ð = \partial + \partial^* \tag{1}$$

Thus for $f \in \boldsymbol{B}(s)$,

$$ð f \equiv P^{-s+1}\, \partial(P^s f) \tag{2}$$

defines a function of spin-weight $s + 1$.

Spin-weighted harmonics functions

See http://en.wikipedia.org/wiki/Spin-weighted_spherical_harmonics for more details.
Just as conventional spherical harmonics are the eigenfunctions of the Laplace-Beltrami operator on the sphere, the spin-weight s harmonics are the eigensections for the Laplace-Beltrami operator acting on the bundles $\mathcal{E}(s)$ of spin-weight s functions.
The spin-weighted harmonics can be represented as functions on a sphere once a point on the sphere has been selected to serve as the North Pole. By definition, a function η with *spin weight* s transforms under rotation about the pole via

$$\eta \rightarrow \exp(i\, s\, \psi)\, \eta \tag{1}$$

Working in standard spherical coordinates, we can define a particular operator ð acting on a function η as:

$$ð\eta = -\sin^s(\theta)\, \{\frac{\partial}{\partial\theta} + \frac{i}{\sin(\theta)}\, \frac{\partial}{\partial\varphi}\}[\sin^{-s}(\theta)\, \eta] \tag{2}$$

This gives us another function of θ and φ. [The operator ð is effectively a covariant derivative operator in the sphere.]
An important property of the new function $ð\eta$ is that if η had spin weight s, $ð\eta$ has spin weight $s + 1$. Thus, the operator raises the spin weight of a function by 1. Similarly, we can define an operator which will lower the spin weight of a function by 1:

$$\bar{ð}\eta = -\sin^{-s}(\theta) \left\{\frac{\partial}{\partial\theta} - \frac{i}{\sin(\theta)} \frac{\partial}{\partial\varphi}\right\} [(\sin^s(\theta)\, \eta] \tag{3}$$

We extend the function Y_l^m to ${}_sY_l^m$ according to

$${}_0Y_l^m\,(\theta, \varphi) = Y_l^m(\theta, \varphi) \tag{4}$$

$$l = 0, 1, 2, \ldots;\ m = -l, \ldots 0, \ldots l \tag{5}$$

The spin-weighted spherical harmonics are then defined in terms of the usual spherical harmonics as:

$$ {}_sY_l^m = \sqrt{\frac{(l-s)!}{(l+s)!}}\,\eth^s\, Y_l^m;\ 0 \le s \le l \tag{6} $$

$$ {}_sY_l^m = \sqrt{\frac{(l+s)!}{(l-s)!}}\ (-1)^s\,\eth^s\, Y_l^m;\ -l \le s \le 0 \tag{7} $$

$$ {}_sY_l^m = 0;\ l < |s|; \tag{8} $$

The functions ${}_sY_l^m$ then have the property of transforming with spin weight s.
Other important properties include the following:

$$ \eth({}_sY_l^m) = +\sqrt{(l-s)\,(l+s+1)}\ {}_{s+1}Y_l^m \tag{9} $$

$$ \eth({}_sY_l^m) = -\sqrt{(l+s)\,(l-s+1)}\ {}_{s-1}Y_l^m \tag{10} $$

Hilbert field equations

Statics and dynamics

In this section we mix statics and dynamics. When we confine to the static status quo we get the two sets of equations. The first set concerns the rotation free part of the field. The second set concerns the divergence free part. When dynamics is supported as well, then these parts get coupled. A single separable Hilbert space $\mathbf{H}$ only meets the static parts of the Hilbert field. This means that in this Hilbert space the two parts do not couple. Coupling only takes place during the step from one Hilbert space to the next member in the sequence. Continuity equations describe the coupling between the parts.

The parameter that controls dynamics in these differential equations is the progression parameter. This parameter stands for the counter of the progression steps. For mathematical convenience we take an approach that is similar to the move from a separable Hilbert space $\mathbf{H}$ to a rigged Hilbert space $\mathbf{\mathfrak{H}}$. We move from a fundamentally granular progression step counter to a continuous progression parameter. This progression parameter is not our common notion of time. The derivative of the field f for this parameter is defined as $\nabla_0 f$.
In order to make the step from the integer progression step to the continuous progression parameter possible. There must be a mechanism that reduces change, such that no violent steps are taken. On the other hand the mechanism must not be so strong that only a few steps are taken after which the universe is put to an eternal hold. How this in practice is regulated is shown by the phenomenon inertia. Inertia is installed by the community of all particles. Locally this community generates an enormous potential. This potential works the same in all directions, so when nothing happens it has no influence on a local particle. A uniform movement of a local particle corresponds with the existence of a local vector potential. Also this vector potential does not apply any action. However, when the particle

accelerates, then this goes together with the existence of a vector field that counteracts the acceleration. Thus, inertia does not counteract uniform movement. This uniform movement causes redistribution of the particles and with it a reconfiguration of the field. This disturbance of the static status quo is the motor that keeps dynamics going. The tolerance with respect to uniform movement is the reason that the movement does not get killed.

The continuity equation for charges

Despite the obvious similarity, Hilbert field equations are not Maxwell field equations. First of all, the Hilbert field is a skew field and it carries the properties of the quaternions and the properties of the quaternionic Fourier transform. Next Hilbert fields are mathematical (skew) fields, while Maxwell fields are physical fields. Finally the Hilbert fields consist of a collection of member fields.
The law of charge conservation in configuration space is a continuity equation:

$$s_0(q) = \nabla_0 \rho_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle \tag{1}$$

In Fourier space this becomes:

$$\tilde{s}_0(p) = p_0 \tilde{\rho}_0(p) \mp \langle \boldsymbol{p}, \tilde{\boldsymbol{\rho}}(q) \rangle \tag{2}$$

We reserve the character $\boldsymbol{E}$ for the imaginary longitudinal static part of the Hilbert fields. This differs from the approach in Maxwell equations.
We reserve the character $\boldsymbol{B}$ for the imaginary transverse static part of the Hilbert fields.

The quaternionic nabla

The quaternionic nabla operator is defined by:

$$\nabla f(q) = \nabla_0 f_0(q) \pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q) \right) \tag{3}$$

The third and fourth term on the right treat imaginary divergences. The last term treats the rotation.
In Fourier space:

$$p\tilde{f}(p) = p_0 \tilde{f}_0(p) \pm p_0 \tilde{\boldsymbol{f}}(p) + \boldsymbol{p} \tilde{f}_0(p) \mp \langle \boldsymbol{p}, \tilde{\boldsymbol{f}}(p) \rangle \pm \left(\pm \boldsymbol{p} \times \tilde{\boldsymbol{f}}(p) \right) \tag{4}$$

Due to the fact that the Fourier transform transfers a divergence ∇ in a simple multiplication, the split divides differentiable quaternionic functions into a (longitudinal) rotation free part and a (transverse) divergence free part. The longitudinal part is complex. The transverse part is imaginary.

$$f(q) = f_{\parallel}(q) + f_{\perp}(q) \tag{5}$$

$$\nabla f_{\perp}(q) = 0 \tag{6}$$

$$\nabla \times f_{\parallel}(q) = 0 \tag{7}$$

$$\tilde{f}(p) = \tilde{f}_{\parallel}(p) + \tilde{f}_{\perp}(p) \tag{8}$$

$$\langle \hat{\imath}, \tilde{f}_{\perp}(p) \rangle = 0 \tag{9}$$

$$\hat{\imath} \times \tilde{f}_{\parallel}(p) = 0 \tag{10}$$

Special Fourier transform pairs

Functions that keep the same form through Fourier transformation are:

$$f(q) = \exp(-|q|^2) \tag{1}$$

$$f(q) = \frac{1}{|q|} \tag{2}$$

$$f(q) = comb(q) \tag{3}$$

The comb function consists of a set of equidistant Dirac delta functions.

Other examples of functions that are invariant under Fourier transformation are the linear and spherical harmonic oscillators and the solutions of the Laplace equation.

See invariant functions.

Convolution

Through Fourier transformation a convolution changes into a simple product and vice versa.

$$\mathcal{F}(f(q) \circ g(q)) = \tilde{f}(p) \cdot \tilde{g}(p) \tag{3}$$

Parceval's theorem

Parceval's theorem runs:

$$\int f^*(q) \cdot g(q) \cdot dV_q = \int \tilde{f}^*(p) \cdot \tilde{g}(p) \cdot dV_p \tag{1}$$

This leads to

$$\int |f(q)|^2 \cdot dV_q = \int \left|\tilde{f}(p)\right|^2 \cdot dV_p \tag{2}$$

Source conservation

The law of charge conservation in configuration space runs:

$$s_0(q) = \nabla_0 \rho_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle \tag{1}$$

In configuration space the distribution $\rho(q)$ becomes a potential $\phi(q)$ through convolution with a blur $\phi(q)$.

$$\phi(q) = \rho(q) \circ \varphi(q) \tag{2}$$

In Fourier space the convolution becomes multiplication:

$$\tilde{s}_0(p) = p_0\tilde{\rho}_0(p) \mp \langle \boldsymbol{p}, \tilde{\boldsymbol{\rho}}(p)\rangle \tag{3}$$

$$\tilde{s}_0(p)\tilde{\varphi}(p) = p_0\tilde{\rho}_0(p)\tilde{\varphi}(p) \mp \langle \boldsymbol{p}, \tilde{\boldsymbol{\rho}}(q)\rangle\tilde{\varphi}(p)) = p_0\tilde{\rho}_0(p)\tilde{\varphi}(p) \mp \langle \boldsymbol{p}, \tilde{\boldsymbol{\rho}}(q)\tilde{\varphi}(p)\rangle \tag{4}$$

$$\tilde{\phi}(p) = \tilde{\rho}(p)\tilde{\varphi}(p) \tag{5}$$

$$\tilde{\phi}_0(p) = \tilde{\rho}_0(p)\tilde{\varphi}(p) \tag{6}$$

$$\tilde{\boldsymbol{\phi}}(p) = \tilde{\rho}_0(p)\tilde{\boldsymbol{\varphi}}(p) \tag{7}$$

In configuration space, with $s_0(q) = 0$ this means:

$$\nabla_0\phi_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle = 0 \tag{8}$$

Potential

If there is a static spherically symmetric Gaussian charge density ϱ (*r*):

$$\rho(q) = \frac{Q}{\sqrt{2\pi\sigma^2}^3}\exp(-|q|^2/(2\sigma^2)) \tag{1}$$

where *Q* is the total charge, then the solution ϕ (*r*) of Poisson's equation,

$$\nabla^2\varphi(q) = -\frac{\rho(q)}{\varepsilon} \tag{2}$$

is given by

$$\varphi(q) = \frac{Q}{4\pi\varepsilon|q|}\mathrm{erf}\left(\frac{|q|}{\sqrt{2\sigma}}\right) \tag{3}$$

where erf(*x*) is the error function.

In fact the quaternionic Poisson's equation represents two separate equations:

$$({\nabla_0}^2 - \boldsymbol{\nabla}^2)\varphi_0(q) = -\frac{\rho_0(q)}{\varepsilon} \tag{4}$$

$$({\nabla_0}^2 - \boldsymbol{\nabla}^2)\boldsymbol{\varphi}(q) = -\frac{\boldsymbol{\rho}(q)}{\varepsilon} \tag{5}$$

Note that, for $|q|$ much greater than σ, the erf function approaches unity and the potential ϕ (*r*) approaches the point charge potential $\frac{Q}{4\pi\varepsilon|q|}$, as one would expect. Furthermore the erf function approaches 1 extremely quickly as its argument increases; in practice for $|q| > 3\sigma$ the relative error is smaller than one part in a thousand. See: http://en.wikipedia.org/wiki/Poisson's_equation#Potential_of_a_Gaussian_charge_density.

The definition of the quaternionic potential $\phi(q)$ is based on the convolution of a quaternionic distribution $\rho(q)$ with the real function $\varphi(q)$ See Newton potential and Bertrand's theorem in Wikipedia. The real part $\rho_0(q)$ of the distribution $\rho(q)$ can be interpreted as a charge distribution. The imaginary part $\boldsymbol{\rho}(q)$ can be interpreted as a current distribution.
The convolution blurs the distribution such that the result becomes differentiable.

In configuration space holds:

$$\phi(q) = \rho(q) \circ \frac{1}{|q|}. \tag{6}$$

Reversely, according to Poisson's equation:

$$\rho(q) = = -\Delta\phi(q) \tag{7}$$

The real part of $\phi(q)$ presents a scalar potential. The imaginary part presents a vector potential.

$$\phi(q) = \phi_0(q) + \boldsymbol{\phi}(q) \tag{8}$$

The scalar potential is a blurred charge distribution.

The vector potential is a blurred current distribution.

Current is moving charge.

Mass is a form of charge.

(The selected blurring function has striking resemblance with the ground state of the quantum harmonic oscillator).

In Fourier space holds:

$$\tilde{\phi}(p) = \tilde{\rho}(p) \cdot \frac{1}{|p|}. = \tilde{\phi}_0(p) + \tilde{\boldsymbol{\phi}}(p) \tag{9}$$

In Fourier space the frequency spectrum of the Hilbert distribution is multiplied with the Fourier transform of the blurring function. When this falls off when the frequencies go to infinity, then as a consequence the frequency spectrum of the potential is bounded. This is valid independent of the fact that the frequency spectrum of the Hilbert distribution might be unbounded.

The equation for the conservation of charge:

$$s_0(q) = \nabla_0\rho_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle \tag{10}$$

Translates in the source free case $s_0(q) = 0$ into:

$$\nabla_0 \rho_0(q) = \pm\langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q)\rangle \tag{11}$$

And in the Lorentz Lorentz Gauge condition:

$$\nabla_0 \boldsymbol{\phi}_0(q) = \pm\langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle \tag{12}$$

$$\mathfrak{F}(\mathrm{q}) = \nabla\boldsymbol{\phi}(q) \tag{13}$$

$$\mathfrak{F}_0(q) = \nabla_0 \boldsymbol{\phi}_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle = 0$$

In the source divergence free case $\boldsymbol{\nabla} s_0(q) = 0$ this means:

$$\nabla_0 \boldsymbol{\nabla} \rho_0(q) = \pm\boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q)\rangle \tag{14}$$

$$\nabla_0 \boldsymbol{\nabla} \boldsymbol{\phi}_0(q) = \pm\boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle \tag{15}$$

$$\boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{\phi}(q) + \boldsymbol{\nabla}^2 \boldsymbol{\phi}(q) \tag{16}$$

Due to the fact that there are other charges present, the divergence of the scalar potential need be in the direction of the current $\boldsymbol{\rho}(q)$, which for a spherical symmetric blur is also in the direction of the vector potential $\boldsymbol{\phi}(q)$. However, a tendency exists to minimize that difference. Thus $\nabla_0 \boldsymbol{\nabla} \boldsymbol{\phi}_0(q)$ is parallel to $\boldsymbol{\phi}(q)$. With other words:

$$\boldsymbol{\phi}(q) \times \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle = 0 \tag{17}$$

Reckoning the sign selections for the sign $\pm$ of the conjugation and the handedness $\pm$ of the cross product will provide four different sets of equations. This will provide four different Hilbert fields.

Discrete distribution

If $\rho(q)$ is discrete, such that

$$\rho(q) = \sum_i q_{E\,i} \cdot \delta(q - q_i) \tag{1}$$

where q'_E is a point charge at location q′, then the contribution to the field $\boldsymbol{E}(q)$ that is generated by a point charge at location q_i is given by:

$$d\boldsymbol{E}(q) = q_{E\,i} \cdot \frac{q_i - q}{|q_i - q|^3} = -q_{E\,i} \cdot \nabla \cdot \frac{1}{|q_i - q|} \tag{2}$$

Differential potential equations

The gradient and curl of $\boldsymbol{\phi}(q)$ are related. In configuration space holds:

$$\nabla\boldsymbol{\phi}(q) = \nabla_0\phi_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle \pm \nabla_0\boldsymbol{\phi}(q) \pm \boldsymbol{\nabla}\phi_0(q) \pm \boldsymbol{\nabla} \times \boldsymbol{\phi}(q) \tag{1}$$

$$\mathfrak{E}(q) \equiv -\boldsymbol{\nabla}\phi_0(q) \tag{2}$$

$$\mathfrak{B}(q) \equiv \nabla \times \phi(q) \tag{3}$$

$$\mathfrak{F}(q) = \nabla\phi(q) = \mathfrak{F}_0(q) \mp \mathfrak{E}(q) \pm \mathfrak{B}(q) \pm \nabla_0\boldsymbol{\phi}(q) \tag{4}$$

$$\mathfrak{F}_0(q) = \nabla_0\phi_0(q) \mp \langle\nabla, \boldsymbol{\phi}(q)\rangle \tag{5}$$

$$\boldsymbol{\mathfrak{F}}(q) = \mp\mathfrak{E}(q) \pm \mathfrak{B}(q) \pm \nabla_0\boldsymbol{\phi}(q) \tag{6}$$

$$\langle\nabla, \mathfrak{E}(q)\rangle = -\nabla^2\phi_0(q) = \rho_0(q) \tag{7}$$

$$\nabla \times \mathfrak{E}(q) = 0; \text{ Rotation free field} \tag{8}$$

$$\langle\nabla, \mathfrak{B}(q)\rangle = 0; \text{ Divergence free } B \text{ field} \tag{9}$$

$$\nabla \times \mathfrak{B}(q) = \nabla\langle\nabla, \boldsymbol{\phi}(q)\rangle - \nabla^2\boldsymbol{\phi}(q) = \nabla\langle\nabla, \boldsymbol{\phi}(q)\rangle + \boldsymbol{\rho}(q) + {\nabla_0}^2\boldsymbol{\phi}(q) \tag{10}$$

$$\nabla \times \mathfrak{B}(q) = \pm\nabla_0\nabla\phi_0(q) + \boldsymbol{\rho}(q) + {\nabla_0}^2\boldsymbol{\phi}(q) \tag{11}$$

$$= \pm\nabla_0\mathfrak{E}(q) + \boldsymbol{\rho}(q) + {\nabla_0}^2\boldsymbol{\phi}(q)$$

Since $\nabla_0\boldsymbol{\phi}(q)$is supposed to be parallel to $\nabla\phi_0(q)$, it is sensible to define $\boldsymbol{E}(q)$as the total field in longitudinal direction:

$$\boldsymbol{E}(q) = -\nabla\phi_0(q) - \nabla_0\boldsymbol{\phi}(q) = \mathfrak{E}(q) - \nabla_0\boldsymbol{\phi}(q) \tag{12}$$

And

$$\boldsymbol{B}(q) = \mathfrak{B}(q) \tag{13}$$

With this definition:

$$\nabla \times \boldsymbol{E}(q) = -\nabla_0\boldsymbol{B}(q) \tag{14}$$

$$\langle\nabla, \boldsymbol{B}(q)\rangle = 0 \tag{15}$$

$$\nabla \times \boldsymbol{B}(q) = \boldsymbol{\rho}(q) + \nabla_0\boldsymbol{E}(q) \tag{16}$$

In Fourier space

In Fourier space holds:

$$\tilde{\mathfrak{F}}(p) = p_0\tilde{\phi}_0(p) - \langle\boldsymbol{p}, \tilde{\boldsymbol{\phi}}(p)\rangle \pm p_0\tilde{\boldsymbol{\phi}}(p) \pm \boldsymbol{p}\tilde{\phi}_0(p) \pm \boldsymbol{p} \times \tilde{\boldsymbol{\phi}}(p) \tag{1}$$

$$\tilde{\mathfrak{F}}(p) = p\tilde{\phi}(p) = \mathfrak{F}_0(p) \mp \tilde{\mathfrak{E}}(p) \pm \tilde{\mathfrak{B}}(p) \pm p_0\tilde{\boldsymbol{\phi}}(p) \tag{2}$$

$$\tilde{\mathfrak{F}}_0(p) = p_0\tilde{\phi}_0(p) - \langle\boldsymbol{p}, \tilde{\boldsymbol{\phi}}(p)\rangle \tag{3}$$

$$\widetilde{\mathfrak{E}}(p) = -\boldsymbol{p}\tilde{\phi}_0(p) \tag{4}$$

$$\widetilde{\boldsymbol{E}}(p) = -\boldsymbol{p}\tilde{\phi}_0(p) \pm p_0\widetilde{\boldsymbol{\phi}}(p)$$

$$\widetilde{\mathfrak{B}}(p) = \boldsymbol{p} \times \widetilde{\boldsymbol{\phi}}(p) \tag{5}$$

$$\widetilde{\mathfrak{F}}(p) = \mp\widetilde{\mathfrak{E}}(p) \pm \widetilde{\mathfrak{B}}(p) \pm p_0\widetilde{\boldsymbol{\phi}}(p) \tag{6}$$

$$\langle \boldsymbol{p}, \widetilde{\mathfrak{E}}(p)\rangle = -\boldsymbol{p}^2\tilde{\phi}_0(p) = \tilde{\rho}_0(p) \tag{7}$$

$$\boldsymbol{p} \times \widetilde{\mathfrak{E}}(p) = 0; \text{ Rotation free field} \tag{8}$$

$$\langle \boldsymbol{p}, \widetilde{\mathfrak{B}}(p)\rangle = 0; \text{ Divergence free } B \text{ field} \tag{9}$$

$$\boldsymbol{p} \times \widetilde{\mathfrak{B}}(p) = \boldsymbol{p}\langle \boldsymbol{p}, \widetilde{\boldsymbol{\phi}}(q)\rangle - \boldsymbol{p}^2\widetilde{\boldsymbol{\phi}}(q) = \boldsymbol{p}\langle \boldsymbol{p}, \widetilde{\boldsymbol{\phi}}(p)\rangle + \tilde{\boldsymbol{\rho}}(p) \tag{10}$$

$$\boldsymbol{p} \times \widetilde{\mathfrak{B}}(p) = \pm p_0\boldsymbol{p}\tilde{\phi}_0(p) + \tilde{\boldsymbol{\rho}}(p) = \pm p_0\widetilde{\mathfrak{E}}(p) + \tilde{\boldsymbol{\rho}}(p) \tag{11}$$

If the distribution $\rho(q)$ is differentiable, then the same equations that hold for fields $\phi(q)$ and $\tilde{\phi}(p)$ hold for the non-blurred distributions $\rho(q)$ and $\tilde{\rho}(q)$.

Maxwell

First it must be noted that the derived field equations hold for general quaternionic fields.
The resemblance with physical fields hold for electromagnetic fields as well as for gravitational fields and for any fields whose blurring function approximates

$$f(q) \approx \frac{1}{|q|}.$$

In Maxwell equations, $\boldsymbol{E}(\mathbf{r})$ is defined as:

$$\boldsymbol{E}(\boldsymbol{r},t) \equiv -\boldsymbol{\nabla}\phi_0(\boldsymbol{r},t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = \mathfrak{E}(\boldsymbol{r},t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t}$$

Further:

$$\langle \boldsymbol{\nabla}, \boldsymbol{E}(\boldsymbol{r},t)\rangle = -\boldsymbol{\nabla}^2\phi_0(\boldsymbol{r},t) - \frac{\partial\langle \boldsymbol{\nabla}, \boldsymbol{A}(\boldsymbol{r},t)\rangle}{\partial t}$$

$$= \frac{\rho_0(\boldsymbol{r},t)}{\varepsilon_0} - \frac{\partial\langle \boldsymbol{\nabla}, \boldsymbol{A}(\boldsymbol{r},t)\rangle}{\partial t}$$

In Maxwell equations, $\boldsymbol{B}(\boldsymbol{r})$ is defined as:

$$\boldsymbol{B}(\boldsymbol{r},t) \equiv \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r},t) = \mathfrak{B}(\boldsymbol{r},t)$$

Further:

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{B}(\boldsymbol{r},t)\rangle = 0$$

$$\boldsymbol{\nabla} \times \boldsymbol{B}(\boldsymbol{r},t) = \mu_0\left(\boldsymbol{j} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right)$$

Differentiable distribution

If the distribution $\rho(q)$ is differentiable, then the same equations that hold for fields $\phi(q)$ and $\tilde{\phi}(p)$ hold for the non-blurred distributions $\rho(q)$ and $\tilde{\rho}(q)$.
Using:

$$\boldsymbol{B} = \nabla \times \boldsymbol{\phi} = \boldsymbol{i}(\nabla_2\phi_{\|} - \nabla_{\|}\phi_2) + \boldsymbol{j}(\nabla_{\|}\phi_1 - \nabla_1\phi_{\|}) + \boldsymbol{k}(\nabla_1\phi_2 - \nabla_2\phi_1) \quad (1)$$

gives

$$\nabla_0\phi_\parallel(q) = \mp\nabla_\parallel\phi_0(q) \tag{2}$$

$$\nabla_0\phi_1(q) = \mp\left(\nabla_2\phi_\parallel(q) - \nabla_\parallel\phi_2(q)\right) \tag{3}$$

$$\nabla_0\phi_2(q) = \pm\left(\nabla_1\phi_\parallel(q) - \nabla_\parallel\phi_1(q)\right) \tag{4}$$

$$\nabla_0\phi_0(q) = \langle\boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle = \nabla_\parallel\phi_\parallel(q) + \nabla_1\phi_1(q) + \nabla_2\phi_2(q) \tag{5}$$

And correspondingly in Fourier space

$$p_0\tilde{\phi}_\parallel(p) = \mp p_\parallel\tilde{\phi}_0(p) \tag{6}$$

$$p_0\tilde{\phi}_1(p) = \pm\left(p_\parallel\tilde{\phi}_2(p) - p_2\tilde{\phi}_\parallel(p)\right) \tag{7}$$

$$p_0\tilde{\phi}_2(p) = \mp\left(p_\parallel\tilde{\phi}_1(p) - p_1\tilde{\phi}_\parallel(p)\right) \tag{8}$$

$$p_0\tilde{\phi}_0(p) = \langle\boldsymbol{p}, \tilde{\boldsymbol{\phi}}(p)\rangle = p_\parallel\tilde{\phi}_\parallel(p) + p_1\tilde{\phi}_1(p) + p_2\tilde{\phi}_2(p) \tag{9}$$

Conservation laws

Flux vector

The longitudinal direction **k** of $\boldsymbol{E}(q)$ and the direction **i** of $\boldsymbol{B}(q)$ fix two mutual perpendicular directions. This generates curiosity to the significance of the direction $\mathbf{k}\times\mathbf{i}$. With other words what happens with $\boldsymbol{E}(q)\times\boldsymbol{B}(q)$.

The **flux vector** $\mathfrak{S}(q)$ is defined as:

$$\mathfrak{S}(q) \equiv \boldsymbol{E}(q)\times\boldsymbol{B}(q) \tag{1}$$

Conservation of energy

Field energy density

$$\begin{aligned}\langle\boldsymbol{\nabla}, \mathfrak{S}(q)\rangle &= \langle\boldsymbol{B}(q), \boldsymbol{\nabla}\times\boldsymbol{E}(q)\rangle - \langle\boldsymbol{E}(q), \boldsymbol{\nabla}\times\boldsymbol{B}(q)\rangle \\ &= -\langle\boldsymbol{B}(q), \nabla_0\boldsymbol{B}(q)\rangle - \langle\boldsymbol{E}(q), \boldsymbol{\phi}(q)\rangle - \langle\boldsymbol{E}(q), \nabla_0\boldsymbol{E}(q)\rangle \\ &= -½\nabla_0(\langle\boldsymbol{B}(q), \boldsymbol{B}(q)\rangle + \langle\boldsymbol{E}(q), \boldsymbol{E}(q)\rangle) - \langle\boldsymbol{E}(q), \boldsymbol{\phi}(q)\rangle\end{aligned} \tag{1}$$

The **field energy density** is defined as:

$$u_{field}(q) = ½(\langle\boldsymbol{B}(q), \boldsymbol{B}(q)\rangle + \langle\boldsymbol{E}(q), \boldsymbol{E}(q)\rangle) = u_B(q) + u_E(q) \tag{2}$$

$\mathfrak{S}(q)$ can be interpreted as the **field energy current density**.

The continuity equation for field energy density is given by:

$$\nabla_0 u_{field}(q) + \langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle = -\langle \boldsymbol{E}(q), \boldsymbol{\phi}(q) \rangle = -\phi_0(q) \langle \boldsymbol{E}(q), \boldsymbol{v}(q) \rangle \tag{3}$$

This means that $\langle \boldsymbol{E}(q), \boldsymbol{\phi}(q) \rangle$ can be interpreted as a source term.
$\phi_0(q)\boldsymbol{E}(q)$ represents **force** per unit volume.
$\phi_0(q)\langle \boldsymbol{E}(q), \boldsymbol{v}(q) \rangle$ represents **work** per unit volume, or, in other words, the power density. It is known as the Lorentz power density and is equivalent to the time rate of change of the mechanical energy density of the charged particles that form the current $\boldsymbol{\phi}(q)$**.**

$$\nabla_0 u_{field}(q) + \langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle = -\nabla_0 u_{mechanical}(q) \tag{4}$$

$$\nabla_0 u_{mechanical} = \langle \boldsymbol{E}(q), \boldsymbol{\phi}(q) \rangle = \phi_0(q) \langle \boldsymbol{E}(q), \boldsymbol{v}(q) \rangle \tag{5}$$

$$\nabla_0 u_{field}(q) + \langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle = -\nabla_0 u_{mechanical}(q) \tag{6}$$

$$\nabla_0 \Big(u_{field}(q) + u_{mechanical}(q) \Big) = -\langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle \tag{7}$$

Total change within V = flow into V + production inside V

$$u(q) = u_{field}(q) + u_{mechanical}(q) = u_B(q) + u_E(q) + u_{mechanical}(q) \tag{8}$$

$$U = U_{field} + U_{mechanical} = U_B + U_E + U_{mechanical} = \int_V u \, dV \tag{9}$$

$$\frac{d}{dt} \int_V u \, dV = \oint_S \langle \hat{\boldsymbol{n}}, \mathfrak{S} \rangle dS + \int_V s_0 \, dV \tag{10}$$

Here the source s_0 is zero.

Conservation of linear momentum

Field linear momentum

$\mathfrak{S}(q)$ can also be interpreted as the **field linear momentum density**. The time rate change of the field linear momentum density is:

$$\nabla_0 \mathfrak{S}(q) = \boldsymbol{g}_{field}(q) = \nabla_0 \, \boldsymbol{E}(q) \times \boldsymbol{B}(q) + \boldsymbol{E}(q) \times \nabla_0 \boldsymbol{B}(q) \tag{1}$$

$$= \big(\boldsymbol{\nabla} \times \boldsymbol{B}(q) - \boldsymbol{\rho}(q) \big) \times \boldsymbol{B}(q) - \boldsymbol{E}(q) \times \boldsymbol{\nabla} \times \boldsymbol{E}(q) \tag{2}$$

$$\boldsymbol{G}(\boldsymbol{E}) = \boldsymbol{E} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = \langle \boldsymbol{\nabla E}, \boldsymbol{E} \rangle - \langle \boldsymbol{E}, \boldsymbol{E} \rangle = \tfrac{1}{2} \boldsymbol{\nabla} \langle \boldsymbol{E}, \boldsymbol{E} \rangle - \langle \boldsymbol{E}, \boldsymbol{E} \rangle \tag{3}$$

$$= -\boldsymbol{\nabla}(\mathbf{EE}) + \tfrac{1}{2} \boldsymbol{\nabla} \langle \boldsymbol{E}, \boldsymbol{E} \rangle + \langle \boldsymbol{\nabla}, \boldsymbol{E} \rangle \boldsymbol{E}$$

$$= -\boldsymbol{\nabla}(\mathbf{EE} + \tfrac{1}{2} \mathbf{1_3} \langle \boldsymbol{E}, \boldsymbol{E} \rangle) + \langle \boldsymbol{\nabla}, \boldsymbol{E} \rangle \boldsymbol{E}$$

$$\boldsymbol{G}(\boldsymbol{B}) = \boldsymbol{B} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = -\boldsymbol{\nabla}(\mathbf{BB} + \tfrac{1}{2}\mathbf{1_3}\langle \boldsymbol{B}, \boldsymbol{B}\rangle) + \langle \boldsymbol{\nabla}, \boldsymbol{B}\rangle \boldsymbol{B} \tag{4}$$

$$\boldsymbol{H}(\boldsymbol{B}) = -\boldsymbol{\nabla}(\mathbf{BB} + \tfrac{1}{2}\mathbf{1_3}\langle \boldsymbol{B}, \boldsymbol{B}\rangle) \tag{5}$$

$$\begin{aligned} \nabla_0 \mathfrak{S}(q) &= \boldsymbol{G}(\boldsymbol{B}) + \boldsymbol{G}(\boldsymbol{E}) - \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) \\ &= \boldsymbol{H}(\boldsymbol{E}) + \boldsymbol{H}(\boldsymbol{B}) - \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) + \langle \boldsymbol{\nabla}, \boldsymbol{B}\rangle \boldsymbol{B} + \langle \boldsymbol{\nabla}, \boldsymbol{E}\rangle \boldsymbol{E} \\ &= \boldsymbol{H}(\boldsymbol{E}) + \boldsymbol{H}(\boldsymbol{B}) - \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) - \rho_0(q)\, \boldsymbol{E}(q) \\ &= \boldsymbol{H}(\boldsymbol{E}) + \boldsymbol{H}(\boldsymbol{B}) - \boldsymbol{f}(q) = \boldsymbol{\mathcal{T}}(q) - \boldsymbol{f}(q) \end{aligned} \tag{6}$$

$\mathcal{T}$(q) is the linear momentum flux tensor.
The linear momentum of the field contained in volume V surrounded by surface S is:

$$\boldsymbol{P}_{field} = \int_V \boldsymbol{g}_{field}\, dV = \int_V \rho_0 \boldsymbol{\phi}\, dV + \int_V \langle \nabla \boldsymbol{\phi}, \boldsymbol{E}\rangle\, dV + \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{EA}\rangle dS \tag{7}$$

$$\boldsymbol{f}(q) = \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) + \rho_0(q)\, \boldsymbol{E}(q) \tag{8}$$

Physically, $\boldsymbol{f}(q)$ is the Lorentz force density. It equals the time rate change of the mechanical linear momentum density $\boldsymbol{g}_{mechanical}$.

$$\boldsymbol{g}_{mechanical}(q) = \rho_{0m}(q)\boldsymbol{v}(q) \tag{9}$$

The force acted upon a single particle that is contained in a volume V is:

$$\boldsymbol{F} = \int_V \boldsymbol{f}\, dV = \int_V (\boldsymbol{\rho} \times \boldsymbol{B} + \rho_0\, \boldsymbol{E})\, dV \tag{10}$$

Brought together this gives:

$$\nabla_0 \left(\boldsymbol{g}_{field}(q) + \boldsymbol{g}_{mechanical}(q) \right) = -\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{T}}(q)\rangle \tag{11}$$

This is the continuity equation for linear momentum.
The component $\mathcal{T}_{ij}$ is the linear momentum in the i-th direction that passes a surface element in the j-th direction per unit time, per unit area.

Total change within V = flow into V + production inside V

$$\boldsymbol{g}(q) = \boldsymbol{g}_{field}(q) + \boldsymbol{g}_{mechanical}(q) \tag{12}$$

$$\boldsymbol{P} = \boldsymbol{P}_{field} + \boldsymbol{P}_{mechanical} = \int_V \boldsymbol{g}\, dV \tag{13}$$

$$\frac{d}{dt}\int_V \boldsymbol{g}\, dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{\mathcal{T}} \rangle dS + \int_V \boldsymbol{s_g}\, dV \tag{14}$$

Here the source $\boldsymbol{s_g} = 0$.

Conservation of angular momentum

Field angular momentum

The angular momentum relates to the linear momentum.

$$\boldsymbol{h}(\boldsymbol{q_c}) = (\boldsymbol{q} - \boldsymbol{q_c}) \times \boldsymbol{g}(q) \tag{1}$$

$$\boldsymbol{h}_{field}(\boldsymbol{q_c}) = (\boldsymbol{q} - \boldsymbol{q_c}) \times \boldsymbol{g}_{field}(q) \tag{2}$$

$$\boldsymbol{h}_{mechanical}(q) = (\boldsymbol{q} - \boldsymbol{q_c}) \times \boldsymbol{g}_{mechanical}(q) \tag{3}$$

$$\boldsymbol{\mathcal{K}}(\boldsymbol{q_c}) = (\boldsymbol{q} - \boldsymbol{q_c}) \times \boldsymbol{\mathcal{T}}(q) \tag{4}$$

This enables the balance equation for angular momentum:

$$\nabla_0 \left(\boldsymbol{h}_{field}(\boldsymbol{q_c}) + \boldsymbol{h}_{mechanical}(\boldsymbol{q_c}) \right) = -\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{K}}(\boldsymbol{q_c}) \rangle \tag{5}$$

Total change within V = flow into V + production inside V

$$\boldsymbol{J} = \boldsymbol{J}_{field} + \boldsymbol{J}_{mechanical} = \int_V \boldsymbol{h}\, dV \tag{6}$$

$$\frac{d}{dt}\int_V \boldsymbol{h}\, dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{\mathcal{K}} \rangle dS + \int_V \boldsymbol{s_h}\, dV \tag{7}$$

Here the source $\boldsymbol{s_h} = 0$.

For a localized charge density contained within a volume V holds for the mechanical torque:

$$\tau(\boldsymbol{q_c}) = \int_V (\boldsymbol{q'} - \boldsymbol{q_c}) \times \boldsymbol{f}(q') dV \tag{8}$$

$$= \int_V (\boldsymbol{q'} - \boldsymbol{q_c}) \times \left(\rho_0(q') \boldsymbol{E}(q') + \boldsymbol{j}(q') \times \boldsymbol{B}(q') \right) dV$$

$$= Q(\boldsymbol{q} - \boldsymbol{q_c}) \times (\boldsymbol{E}(q) + \boldsymbol{v}(q) \times \boldsymbol{B}(q))$$

$$\boldsymbol{J}_{field}(\boldsymbol{q_c}) = \boldsymbol{J}_{field}(\boldsymbol{0}) + \boldsymbol{q_c} \times \boldsymbol{P}(q) \tag{9}$$

Using

$$\langle \boldsymbol{\nabla a}, \boldsymbol{b} \rangle = \boldsymbol{n}_\nu \frac{\partial a_\mu}{\partial q_\nu} b_\mu \tag{10}$$

$$\langle \boldsymbol{b}, \boldsymbol{\nabla a} \rangle = \boldsymbol{n}_\mu \frac{\partial a_\mu}{\partial q_\nu} b_\mu \tag{11}$$

holds

$$\boldsymbol{J}_{field}(\boldsymbol{0}) = \int_V \boldsymbol{q}' \times \mathfrak{S}(q') dV = \int_V \boldsymbol{q}' \times \boldsymbol{E}(q') \times \boldsymbol{\nabla} \times \boldsymbol{\phi}(q')\, dV \tag{12}$$

$$= \int_V (\boldsymbol{q}' \times \langle (\boldsymbol{\nabla \phi}), \boldsymbol{E} \rangle - \langle \boldsymbol{q}' \times \boldsymbol{E}, (\boldsymbol{\nabla \phi}) \rangle)\, dV$$

$$= \int_V \boldsymbol{q}' \times \langle (\boldsymbol{\nabla \phi}), \boldsymbol{E} \rangle dV$$

$$+ \int_V \boldsymbol{E} \times \boldsymbol{\phi}\, dV - \int_V \langle \boldsymbol{\nabla}, \boldsymbol{E q}' \times \boldsymbol{\phi} \rangle dV + \int_V (\boldsymbol{q}' \times \boldsymbol{\phi}) \langle \boldsymbol{\nabla}, \boldsymbol{E} \rangle dV$$

Define the non-local spin term, which does not depend on $\boldsymbol{q}'$ is:

$$\boldsymbol{\Sigma}_{field} = \int_V \boldsymbol{E}(q) \times \boldsymbol{\phi}(q) dV \tag{13}$$

And

$$\boldsymbol{L}_{field}(\boldsymbol{0}) = \int_V \boldsymbol{q}' \times \langle (\boldsymbol{\nabla \phi}), \boldsymbol{E} \rangle dV + \int_V \boldsymbol{q}' \times \rho_0 \boldsymbol{\phi} dV \tag{14}$$

Using Gauss:

$$\int_V \langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{a} \rangle dS \tag{15}$$

And

$$\rho_0 = \langle \boldsymbol{\nabla}, \boldsymbol{E} \rangle \tag{16}$$

Leads to:

$$\boldsymbol{J}_{field}(\boldsymbol{0}) = \boldsymbol{\Sigma}_{field} + \boldsymbol{L}_{field}(\boldsymbol{0}) + \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{E q}' \times \boldsymbol{\phi} \rangle dS \tag{17}$$

The consequence of granularity

In separable Hilbert space, Fourier transformation means more than just taking the Fourier transfer of the analyzed function. It also means a switch from the eigenvalues of a coordinate operator to the eigenvalues of its canonical conjugate. In fact it means the transfer from the Hilbert space with a Q coordinate system to the same Hilbert space with a P coordinate system or vice versa. It means detaching the field to be analyzed from the vectors of the separable Hilbert space, then performing the Fourier transform on the retrieved field and as a next step the attachment of the field values back to the vectors of the separable Hilbert space **H**.This corresponds to **a double sampling process**. The Fourier transformation of the field is performed as if it was done on the sampled version of a continuous function.

Vacuum expectation value

The **vacuum expectation value** (also called **condensate** or simply VEV) of an operator is its average, expected value in the vacuum. The vacuum corresponds to a ground state. The vacuum expectation value of an operator O is usually denoted by $\langle O \rangle$.

States

Where a unique closed Hilbert subspace *represents* a given physical item, its **state** *characterizes* the probabilistic properties of that item. In quantum physics, a quantum state is a set of mathematical variables that as far as is possible describes the corresponding physical item. For example, the set of 4 numbers $\{n, l, m_l, m_s\}$ define part of the state of an electron within a hydrogen atom and are known as the electron's quantum numbers. The observables that determine the state are mutually compatible. The position of the electron within the atom is a hidden property. If two operators are each other's canonical conjugate, then only one of them can participate in the state, or the state must contain an account of the combination of both values. An example of such a combination is the ladder operator.

Quantum states can be either pure or mixed. **Pure states** cannot be described as a mixture of others. **Mixed states** correspond to a random process that blends pure states together. Realizations of elementary types are characterized by pure states.

This indicates that the notion of state is closely related to the basic constituents of Hilbert fields. Stated in other words: the probability amplitude distribution that represents the private field of an elementary particle also represents the state of that particle.

In relation to the concepts defined earlier, a pure state corresponds to the blur of an elementary Hilbert distribution, while the blur itself can be interpreted as a probability amplitude distribution. The state then corresponds to the squared modulus of this probability amplitude distribution, which is a probability density function. The blur is defined with respect to a coordinate operator. This coordinate operator can be a GPS operator or a position operator ($\breve{\mathrm{Q}}$ or $\breve{Q}$), or it is represented by the canonical conjugate, the GMS operator or the momentum operator ($\breve{\mathcal{P}}$ or $\breve{P}$). With other words, several different states correspond with the same physical item.

When performing an observation on a quantum state, the result is generally described by a probability distribution, and the form that this distribution takes is completely determined by the quantum state and the operators that are related to the observation of the quantum state. The result of an observation is only determined probabilistically. In relation to the observables that determine the state, a pure state is characterized by the blur of a single elementary Hilbert distribution and that distribution corresponds in relation to these observables to a mathematical object known as a wave function. If another observable concerns the canonical conjugate, then the corresponding Fourier transform of the wave function must be considered.

The result of a sharp observation equals one of the eigenvalues that corresponds to the set of eigenvectors over which the probability distribution is non-zero. It means that the observable must have the same eigenvectors as the operator that is used to define the wave function. The probability of getting this vector and the corresponding value is given by the probability density that corresponds to the wave function value. The probalistic nature of observations reflects a core difference between classical and quantum physics.

Linear combinations (superpositions) of states can describe interference phenomena. A mixed state cannot be characterized by a single blurred elementary Hilbert distribution.

Instead, it is described by its associated density operator. It is still represented by a (blurred) closed Hilbert subspace, but that is no longer the subspace that is spanned by a single elementary Hilbert distribution.

Pure states can be represented by a single blurred elementary Hilbert distribution.

State definition

A measure μ on the closed subspaces of a Hilbert space obeys the rule:

$$\mu(B) = \sum_{i=1}^{i=n} A_i, \text{ for each set } \{A_i\} \text{ of closed subspaces}$$

Each Hilbert vector $|v>$ generates a measure $\mu_v(A)$ via the projection P_A of $|v>$ on A

$$\mu_v(A) = \|P_A(v)\|^2$$

Gleason's theorem states:
Let $\mu(A)$ be a measure on the closed subspaces A of a separable Hilbert space **H**, with dimension ≥ 3, then there exists a positive definite self adjoint operator T of the trace class, such that

$$\mu(A) = trace(TP_A)$$

Given a state P on a space of dimension ≥ 3, there is an Hermitian, non-negative operator ρ on **H**, whose trace is unity, such that

$$\forall_{|x>\in H}\{P(x) = <x|\,\rho\, x>\}, \text{ where } x \text{ is a ray spanned by } |x> \quad (1)$$

With each compact normal operator Q corresponds an orthonormal base of eigenvectors $\{|q>\}_q$ with eigenvalues q. As a consequence a notion of state is attached to each physical item combined with one or more mutually compatible compact normal operators $\{Q_j\}_j$.

In Hilbert space a state, or probability function, is a real function P on the Hilbert subspaces, with the following properties:

1. $P(0) = 0$
2. $\forall_{y\in H}\{P(y) \geq 0\}$, y is a Hilbert subspace
3. $\sum_j P(x_j) = 1$, where $|x_j>$ form an orthonormal base of **H**, and x_j is the ray spanned by $|x_j>$
4. $P(y) = \sum_{j=1}^{r} P(x_j)$ where x_j are mutually orthogonal rays spanning subspace y

Pure state

In particular, if some ray x_0 satisfies $P(x_0) = 1$, then according to Born's rule:

$$\forall_{|x>\in H}\{P(x) = |<x_0, x>|^2\} \quad (1)$$

This happens when $|x_0>$ represents an unary pure state.

Item state

The state $P(\{q_j\}_j)$ is connected to a **wave function** $\psi(\{q_j\}_j)$, where

$$P(\{q_j\}_j) = |\psi(\{q_j\}_j)|^2 \tag{2}$$

and $\{q_j\}_j$ are the eigenvalues of eigenvector$\{|q_j>\}$ of the operators $\{Q_j\}_j$. Two operators A and B are compatible when their commutator is zero:

$$[A,B] \equiv AB - BA = 0 \tag{3}$$

If the state is characterized by a *set* of independent properties, then each of these properties corresponds with a corresponding operator. These operators must be normal, but they need not be compact. It must be possible to construct a spectral decomposition for each of the operators. Further, the operators that together determine the state must be mutually compatible. The wave function is then the product of the probability amplitudes that correspond to the separate operators. Thus the resulting wave function is a characteristic that represents the probability amplitudes of a set of mutually compatible observables that correspond to the normal operators that determine the state.

The squared modulus of the probability amplitude is the probability density. The wave function can also be a function of a progression parameter. Position can be a state characterizing observable. However, like the progression parameter, spacetime does not occur as an eigenvalue of a Hilbert space operator. The operators may vary. For example an operator may be replaced by its canonical conjugate. In that case, care must be taken that the operators that form the changed state are still compatible. Thus, even with the same physical item, the wave function is not unique.

For the operator Q with eigenfunctions |q> and eigenvalues q the probability amplitude function $\psi(q)$ is given by the smoothed version of $<\psi|q>$

$$\psi(q) \cong <\psi|q> \tag{4}$$

When Q is a compact normal operator then the smoothed version of $\psi(q)$ is a continuous function. In that case the eigenvalues of the corresponding operator $\check{Q}$ that resides in rigged Hilbert space $\mathbf{H}$ are used. Then $\psi(q)$ has a Fourier transform $\phi(p)$, where the operator $\check{P}$ with eigenvectors |p> and eigenvalues p is the canonical conjugate of $\check{Q}$. Like $\psi(q)$, the function $\phi(p)$ is also a function that characterizes the corresponding item and $|\phi>$ is a characterizing vector. The parameters q and p may be quaternionic.

$$\varphi(p) = <\varphi|p> \tag{5}$$

With respect to the correspondence with traditional quantum logic, it is wrong to take any characteristic vector including the locator or any function including the wave function as the *representative* for the item. It is ridiculous to expect that a single vector carries all properties of

a complex physical item, such as a DNA molecule or an elephant. The Heisenberg uncertainty relation also offers objections against this single vector based representation.

In quantum mechanics the wave function can be interpreted as the combination of a stationary vector and a progression operator. The progression operator has the form $A \cdot \exp(S/\hbar)$. A is Hermitian and positive. S is anti-Hermitian. This is reflected in the Hamilton-Jacobi equation.

In quantum field theory the fields are replacing the wave function. Thus a field may be interpreted as the amplitude of the probability to find something at the location of the field value. For bosons that something may be interpreted as a virtual particle. For fermions that something may be interpreted as a pair of virtual particles. Each type of virtual particle has its own type of field.

There are some questions left with wave functions:

- Can it have non-zero values outside the subspace that represents the physical item?
  - Answer: Yes. The private field covers the whole Hilbert space.
- Is the wave function a regular function?
  - Answer: When only compact normal operators are involved, then the wave function is regular.
- What happens to the representing subspace and to the wave function when a measurement is performed?
  - Answer: The subspace is squeezed into an appropriate configuration and the wave function is adapted to this new form.
- Has a system a wave function?
  - Answer: In general a system must not have a wave function, but it has a density operator.

Probability density

Gleason's theorem states that a probability measure $\mu(P)$ on the lattice $L(\mathbf{H})$ of projections P on closed subspaces of a Hilbert space $\mathbf{H}$, corresponds to a non-negative Hermitian operator ρ with trace 1, such that $\mu(P) = \mathrm{tr}(\rho P)$. When the projections P_q correspond to the rays formed by the eigenvectors $|q\rangle$ of operator Q and $\mu_i(P_q)$ corresponds to the considered physical item, then $\mu_i(P_q) = \langle q, \rho_i\, q\rangle$ corresponds to the square of the modulus of the wave function $\psi_i(q)$. ρ_i is the probability density operator corresponding to μ_i. The probability measure μ is a regular function. $\mu_i(P_q)$ is zero outside the subspace that represents the considered physical item.

The probability density function $P(q) = |\psi(q)|^2$ of an absolutely continuous random variable q is a function that describes the relative chance for this random variable to occur at a given point in the Q observation space. The probability for a random variable to fall within a given set is given by the integral of its density over the set.

The probability density operator ρ is positive-semi-definite ($\forall_{|f\rangle \in H} \{\langle f|\rho\, f\rangle \geq 0\}$), self-adjoint ($\rho = \rho^\dagger$), and has trace one ($\mathrm{tr}(\rho) = 1$). For the operator Q with eigenfunctions $|q\rangle$ and eigenvalues q with probability amplitude $\psi(q)$, the density operator ρ is given by

$$\rho = \sum_q \{|< \psi|q >|^2 \cdot |q >< q|\} \quad (1)$$

Von Neumann entropy is defined using the density operator of physical items.

The entropy $S(\varrho)$ describes the departure of the system from a pure state. In other words, it measures the degree of mixture (entanglement) of the state $|\psi>$.

The operator A can be decomposed

$$A = \sum_a |a > a < a| \quad (2)$$

For the state $|\psi>$ the **expectation value** $\langle A \rangle$ for the observable A is

$$\langle A \rangle \equiv < \psi|A\,\psi > = \sum_q \{|< \psi|q >|^2 \cdot < q|\, Aq > \} = tr(\rho A) \quad (3)$$

A Hilbert field is a blurred Hilbert distribution. The blur represents a probability amplitude distribution.

States and blurs

Apparently a state is the same stuff as the basic constituent of a Hilbert field. Both can be characterized as probability amplitude distributions. The squared modulus of a probability amplitude distribution is a probability density distribution. The state corresponds with a wave function or with a probability density operator.

- The state of a physical item can be interpreted as the probability of finding the parameter value when an observation is done that corresponds to the corresponding coordinate operator.
- The squared modulus of the blur can be interpreted as the probability of detecting a quantum at the location specified by the parameter value that corresponds to the corresponding coordinate operator.

Blurs are the building stones of Hilbert fields. In a similar way wave functions must be interpretable as the building stones of fields.
Blurs are private fields of elementary Hilbert distributions. Thus, wave functions must also be related to elementary Hilbert distributions.

Observables and field values

In Hilbert space observables are represented by operators. The observed value is represented by an eigenvalue or by the expectation value of the operator that represents the observable. Scalar physical fields have numeric values. Vector and tensor fields consist of vectors and tensors that are constructed using numbers. Both the eigenvalues of operators and the values of fields may be hyper complex 2^n-ons.

Numbers

The Hilbert space can be specified by using a number space that allows the mutual orthogonalization and the closure of subspaces. The real's, the complex numbers and the quaternions can perform that job. Horwitz showed that even the octonions with some trouble can achieve this (see: http://arxiv.org/abs/quant-ph/9602001). The real's, the complex numbers, the quaternions and the octonions are the only normed division algebras and they are the only alternative division algebras. In general the octonions are not associative, but the product of two octonions that belong to the same quaternionic subfield is associative. The alternative property of the octonions admits the closure of the subspace generated by (successively associated) products of the vector with octonion elements to order seven, i.e., after multiplication seven times by octonions, the subspace no longer grows.
Neither all quaternions nor all octonions commute. However, within complex subspaces the numbers commute.

We will take the following freedom. The fact that a given number space is used for specifying linear combinations of Hilbert space vectors does not mean that eigenvalues of operators must also be restricted to that same number space. In this sense a Hilbert space specified over the quaternions may allow eigenvalues of operators that are taken from the octonions or even higher 2^n-ons (see http://www.math.temple.edu/~wds/homepage/nce2.pdf or the toolkit). The problem with higher dimension 2^n-ons is that their number characteristics deteriorate with n. However, as long as the (full) eigenvalues are not used to construct linear combinations of vectors, or to specify the inner products of the Hilbert space, there is no problem. All higher dimensional 2^n-ons contain several subspaces that are lower dimensional 2^m-on number spaces. Further, 2^n-ons behave like 2^m-ons in their lower 2^m dimensions.

In general the elements of curves or curved manifolds are themselves not numbers. So, in general they cannot be used as eigenvalues. However, locally the elements of a curved manifold may resemble numbers of a 2n-on number space. Smoothly curved trails of objects that locally resemble 2^n-ons can be treated with the Frenet-Serret frame toolkit. Number spaces can be **attached** as tangent spaces to smoothly curved manifolds. In that way the elements of the curves and the manifolds obtain number characteristics in a small enough environment. Sequences or sets of operators can locally have eigenvalues that are numbers which can be considered as member of smooth curves or of the tangent space of a curved manifold at that location. In that way the elements of smooth curves or of curved manifolds can be related to the corresponding eigenvalues. 2^n-ons are ideally suited for this purpose. This means that the eigenspaces of the subsequent operators in a trail need not overlap. These eigenspaces are only used locally. When curvature and bending of the operator trail diminish, the dimension of the local number space can be lower. When the curvature and the bending increase, the dimension must be higher. This will be reflected in the dimensionality

of the local eigenvalues. Apart from the application as eigenvalues of operators the 2^n-ons are suited as values of physical fields.

We will restrict to the 2^n-ons as extensions of the quaternions. As we stated, the numbers created with the Cayley-Dickson construction are not so well behaved. Alternatives are the use of Clifford algebras, Jordan algebras or Grassmann algebras. We will show that in the Hilbert space the 2^n-ons for $n > 1$ automatically introduce these latter algebras through their number waltz.

The **niners** are the most extensive 2^n-on numbers that still keep a reasonable set of number characteristics. More precisely said the 2^n-ons, even those that have a higher dimension than the octonions, keep reasonable number characteristics in the space spanned by their coordinates that have an index lower than nine. The real numbers, the complex numbers, the quaternions and the octonions completely fall within these boundaries. The above hyperlink describes exactly what characteristics the niners retain.

The subspace of the 2^n-on field that is spanned by the first 2^m dimensions acts as a 2^m-on number space. Thus in a dynamic situation, an octionic operator acts locally as a quaternionic operator. In a smaller or more flat region it acts as a complex operator and at "nano"-locality as a real (or as an imaginary) operator.

2^n-on construction

The 2^n-ons use the following doubling formula

$$(a, b)\,(c, d) = (a\cdot c - (b\cdot d^*)^*, (b^*\cdot c^*)^* + (b^*\cdot(a^*\cdot((b^{-1})^*\cdot d^*)^*)^*)^*) \tag{1}$$

Up until the 16-ons the formula can be simplified to

$$(a, b)\,(c, d) = (a\cdot c - b\cdot d^*,\ c\cdot b + (a^*\cdot b^{-1})\cdot(b\cdot d)) \tag{2}$$

Up to the octonions the Cayley Dickson construction delivers the same as the 2^n-on construction. From n>3 the 2^n-ons are 'nicer' than the Cayley Dickson numbers. They keep more useful number characteristics. The 2^{n+1}-ons contain the 2^n-ons as the sub-algebra of elements of the form (a, 0)

Waltz details

The 16-ons lose the continuity of the map x => xy. Also, in general holds $xy\cdot x \neq x\cdot yx$ for 16-ons. However, for all 2^n-ons the base numbers fulfill $e_ie_j\cdot e_i = e_i\cdot e_je_i$. All 2^n-ons feature a conjugate and an inverse. The inverse only exists for non-zero numbers. The 2^n-ons support the **number waltz**

$$c = a\cdot ba^{-1}. \tag{1}$$

Often the number waltz appears as a unitary number waltz

$$c = u^*\cdot bu \tag{2}$$

where u is a unit size number and u^* is its conjugate $u \cdot u^* = 1$.

In quaternion space the **quaternion waltz** $a \cdot b \cdot a^{-1}$ can be written as

$$a \cdot b \cdot a^{-1} = \exp(2 \cdot \pi \cdot \underline{\tilde{\mathbf{i}}} \cdot \phi) \cdot b \cdot \exp(-2 \cdot \pi \cdot \underline{\tilde{\mathbf{i}}} \cdot \phi) \tag{3}$$

$$= b - \underline{\mathbf{b}}_\perp + \exp(2 \cdot \pi \cdot \underline{\tilde{\mathbf{i}}} \cdot \phi) \cdot \underline{\mathbf{b}}_\perp \cdot \exp(-2 \cdot \pi \cdot \underline{\tilde{\mathbf{i}}} \cdot \phi)$$

$$= b - \underline{\mathbf{b}}_\perp + \exp(4 \cdot \pi \cdot \underline{\tilde{\mathbf{i}}} \cdot \phi) \cdot \underline{\mathbf{b}}_\perp$$

$$\Delta b = (\exp(4 \cdot \pi \cdot \underline{\tilde{\mathbf{i}}} \cdot \phi) - 1) \cdot \underline{\mathbf{b}}_\perp \tag{4}$$

$$= (\cos(4 \cdot \pi \cdot \phi) + \underline{\tilde{\mathbf{i}}} \cdot \sin(4 \cdot \pi \cdot \phi) - 1) \cdot \underline{\mathbf{b}}_\perp$$

$$= \exp(2 \cdot \pi \cdot \underline{\tilde{\mathbf{i}}} \cdot \phi) \cdot 2 \cdot \underline{\tilde{\mathbf{i}}} \cdot \sin(2 \cdot \pi \cdot \phi) \cdot \underline{\mathbf{b}}_\perp$$

$$||\Delta b|| = ||2 \cdot \sin(2 \cdot \pi \cdot \phi) \cdot \underline{\mathbf{b}}_\perp|| \tag{5}$$

Another way of specifying the difference is:

$$\Delta b = (a \cdot b - b \cdot a)/a = 2 \cdot (\underline{\mathbf{a}} \times \underline{\mathbf{b}})/a \tag{6}$$

$$||\Delta b|| = 2 \cdot ||\underline{\mathbf{a}} \times \underline{\mathbf{b}}|| / ||a|| \tag{7}$$

Infinitesimal number transformation

The number v is close to 1. Thus $v = 1 + \Delta s$. Let us investigate the transform $c = v^* \cdot b \cdot v$.

$$\begin{aligned} c &= (1 + \Delta s^*) \cdot b \cdot (1 + \Delta s) \\ &= b + \Delta s^* \cdot b + b \cdot \Delta s + \Delta s^* \cdot b \cdot \Delta s \\ &\approx b + \Delta s^* \cdot b + b \cdot \Delta s \\ &= b + \Delta s_0 \cdot b + 2 \cdot \underline{\mathbf{b}} \times \boldsymbol{\Delta s} \end{aligned} \tag{1}$$

$$\Delta b = \Delta s_0 \cdot b + 2 \cdot \underline{\mathbf{b}} \times \boldsymbol{\Delta s} \tag{2}$$

This comes close to the effect of an infinitesimal number waltz, especially when $\Delta s_0 = 0$ In that case $\Delta b_0 = 0$ and $\boldsymbol{\Delta b}$ is perpendicular to $\boldsymbol{\Delta s}$.

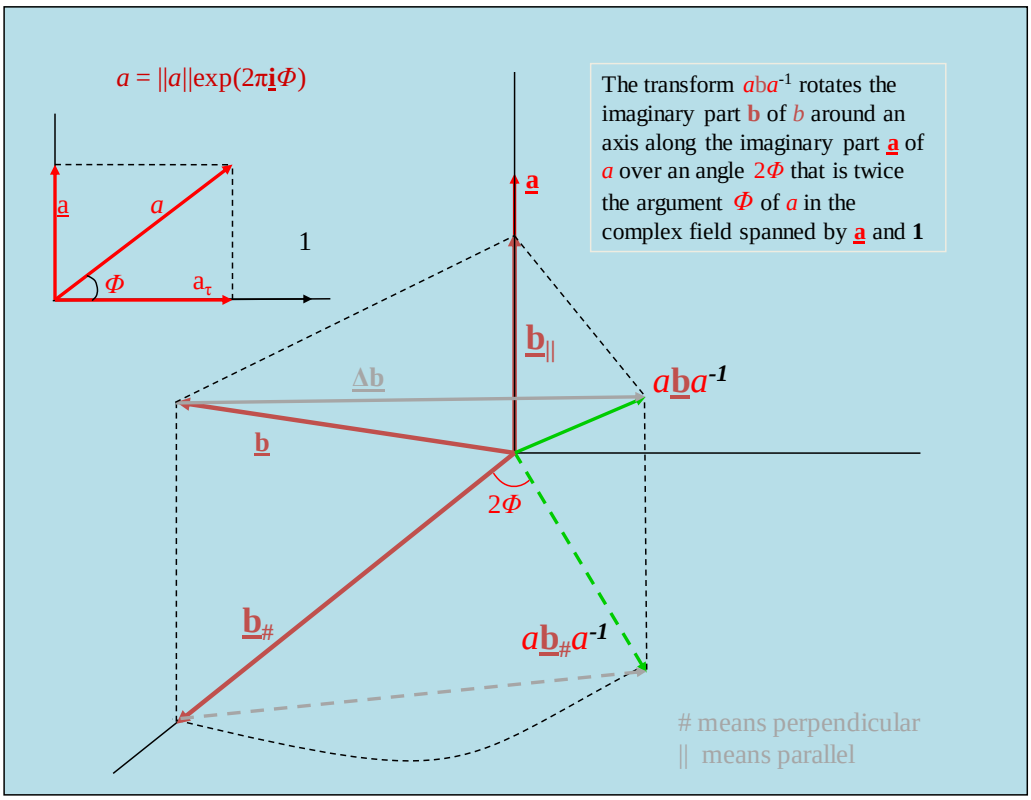


Figure 1. The rotation of a quaternion by a second quaternion.

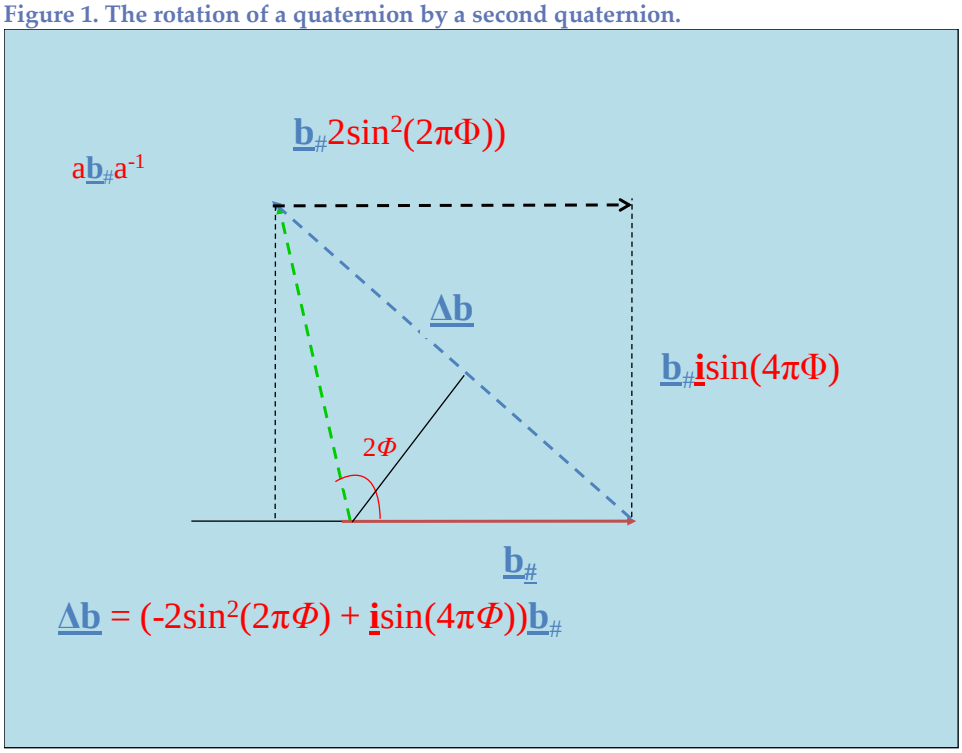


Figure 2: The difference after rotation

For 2^n-ons with n > 1, $a \cdot ba^{-1}$ in general does not equal b. This effect stays unnoticed when quantum mechanics sticks to a complex Hilbert space.

Sign selections

The paper that describes 2^n-ons does not describe the choice for right or left handedness of the external vector product. So, we do it here. The generally accepted convention is to let the handedness depend on the orientation of the underlying $\mathbb{R}^n$ space. However, when numbers

are constructed via the Cayley-Dickson construction or the 2^n-on construction then the handedness follows from the applied construction formula. We want to get rid of these restrictions, because we want to give operators the freedom to select the handedness and other sign selections of their eigenvalues.

The 2^n-ons have n independent binary base numbers and n sign selections. The real numbers do not offer a sign selection. The complex numbers offer the selection of the sign of the real or the imaginary axis. This is inherited by all higher 2^n-ons. The quaternions have two independent imaginary base numbers and offer an extra sign selection that represents the handedness of its external product. The octonions have three independent imaginary base numbers and offer an extra sign selection for the handedness in external products that involve this new base number.

Need for spinors

In the number waltz the current manipulator only needs an argument α in order to turn the subject over 2α. This is typical behavior for spinors. Spinors also have a storage place for the handedness of rotations. By using the number waltz and the sign selections the 2^n-ons can perform the same act as the spinors. Spinors are only required when quantum mechanics is restricted to complex Hilbert spaces. Spinors are the carriers of the spin phenomenon. Thus, in our model the sign selections in combination with the number waltz form the carriers of spin. Because a strange trick is played with the real parts of eigenvalues, the influence of the selection of the sign of the real axis will be revealed later.

The approach taken in this paper might cause a revival of the importance of the hyper complex numbers that turned in oblivion when Gibbs introduced his vector analysis.

Influence

The original proposition (♠) talks about influencing the position of an item. This implies that the position of the item changes due to the mentioned influence. Thus when the influence occurs, the eigenvector that represents the position of the item is exchanged against another eigenvector. That other eigenvector corresponds to another environment inside the eigenspace of the position operator. The new eigenvector takes the role of the old eigenvector and is the new characteristic for the item's position. This replacement may take place inside the subspace, which represents the considered item, or the original eigenvector moves outside the subspace, while the new eigenvector moves in or stays in the subspace. In both cases the eigenvectors of the position operator move with respect to the vector that characterizes the subspace of the item. The movement is relative and takes place inside the Hilbert space. Another possibility is that the eigenvectors stay, but the corresponding eigenvalues change while the Hilbert subspace moves. See Heisenberg picture versus Schrödinger picture.

Thus, there is a way to implement influence in Hilbert space. The influence causes a move of the item's subspace relative to one or more eigenvectors of the position operator. The original proposition (♠) claims that this movement is caused by other items. We must check whether this is true.

If this is true then influences are the motor behind the dynamics of the items.

The universe of items

The original proposition (♠) states that all items influence each other's position. This includes that all items influence the considered item. Part of the items compensates each other's influences on the currently considered item. It will be shown that this holds for the largest part.

Inertia

The influence may decrease with distance according to some function $f(r)$ of the distance r. However the number of contributing items increases with the distance. Depending on function $f(r)$ the most probable result is that the strongest influence comes from the cooperative activity of the most distant items. Due to the enormous number of items in the universe, any variation of the influences of the distant items averages away. This also holds for the density distribution of the items. So there exists a fairly uniform background influence caused by the universe of items. What will happen, can be deduced from an equivalent of Denis Sciama's analysis. We will take his analysis as a guide. Sciama's analysis uses a different setting: the (observed) 3D space and coordinate time. (See: http://arxiv.org/abs/physics/0609026v4.pdf and "On the origin of inertia", by Denis Sciama (http://www.adsabs.harvard.edu/abs/1953MNRAS.113...34S). This setting raised critique because the approach involves instantaneous action on large distances. In Sciama's setting this is in conflict with special relativity. In our setting we do not (yet) encounter special relativity. We use the coordinate space defined by an appropriate coordinate operator and the progression parameter t that relates to the progression step counter as our setting. A location in coordinate space represents a location on the unit sphere of Hilbert space. As stated before this is an affine space. This means that we may treat position as relative data. The eigenspace of the coordinate operator has no absolute origin.

The most important aspects of the analysis are:

The total potential at the location of the influenced subject is

$$\Phi = -\int_V \frac{\rho}{r} dV = -\rho \int_V \frac{dV}{r} \tag{1}$$

(See: http://en.wikipedia.org/wiki/Newtonian_potential). This conforms to a Gaussian blur as a representative of the average blur function. The integral is taken over the coordinate space volume V. Indirectly, the integral is taken over the unit sphere of Hilbert space. This is an affine space. The parameter r is the length of the vector from the actor to the location of the subject. The considered subject is located somewhere in the affine coordinate space. All other subjects have positions relative to that considered subject. At large distances, the density ϱ of the contributing items can be considered to be uniformly distributed. Also any variance in strength other than the dependence on r becomes negligible because the differences are blurred and averaged away. We already assumed that the average blur of the distributed matter in universe is a Gaussian blur. We take the average of the strength as the significant parameter. We combine it with ρ. Therefore the average of ρ can be taken out of the integral. Thus, apart from its dependence on the average value of ρ, Φ is a huge constant. (Sciama relates Φ to the gravitational constant). As a consequence we can consider the universe as a very large rigid body. If nothing else happens then all influences compensate each other.

In the following equations we use imaginary quaternions rather than 3D vectors. In this way we can avoid the distracting factor i.

If the considered subject moves relative to the universe with a uniform speed **v**, then a vector potential **A** is generated.

$$\mathbf{A} = -\int_V \frac{\mathbf{v} \cdot \rho}{c \cdot r} dV \tag{2}$$

Both ρ and **v** are independent of r. Together with the constant c they can be taken out of the integral. Thus

$$\mathbf{A} = \Phi \cdot \mathbf{v}/c \tag{3}$$

What we have here is the reverse of the definition of the potential that goes together with a charge distribution. When we defined a Hilbert field we started from a charge distribution and a current distribution and considered the blurred influence of these distributions on the universe. Here we consider the influence of the universe on a local charge or current. For this purpose we use the same volume integrals! The field that we treat in studying inertia is a curvature field rather than a covering field. The curvature field derives from the covering field by taking the curvature that is caused by the decomposition of the covering field as its charge.

The notions of charge and current correspond to equivalent notions in Noether's theorem. Here we talk about inertia and curvature fields. Thus charge may symbolize mass.

Here the progression parameter t plays the role of "time". Be aware, this is not our usual notion of time.

According to Helmholtz theorem the Hilbert field derived from the above potentials can be split into a divergence free part and a rotation free part. The Helmholtz decomposition theorem only concerns the static versions of the derived field. It is related to the fact that the Fourier transform of a vector field can be split in a longitudinal and a transversal version. There also exists a corresponding split of the multi-dimensional Dirac delta function in a longitudinal and a transversal version. If we use the position operator $\check{Q}$ as the coordinate operator, then the decomposition runs along straight lines. If we use the GPS operator $\check{\Omega}$ then the decomposition runs along curved lines. In curved manifolds the Helmholtz decomposition theorem should be replaced by the Hodge decomposition theorem.

A variation of **v** goes together with a variation of **A**. On its turn this goes together with a non-zero field $\dot{\boldsymbol{A}}(\boldsymbol{r},t)$ which is a **dynamical** part of the derived Hilbert field.

Sciama uses a Maxwell equation to explain the relation between ∂**v**/∂t and $\dot{\boldsymbol{A}}(\boldsymbol{r},t)$. Our setting differs, but the main reasoning is the same.

$$\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\Phi(\boldsymbol{r},t) - \frac{1}{c}\cdot\dot{\boldsymbol{A}}(\boldsymbol{r},t) \tag{4}$$

$$\widetilde{\boldsymbol{E}}(\boldsymbol{k},\omega) = -\boldsymbol{k}\cdot\widetilde{\Phi}(\boldsymbol{k},\omega) - \frac{1}{c}\cdot\omega\widetilde{\boldsymbol{A}}(\boldsymbol{k},\omega) \tag{5}$$

If we exclude the first term because it is negligible small, we get:

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\Phi}{c^2}\cdot\frac{\partial\mathbf{v}}{\partial\mathrm{t}} = G\cdot\frac{\partial\mathbf{v}}{\partial\mathrm{t}} \tag{6}$$

Remark: As soon as we turn to the dynamic version (4) an extra component $\dot{\boldsymbol{A}}$ of field $\boldsymbol{E}$ appears that corresponds to acceleration ∂**v**/∂t. (See for derivation of Maxwell equations e.g. the online book http://www.plasma.uu.se/CED/Book; formula 3.25 or the section on Hilbert field equations in this e-paper)

As already claimed, in our setting the component $\boldsymbol{\nabla}\Phi$ of the field $\boldsymbol{E}$ is negligible. With respect to this component the items compensate each other's influence. This means that if the influenced subject moves with uniform speed **v**, then $\boldsymbol{E} \approx 0$. However, a vector potential $\boldsymbol{A}$ is present due to the movement of the considered item. Any acceleration of the considered item goes together with an extra non-zero $\boldsymbol{E}$ field. In this way the universe of items causes inertia in the form of a force that acts upon the accelerating item's charge.

We have used the coordinate space as a playground to implement an equivalent of Sciama's analysis. The analysis uses the fact that every item in universe causes an influence and that

this influence reduces according to $f = -k/r$. (Compare this with Bertrand's theorem in Wikipedia)

A uniform movement in Hilbert space does not on itself generate a reaction of the universe of items. Any alteration of that uniform movement will cause as reaction **a field**. The physical name for this reaction is **action**. It usually gets the symbol **S**. When the path of the item coincides with a **geodesic**, then it can be travelled field free.

Uniform movements do cause displacement of charges. On its turn it changes the configuration of the local field. Thus, indirectly the field will also act on uniform displacements. As we see from inertia, any field change goes together with a corresponding acceleration.

It must be noticed that the original analysis of Sciama uses observable position space rather than Hilbert space, coordinate space or action space and it uses a different notion of time. However, the general conclusion stays the same. Sciama's analysis is criticized because it uses infinite speed of information transfer. Since we do not work in observable position space, we do not encounter coordinate time. So for us, this criticism is misplaced. (Coordinate time relates to observations of position. It is a local player in the game, where the progression parameter is a global player.)

The situation with electromagnetic fields is different, because with this field positive and negative charges compensate each other's long range influence. For that reason there exists no electromagnetic background influence or it is far less important. The masses of the gravitational and inertial fields only compensate each other's long range influences through geometrical circumstances. Still in combination, they create gigantic potentials.

Nearby items

Items that are located nearby have a different effect. In general their influence will not have strength equal to the average strength. Further these items are not uniformly distributed. Still their influence depends on inter-distance as $f = -k/r$. As a consequence their influences form a landscape of which the effects will become sensible in the action of the fields that surround the considered item. This landscape will form a curved action space. The considered item will try to follow a geodesic through that curved space.

Rotational inertia

Besides linear inertia there exists rotational inertia. In a non-rotating universe hold near the origin $\mathbf{A} = 0$ and $\Phi = -c^2/G$. We choose units such that c=G=1. In a universe rotating slowly with angular speed ω hold

$$A_x = \omega \cdot y \tag{1}$$

$$A_y = -\omega \cdot x \tag{2}$$

$$A_z = 0 \tag{3}$$

(4)

$$\Phi = -\sqrt{1 + (\omega \cdot r)^2}$$

A constant angular movement meets the fields that correspond to a centripetal force.

The field $\boldsymbol{E}$ has the form

$$\boldsymbol{E} = \frac{\omega^2 \boldsymbol{r}}{\sqrt{1 + \omega^2 r^2}} \tag{5}$$

An added uniform speed v meets the fields corresponding to a Coriolis force.

$$\boldsymbol{H} = \boldsymbol{\nabla} \times \boldsymbol{A} = 2 \cdot \boldsymbol{\omega} \tag{6}$$

$$\boldsymbol{v} \times \boldsymbol{H} = 2 \cdot \boldsymbol{v} \times \boldsymbol{\omega} \tag{7}$$

The forces are usually considered as *fictitious* but they are actually caused by inertia. Sciama treats them in section 5 of his paper. Like fields of linear inertia these rotation related fields correspond to actions of the manipulator.

Storage, sign selections and virtual items.

The static fields act as storage media for the location and the speed of the charges of the physical items.

When the values of the fields are stored in hyper complex numbers, then the sign choices for these numbers will also be reflected in these fields. For example the handedness will show in the transverse fields. A right handed and a left handed version of the field will exist. The sign selection of the real part of the hyper complex numbers may also cause extra versions of fields.

The fields can be interpreted as reflections of the presence of virtual items that are ready to exchange roles with actual items.

The proposition

This finding indicates that when our interpretation of Sciama's analysis is correct, the original proposition

All items in universe influence each other's position.

is not generally true. The universe of items does not influence position. It counteracts acceleration of individual items. Position is only influenced in an indirect way and presupposes an observation. If the item moves in a geodesic with uniform speed, then the position changes while the influences of all other items compensate each other. In such cases the summed influence is zero.

We may alter the original proposition (♠). If our analysis is correct, then the proposition

All items in universe influence each other's acceleration.

is true.

The origin of dynamics

If we want to discover the origin of dynamics, we must first determine what the static structure of nature is. We already found an important ingredient of this skeleton: the lattice structure of quantum logic and the corresponding lattice structure of the closed subspaces of a Hilbert space. Both structures are only defined in a static way. Nothing is said about their dynamics. Besides of these static relations the concept of wave functions and density operators offer insight in the probability and information content of these relations. These subjects correspond to private fields, which are the constituents of a covering field. This covering field can be seen as the superposition of all private fields. Its static decomposition into rotation free and divergence free parts defines a curvature that depends on the selection of a coordinate system. That curvature can be used to define a curvature field. This together defines the ingredients of the static status quo. It all fits in a model that we call *extended quantum logic* or equivalently *extended Hilbert space*.

In the previous part of the paper the added component of the static structure of nature is investigated: the static structure of the influences. It appears that this structure is identical with the structure of static Hilbert fields. Both the analysis of inertia and the study of Hilbert fields showed the static relation between divergence free fields and rotation free fields. These analyses also showed the influence of dynamics on the coupling of these static fields. The analysis of Hilbert fields explained how these fields change as a function of q_0. Inertia showed how these fields get coupled when the uniform movement of a physical item is disturbed. We also explained that uniform movement may cause a reconfiguration of the field. This change on its turn will stimulate movement.

Next we try to find a more precise formulation for these origins of dynamics.

Extended quantum logic

Wave functions represent the probability of finding a property of realistic items. For example if a GPS type coordinate system is selected as its parameter space, then this property can be the position of the item. If it is a GMS type system, then the property can be momentum.

In quantum field theory the values of fields are treated in equations of motion in a similar way as the wave functions of realistic items are treated in such equations. In fact the wave function itself can be interpreted as part of the field.

When fields in general can be considered as representations of the probability of finding properties of actual as well as virtual items, then the fields get an interesting interpretation. In quantum logic the realistic physical item is represented by a proposition in the form of a statement that says everything about that item. For virtual items the new interpretation would mean that in extended quantum logic the virtual items are represented by potential propositions that are ready to become actual propositions.

This would mean that traditional quantum logic is embedded in extended quantum logic such that it apart from propositions about actual physical items also contains potential proposition that represent virtual particles.

In this way, the set of propositions of extended quantum logic is much larger than the set of propositions of quantum logic.

Interpretation in logical terms

The results of the analysis of inertia mean that when the redefinition of the set of vectors that belongs to the representation of the item occurs such that this corresponds to a uniform movement of the physical item, then the influences of the universe of items tend to compensate each other. Otherwise, the universe of items reacts with a corresponding field. That field manifests as an action of the current manipulator. Besides of the universe wide response, a local variance in the distribution of items causes a variation in the influences. This local variance can partly be the consequence of a uniform displacement of particles.

It seems that quantum logic and Helmholtz decomposition together define an important part of the static relations that exist in physics. The fields appear to resist the disturbance of the interrelations in the lattice of quantum propositions. In dynamical sense this lattice might step from one static status quo to the next. After a step new conditions are established that again must fulfill the laws that govern the static situation. If this is a proper interpretation, then it is likely that the progression step is taken universe wide. After each step the positions of the physical items relative to the fields have changed, thus when the fields are not uniformly distributed, the items meet a different field configuration. The next step is taken with and due to these new conditions.

Quantum logic only defines a static skeleton in which the dynamics of quantum physics takes place. To make it a dynamic logic, the set of axioms must be extended. The new axioms must state that all propositions influence each other. The influence depends on their mutual (coordinate) distance. In stationary conditions, which include uniform motion, these influences compensate each other. When an atomic predicate that concerns an element of an ordered set is replaced in a non-ordered fashion, meaning that the distance between the replaced elements does not stay the same, then the universe of all propositions will react such that the influences of the other propositions no longer compensate each other. The disordered influences counteract the disordered replacement.

Besides of that the local variance in the distribution of the propositions, which corresponds to a variance of the distribution of the corresponding physical items, also cause a variation in the influences that propositions have with respect to each other.

In Hilbert space these influences are implemented in the actions of Hilbert fields. In quantum physics the influence appears as a set of physical fields.

Redefiner action

One important step must still be taken. In physics observed spacetime has a Minkowski signature. Further we observe that space corresponds with the imaginary part of a position quaternion for which the real part seems to have no direct physical meaning. We must find an explanation for these facts. The Minkowski signature defines the following time-like relation between the proper time $\boldsymbol{\Delta t}$, the space step $\boldsymbol{\Delta q}$ and the coordinate time step $\boldsymbol{\Delta \tau}$

$$\boldsymbol{\Delta t}^2 = \boldsymbol{\Delta \tau}^2 - \boldsymbol{\Delta q}^2/c^2 \qquad (1)$$

A possible explanation can be given by the action of the redefiner when the infinitesimal action step is perpendicular to the space step and the coordinate time step is used to close the rectangular triangle. The action step $\boldsymbol{\Delta s}$ equals $\boldsymbol{\Delta t}$. $\boldsymbol{\Delta t}$, $\boldsymbol{\Delta s}$ and $\boldsymbol{\Delta t}$ are 3D vectors.

$$\boldsymbol{\Delta\tau} = \boldsymbol{\Delta s} + \boldsymbol{\Delta q}/c \tag{2}$$

Dynamics

Schrödinger or Heisenberg picture

For global rotations around its origin the Hilbert unit sphere acts as an affine space. It does not matter whether the eigenvectors of operators or the subspace that represents the item is moved. We can take the picture in which the subspace stays fixed, while the vectors move and the operators change with them. This is the **Heisenberg picture.**
We can also take the picture in which the vectors and operators stay fixed and the subspace moves. This is the **Schrödinger picture**.

We are only interested in the consequences. These are determined by the relative movement, not by the absolute movement. For a given physical item, in both pictures the expectation values of the operators vary in the same way.

Unitary transform

A unitary transform is a bounded normal operator. Unitary transforms keep the value of inner products untouched. If a unitary transform is applied to two vectors, then their inner product stays the same.

Unitary transforms need not have eigenvectors. For example Fourier transforms do not possess eigenvectors. In the rigged Hilbert space $\mathbf{H}$ functions exist that apart from a scaling factor are invariant under Fourier transformation.

If a unitary transform has eigenvectors then it has unit sized eigenvalues and to each of these eigenvalues correspond one or more eigenvectors that are mutually orthogonal. Unitary transforms are completely determined by their vector replacement characteristics, their eigenvectors and the corresponding eigenvalues. An extra characteristic is for example the smoothness of their eigenspace.

When a unitary operator U is applied to the eigenvector $|q>$ of an operator Q with eigenvalue q, then the eigenvector is transferred into another vector $|U\ q>$. In general $|U\ q>$ is not another eigenvector of Q. The expectation value for $|QU\ q>$ is no longer q, but

$$< q\ U|QU\ q > = < q|U^{\dagger}QU\ q > \qquad (1)$$

Or, with other words the operator Q is redefined to $U^{\dagger}QU$.

The norm of the expectation value $< f\ U|QU\ f >$ for an arbitrary vector $|f>$ does not depend on U. It only depends on Q and $|f>$.

Trail of infinitesimal transforms

The effect of a unitary transform U can also be achieved by a trail of infinitesimal unitary transformations $\{U_t\}_t$. This also holds for a set of unitary operators that operate in parallel.

The situation sketched above can be refined for any instant t occurring after t=0. We can treat it more generally by chopping the path from $\{|f_s>\}_s$ to $\{|g_{st}>\}_s$ into a **trail** of infinitesimal steps of size Δt that is achieved by a set of infinitesimal transforms $\{U_{st}\}_{st}$, where

$$|g_{st}>=|\prod_s U_{st}\, f_s> \tag{1}$$

and

$$U_{st} \approx 1 + \Delta S_{st} \tag{2}$$

The parameter t acts as the trail progression parameter. It is not identical with our common notion of time. The infinitesimal transforms U_{st} work in parallel as well as in sequence. ΔS_{st} represents the current local infinitesimal action step. It is an imaginary operator.

The Heisenberg picture conforms to the description with unitary transforms where operators are redefined. When this is done in small steps, then the redefined operator becomes a function of progression parameter t.

Unitary transform with full set of eigenvectors

When a unitary transformation U is applied to an arbitrary vector $|f>$, which is not an eigenvector, then that vector is transferred into another vector $|g> = |U\, f>$, which has the same norm. If $|f>$ is an eigenvector of U then $|f>$ is **not** transferred to a different vector, but it is multiplied with the corresponding eigenvalue. Also in this case the norm stays the same.

If a unitary transform contains a full set of eigenvectors, then multidimensional subspaces usually contain one or more eigenvectors of that unitary transform. In that case the transfer of a multidimensional closed subspace requires a set of parallel unitary transforms.

If we take a set of vectors $\{|f_s>\}_s$ that together span a closed subspace, then a set of suitable unitary transforms $\{U_s\}_s$, can in parallel transfer all vectors of this set such that after the transform $|g_s> = |U_s\, f_s>$ the set $\{|g_s>\}_s$ spans the new subspace. Each of the members U_s of the set $\{U_s\}_s$ can be split in a trail. $\{U_{st}\}_t$

Fourier transform as unitary transform

A Fourier transform is an example of a unitary transform. However, it does not leave a single Hilbert vector untouched. In Hilbert space a Fourier transform has NO eigenvectors!

There exist Hilbert fields that apart from a scaling factor are invariant under Fourier transformation. They keep their form through Fourier transformation. For that reason they are often called eigenfunctions, but they do not correspond to eigenvectors. The Hermite functions are notorious examples of Fourier "eigenfunctions". Even and odd functions have an indirect relation to functions that are invariant under Fourier transformation.

An invariant function is not an eigenfunction. In separable Hilbert space, every Fourier transform causes a resampling of the analyzed field or function.

Each Fourier transform means a complete replacement of the current orthonormal base. For that reason, a Fourier transform that resides in separable Hilbert space can never be an infinitesimal Fourier transform. However, Fourier transforms $U_{\mathcal{F}}$ and reverse Fourier

transforms $U_{\mathcal{F}}^{\dagger}$ can be member of a trail of unitary transforms, where each trail step contains a move up and down to Fourier space, while in Fourier space only an infinitesimal action is taken.

$$|g_{st}>=|\prod_s U_{\mathcal{F}} U_{st} U_{\mathcal{F}}^{\dagger} f_s> \tag{1}$$

Stepping through the Fourier space has the advantage that there derivation turns into multiplication and multiplication with a factor close to unity reduces to addition.

Single infinitesimal step

The success of the path integral formalism gives us guidance. We analyze an arbitrary trail consisting of infinitesimal trajectory steps:

$$< q_{t_1}|U_{s_{t_1,t_0}}\, q_{t_0}> = \left\{\prod_{t=t_0}^{t=t_1}(< q_t{}'|p_t >< p_t\, U_{s_t}|q_t >)\right\} < q_{t_1}|p_{t_1} > \tag{1}$$

What happens during a single step when the system moves from position q to $q' = q + \Delta q$? Let us evaluate $< q'|p >< p\, U_{st}|q >$ for a single trajectory. Here U_{st} is an infinitesimal unitary transform. It is a member of the set of parallel unitary transforms that act on a target subspace. In the following text we leave the parallel trajectory index s in U_{st} unspecified. We concentrate on the sequence index t, which represents the progression parameter. The infinitesimal sequence step comprises three sub-steps:

1. Goto Fourier space. This is achieved by step $< q'|p >$.
2. Perform the action. This is done by $< p\ U_t^{\dagger}| = \exp(\Delta \boldsymbol{s}_t) < p|$.
3. Go back to configuration space. This is achieved by $< p|q >$.

The sense behind the first and the last step is a travel to and back from Fourier space, where the differential operators appear as simple multiplications.

First we split $< q'|p >$.

$$< q'|p > = \exp\left(\frac{\boldsymbol{qp}}{\hbar}\right) \exp\left(\frac{\Delta \boldsymbol{q}\, \boldsymbol{p}}{\hbar}\right) \approx \exp\left(\frac{\boldsymbol{qp}}{\hbar}\right)\left(1+\frac{\Delta \boldsymbol{q}\, \boldsymbol{p}}{\hbar}\right) \tag{2}$$

$$< p|q > = \exp\left(\frac{\boldsymbol{pq}}{\hbar}\right) = < q|p >^{*} = \exp\left(-\frac{\boldsymbol{qp}}{\hbar}\right) \tag{3}$$

$$< q'|p > \exp(\Delta \boldsymbol{s_t}) < p|q > \approx \exp\left(\frac{\boldsymbol{qp}}{\hbar}\right)\left(1+\frac{\Delta \boldsymbol{q}\, \boldsymbol{p}}{\hbar}\right)(1+\Delta \boldsymbol{s_t}) \exp\left(-\frac{\boldsymbol{qp}}{\hbar}\right) \tag{4}$$

$$= \exp\left(\frac{\boldsymbol{qp}}{\hbar}\right) \mathrm{C} \exp\left(-\frac{\boldsymbol{qp}}{\hbar}\right)$$

This is a quaternionic rotation of the central term C, which is close to unity. The quaternionic rotation manipulators stands for the route to Fourier space and the route back from Fourier

space. The central term C stands for what is done during a single step by the action in Fourier space.

$$\mathrm{C} \approx \left(1 + \frac{\Delta \boldsymbol{q}\, \boldsymbol{p}}{\hbar}\right)(1 + \Delta \boldsymbol{s_t}) \approx 1 + \frac{\Delta \boldsymbol{q}\, \boldsymbol{p}}{\hbar} + \Delta \boldsymbol{s_t} = 1 + \Delta \mathbf{C} \tag{5}$$

$$\Delta \mathbf{C} = \frac{\Delta \boldsymbol{q}\, \boldsymbol{p}}{\hbar} + \Delta \boldsymbol{s_t} \tag{6}$$

In configuration space a similar step occurs, but then rotated with the transformation

$$\Delta \mathrm{C}_{config} = \exp\left(\frac{\boldsymbol{qp}}{\hbar}\right) \Delta \mathrm{C}_{Fourier} \exp\left(-\frac{\boldsymbol{qp}}{\hbar}\right) \tag{7}$$

We study the step $\Delta \mathbf{C}$ somewhat deeper. As in the case of Hilbert fields we will split the analysis in a longitudinal part and a transverse part. The longitudinal part treats the part of p that is parallel to Δq. The transverse analysis treats the part of p that is perpendicular to Δq The longitudinal equation is:

$$\Delta \boldsymbol{s}_{\parallel_t} = \frac{\langle \boldsymbol{p}, \Delta \boldsymbol{q} \rangle}{\hbar} + \Delta \mathbf{C}_{\parallel} = \frac{\langle \boldsymbol{p}, \Delta \boldsymbol{q} \rangle}{\hbar} - \mathbf{H}_{\parallel}\, \Delta \tau \tag{8}$$

$$ds_{\parallel_t} = \frac{p_{\parallel}}{\hbar} dq - \mathrm{H}_{\parallel}\, d\tau \tag{9}$$

$$\frac{\partial s_{\parallel_t}}{\partial q} = \frac{p_{\parallel}}{\hbar} \tag{10}$$

$$\frac{\partial s_{\parallel_t}}{\partial \tau} = -\mathrm{H}_{\parallel} \tag{11}$$

$$\dot{s}_{\parallel_t} - \frac{p_{\parallel}}{\hbar} \dot{q} = \mathrm{H}_{\parallel} \tag{12}$$

The transversal equation is:

$$\Delta \boldsymbol{s}_{\perp_t} = \frac{\boldsymbol{p} \times \Delta \boldsymbol{q}}{\hbar} + \Delta \mathbf{C}_{\perp} = \frac{\boldsymbol{p} \times \Delta \boldsymbol{q}}{\hbar} - \mathbf{H}_{\perp} \Delta \tau \tag{13}$$

$$ds_{\perp_t} = \frac{p_{\perp}}{\hbar} dq - \mathrm{H}_{\perp} d\tau \tag{14}$$

$$\frac{\partial s_{\perp_t}}{\partial q_{\perp}} = \frac{p_{\perp}}{\hbar} \tag{15}$$

$$\frac{\partial s_{\perp_t}}{\partial \tau} = -\mathrm{H}_{\perp} \tag{16}$$

(17)

$$\dot{s}_{\perp t} - \frac{p_\perp}{\hbar}\dot{q}_\perp = \mathrm{H}_\perp$$

In fact there exists a third dimension along $\boldsymbol{p} \times \boldsymbol{p} \times \Delta\boldsymbol{q}$, which is along $\Delta\boldsymbol{q}$ when $\boldsymbol{p}$ is perpendicular to $\Delta\boldsymbol{q}$.

The trail corresponds to a sum:

$$< q_{t_1}|U_{t_1,t_0}\, q_{t_0} > = \left\{\prod_{t=t_0}^{t=t_1}(< q_t{}'|p_t >< p_t\, U_t|q_t >)\right\} < q_{t_1}|p_{t_1} > \tag{18}$$

$$= 1 + \sum_{t=t_0}^{t=t_1}\left\{\frac{\Delta\boldsymbol{q}\,\boldsymbol{p}}{\hbar} + \Delta\boldsymbol{s_t}\right\}$$

Depending on the choice of the coordinate operator, the longitudinal direction $\boldsymbol{k}_t$varies with the trajectory parameter t. The steps Δq_t and Δs_t depend on the step Δt of the trajectory parameter t that is used to chop the unitary transform U_{t_1,t_0}.

Following a trail has much in common with ray tracing in optics. However in optics the use of characteristics that have their base in Fourier analysis seems to be more fruitful than ray tracing. Ray tracing follows the path of a sharp particle, while Fourier analysis is capable of following the life path of a blurred particle and include more of the influences of the environment in the analysis. It is sensible to expect that the advantages of Fourier analysis also hold for wave mechanics. As shown here, to a certain extent the path integral approach also makes use of Fourier analysis.

Relativity

Einstein's own explanation of the origin of relativity was: "*There is no logical way to the discovery of these elementary laws. There is only the way of intuition.*" Read more in: http://www.time.com/time/magazine/article/0,9171,878733,00.html#ixzz15NlhpWDu

Thus, Einstein never gave a proper explanation for the existence of special relativity. He just provided a set of formulas that work properly. He left us the choice of finding the origin of special relativity or otherwise to follow his intuition. Let us give it a try:

The position operator $\boldsymbol{Q}$ is modified by the unitary operators of the trail into another operator $\boldsymbol{Q}_t$ that has different eigenvectors and different eigenvalues.

$$\boldsymbol{Q}_{t+\Delta t} = U_t \boldsymbol{Q}_t U_t^\dagger \tag{1}$$

$$U_t \approx 1 + \Delta\boldsymbol{S}_t \tag{2}$$

$$U_t^\dagger \approx 1 - \Delta \boldsymbol{S}_t \tag{3}$$

$$\boldsymbol{Q}_{t+\Delta t} \approx \boldsymbol{Q}_t + [\Delta \boldsymbol{S}_t, \boldsymbol{Q}_t] \tag{4}$$

$$\Delta\langle \boldsymbol{q} \rangle = \langle \boldsymbol{Q}_{t+\Delta t} \rangle - \langle \boldsymbol{Q}_t \rangle \approx [\Delta \boldsymbol{S}_t, \boldsymbol{Q}_t] = 2\, \boldsymbol{Q}_t \times \Delta \boldsymbol{S}_t \tag{5}$$

This indicates that the step $\Delta\langle \boldsymbol{q} \rangle$ in the expectation value $\langle \boldsymbol{Q}_t \rangle$ of $\boldsymbol{Q}_t$ is perpendicular to both $\boldsymbol{Q}_t$ and $\Delta \boldsymbol{S}_t$. The steps $\Delta\langle \boldsymbol{q} \rangle$ and $\Delta \boldsymbol{S}_t$ form a right angular triangle with a hypotenuse: $c\, \Delta \boldsymbol{\tau}$, such that:

$$c\, \Delta \boldsymbol{\tau} = \Delta\langle \boldsymbol{q} \rangle + \Delta \boldsymbol{S}_t \tag{6}$$

With $\Delta \boldsymbol{\sigma} = \Delta \boldsymbol{S}_t / c$ the Minkowski signature of a new "observable" spacetime becomes visible.

$$\Delta \boldsymbol{\tau} = \frac{\Delta\langle \boldsymbol{q} \rangle}{c} + \Delta \boldsymbol{\sigma} \tag{7}$$

$$\Delta \boldsymbol{\sigma} = \Delta \boldsymbol{\tau} - \Delta\langle \boldsymbol{q} \rangle / c \tag{8}$$

$$|\Delta \boldsymbol{\sigma}|^2 = |\Delta \boldsymbol{\tau}|^2 - |\Delta\langle \boldsymbol{q} \rangle|^2 / c^2 \tag{9}$$

Thus, the analysis of what occurs during a single infinitesimal step gives us an indication how relativity enters physics. However, it asks for the introduction of a local notion of time $\boldsymbol{\tau}$ that differs considerably from the (global) progression parameter t. This new parameter is the **coordinate time** t_c.

$$\Delta \boldsymbol{\tau} = \boldsymbol{e}_{\boldsymbol{\tau}}\, \Delta t_c \tag{10}$$

Proper time

In relativity, **proper time** t_p is time measured by a single clock between events that occur at the same place as the clock. It depends not only on the events but also on the motion of the clock between the events. An accelerated clock will measure a proper time between two events that is shorter than the coordinate time measured by a non-accelerated (inertial) clock between the same events.

$$\left|\Delta t_p\right|^2 = |\Delta \boldsymbol{\tau}|^2 - |\Delta\langle \boldsymbol{q} \rangle|^2 / c^2 \tag{1}$$

$$\Delta \boldsymbol{\sigma} = \boldsymbol{e}_{\boldsymbol{\sigma}}\, \Delta t_p \tag{2}$$

Thus, proper time is, upon a proportionality factor, identical with our notion of progression parameter t.

Discussion

We have successfully introduced special relativity into our model.
By introducing relativity the way we did we played a few tricks.

- We neglect the real part of the position observable. It appears to play no part in dynamics.
- We shift from the global progression parameter t to the local coordinate time t_c.
- We shift from Hilbert space via quaternionic 1+3D coordinate space to observed space, thereby losing one dimension.
- We combine the resulting observed space with coordinate time into a Minkowski/Lorentzian space.

As a consequence

- We shift from 2^n-on/Riemannian space to Minkowski/Lorentzian space.
- Most physicists will use Clifford, Jordan and Grassmann algebras rather than 2^n-on algebras.
- With these algebras they can use complex analysis instead of the more complicated 2n-on analysis.
- But if they do so, they are confronted with unintuitive selection features.
- In the new space the quaternion waltz becomes an odd operation.
- Spinors can help in order to cope with these changes.

Can we do without relativity?

Yes.

- Skip coordinate time.
- Use clocks that measure the proper time.

However, you would have to fight existing conventions. Specifying causality might become a problem.

Speed along the live path

For the speed υ_{trail} along the action trail measured in coordinate time units holds:

$$\frac{ds}{dt_c} = v_{trail} = \sqrt{1 - (v_{Qspace}/c)^2} \tag{1}$$

Where

$$v_{Qspace} = dq/dt_c \tag{2}$$

is the speed measured in coordinate time units in the Q space along the observed life path of the item.

Inertia and progression step

The covering field represents the influence of the universe of all particles. According to the findings about inertia, the change $\Delta \boldsymbol{E}$ since the last progression step of the corresponding curvature field $\boldsymbol{E}$ determines the acceleration that a local particle senses during the current progression step.

This results in the acceleration $\frac{\partial \mathbf{v}}{\partial \mathrm{t}}$ of the particle.

$$\Delta \boldsymbol{E}(\boldsymbol{r}, t) = G \cdot \frac{\partial \mathbf{v}}{\partial \mathrm{t}}$$

Redefinition

If we want to use the Schrödinger picture, rather than the Heisenberg picture, then it is better not to use unitary transforms, because they change the operators by acting on the eigenvectors of the operators. Instead the subspace should be redefined without touching eigenvectors.

Let us suppose that there exists a dynamical equivalent of the traditional quantum logic. The equivalent of a move of a physical item in the lattice of propositions is a redefinition of a subset of the propositions. The redefinition occurs in terms of atomic predicates that describe the properties of the physical items. In the Hilbert space this corresponds with a redefinition of a relevant part of the Hilbert subspace in terms of the eigenvectors that belong to the new eigenvalues.

The redefinition concerns the Hilbert space which represents the current status quo. The step transforms the current version of the Hilbert space into a past version of the Hilbert space and it transforms a future version of the Hilbert space into the new current version. This is interesting in the light that a Hilbert field exists that controls the relation between the past, the current and the future versions of the Hilbert fields. For that reason we will call this special Hilbert field the **adventure field**. A transform that controls dynamics converts a future Hilbert space into the new current Hilbert spaces and it converts the current Hilbert spaces into a past Hilbert space. This transform will be called **progression transform**. The local blurs that characterize the adventure field form boundary conditions for the local transfer characteristics of the progression transform. Each item type is surrounded by a characteristic blur.

A progression transform that moves Hilbert subspaces without touching the eigenvectors of normal operators will be called a **redefiner**. The effect of the action of the redefiner on expectation values of operators must be similar to the effect of the trail of parallel unitary transforms treated in the previous paragraphs. While the set of parallel trails of unitary transforms act in the Heisenberg picture, the redefiner acts in the Schrödinger picture. As indicated earlier, the redefiner has an equivalent in the dynamic version of quantum logic.

In order to achieve the same effect as the Heisenberg picture, the Hilbert subspace redefiner must to a large degree have similar properties as the trails of parallel infinitesimal unitary transformations that are used to move the subspace in the Heisenberg picture. The redefinition keeps the inner products of vectors intact. Where unitary transforms rotate vectors around the origin of a Hilbert space, the redefiner takes subspaces of a potential future Hilbert space in order to redefine them into subspaces of the new current Hilbert space. In contrast to a unitary transform the redefiner does not change the eigenvectors of normal operators. Thus, it leaves the operators untouched. Like the trails of unitary transforms the redefinition works in infinitesimal steps. These infinitesimal actions also form

trails. In this way the manipulated subspace can move continuously through Hilbert space. Where the redefiners act on subspaces, the trails of unitary transforms redefine operators.

During this process the subspace may change its configuration. This may include a change that corresponds to the change of type definitions of atomic predicates. The redefiner steps from one stationary situation to the next. The Schrödinger picture conforms to the description with a redefiner. The result for the position of the locator must be the same as it was under the influence of the set of parallel infinitesimal unitary operators in the Heisenberg picture. The redefiner moves the subspace such that the new locator position is similar to the value as was established by the redefined position operator. It means that during the redefiner step the position of the locator undergoes an infinitesimal number transform that is equivalent to the infinitesimal transform that is established by the redefined position operator. That redefinition was caused by the parallel infinitesimal unitary transforms.

Trails

In fact the *t* step characterizes the redefinition step. The subsequent replacement of vectors and the replacement of the corresponding eigenvalues can be interpreted as a rather continuous movement of the corresponding characteristic subjects. Here we encountered ten different trails.

1. The trail of subsequent manipulators (infinitesimal unitary transforms or infinitesimal redefiners) that each perform an infinitesimal action.
2. The trail of subspaces, which with respect to the manipulators are characteristic for the considered item.
3. The trail of corresponding "action values" of the redefiner.
4. The trails of corresponding "action values" of the unitary transforms.
5. The trail of eigenvectors $|q_t\rangle$
6. The trail of corresponding observables Q_t.
7. The trail of corresponding observed expectation values q_t.
8. The trail of values $\psi(q_t)$ of a wave function.
9. This, on its turn corresponds to a trail of a state in coordinate space
10. And a trail of that state in Hilbert space.

Cycles

It is quite possible that subsequent steps are done in cycles of two or more steps. It is obvious that movements inside an item are cyclic. In ideal circumstances these movements are harmonic.

Redefiner

The concept of dynamic manipulator gives us reason to introduce a **new type of actuator**: the redefiner ℜ. This actuator moves subspaces, but leaves vectors untouched. It works in infinitesimal steps. It is easily interpreted as a function $\Re_t$ of the progression parameter *t*. Its scope spans the Hilbert space. The effect of each step on an item is similar to the effect of a set of parallel infinitesimal unitary transforms $\{U_{ts}\}_s$. The current "action value" of the redefiner is a number, which is close to unity. It is an "average" of the "actions values" of the parallel infinitesimal uniforms that are active in the same step. The redefiner accepts 2^n-ons as "action values".

The redefiner has an equivalent in a dynamic quantum logic, where it redefines propositions that concern the same objects as are represented by the closed subspaces of the Hilbert space that are moved by $\Re_t$. There seems to be no objection against the assumption that $\Re_t$ has a global scope. If we take that point of view, then the progression parameter t also has a global scope.

With this interpretation, the redefiner is a universe-wide stepper. It transforms the universe from one static situation to the next static situation. These static situations are governed both by traditional quantum logic and by the Helmholtz/Hodge decomposition theorems. After each step the status quo of subspaces and fields is reestablished. However, after the step the conditions have been changed. After each step the position of the physical item relative to the fields has changed, thus when the fields are not uniformly distributed, the item meets a different field configuration. On the other hand the fields represent the blurs of the individual items. Thus, when the position or the type of the item has changed, then the local configuration of the field has changed. This is the way that macroscopic dynamics takes place in quantum physics.

Equations of motion

Private continuity equation

Existence, transport, generation and annihilation of information carrying quanta is governed by a continuity equation. In short this equation runs:

Total change within V = flow into V + production inside V

This integral equation corresponds to differential equations in which how an information quantum density ρ_0, an information quantum current $\boldsymbol{\rho}$ and an information quantum source s will play a role. Particles act as sources and drains. Private fields represent the currents and the static density distributions. Wave functions are private fields that represent the situation in the direct environment of particles.

The rotation free part of the private field corresponds to the information probability amplitude distribution, whose squared modulus corresponds to the probability density. The transverse part of the private field corresponds to the information current probability amplitude distribution, whose squared modulus corresponds to the information current. Together the private fields form the covering field. The curvature field is derived from the decomposition properties of the covering field.

Particles

Fields are superpositions of probability amplitude distributions. These probability amplitude distributions are typical for corresponding particles and are attached to one or a small set of Hilbert vectors. The leptons are attached to three Hilbert vectors. Together they carry a centralized probability distribution that represents the influence of a single charge. The quarks are attached to two Hilbert vectors. Bosons are attached to a single Hilbert vector. However, bosons do not carry a charge. They are the messengers that transfer interactions.

Interactions

Probability amplitude distributions, which represent particles, move and rotate. That is interpreted as a movement / rotation of the corresponding item. Interactions may change the form of the probability amplitude distributions. Three types of change are discerned:

In strand theory the ***first Reidemeister move***, or ***type I move***, or ***twist***, is the addition or removal of a twist in a corresponding strand. In Hilbert space it involves the approach of a single Hilbert vector into the realm of a particle. The twist, is related to the electromagnetic interaction. Two twist directions are possible. The twists form an SU(1) group.

In strand theory the ***second Reidemeister move***, or ***type II move***, or ***poke***, is the addition or removal of a bend of one strand under (or over) a second strand. In Hilbert space it involves the interaction of two Hilbert vectors in the realm of a particle, where one Hilbert vector approaches the particle. The poke is related to electro-weak interaction. Three basic pokes exist. The pokes form an SU(2) group.

In strand theory the ***third Reidemeister move***, or ***type III move***, or ***slide***, is the displacement of one strand segment under (or over) the crossing of two other strands. In Hilbert space it involves the simultaneous interaction of three Hilbert vectors in the realm of a particle, where one Hilbert vector approaches the particle. The slide is related to electro-strong interaction. The slides form an SU(3) group.

Each Reidemeister move generates a single corresponding observable quant or annihilates a single potentially observable quant.

Schrödinger equation

When the spin has a constant direction:
The first term on the left side signifies the quantum generation rate per time step.
The second term indicates the influence of the electric field on this rate.
The first term on the right signifies the generation rate per path length.
The second term indicates the influence of the vector potential on this rate.
The square dependence indicates the increasing alignment of spin with the movement.

$$(\hbar\omega - q\tilde{V})\tilde{\psi}(x,t) = (\hbar\boldsymbol{k} - q\tilde{\boldsymbol{A}})\tilde{\psi}(x,t) \tag{1}$$

$$(i\hbar\partial_t - qV)\psi = \frac{1}{2m}(-i\hbar\boldsymbol{\nabla} - q\boldsymbol{A})^2\psi \tag{2}$$

Pauli equation

When the spin has no constant direction:
The density $\rho(x,t)$ and the Euler angles α, β, γ define the Pauli equation:

$$\Psi(x,t) = \sqrt{\rho}\, e^{\frac{i\alpha}{2}} \begin{bmatrix} \cos\left(\frac{\beta}{2}\right) e^{\frac{i\gamma}{2}} \\ i\sin\left(\frac{\beta}{2}\right) e^{-\frac{i\gamma}{2}} \end{bmatrix} \tag{1}$$

Due to the half angles, the two-component matrix is not a vector, but a spinor.

$$\boldsymbol{\sigma} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) \tag{2}$$

$$1 \mapsto I, \qquad \boldsymbol{i} \mapsto i\,\sigma_1, \qquad \boldsymbol{j} \mapsto i\,\sigma_2, \qquad \boldsymbol{k} \mapsto i\,\sigma_3 \tag{3}$$

Pauli's equation for the evolution of a free quantum particle with spin ½ is:

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}(\boldsymbol{\sigma}\boldsymbol{\nabla})^2\Psi \tag{4}$$

$$(i\hbar\partial_t - qV)\Psi = \frac{1}{2m}(-i\hbar\boldsymbol{\nabla} - q\boldsymbol{A})^2\Psi - \frac{q\hbar}{2m}\boldsymbol{\sigma}\boldsymbol{B}\Psi \tag{5}$$

The last term shows the influence of spin.

Dirac equation

The final and most detailed description of elementary fermions, the Dirac equation, results from combining all three ingredients:

1. the relation between the quantum of action and the phase of the wave function,
2. the relativistic mass–energy relation,
3. spin 1/2.

$$i\frac{\partial\psi}{\partial t} = (\boldsymbol{\alpha}\boldsymbol{\nabla} \pm i\gamma_5 m)\psi \tag{1}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{2}$$

$$\gamma_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3}$$

$$\psi = \sqrt{\rho}\, e^{i\delta}\, L(\boldsymbol{v}) R\left(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}\right) \tag{4}$$

$\rho(\boldsymbol{r}, t)$ is the probability density.
δ is a phase which represents the relative importance of particle and antiparticle density.
$\alpha, \beta \; and \; \gamma$ are Euler angles. They describe the average local orientation and phase of the spin axis. (this defines a rotating spin vector)
$\boldsymbol{v}(r, t)$ is the average local Lorentz boost.
LR is an abbreviation for the boosted and rotated unit spinor.(quantum)

The probability amplitude ψ moves and rotates and individually the quanta carry position, momentum and angular momentum (including spin) information.

Fields

It is clear that the physical fields play an important role in nature. They form an indispensible ingredient in the establishment of dynamics. Each physical item follows a path through a set of universe wide fields. The static gravitational field, the electrostatic field and the electromagnetic field are all subjected to the Helmholtz decomposition theorem. The difference between the gravitational field and the electromagnetic field is that the masses are non-negative and the electric charges are, apart from a sign, always the same. The gravitation field can be seen as being derived from the curvature set by the decomposition of the covering field. The covering field is the superposition of all fields but the gravitation field. When the path with respect to the gravitation field corresponds to a unit speed curve then that field executes no action onto that item.

More fields

There exists a list of fields with shorter ranges than the range of the gravitation field and the range of the electromagnetic fields. The electro-weak field and the electro-strong field are not treated here in detail.
The action represented by a complete Lagrangian indicates how fields appear in the argument of a manipulator. See Lagrangian of the world for a complete survey of terms. Mendel Sachs has found a way to bring all terms under the same hood.

Lagrangian

$$\mathcal{L} = \bar{\varphi}(i\hbar c \not{D} - mc^2)\varphi - \frac{1}{4\mu_0} F_{\mu\upsilon} F^{\mu\upsilon} - ¼ \sum_{a=1}^{3} W_{\mu\upsilon}^{a} W_{a}^{\mu\upsilon} - ¼ \sum_{a=1}^{8} G_{\mu\upsilon}^{a} G_{a}^{\mu\upsilon}$$

Where $\mu, \upsilon = 1, 2, 3$
The first term concerns the affected particle.
The second term concerns electromagnetic interactions. Reidemeister twists. SU(1).
The third term concerns unbroken weak interactions. Reidemeister moves. SU(2).
The fourth term concerns unbroken strong interactions. Reidemeister slides. SU(3).

$$\not{D} = \gamma^{\sigma} D_{\sigma} = \gamma^{\sigma}(\partial_{\sigma} - iqA_{\sigma})$$

$$F_{\mu\upsilon} = \partial_{\mu} A_{\upsilon} - \partial_{\upsilon} A_{\mu}$$

$$W_{\mu\upsilon}^{a} = \partial_{\mu} W_{\upsilon}^{a} - \partial_{\upsilon} W_{\mu}^{a} - g\, f^{abc}\, W_{\mu}^{b} W_{\upsilon}^{c}$$

$$G_{\mu\upsilon}^{a} = \partial_{\mu} G_{\upsilon}^{a} - \partial_{\upsilon} G_{\mu}^{a} - g\, f^{abc}\, G_{\mu}^{b} G_{\upsilon}^{c}$$

φ is the private field of the affected particle.
A_{υ}, W_{υ}^{a} and W_{υ}^{a} are vector potentials of the corresponding subfields
g is the gauge coupling constant. The quantity f^{abc} is the structure constant of the gauge group.

Path through field

The text in this section is borrowed from Wikipedia.
In a Riemannian manifold M with metric tensor g, the length of a continuously differentiable curve $\gamma\colon [a, b] \to M$ is defined by

$$L(\gamma) = \int_{a}^{b} \sqrt{g_{\gamma(t)(\dot{\gamma}(t),\dot{\gamma}(t))}}\, dt \tag{1}$$

The distance $d(p, q)$ between two points p and q of M is defined as the infimum of the length taken over all continuous, piecewise continuously differentiable curves $\gamma\colon [a, b] \to M$ such that $\gamma(a) = p$ and $\gamma(b) = q$. With this definition of distance, geodesics in a Riemannian manifold are then the locally distance-minimizing paths, in the above sense.
The minimizing curves of L in a small enough open set of M can be obtained by techniques of calculus of variations. Typically, one introduces the following action or energy functional

$$E(\gamma) = ½ \int_{a}^{b} g_{\gamma(t)(\dot{\gamma}(t),\dot{\gamma}(t))}\, dt \tag{2}$$

It is then enough to minimize the functional E, owing to the Cauchy–Schwarz inequality

$$L(\gamma)^2 \leq 2(b-a)\, E(\gamma) \tag{3}$$

with equality if and only if $|d\gamma/dt|$ is constant.
The Euler–Lagrange equations of motion for the functional E are then given in local coordinates by

$$\frac{d^2 x^{\lambda}}{dt^2} + \Gamma_{\mu\nu}^{\lambda} \cdot \frac{dx^{\mu}}{dt} \cdot \frac{dx^{\nu}}{dt} = 0 \tag{4}$$

where $\Gamma_{\mu\nu}^{\lambda}$ are the Christoffel symbols of the metric. This is the **geodesic equation**.

Calculus of variations

Techniques of the classical calculus of variations can be applied to examine the energy functional E. The first variation of energy is defined in local coordinates by

$$\delta E(\gamma)(\varphi) = \frac{\partial}{\partial t}\Big|_{t=0} E(\gamma + t\,\varphi) \tag{1}$$

The critical points of the first variation are precisely the geodesics. The second variation is defined by

$$\delta^2 E(\gamma)(\varphi,\psi) = \frac{\partial^2}{\partial t^2}\bigg|_{t=0} E(\gamma + t\,\varphi + s\psi) \tag{2}$$

In an appropriate sense, zeros of the second variation along a geodesic γ arise along Jacobi fields. Jacobi fields are thus regarded as variations through geodesics.
By applying variational techniques from classical mechanics, one can also regard geodesics as Hamiltonian flows. They are solutions of the associated Hamilton–Jacobi equations, with (pseudo-)Riemannian metric taken as Hamiltonian.

Affine geometry

A **geodesic** on a smooth manifold M with an affine connection ∇ is defined as a curve $\boldsymbol{\gamma}(t)$ such that parallel transport along the curve preserves the tangent vector to the curve, so

$$\nabla_{\dot{\boldsymbol{\gamma}}}\dot{\boldsymbol{\gamma}}(t) = 0 \tag{1}$$

at each point along the curve, where $\dot{\boldsymbol{\gamma}}$ is the derivative with respect to t. More precisely, in order to define the covariant derivative of $\dot{\boldsymbol{\gamma}}$ it is necessary first to extend $\dot{\boldsymbol{\gamma}}$ to a continuously differentiable imaginary Hilbert field in an open set. However, the resulting value of the equation is independent of the choice of extension.
Using local coordinates on M, we can write the **geodesic equation** (using the summation convention) as

$$\frac{d^2x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu}\cdot\frac{dx^\mu}{dt}\cdot\frac{dx^\nu}{dt} = 0 \tag{2}$$

where $x^\mu(t)$ are the coordinates of the curve $\boldsymbol{\gamma}(t)$ and $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols of the connection ∇. This is just an ordinary differential equation for the coordinates. It has a unique solution, given an initial position and an initial velocity.
From the point of view of classical mechanics, geodesics can be thought of as trajectories of free particles in a manifold. Indeed, the equation $\nabla_{\dot{\boldsymbol{\gamma}}}\dot{\boldsymbol{\gamma}}(t) = 0$ means that the acceleration of the curve has no components in the direction of the surface (and therefore it is perpendicular to the tangent plane of the surface at each point of the curve). So, the motion is completely determined by the bending of the surface. This is also the idea of the general relativity where particles move on geodesics and the bending is caused by the gravity.

Christoffel symbols

If x^i, $i = 1,2,\ldots,n$, is a local coordinate system on a manifold M, then the tangent vectors

$$e_\mu = \frac{\partial}{\partial x_\mu}, \qquad \mu = 1, 2, \ldots, n \tag{1}$$

define a basis of the tangent space of M at each point. The Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ are defined as the unique coefficients such that the equation

$$\nabla_\mu \, e_\nu \; = \; \Gamma^{\lambda}_{\mu\nu} \cdot e_\lambda \tag{2}$$

holds, where ∇_μ is the Levi-Civita connection on M taken in the coordinate direction e_μ.
The Christoffel symbols can be derived from the vanishing of the covariant derivative of the metric tensor g_{ik}:

$$0 \; = \; \nabla_\lambda \, g_{\mu\nu} \; = \frac{\partial g_{\mu\nu}}{\partial x_\lambda} - g_{\eta\mu} \cdot \Gamma^{\eta}_{\mu\lambda} - \, g_{\mu\eta} \cdot \Gamma^{\eta}_{\nu\lambda} \tag{3}$$

By permuting the indices, and re-summing, one can solve explicitly for the Christoffel symbols as a function of the metric tensor:

$$\Gamma^{\mu}_{\nu\lambda} \; = \; ½ \cdot g^{\mu\nu} \cdot (\frac{\partial g_{\eta\nu}}{\partial x^\lambda} + \frac{\partial g_{\eta\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\eta}) \tag{4}$$

where the matrix $(g^{\mu\nu})$ is an inverse of the matrix $(g_{\mu\nu})$, defined as (using the Kronecker delta, and Einstein notation for summation)

$$g^{\lambda\mu} \cdot g_{\mu\nu} = \delta^{\lambda}_{\nu} \tag{5}$$

Although the Christoffel symbols are written in the same notation as tensors with index notation, they are **not** tensors, since they do not transform like tensors under a change of coordinates.
Under a change of variable from $(x^1, \ldots, x^n)$ to $(y^1, \ldots, y^n)$, vectors transform as

$$\frac{\partial}{\partial y^i} \; = \; \frac{\partial x^k}{\partial y^i} \cdot \frac{\partial}{\partial x^k} \tag{6}$$

and so

$$\Gamma^{k}_{ij} \; = \frac{\partial x^p}{\partial y^i} \cdot \frac{\partial x^q}{\partial y^j} \cdot \Gamma^{r}_{pq} \cdot \frac{\partial y^k}{\partial x^r} + \frac{\partial y^k}{\partial x^m} \cdot \frac{\partial^2 x^m}{\partial y^i \partial y^j} \tag{7}$$

where the underline denotes the Christoffel symbols in the y coordinate frame. Note that the Christoffel symbol does **not** transform as a tensor, but rather as an object in the jet bundle.
At each point, there exist coordinate systems in which the Christoffel symbols vanish at the point. These are called (geodesic) normal coordinates, and are often used in Riemannian geometry.
The Christoffel symbols are most typically defined in a coordinate basis, which is the convention followed here. However, the Christoffel symbols can also be defined in an arbitrary basis of tangent vectors e_μ by

$$\nabla_{e_\mu} e_\nu \; = \; \Gamma^{\lambda}_{\mu\nu} \cdot e_\lambda \tag{8}$$

The action along the live path

The integrated action S_{ab} is performed over a distance along the action trail or equivalently over a period of coordination time

$$S_{ab} = -\int_a^b m \cdot c^2 \cdot ds + matter\ terms \tag{1}$$

$$= -\int_{\tau_a}^{\tau_b} m \cdot c^2 \cdot \sqrt{1 - \left(\frac{v}{c}\right)^2} \cdot d\tau + matter\ terms$$

$$= \int_{\tau_a}^{\tau_b} \mathcal{L} \cdot d\tau$$

m is the mass of the considered item.
v is the speed in Q space.
$\mathcal{L}$ is the Lagrangian.

The first line of this formula can be considered as an integral along the trail in coordinate space or equivalently over the trail in Hilbert space. The next lines concern integrals over the corresponding path in observed space combined with coordinate time. It must be noticed that these spaces have different signature.

$$\mathcal{L} = -m \cdot c^2 \cdot \frac{ds}{d\tau} + \text{matter terms} \tag{2}$$

In general relativity, the first term generalizes (includes) both the classical kinetic energy and interaction with the Newtonian gravitational potential. It becomes:

$$m \cdot c^2 \cdot \frac{ds}{d\tau} = -m \cdot c \cdot \sqrt{g_{\alpha\beta} \cdot \dot{q}_\alpha \cdot \dot{q}_\beta} \tag{3}$$

$g_{\alpha\beta}$ is the rank 2 symmetric metric tensor which is also the gravitational potential. Notice that a factor of c has been absorbed into the square root.
The matter terms in the Lagrangian $\mathcal{L}$ differ from those in the integrated action S_{ab}.

$$S_{ab\_matter} = -\int_a^b e \cdot A_\gamma \cdot dq^\gamma + \text{other matter terms} \tag{4}$$

The matter term in the Lagrangian due to the presence of an electromagnetic field is given by:

$$\mathcal{L} = -m \cdot c^2 \cdot \frac{ds}{d\tau} + e \cdot \dot{q^\gamma} \cdot A_\gamma + \text{other matter terms} \tag{5}$$

A_γ is the electromagnetic 4-vector potential.

Optics

The optical Fourier transform (OTF) is an objective imaging quality characteristic for imaging devices in a similar way as the frequency transfer function qualifies the signal transfer function of a linearly operating electronic device. The transfer quality of a chain of linear signal transforming devices is characterized by the product of the frequency transfer functions of the elements of the chain. In a similar way the OTF of a chain of imaging devices is given by the product of the OTF's of the elements of the chain. However, this is a profound simplification of reality. The product rule only holds when the transfer characteristics of the imaging devices are spatially uniform over the complete input field of the separate imaging components. Further, the conditions in which the OTF's of the components are determined must be similar to the conditions in the chain. More in detail, this means that the angular distribution, the chromatic distribution and the homogeneity of the radiation must be identical.

In optics, the image sided spread function equals the convolution of the object sided spread function and the point spread function (PSF, the image of a point). The Fourier transform of the image sided spread function is equal to the product of the Fourier transform of the object sided spread function and the optical Fourier transforms (OTF's) of the imaging devices. When several imaging devices work in sequence, then the total optical transfer function of the imaging system equals the product of the transfer functions of the components.

If we restrict to a static situation and include the "depth" of the image, the static PSF is a three parametric function. Thus the OTF must have the same number of parameters. Like the PSF the three dimensional OTF has a longitudinal component and a two dimensional transverse component. In most cases only the transverse component is used as an imaging quality characteristic. On-axis the transverse component is rotationally symmetric. Off-axis its modulus, the MTF, is symmetric but not rotationally symmetric. On-axis the MTF is presented as a one parametric curve in which only the positive axis is given. Off-axis the two extremes of the MTF are given. They correspond to radial and tangential directions.

Due to the fact that Hilbert fields are blurred Hilbert distributions, wave mechanics has much in common with wave optics. For each compact normal operator the Hilbert subspace that represents a physical item corresponds to a spread in Hilbert space and a corresponding spread in the eigenspaces of that normal operator. The distribution of this spread is represented in a wave function, or more correctly, in a probability distribution. For example the wave function that has the position as a variable corresponds to the triple consisting of a physical item, its Hilbert subspace representation and the position operator.

After a move of a physical item its position related wave function has much in common with the spread function that characterizes the blur of the image sided pictures in a linear operating imaging system. The physical fields that influence the physical item have an equivalent in the chain of imaging devices that transfers the image.

The product formula for the transfer functions relies on several preconditions. First of all it relies on the fulfillment of the requirement for sufficient spatial uniformity of the transfer. At all places where information is passed, the transfer characteristics must be sufficiently

identical. The product formula has only validity in the spatial area where this requirement is fulfilled.

The transfer characteristics will be different for each Fourier component. Their quality will reduce with higher spatial frequencies.

The final result can be computed in longitudinal direction by multiplication. In lateral direction these regions are restricted to areas where the transfer is locally sufficiently uniform. In the summation that is used to compute a sensible average the angular and chromatic distribution of the transferred information play a role. These distributions determine the summation coefficients. The extent of the region in which the considered transfer function is considered valid depends on the accuracy that is required for the result of the computation. Sign selections inside the radiation determine the polarization. Often in optics this feature and its influence is ignored. Coherence plays a role as well, but in practice optical imaging uses either nearly completely incoherent light or nearly completely coherent light.

In wave mechanics the wave function, which is taken just before the item moves, gets the role of the object. After a movement through a region of the fields the wave function has been changed. Its Fourier transform then equals the product of the Fourier transform of the original wave function and the wave transfer functions (WTF's) of the fields that influence the item. If several steps are taken in sequence, then the transfer functions of the passed field pieces must be multiplied in order to get the overall result. This transfer is affected in a similar way by spatial non-uniformity as the optical case.

In cylindrical imaging systems Seidel aberrations take their toll. When the system is folded or when lenses are not perfectly in line, also non-cylindrical influences will influence the imaging quality. The measurement and the specification of the OTF must cope with the spatial non-uniformity of the imaging characteristics of the imaging devices and with the angular and chromatic distribution of the radiation. The OTF also depends on the longitudinal location of the object and where the image is detected. This also occurs with the WTF of physical fields. Both in optics and in wave mechanics the precise locations of the "object" and the "image" are often not well determined. They are defined by spatial distributions in three dimensions. In both cases the angular and chromatic distributions of the contributing radiation influence the transfer. The final result is constituted by the weighted sum of all contributions.

With inhomogeneous (= incoherent) imaging the phases are ignored. These facts indicate the difference between the particle view and the wave view. From optics it is known that the modulation transfer function (MTF) is a proper imaging qualifier for inhomogeneous light imaging. In inhomogeneous imaging the imaging process can be properly described by ray tracing. Ray tracing has much similarities with the application of the path integral. However, ray tracing normally does not use arbitrary paths. In inhomogeneous imaging phases are scrambled. For holographic imaging the phase transfer function (PTF) or the whole OTF is the better measure. With holographic imaging the phases carry the depth information. Feynman's path integral can cover arbitrary paths because, according to Feynman's claim,

interference via the phases eliminates the contributions of non-realistic paths. That is why in the path integral the angular distribution of the radiation plays no role.

In optics the image space is often a surface. In optics the OTF depends on the position in the object space. Off axis the OTF is not rotationally symmetric. The OTF also depends on the angular distribution and the chromatic distribution of the radiation. These dependencies also hold for the WTF in wave mechanics.

A longitudinal displacement of the image spread function with respect to the object spread function corresponds to an extra phase term in the longitudinal component of the Fourier transform of the image spread function. A lateral displacement corresponds to an extra phase term in the transverse component of the Fourier transform. In wave mechanics this holds for the respective components of the Fourier transform of the wave function after the move.

The resemblance between optics and wave mechanics becomes striking when the discrete lens pack is replaced by a medium with a continuously varying refraction. In optics this happens with electron optical lenses that are used in imaging with charged particles.

When the point spread function is a function of three-dimensional position, then the OTF is also a three-parametric function of spatial frequency. The MTF is a symmetric function. However, the MTF is not rotationally symmetric (in 2D) or spherical symmetric (in 3D). On its vertical axis the MTF indicates the part of the energy of the radiation that is transferred by a given spatial frequency.

Veiling glare and halo

Due to reflections on refracting surfaces some of the energy of the radiation loses much of its spatial information content. As a consequence the MTF shows a sharp peak near zero spatial frequency. This phenomenon is called **veiling glare**. When the drop is not so fast the phenomenon is called **halo**.

Equivalents of veiling glare and halo can also occur in wave mechanics. In this way spurious radiation and a spurious halo can enter space. This can happen in the form of energy or in the form of matter. Spurious radiation contains no spatial information.

Strands

Schiller's strands

A strand is a notion introduced by Christoph Schiller. Here I try to embed the notion of a strand into the context of a sequence of Hilbert spaces or similarly into the context of a sequence of traditional quantum logics. The strand interpretation used in this e-paper is a mixture of the interpretation of Christoph Schiller and my personal interpretation. Any difference with the original interpretation is for my account. The reader must take Schiller's interpretation as the most fundamental and the most original. See also braid theory.

The fundamental principle of the strand theory is:

> ***Planck units*** are defined through crossing switches of strands.

An alternative fundamental postulate is:

> An ***event*** is the switch of a crossing between two strand segments.

The original interpretation of strands can be found in Motion Mountain, volume VI. In Schiller's words:

> ***Strands*** are one-dimensional curves in three-dimensional space that reach the border of space.
> Strands exist in relation to the mentioned 3D background space.
> Strands are fluctuating.

In my interpretation:

> Strands are the equivalent of a particular kind of elementary type definition. The photon is the simplest boson.
>
> Strands are defined in relation to a Hilbert GPS operator that resides in rigged Hilbert space. That operator has a canonical conjugate which is the Hilbert GMS operator. In this e-paper strands are elements in the eigenspace of a special strand operator. The reason of the better suitability of the strand operator is the fact that the granularity of a GPS like coordinate operator in separable Hilbert space would cause unnatural preferred directions. The strand operator does not generate those anomalies.
>
> The one-dimensional curve fluctuates under the influence of the fields in its surround. Fluctuation of a strand becomes apparent when the strand is followed over a sequence of Hilbert spaces. Schiller uses the reverse picture: in his strand theory the strand fluctuations cause the fields. This interpretation means that the strand tries to follow a direction in which the surrounding Hilbert field is rotation free. Hilbert fields can reach the borders of space.
>
> A crossing switch event may correspond to a creation/annihilation operator. The GPS and GMS related parts of this operator deliver the information that the created quanta contain.

Simple elementary types are bosons. All tangles made of one strand are elementary particles of spin 1, thus, they are elementary vector bosons. Conversely, all elementary spin-1 particles are made of one strand, because other tangles do not reproduce the spin-1 behavior under rotations: only one-stranded tangles return to the original strand after a core rotation by 2π.

According to the strand model no Higgs boson are required.

The strand model predicts that apart from the six quarks and the graviton, no other two-stranded elementary particles exist in nature.

Fermions correspond with triples of tangled strands. The strand model predicts that apart from the six leptons, no other elementary particles made of three strands exist in nature.

More complex types are composed of the above mentioned elementary types.

> **The strand events become observable as quanta that have a position, a direction and a handedness (chirality).**

The clouds of quanta that belong to a free particular elementary particle type are characterized via a probability amplitude distribution. When a quaternionic function is used for this function, then it can relate probability with position direction and chirality.

Interaction is caused by one of three processes.

- The first process involves a single strand. It corresponds with normal electromagnetic interaction. It is characterized by the first Reidemeister move.
- The second process involves two strands. It corresponds with the electro-weak force. It is characterized by the second Reidemeister move.
- The third process involves three strands. It corresponds with the electro-strong force. It is characterized by the third Reidemeister move.

These interactions play in the direct environment of strand cores. According to Schiller, gravitational forces have their origin in the tails, away of the cores. That is also the region where masses get their influence.

Strands and their fluctuations are *unobservable*. The only things that become observable from a strand are its crossing switches with itself or with other strands. Drawing strands makes little sense because its dynamic behavior takes simultaneously place in two completely different spaces: in configuration space and in Fourier space. Still drawings are made in order to clarify strand behavior. In that case strands are pictured in 3D space and the rotations are represented by rotating cores or knots.

The **tangle function** – the (short) time average of strand crossings – corresponds with a complex **probability amplitude distribution**.

The crossing switch events constitute a **quantum cloud**, where each event is represented by a time interval, a location and a direction. The quaternionic probability amplitude distribution that describes this cloud is used to define the **blur** on which a Hilbert field is based.

The strand theory does not say anything about the transfer of information to the quanta.

Strands and their fluctuations are *unobservable*. The only things that become observable from a strand are its crossing switches with itself or with other strands. Drawings are made in order to clarify strand behavior. In that case strands are pictured in 3D space and the rotations are represented by rotating cores or knots.

The **tangle function** – the (short) time average of strand crossings – corresponds with a complex **probability amplitude distribution**.

The crossing switch events constitute a **quantum cloud**, where each event is represented by a time interval, a location and a direction. The (quaternionic) probability amplitude distribution that describes this cloud is similar to the probability amplitude distribution that is used to define the **blur** on which a Hilbert field is based.

Planck values

Schiller: Up to a numerical factor, the limit for every physical observable corresponds to the Planck value. (The limit values are deduced from the commonly used Planck values simply by substituting 4G for G.) These limit values are the true natural units of nature. In fact, the ideal case would be to redefine the usual Planck values for all observables to these extreme values, by absorbing the numerical factor 4 into the respective definitions. In the following, we call the limit values the corrected Planck units and assume that the factors have been properly included.

Strand basics

A **crossing** between two strands has a position and a direction. It is the position where the distance between the strands has a minimum. The distance is measured in terms of the potential eigenvalues of the GPS operator. The distance is measured in Planck length units.

A **crossing switch** is a turn of the crossing over π radians. Via its infinitesimal geometry the crossing switch defines the action $\hbar/2$, the Planck length l_{Pl}, the Planck time t_{Pl} and the Boltzmann constant k.

Events are observable crossing switches of unobservable strands. Every event in nature is characterized by the (corrected) Planck time, the (corrected) Planck length, the Planck entropy, i.e., the Boltzmann constant k, and Planck's quantum of action $\hbar$ (for a full turn)

The ***distance*** between two particles is the maximum number of crossing switches that could appear between them. Length measurement is thus defined as counting Planck lengths.

The ***time interval*** between two events is the maximum number of crossing switches that could appear between them. Time measurement is thus defined as counting Planck times.

The physical ***action*** of a physical system evolving from an initial to a final state is the number of crossing switches that take place. Action measurement is thus defined as counting crossing switches. Physical action is thus a measure for the change that a system undergoes.

The ***entropy*** of any physical system is related to the total number of crossing switches that are possible. Entropy measurement is thus defined through the counting of potential crossing switches.

The strand model thus states that any large physical system – be it made of matter, radiation, empty space or horizons – has entropy.

Strand table

Typical strand configurations:

Physical system	Strands	Tangle type
Vacuum	many infinite unknotted strands	unlinked
Dark energy	many fluctuating infinite strands	unlinked
Elementary vector boson	one infinite strand	knotted or unknotted curve
Quark	two infinite strands	rational tangle
Lepton	three infinite strands	braided tangle
Meson,baryon	three or more infinite strands	rational tangle
Higher-order propagating fermion	two or more infinite strands	locally knotted or prime tangle
Virtual particles	open, unlinked and closed strands	trivial tangles, knots, links
Composed systems	many strands	separable tangles
Graviton	two infinite twisted strands	specific rational tangle
Gravity wave	many infinite twisted strands	many graviton tangles
Horizon	many tightly woven infinite strands	web-like tangle
Earliest form of the universe	Single closed strand	No tangles

The most elementary strand conforms to the type of an elementary vector boson.

See the Toolkit for more details on strands.

Unique aspects of the model

Fundament

- The model takes as basis the axioms of traditional quantum logic.
- It exploits the isomorphism between the set of propositions in this logic and the set of closed subspaces of an infinite dimensional separable Hilbert space $\mathbf{H}$, in which the inner product is defined over the division ring of the quaternions.
- A proposition that treats everything that can be said about a physical item represents that item. Thus, the model represents physical items.
- Traditional quantum logic and its partner the separable Hilbert space $\mathbf{H}$, cannot represent physical fields and they cannot represent dynamics.
- However, this basic model can be extended such that fields are attached to it. However, this extended model only represents a static status quo.
- A sequence of such extended models can represent dynamics.
- The separable Hilbert space $\mathbf{H}$, does not contain a useable GPS coordinate operator. Due to the granularity of its eigenspace, such a normal operator would introduce preferred directions in the imaginary part of that eigenspace.
- Instead the corresponding continuous GPS operator that resides in the corresponding rigged Hilbert space $\mathbf{\mathbb{H}}$ can act as a background coordinate operator. Its eigenspace can be used to indicate the location of the field values. However, this operator cannot be used to locate the Hilbert vectors that represent particles.
- Instead a special normal operator whose eigenspace contains a set of freely located chains of granules can deliver the position observables. This operator is a strand operator.
- In each chain one granule represents the current position. It divides the chain in a past part and a future part.
- A probability amplitude distribution takes care of the smoothness in the surround of the current granule. This attachment extends the separable Hilbert space.
- Particles are represented by a single Hilbert vector or by a small set of Hilbert vectors. These vectors are eigenvectors of the strand operator and are blurred by a spread function that can be interpreted as a probability amplitude distribution. The blur of the set of Hilbert vectors represents the private field of the particle and describes the cloud of quanta that carry the observable information about the particle.
- The particle acts as the source or as the drain of these quanta. The cloud moves and rotates around a rotation axis.
- The superposition of all private fields constitutes a covering field.
- For a given coordinate system the static decomposition of the covering field into a rotation free part and one or two divergence free parts runs along curved lines. The local curvature value can be used to define a derived partner field of the covering field. This curvature field has all the characteristics of the gravitation field.

- The private fields of bosons are attached to a single unit size Hilbert vector and touch all other unit size Hilbert vectors.
- The private fields of quarks are attached to a pair of unit size Hilbert vectors and touch all other unit size Hilbert vectors.
- The private fields of leptons are attached to a triple of unit size Hilbert vectors and touch all other unit size Hilbert vectors.
- In interactions bosons take care of the transfer.
- Each electromagnetic interaction involves only one extra Hilbert vector.
- Each weak interaction involves an internal and an extra Hilbert vector.
- Each strong interaction involves two internal and one extra Hilbert vector.
- There are no more elementary kinds of interactions.
- The progression parameter that counts the subsequent Hilbert spaces is not our common notion of time, but it has certainly some relation with it.

Insights

- The Minkowski signature of spacetime must have its explanation in what occurs during a progression step.
- The Minkowski signature of spacetime forbids that coordinate time acts as the fourth dimension that goes together with 3D coordinate space.
- Momentum acts as the precondition of the next displacement step.
- The fourth dimension must be as granular as the 3D displacement.
- Fields act as the precondition for the next action step.
- The action step may function as the fourth dimension.
- The displacement, measured in Planck length units, the progression step measured in Planck time units, the action step measured in Planck constant sized units and the entropy step in Boltzmann constant sized units form the basic steps during an observable event.
- Action represents change. Entropy represents potential change.

Thoughts

The following texts represent collections of thoughts that still have to be brought in proper order and in mutual consistency.

Harmonic oscillating Hilbert field

Take the ingredients of the complex harmonic oscillator and interpret these as similar ingredients of a harmonic oscillating Hilbert field that is based on a Gaussian blur. The blur delivers the conditions of the ground state.

$$\psi_0(r) = \sqrt[4]{\frac{m\,\omega}{\pi\hbar}}\; e^{-\frac{m\,\omega}{2\hbar}r^2} \tag{1}$$

This means that the ground state corresponds with a Gaussian charge distribution. Higher states correspond to a blurred current. We indicate this current as vector potential $\boldsymbol{\varphi}$. Its time derivative $\dot{\boldsymbol{\varphi}}$ is perpendicular to $\boldsymbol{\varphi}$. The other ingredients are P, Q, $\mathcal{A}$ and $\mathcal{A}^\dagger$.

$$Q \Leftrightarrow \boldsymbol{\varphi} = \varphi_x = \sqrt{\frac{\hbar}{2m\omega}}(\mathcal{A}_x + \mathcal{A}_x^\dagger) \tag{2}$$

$$\mathcal{A}_x - \mathcal{A}_x^\dagger = 0 \tag{3}$$

$$P \Leftrightarrow m\dot{\boldsymbol{\varphi}} = m\dot{\varphi}_y = \sqrt{\frac{m\omega\hbar}{2}}(-\mathcal{A}_y + \mathcal{A}_y^\dagger) \tag{4}$$

$$\mathcal{A}_y + \mathcal{A}_y^\dagger = 0 \tag{5}$$

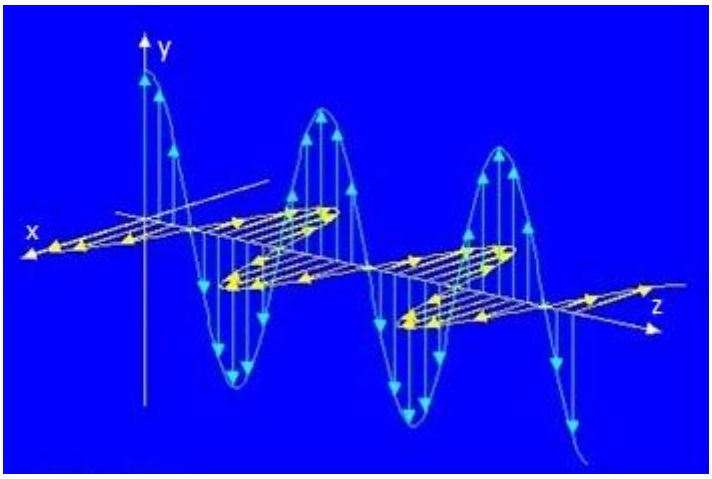
$$\mathcal{A} \Leftrightarrow \boldsymbol{\mathcal{A}} = \boldsymbol{i}\mathcal{A}_x - \boldsymbol{ik}\mathcal{A}_y = \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{\varphi} + \frac{\dot{\boldsymbol{\varphi}}}{\omega}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{i}\varphi_x + \boldsymbol{ik}\frac{\dot{\varphi}_y}{\omega}\right) \tag{6}$$

$$\mathcal{A}^\dagger \Leftrightarrow \boldsymbol{\mathcal{A}}^\dagger = \boldsymbol{i}\mathcal{A}_x^\dagger + \boldsymbol{ik}\mathcal{A}_y^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{\varphi} - \frac{\dot{\boldsymbol{\varphi}}}{\omega}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{i}\varphi_x - \boldsymbol{ik}\frac{\dot{\varphi}_y}{\omega}\right) \tag{7}$$

The $\boldsymbol{\varphi}$ field and the $\dot{\boldsymbol{\varphi}}$ field are mutually perpendicular. If both fields are subjected to a synchronized quantum harmonic oscillation, then an oscillating wave results. We take the same ground state for each of the fields. These ground states correspond to a spherical symmetric Gaussian blur.

When bounds of the cavity are removed or relaxed, then the higher order modes may differ in a phase shift. The sign selections set the eigenvalues of the spin operator. The result is an elliptically polarized wave that moves in directions along $\boldsymbol{\varphi} \times \dot{\boldsymbol{\varphi}}$.

$\boldsymbol{\varphi}$ no longer stands for a single position, but instead for a Gaussian distribution of positions. Similarly $\dot{\boldsymbol{\varphi}}$ does not stand for a single moving particle, but for a moving Gaussian cloud of virtual particles.



Annihilator and creator

The annihilator $\boldsymbol{\mathcal{A}}$ and the creator $\boldsymbol{\mathcal{A}}^{\dagger}$ are examples of boson operators. This is a consequence of their commutation relations.

$$\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{A}}^{\dagger} = \alpha\, \boldsymbol{\varphi} \tag{1}$$

$$\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}^{\dagger} = \beta\, \dot{\boldsymbol{\varphi}} \tag{2}$$

$$\boldsymbol{\mathcal{A}} = \tfrac{1}{2}\alpha\, \boldsymbol{\varphi} + \tfrac{1}{2}\beta\, \dot{\boldsymbol{\varphi}} \tag{3}$$

$$\boldsymbol{\mathcal{A}}^{\dagger} = \tfrac{1}{2}\alpha\, \boldsymbol{\varphi} - \tfrac{1}{2}\beta\, \dot{\boldsymbol{\varphi}} \tag{4}$$

$$\left[\boldsymbol{\mathcal{A}}(f), \boldsymbol{\mathcal{A}}^{\dagger}(g)\right] = < f|g > \tag{5}$$

$$[\boldsymbol{\mathcal{A}}(f), \boldsymbol{\mathcal{A}}(g)] = 0 \tag{6}$$

$$\left[\boldsymbol{\mathcal{A}}^{\dagger}(f), \boldsymbol{\mathcal{A}}^{\dagger}(g)\right] = 0 \tag{7}$$

The corresponding fermion operators are:

$$\left\{\boldsymbol{\mathcal{B}}(f), \boldsymbol{\mathcal{B}}^{\dagger}(g)\right\} = < f|g > \tag{8}$$

$$\{\boldsymbol{\mathcal{B}}(f), \boldsymbol{\mathcal{B}}(g)\} = 0 \tag{9}$$

$$\left\{\boldsymbol{\mathcal{B}}^{\dagger}(f), \boldsymbol{\mathcal{B}}^{\dagger}(g)\right\} = 0 \tag{10}$$

The fermion operators can be represented by imaginary quaternionic base numbers:

$$\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{B}}^{\dagger} = \boldsymbol{i} \tag{11}$$

$$\boldsymbol{\mathcal{B}} - \boldsymbol{\mathcal{B}}^{\dagger} = \boldsymbol{j} \tag{12}$$

$$\boldsymbol{\mathcal{B}} = \tfrac{1}{2}(\boldsymbol{i} + \boldsymbol{j}) \tag{13}$$

$$\boldsymbol{\mathcal{B}}^{\dagger} = \tfrac{1}{2}(\boldsymbol{i} - \boldsymbol{j}) \tag{14}$$

$$\left(\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{B}}^{\dagger}\right)\left(\boldsymbol{\mathcal{B}} - \boldsymbol{\mathcal{B}}^{\dagger}\right) = \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{B}} - \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{B}}^{\dagger} + \boldsymbol{\mathcal{B}}^{\dagger}\boldsymbol{\mathcal{B}} - \boldsymbol{\mathcal{B}}^{\dagger}\boldsymbol{\mathcal{B}}^{\dagger} = \boldsymbol{\mathcal{B}}^{\dagger}\boldsymbol{\mathcal{B}} - \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{B}}^{\dagger} = \boldsymbol{i}\,\boldsymbol{j} \tag{15}$$

Spin

The spin term is defined by:

$$\boldsymbol{\Sigma}_{field} = \int_{V} \boldsymbol{E}(q) \times \boldsymbol{\phi}(q) dV \tag{1}$$

In free space the charge density ρ_0 vanishes and the scalar potential ϕ_0 shows no variance. Only the vector potential $\boldsymbol{\phi}$ may vary with q_0. Thus:

$$\boldsymbol{E} = \boldsymbol{\nabla}\phi_0 - \nabla_0\boldsymbol{\phi} \approx -\nabla_0\boldsymbol{\phi} \tag{2}$$

$$\boldsymbol{\Sigma}_{field} \approx \int_{V} (\nabla_0\boldsymbol{\phi}(q)) \times \boldsymbol{\phi}(q) dV \tag{3}$$

If $\frac{\phi(q)}{|\phi(q)|}$ can be interpreted as tantrix and $\frac{\nabla_0\boldsymbol{\phi}(q)}{|\nabla_0\boldsymbol{\phi}(q)|}$ can be interpreted as the principle normal, then $\frac{(\nabla_0\boldsymbol{\phi}(q))\times\boldsymbol{\phi}(q)}{|(\nabla_0\boldsymbol{\phi}(q))\times\boldsymbol{\phi}(q)|}$ can be interpreted as the binormal.

Depending on the selected field $\boldsymbol{\Sigma}_{field}$ has two versions that differ in their sign. These versions can be combined in a single operator:

$$\boldsymbol{\Sigma}_{field} = \begin{bmatrix} \boldsymbol{\Sigma}^{+}{}_{field} \\ \boldsymbol{\Sigma}^{-}{}_{field} \end{bmatrix} \tag{4}$$

Spin and dyadic product

As factors of the dyadic product we consider imaginary quaternionic numbers or vectors in $\mathbb{R}_3$. The product corresponds to a matrix. This matrix acts as an operator.

$$u \otimes v \to \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix} \tag{1}$$

The product of quaternions contains sign selections. For the imaginary parts this selection has to do with the handedness of the external product. Dyadic products are well suited to store the product such that the sign selections are stored as well. The sign selection plays its role in the dyad **ij**, which consists of two imaginary base numbers. The dyad **ij** =−**ji**, and **k** can be ± **ij**. Let us apply this to the definition of S_z.

$$S_{z=} - i\hbar \begin{bmatrix} 0 & \boldsymbol{e}_x\boldsymbol{e}_y - \boldsymbol{e}_y\boldsymbol{e}_x & 0 \\ \boldsymbol{e}_y\boldsymbol{e}_x - \boldsymbol{e}_x\boldsymbol{e}_y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = i\hbar \begin{bmatrix} 0 & -2\boldsymbol{e}_x\boldsymbol{e}_y & 0 \\ 2\boldsymbol{e}_x\boldsymbol{e}_y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

This shows that the definition of S_z via the dyadic product reflects the choice in handedness of the external product of $\mathbf{e}_x$ and $\mathbf{e}_y$.

Wave package

The linear momentum is interpretable as a displacement operator. This operator is better treated in Fourier space than in configuration space. In Fourier space a particle becomes a wave package. The Fourier transforms of the fields describe the wave package.
Operator $\boldsymbol{P}$ has eigenfunctions $\tilde{f}(p)$ with eigenvalues p:

$$< q|p > = \tilde{f}(p) = < p|q >^* = f^*(q) = \exp\left(\boldsymbol{k} \cdot \frac{p\, q}{\hbar}\right)) \tag{1}$$

$$< p|\boldsymbol{P}\, p > = p \tag{2}$$

A pure particle can be represented by a single Hilbert vector |f>. Its wave function is given by:

$$\psi(q) = < \psi|q > \tag{3}$$

Or by:

$$\tilde{\psi}(p) = < p|\psi > \tag{4}$$

A mixed particle takes a Hilbert distribution in order to define its presence.

$$\rho(q) = < \rho|q > \tag{5}$$

A blurred Hilbert distribution is a Hilbert field.

$$\phi(q) = \rho(q) \circ \varphi(q) \tag{6}$$

A different type of blur gives a different type of Hilbert field.
The wave functions and private Hilbert fields represent particles. Their Fourier transforms represent wave packages. A very particular Hilbert field is a probability density that is based on a probability density operator.
A single wave mode represents a plane wave. Look at the linear momentum of the field contained in a volume V surrounded by surface S:

$$\boldsymbol{P}_{field} = \int_V \boldsymbol{g}_{field}\, dV = \int_V \rho_0 \boldsymbol{\phi}\, dV + \int_V \langle \nabla \boldsymbol{\phi}, \boldsymbol{E} \rangle\, dV + \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{E}\boldsymbol{\phi} \rangle dS \tag{7}$$

For each temporal Fourier mode of the field in free space (vanishing charge density ρ_0, no variance of scalar potential ϕ_0), where $\boldsymbol{E\phi}$ falls off rapidly, we can neglect the first and the third term.

$$\boldsymbol{P}_{field} \approx \int_V \langle \boldsymbol{\nabla\phi}, \boldsymbol{E} \rangle \, dV \tag{8}$$

Further:

$$\boldsymbol{E} = \boldsymbol{\nabla}\phi_0 - \nabla_0\boldsymbol{\phi} \approx -\nabla_0\boldsymbol{\phi} \tag{9}$$

$$\boldsymbol{P}_{field} \approx -\int_V \langle \boldsymbol{\nabla\phi}, \nabla_0\boldsymbol{\phi} \rangle \, dV \tag{10}$$

$$= \int_{V_p} \langle \mathbf{p}\tilde{\boldsymbol{\phi}}, \nabla_0\tilde{\boldsymbol{\phi}} \rangle \, dV_p$$

$$= \int_{V_p} \mathbf{p}\langle \tilde{\boldsymbol{\phi}}, \nabla_0\tilde{\boldsymbol{\phi}} \rangle \, dV_p$$

$$= \int_{V_p} \omega(p)\, \mathbf{p}\langle \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}} \rangle \, dV_p$$

If the function $\langle \tilde{\boldsymbol{\phi}}(p), \tilde{\boldsymbol{\phi}}(p) \rangle$ gives the probability density for eigenvalue $\boldsymbol{p}$. Then, this gives reason to interpret $\langle \boldsymbol{\phi}(q), \boldsymbol{\phi}(q) \rangle$ as probability density for the position $\boldsymbol{q}$ of the particle.

Fourier mode

A Fourier mode is a single frequency wave. It can be interpreted as a "particle" or as a train of particles whose charge is blurred by a very wide spread function. The corresponding current is blurred by that same spread function. It means that the divergence along the wave reduces to zero.

Often waves of the same frequency that belong to different mutually perpendicular fields combine to form polarized waves. The waves may differ in their phase shifts. The combination then forms a polarized wave. Depending on the phase difference it may be an elliptical polarized wave, a circular polarized wave or a linearly polarized wave.

Systems

A system is a local assembly of physical items that act as a single physical item. Its state is mixed. When a redefinition of physical items in terms of atomic predicates goes together with influences between items in the form of fields, then a redefinition of a system in terms of its components will certainly also have such effects. The redefinition may take different forms. It may be represented by an emission or absorption of a component or it may be a reshuffling of the components. The simplest case of reshuffling is a permutation of items that belong to the same category. A more complex situation is a periodic movement of one or more components within the realm of a system. In addition each sequence of creation and annihilation is a form of redefinition.

The system has its own characteristic vectors. The wave function may depend on the permutation state of the system. For example for fermions an odd permutation changes the sign of the (position related) wave function. For bosons a permutation does not affect the wave function. Permutations of different categories of components go together with their own type of influence. Thus, there are fermionic fields and there are bosonic fields. Each of these fields has its own type of creation and annihilation. Being fermion or boson relates to the spin type of the component. The annihilation and creation operators are closely related to the type of components involved and are also closely related to the type of fields involved. The annihilation/creation operators of fermions anti-commute and the annihilation/creation operators of bosons commute.

Entropy

A system is a local assembly of physical items that act as a single physical item. The Density operator ϱ relates to the currently considered observable Q. A pure state is a ray spanned by an eigenvector of the operator Q.

The von Neumann entropy $S(\varrho)$ of a physical system that is characterized by a state $|\psi\rangle$ is given by

$$\rho = \sum_q \{|q> \lambda_q <q|\} = \sum_q \{\lambda_q \cdot \rho_q\} \tag{1}$$

$$\rho_q = |q><q| \tag{2}$$

$$\lambda_q = |<\psi|q>|^2$$

$$S(\rho) = -k_B \cdot \sum_q \{\lambda_q \cdot \ln(\lambda_q)\} \tag{3}$$

The entropy $S(\varrho)$ describes the departure of the system from a pure state. In other words, it measures the degree of mixture (entanglement) of the state $|\psi\rangle$.

Some properties of the von Neumann entropy:

- $S(\varrho)$ is only zero for pure states.
- $S(\varrho)$ is maximal and equal to $\log_2 N$ for a maximally mixed state, N being the dimension of the Hilbert space.
- $S(\varrho)$ is invariant under changes in the basis of ϱ, that is, $S(\varrho) = S(U\varrho U^\dagger)$, with U a unitary transformation.
- $S(\varrho)$ is concave, that is, given a collection of positive numbers λ_q which sum to unity ($\Sigma_q \lambda_q = 1$) and density operators ϱ_q, we have

$$S\left(\sum_q \lambda_q \rho_q\right) \geq \sum_q \lambda_q S(\rho_q) \tag{4}$$

- $S(\varrho)$ is additive. Given two density matrices ϱ_A,ϱ_B describing independent systems A and B, then

$$S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B) \tag{5}$$

Instead, if ϱ_A,ϱ_B are the reduced density operators of the general state ϱ_{AB}, then

$$|S(\rho_A) - S(\rho_B)| \leq S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B) \tag{6}$$

While in Shannon's theory the entropy of a composite system can never be lower than the entropy of any of its parts, in quantum theory this is not the case, i.e., it is possible that $S(\varrho_{AB}) = 0$ while $S(\varrho_A) > 0$ and $S(\varrho_B) > 0$.

Intuitively, this can be understood as follows: In quantum mechanics, the entropy of the joint system can be less than the sum of the entropy of its components because the components may be entangled. The left-hand inequality can be roughly interpreted as saying that entropy can only be canceled by an equal amount of entropy. If system A and system B have different amounts of entropy, the lesser can only partially cancel the greater, and some entropy must be left over. Likewise, the right-hand inequality can be interpreted as saying that the entropy of a composite system is maximized when its components are uncorrelated, in which case the total entropy is just a sum of the sub-entropies.

- The von Neumann entropy is also *strongly sub-additive.* Given three Hilbert spaces, A,B,C,

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}) \tag{7}$$

Isolated systems

With isolated systems we mean systems in a geometrically compound environment where influences from the environment compensate each other, possibly including the influences on the environment that are caused by the system under consideration. This includes e.g. the gravitation field. The gravitation potential cannot be zero, but the influence of other items can be negligible. Internal influences are internally compensated such that they are not felt by other systems. For example the sum of the charges, which are related to electromagnetic fields is zero. It means that the Fourier transforms of the local fields consist of linear combinations of discrete terms. This holds for the electrostatic fields and the magneto-static fields. It holds for rectangular components as well as for polar components. These components are the germs of quanta and are the source of creations and annihilations.
For example consider the vector potential A. Its Fourier transform can be written as:

$$\boldsymbol{A}(\boldsymbol{r},t) = \sum_{k}\sum_{\mu=-1,1}\{\boldsymbol{e}_\mu \cdot a_{\mu\boldsymbol{k}}(t)\cdot\exp\big(i(\boldsymbol{k},\boldsymbol{r})\big) + \bar{\boldsymbol{e}}_\mu \cdot \bar{a}_{\mu\boldsymbol{k}}(t)\cdot\exp\big(-i(\boldsymbol{k},\boldsymbol{r})\big)\} \tag{1}$$

Where $\mathbf{e}_\mu$ are unit sized polarization vectors. They depend on the orthonormal vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ that represent quaternionic imaginary base numbers. The index μ labels the photon spin.

The product $\mathbf{e}_\mu \cdot a_\mu$ represents a quaternionic imaginary number. The number i can be interpreted as a base imaginary number in the direction of $\boldsymbol{k}$.

$$\boldsymbol{e}_1 \equiv \frac{-1}{\sqrt{2}}\left(\boldsymbol{e}_x + i \cdot \boldsymbol{e}_y\right) \tag{2}$$

$$\boldsymbol{e}_{-1} \equiv \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_x - i \cdot \boldsymbol{e}_y\right) \tag{3}$$

$$(\boldsymbol{e}_x, \boldsymbol{k}) = 0 \tag{4}$$

$$\left(\boldsymbol{e}_y, \boldsymbol{k}\right) = 0 \tag{5}$$

$$\left[a_\mu(\boldsymbol{k}), a_{\mu'}(\boldsymbol{k}')\right] = 0 \tag{6}$$

$$\left[a^\dagger{}_\mu(\boldsymbol{k}), a^\dagger{}_{\mu'}(\boldsymbol{k}')\right] = 0 \tag{7}$$

$$\left[a_\mu(\boldsymbol{k}), a^\dagger{}_{\mu'}(\boldsymbol{k}')\right] = \delta_{\mu\mu'} \cdot \delta_{\boldsymbol{k}\boldsymbol{k}'} \tag{8}$$

Here the $\sqrt{\frac{\hbar}{2\omega V \epsilon_0}}\, a_\mu(\boldsymbol{k})$ are the operator equivalents of the coefficients $a_{\mu\boldsymbol{k}}$ and $\omega = c\ |\mathbf{k}| = ck$.
This results in:

$$\boldsymbol{A}(\boldsymbol{r}, t) = \sum_{\boldsymbol{k},\mu} \sqrt{\frac{\hbar}{2\omega V \epsilon_0}} \left\{\boldsymbol{e}_{\mu(\boldsymbol{k})} \cdot a_\mu(\boldsymbol{k}, t) \cdot \exp\left(i(\boldsymbol{k}, \boldsymbol{r})\right) + \bar{\boldsymbol{e}}_{\mu(\boldsymbol{k})} \cdot a^\dagger{}_\mu(\boldsymbol{k}, t) \cdot \exp\left(-i(\boldsymbol{k}, \boldsymbol{r})\right)\right\} \tag{9}$$

$$\boldsymbol{E}(\boldsymbol{r}, t) = i \cdot \sum_{\boldsymbol{k},\mu} \sqrt{\frac{\hbar}{2\omega V \epsilon_0}} \left\{\boldsymbol{e}_{\mu(\boldsymbol{k})} \cdot a_\mu(\boldsymbol{k}, t) \cdot \exp\left(i(\boldsymbol{k}, \boldsymbol{r})\right) - \bar{\boldsymbol{e}}_{\mu(\boldsymbol{k})} \cdot a^\dagger{}_\mu(\boldsymbol{k}, t) \cdot \exp\left(-i(\boldsymbol{k}, \boldsymbol{r})\right)\right\} \tag{10}$$

$a_\mu(\boldsymbol{k}, t)$ is an annihilation operator and $a^\dagger{}_\mu(\boldsymbol{k}, t)$ is a creation operator.

$$a^\dagger{}_\mu(\boldsymbol{k}, t)\ |n> = |n+1> \sqrt{n+1} \tag{11}$$

$$a^\dagger{}_\mu(\boldsymbol{k}, t)\ |0> = |1> \tag{12}$$

$$a_\mu(\boldsymbol{k}, t)\ |n> = |n-1> \sqrt{n} \tag{13}$$

$$a_\mu(\boldsymbol{k}, t)\ |0> = 0 \tag{14}$$

$$\left[a_\mu(\boldsymbol{k}), \left(a^\dagger{}_\mu(\boldsymbol{k})\right)^n\right] = \left(a_\mu(\boldsymbol{k})\right)^n \tag{15}$$

The Hamiltonian is:

$$\boldsymbol{H}(t) = \hbar\omega \sum_{\boldsymbol{k},\mu} \{a_\mu{}^\dagger(\boldsymbol{k},t) \cdot a_\mu(\boldsymbol{k},t) + ½\} \tag{16}$$

The number operator N_μ gives the number of quanta:

$$N_\mu(\boldsymbol{k},t) = a_\mu{}^\dagger(\boldsymbol{k},t) \cdot a_\mu(\boldsymbol{k},t) \tag{17}$$

The quanta discussed here are bosons. With the electromagnetic field they are photons. Photons have integer spin 1. With the dyadic product $\otimes$ follows:

$$S_z \equiv -i\hbar(\boldsymbol{e}_x \otimes \boldsymbol{e}_y - \boldsymbol{e}_y \otimes \boldsymbol{e}_x) \text{ and cyclically for } x \to y \to z \to x \tag{18}$$

$$[S_x, S_y] = i\hbar S_z \tag{19}$$

$$S_z \cdot \boldsymbol{e}_\mu = \mu \cdot \boldsymbol{e}_\mu \tag{20}$$

Fermions have half integer spin. With fermions the creation and annihilation operators a and $a^\dagger$ have different commutation relations. Instead of commuting, these operators anti-commute.

Measurement

We differentiate between a measurement using a piece of equipment and an observation as is done between items in universe. In the particle view the measuring equipment scrambles the phases. After that scrambling an observation is done. In the wave view the measuring equipment takes care that the phases stay intact, while the amplitudes are ignored during the next observation.

In measurement terms the scramble of the phases is called **de-coherence**. In the same sense the care to keep phases pure and the neglecting of the amplitudes could be called re-coherence. Both actions can be related with the Fourier transforms that convert the wave view into the particle view or visa versa.

Measurement preparation

In a measurement the observation follows after a preparation phase by the measuring equipment. Such a preparation may change the shape of the subspace that represents the item. For example, a preparation for precise position measurement may squeeze the item's subspace such that its range of covered position eigenvectors becomes very short and that its range of covered momentum eigenvectors extends very far. Similarly, when a preparation is made for precise momentum measurement then the item's subspace is squeezed in the other direction, such that it covers a huge range of position eigenvectors. A Fourier transform does not squeeze the item's subspace. It changes the state of the item from position based to momentum based or vice versa.

Squeezing the item's subspace such that its range of covered position eigenvectors becomes very short and that its range of covered momentum eigenvectors extends very far is called **decoherence**. In case of a system it reduces the entanglement of that system.

Hamilton-Jacobi

The Hamilton-Jacobi equation shows how the Hamiltonian relates to the action S of the current manipulator. In this section we consider *t* to be the manipulator time!

$$H \cdot U_t = \tilde{\imath}_t \cdot \hbar \cdot \frac{\partial U_t}{\partial t} \tag{1}$$

For the eigenvalues holds

$$\Delta u_t \approx \Delta \boldsymbol{S}_{lt} \cdot u_t \tag{2}$$

Thus, we can put

$$H \cdot U_t = -\left(\frac{\partial S_t}{\partial t}\right) \cdot U_t \tag{3}$$

$$H = -\left(\frac{\partial S_t}{\partial t}\right) \tag{4}$$

For the expectation values s_t of the action operator S_t holds

$$\tilde{\imath}_t \cdot \hbar \cdot \frac{\Delta s_t}{\Delta t} = \boldsymbol{e}_{t0} + \boldsymbol{e}_{t1} \cdot \Delta t \cdot \frac{\chi_{t1}}{2} + \boldsymbol{e}_{t2} \cdot \Delta t^2 \cdot \frac{\chi_{t1} \cdot \chi_{t2}}{6} \tag{5}$$

$$- \boldsymbol{e}_{t3} \cdot \Delta t^3 \cdot \frac{\chi_{t1} \cdot \chi_{t2} \cdot \chi_{t3}}{24} + \mathcal{O}(\Delta t^3)$$

This derivation is completely independent from the observation of Q. Thus S_t has nothing to do with the Minkowski metric that appears during observations of position.

The Lagrangian

The Lagrangian is equivalent to the local geodesic equation.

The Lagrangian $\mathcal{L}_\tau$ is related with the action s_t.

$$s_t = \int_a^b \mathcal{L}_\tau d\tau \tag{1}$$

The integral is taken over the trail with the observed path. The index *t* of the action S_t is the trail progression parameter. The integration parameter stands for the coordinate time. The right side of the equation plays in Lorentzian space.

The Euler Lagrange equations explicitly use observations. For that reason the Lagrangian is considered to be a function of the observed q, the velocity $\dot{q}$ and the coordinate time $\boldsymbol{\tau}$. The velocity is measured with the coordinate time.

$$\mathcal{L}_\tau = \mathcal{L}_\tau(\tau, q, \dot{q}) \tag{2}$$

$$\dot{q} = \frac{dq}{d\tau} \tag{3}$$

The Euler-Lagrange equations are:

$$\frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial q_i} - \frac{d}{d\tau}\frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial \dot{q}_i} = 0 \tag{4}$$

for i = x, y, z

When the Lagriangian does not vary with one or more of its parameters, then this corresponds with a symmetry of the system. By Noether's theorem, such symmetries of the system correspond to conservation laws. In particular, the invariance of the Lagrangian with respect to time τ implies the conservation of energy.)

By partial differentiation of the above Lagrangian, we find:

$$\frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial q_i} = \frac{\partial U}{\partial q_i} = F_i \tag{5}$$

$$\frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial \dot{q}_i} = m \cdot \dot{q}_i = p_i \tag{6}$$

where the force is $\boldsymbol{F} = -\boldsymbol{\nabla} U$ (the negative gradient of the potential, by definition of conservative force), and $\boldsymbol{p}$ is the momentum. By substituting these into the Euler–Lagrange equation, we obtain a system of second-order differential equations for the coordinates on the particle's trajectory,

$$F_i = \frac{d(m\dot{q}_i)}{dt} = m \cdot \ddot{q}_\imath = \dot{p}_\imath \tag{7}$$

which is Newton's second law.

The world's action

The action S_t represents the influences that the rest of the world via unitary operator U_t release onto the state $\{|f\rangle_s\}_s$.

In his book about quantum gravity Rovelli writes:
"In the general relativistic parlance 'matter' is anything which is not the gravitational field. As far as current physics knows, the world is made up of the gravitational field, Yang Mills fields, fermion fields and, presumably, scalar fields."
(Carlo Rovelli, book: Quantum gravity, 2004, chapter 2, paragraph 2.1.2)

All these fields give a contribution to the action S.

$$S(e, \omega, A, \psi, \phi) \tag{1}$$

$$= S_{GR}[e, \omega] + S_{matter}[e, \omega, A, \psi, \phi] = S_{GR}[e, \omega] + S_{YM}[e, A] + S_f(e, \omega, A, \psi) + S_{sc}[e, A, \phi)$$

e is the gravitational field.
A(q) is the electromagnetic field.
ω(q) is the spin connection. It is a one form in the Lie algebra of the Lorentz group so(3,1)
ψ(q) is a scalar field, possibly with values in the representation of the Yang Mills group.
ϕ(q) is a field in the spinor representation of the Lorenz group.
A(q) has a non Abelian connection to the Yang Mills group.

The local characteristics of these fields must be represented in the eigenvalue of the current manipulator.

Representing multiple fields

Professor Mendel Sachs recently wrote a few books in which he promotes the inclusion of more terms in the metric than Einstein did. Sachs uses a four vector with quaternionic coefficients in order to specify the metric. Sachs uses all sixteen terms, while Einstein skipped six due to symmetry considerations. The argument of Sachs is that the symmetry is broken due to the characteristics of the quaternion number space. See: http://www.compukol.com/mendel/publications/publications.html.

16-ons contain the required 16 real numbers that can be arranged as a four vector with quaternion coefficients. Sachs still uses the Minkowski metric. So, his view concerns observed space.

Discussion

Macro and micro

The treatise up to so far confines to macroscopic dynamics. Micro dynamics concerns movements that occur inside the representation of small particles. Thus, inside the subspace that represent the particle.
In order to stay inside the item, the internal movements must be periodical. They can be combinations of oscillations and rotations. The harmonics oscillator and the spherical harmonics are well known examples.
The current manipulator can be seen as a very complicated (Fourier?) transform. The eigenfunctions of quantum harmonic movements seem to be governed by the functions that keep invariant under the action of this manipulator. Thus micro dynamics occurs via a different process than macro dynamics.

Dynamic logic

The current trend in quantum logic development is to add axioms that change the static character of quantum logic in a more dynamic and operational logic. Logic of quantum actions (LQA) adds unitary transforms as the source of dynamics. As we see in this article these transforms are not the real fundamental causes of dynamics. The fields that are exerted by the items are more fundamental causes of dynamics. They represent potential propositions that may be used to redefine the actual propositions. To my knowledge the influences of physical fields are not yet covered by any dynamic logic theory.

Conclusion

Quantum logic is only a partial description of the fundamentals of quantum physics. It only describes the static skeleton in which the quantum dynamics takes place. It does not treat physical fields. However, traditional quantum logic can be extended into a wider logic, such that fields are also included. When this is done, the fields become representatives of past, current and future versions of quantum logics and dynamics can simply be considered as the simultaneous step from a future version, to a current version and from the current version to a past version. An important ruler of quantum dynamics is the influence that is exposed by the universe of items in the phenomenon inertia. It indicates the laws that govern the exchange of atomic predicates from enveloping propositions. It characterizes the fields as the sticky resistance of the universe of quantum logical propositions against unordered redefinition of their members. This shapes the dynamics of the logic that describes dynamic quantum physics. The fields consist of basic constituents that can be interpreted as probability amplitude distributions. The covering field is the superposition of these basic constituents and the gravitation field is a curvature field that can be derived from the decomposition characteristics of the covering field.

The fact that the set of propositions in traditional quantum logic is lattice isomorphic with the set of closed subspaces of an infinite dimensional separable Hilbert space $\mathbf{H}$, offers the possibility to study the fields and the dynamics of the propositions with mathematical means. When this study is focused on the infinitesimal steps, the equations of motion are revealed.

Blurred Hilbert distributions form private Hilbert fields. Different fields correspond to different blurs. The blurs can be interpreted as probability distributions and as such they are the sources of quantum noise. This means that the blurs also represent the probability of the generation, presence and annihilation of actual and virtual elementary particles. A possible interpretation is that these particles are annihilated and (re)generated at each redefinition step.

The dynamics of the life path of an item can be described by a geodesic equation. The live environment can be considered as sets of 2^n-ons that locally resemble quaternions or in a still smaller region resemble complex numbers. These numbers constitute the fields that influence the dynamics of the items. The analysis of the local infinitesimal dynamic step also reveals the origin of special relativity.

There are strong indications that a universe wide clock exists in our model. This is the progression parameter clock. If this is the case, then the redefinitions are universe wide synchronized. There are also strong indications that in our model universe is controlled by a single dynamic redefiner. However, its actions are locally influenced by fields, which are directly connected to the items that are present in this environment.

Microscopic movements are governed by a different process. They are directly controlled by the current manipulator and relate to its invariant functions.

Trying to implement a complex quantum logical proposition in Hilbert space is indeed an elucidating experience.

Fields have several functions and interpretations:

- From the analysis of inertia you can derive that they represent the sticky resistance of the community of propositions/physical-items against unordered change. A uniform movement is still considered as a well ordered change. Acceleration is considered as unordered change and goes together with field activity.
- Fields are constituted of blurred sets of Hilbert vectors. With other words Hilbert fields are blurred Hilbert distributions. The blur is typical for the field and renders the field differentiable. On the other hand the blur can be interpreted as a probability amplitude distribution. (Wave functions are probability amplitudes)
- Fields can be interpreted as the storage place of the conditions of future, present and past Hilbert spaces or equivalently as the storage place of the conditions of future, present and past versions of quantum logic systems. The Hilbert spaces and the quantum logics describe a static status quo.
- Fields can be interpreted as the housing of annihilation and creation operators that act on actual or virtual particles.
- The probalistic nature of the fields invites their interpretation as clouds of quanta. These quanta represent potential realizations of Hilbert vectors that on their turn represent particles in past, present or future versions of traditional quantum logic propositions.

Appendix

History of quantum logic

Around 1930 John von Neuman and Garrett Birkhoff were searching for an acceptable explanation of the results of experiments that showed that the execution of an observation of a very small object can completely destroy the validity of an earlier observation of another observable of that object. The Schrödinger equation that agreed with the dynamic behaviour of the particles already existed. Not much later Heisenberg's matrix formulation became popular as well. Quite soon the conclusion was made that something was fundamentally wrong with the logic behind the behaviour of small particles. These small objects show particle behaviour as well as wave behaviour and they show quantization effects. It was found that the distribution axiom of classical logic had to be changed. Soon it became apparent that the lattice structure of classical logic must be weakened from an orthocomplementary modular form to an orthocomplementary weakly modular lattice. The quantum logic was born. The next step was to find a useful mathematical presentation of this new logic. A historic review of what happened can be found in: "Quantum Theory: von Neumann" vs. Dirac; http://www.illc.uva.nl/~seop/entries/qt-nvd/. It includes extensions of the concept of Hilbert space and application of these concepts to quantum field theory. Another source is: http://www.quantonics.com/Foulis_On_Quantum_Logic.html.

Quantum logic

Elementary particles behave non-classical. They can present themselves either as a particle or as a wave. A measurement of the particle properties of the object destroys the information that was obtained from an earlier measurement of the wave properties of that object.
With elementary particles it becomes clear that that nature obeys a different logic than our old trusted classical logic. The difference resides in the modularity axiom. That axiom is weakened. The classical logic is congruent to an orthocomplemented modular lattice. The quantum logic is congruent to an orthocomplemented weakly modulare lattice. Another name for that lattice is orthomodular lattice.

Lattices

A lattice is a set of elements a, b, c, …that is closed for the connections $\cap$ and $\cup$. These connections obey:

- The set is partially ordered. With each pair of elements a, b belongs an element c, such that $a \subset c$ and $b \subset c$.
- The set is a $\cap$ half lattice if with each pair of elements a, b an element c exists, such that $c = a \cap b$.
- The set is a $\cup$ half lattice if with each pair of elements a, b an element c exists, such that $c = a \cup b$.
- The set is a lattice if it is both a $\cap$ half lattice and a $\cup$ half lattice.

The following relations hold in a lattice:

$$a \cap b = b \cap a \tag{A1}$$

$$(a \cap b) \cap c = a \cap (b \cap c) \tag{A2}$$

$$a \cap (a \cup b) = a \tag{A3}$$

$$a \cup b = b \cup a \tag{A4}$$

$$(a \cup b) \cup c = a \cup (b \cup c) \tag{A5}$$

$$a \cup (a \cap b) = a \tag{A6}$$

The lattice has a partial order inclusion $\subset$:

$$a \subset b \Leftrightarrow a \cap b = a \tag{A7}$$

A complementary lattice contains two elements n and e with each element a an complementary element a′ such that:

$$a \cap a' = n \tag{A8}$$

$$a \cap n = n \tag{A9}$$

$$a \cap e = a \tag{A10}$$

$$a \cup a' = e \tag{A11}$$

$$a \cup e = e \tag{A12}$$

$$a \cup n = a \tag{A13}$$

An orthocomplemented lattice contains two elements n and e and with each element a an element a″ such that:

$$a \cup a'' = e \tag{A14}$$

$$a \cap a'' = n$$

$$(a'')'' = a \tag{A15}$$

$$a \subset b \Leftrightarrow b'' \subset a'' \tag{A16}$$

e is the unity element; n is the null element of the lattice

A distributive lattice supports the distributive laws:

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \tag{A17}$$

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c) \tag{A18}$$

A modular lattice supports:

$$(a \cap b) \cup (a \cap c) = a \cap (b \cup (a \cap c)) \tag{A19}$$

A weak modular lattice supports instead:

There exists an element d such that

$$a \subset c \Leftrightarrow (a \cup b) \cap c = a \cup (b \cap c) \cup (d \cap c) \tag{A20}$$

where d obeys:

$$(a \cup b) \cap d = d \tag{A21}$$

$$a \cap d = n \tag{A22}$$

$$b \cap d = n \tag{A23}$$

$$[(a \subset g) \text{ and } (b \subset g) \Leftrightarrow d \subset g \tag{A24}$$

In an atomic lattice holds

$$\exists_{p \in L} \forall_{x \in L} \{x \subset p \Rightarrow x = n\} \tag{A25}$$

$$\forall_{a \in L} \forall_{x \in L} \{(a < x < a \cap p) \Rightarrow (x = a \text{ or } x = a \cap p)\} \tag{A26}$$

p is an atom

Both the set of propositions of quantum logic and the set of subspaces of a separable Hilbert space **H**, have the structure of an orthomodular lattice. In this respect these sets are congruent.
In Hilbert space, an atom is a pure state (a ray spanned by a single vector).

Classical logic has the structure of an orthocomplemented distributive modular and atomic lattice.
Quantum logic has the structure of an orthomodular lattice. That is an orthocomplented weakly modular and atomic lattice. The set of closed subspaces of a Hilbert space also has that structure.

Proposition

In Aristotelian logic a proposition is a particular kind of sentence, one which affirms or denies a predicate of a subject. Propositions have binary values. They are either true or they are false.

Propositions take forms like "*This is a particle or a wave*". In quantum logic "*This is a particle.*" is not a proposition.

In mathematical logic, propositions, also called "propositional formulas" or "statement forms", are statements that do not contain quantifiers. They are composed of well-formed formulas consisting entirely of atomic formulas, the five logical connectives, and symbols of grouping (parentheses etc.). Propositional logic is one of the few areas of mathematics that is totally solved, in the sense that it has been proven internally consistent, every theorem is true, and every true statement can be proved. Predicate logic is an extension of propositional logic, which adds variables and quantifiers.

In Hilbert space a vector is either inside or not inside a closed subspace. A proper quantum logical proposition is "*Vector |f> is inside state s*".

In Hilbert space, an atomic predicate corresponds with a subspace that is spanned be a single vector.

Predicates may accept attributes and quantifiers. The predicate logic is also called first order logic. A dynamic logic can handle the fact that predicates may influence each other when atomic predicates are exchanged.

Observation

In physics, particularly in quantum physics, a system **observable** is a property of the system state that can be determined by some sequence of physical operations. This paper distinguishes between measurements and observations.

- With an observation the state is considered as a linear combination of eigenvectors of the observable. An observation returns the statistical expectation value of the eigenvalue of the observable.
- A measurement transforms the observed state to one of the eigenvectors of the observable. What happens depends on the characteristics of the measuring equipment. The measurement can be seen as a combination of a transformation and an observation.

Depending on the characteristics of the measuring equipment a measurement and a clean observation can give the same result.

With this interpretation of the concept of observation it is possible to let states observe other states. A state might do a transformation before doing an observation but in general it fails the equipment to arrange that transformation. In nature observations are far more common than measurements.

Quaternion coordinates

This part of the appendix describes candidates for the coordinates on the coordinate sphere.

Polar coordinates

The equivalent to rectangular coordinates in quaternion space is (a_τ, a_x, a_y, a_z)

$$a = a_\tau + \underline{\mathbf{i}}\cdot a_x + \underline{\mathbf{j}}\cdot a_y \pm \underline{\mathbf{i}}\cdot\underline{\mathbf{j}}\cdot a_z \quad (1)$$

The equivalent to polar coordinates in quaternion space is

$$a_\tau = ||a||\cdot\cos(\psi) \quad (2)$$

$$a_x = ||a||\cdot\sin(\psi)\cdot\sin(\theta)\cdot\cos(\phi) \quad (3)$$

$$a_y = ||a||\cdot\sin(\psi)\cdot\sin(\theta)\cdot\sin(\phi) \quad (4)$$

$$a_z = ||a||\cdot\sin(\psi)\cdot\cos(\theta) \quad (5)$$

$\mathrm{Sin}(\psi)$, where $\psi = (0,\pi)$, is known as the (imaginary) amplitude of the quaternion. Angle $\theta = (0,\pi)$ is the (co-)latitude and angle $\phi = (0,2\pi)$ is the longitude.
For any fixed value of ψ, θ and ϕ parameterize a 2-sphere of radius $\sin(\psi)$, except for the degenerate cases, when ψ equals 0 or π, in which case they describe a point.

This suggests the following structure of the argument $\underline{\mathbf{\Lambda}}$

$$a = ||a||\cdot\exp(\underline{\tilde{\mathbf{i}}}\cdot\psi) \quad (6)$$

$$= ||a||\cdot(\cos(\psi) + \underline{\tilde{\mathbf{i}}}\cdot\sin(\psi)) \quad (7)$$

$$= a_\tau + ||a||\cdot\underline{\tilde{\mathbf{i}}}\cdot\sin(\psi) = a_\tau + \underline{\mathbf{a}} \quad (8)$$

The imaginary number $\underline{\tilde{\mathbf{i}}}$ may take any direction.

3 sphere

A 3-sphere is a compact, connected, 3-dimensional manifold without boundary. It is also simply-connected. What this means, loosely speaking, is that any loop, or circular path, on the 3-sphere can be continuously shrunk to a point without leaving the 3-sphere. The Poincaré conjecture proposes that the 3-sphere is the only three dimensional manifold with these properties (up to homeomorphism).

The round metric on the 3-sphere in these coordinates is given by

$$ds^2 = d\psi^2 + \sin^2(\psi)\,(d\theta^2 + \sin^2(\theta).d\phi^2) \quad (1)$$

The volume form is given by

$$dV = \sin^2(\psi)\cdot\sin(\theta)\cdot d\psi \wedge d\theta \wedge d\phi \quad (2)$$

The 3-dimensional volume (or **hyperarea**) of a 3-sphere of radius r is

$$2\cdot\pi^2\cdot r^3 \quad (3)$$

The 4-dimensional **hypervolume** (the volume of the 4-dimensional region bounded by the 3-sphere) is

$$\tfrac{1}{2}\cdot\pi^2\cdot r^4 \quad (4)$$

The 3-sphere has constant positive sectional curvature equal to $1/r^2$.

The 3-sphere has a natural Lie group structure SU(2) given by quaternion multiplication.

The 3-sphere admits nonvanishing vector fields (sections of its tangent bundle). One can even find three linearly-independent and nonvanishing vector fields. These may be taken to be any left-invariant vector fields forming a basis for the Lie algebra of the 3-sphere. This implies that the 3-sphere is parallelizable. It follows that the tangent bundle of the 3-sphere is trivial.

There is an interesting action of the circle group $\mathbb{T}$ on $\mathbb{S}^3$ giving the 3-sphere the structure of a principal circle bundle known as the Hopf bundle. If one thinks of S^3 as a subset of $\mathbf{C}^2$, the action is given by

$$(z_1, z_2)\cdot\lambda = (z_1\cdot\lambda, z_2\cdot\lambda)\ \forall\lambda \in \mathbb{T}. \quad (5)$$

The orbit space of this action is homeomorphic to the two-sphere S^2. Since S^3 is not homeomorphic to $S^2\times S^1$, the Hopf bundle is nontrivial.

Hopf coordinates

Another choice of hyperspherical coordinates, (η, ξ1, ξ2), makes use of the embedding of S3 in $\mathbf{C}^2$. In complex coordinates $(z_1, z_2) \in \mathbf{C}^2$ we write

$$z_1 = \exp(\tilde{\mathbf{i}}\cdot\xi_1)\cdot\sin(\eta) \quad (1)$$

$$z_2 = \exp(\tilde{\mathbf{i}}\cdot\xi_2)\cdot\cos(\eta) \quad (2)$$

Here η runs over the range 0 to π/2, and ξ_1 and ξ_2 can take any values between 0 and 2π. These coordinates are useful in the description of the 3-sphere as the Hopf bundle

$$\mathbb{S}^1 \rightarrow \mathbb{S}^3 \rightarrow \mathbb{S}^2 \quad (3)$$

For any fixed value of η between 0 and π/2, the coordinates (ξ_1, ξ_2) parameterize a 2-dimensional torus. In the degenerate cases, when η equals 0 or π/2, these coordinates describe a circle.

The round metric on the 3-sphere in these coordinates is given by

$$ds^2 = d\eta^2 + \sin^2(\eta)\,(d\zeta_1^2 + \cos^2(\eta).d\,\zeta_2^2) \tag{4}$$

and the volume form by

$$dV = \sin(\eta)\cdot\cos(\eta)\cdot d\eta \wedge d\zeta_1 \wedge d\zeta_2 \tag{5}$$

Stereographic coordinates

Another convenient set of coordinates can be obtained via stereographic projection of $\mathbb{S}^3$ from a pole onto the corresponding equatorial $\mathbb{R}^3$ hyperplane. For example, if we project from the point (−1, 0, 0, 0) we can write a point p in $\mathcal{S}^3$ as

$$p = ((1-||u||^2)/(1+||u||^2),\ 2\cdot\underline{\mathbf{u}}/(1+||u||^2)) = (1+\underline{\mathbf{u}})/(1-\underline{\mathbf{u}}) \tag{1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is a vector in $\mathbb{R}^3$ and $||u||^2 = u_1^2 + u_2^2 + u_3^2$. In the second equality above we have identified p with a unit quaternion and $\mathbf{u} = u_1\cdot\underline{\mathbf{i}} + u_2\cdot\underline{\mathbf{j}} + u_3\cdot\underline{\mathbf{k}}$ with a pure (imaginary) quaternion. (Note that the numerator and denominator commute here even though quaternionic multiplication is generally non-commutative). The inverse of this map takes $p = (x_0, x_1, x_2, x_3)$ in $\mathcal{S}^3$ to

$$\underline{\mathbf{u}} = 1/(1+x_0)\cdot(x_1, x_2, x_3) \tag{2}$$

We could just as well have projected from the point (1, 0, 0, 0) in which case the point p is given by

$$p = ((-1+||v||^2)/(1+||v||^2),\ 2\cdot\underline{\mathbf{v}}/(1+||v||^2)) = (-1+\underline{\mathbf{v}})/(1-\underline{\mathbf{v}}) \tag{3}$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is another vector in $\mathbb{R}^3$. The inverse of this map takes p to

$$\underline{\mathbf{v}} = 1/(1-x_0)\cdot(x_1, x_2, x_3) \tag{4}$$

Note that the $\mathbf{u}$ coordinates are defined everywhere but (−1, 0, 0, 0) and the $\mathbf{v}$ coordinates everywhere but (1, 0, 0, 0). This defines an atlas on $\mathbb{S}^3$ consisting of two coordinate charts or "patches", which together cover all of $\mathbb{S}^3$. Note that the transition function between these two charts on their overlap is given by

$$\underline{\mathbf{v}} = \underline{\mathbf{u}}/||u||^2 \tag{5}$$

and vice-versa.

Group structure

Because the set of unit quaternions is closed under multiplication, $\mathbb{S}^3$ takes on the structure of a group. Moreover, since quaternionic multiplication is smooth, $\mathbb{S}^3$ can be regarded as a real Lie group. It is a non-abelian, compact Lie group of dimension 3. When thought of as a Lie group $\mathbb{S}^3$ is often denoted Sp(1) or U(1, $\mathbb{H}$).

It turns out that the only spheres which admit a Lie group structure are $\mathbb{S}^1$, thought of as the set of unit complex numbers, and $\mathbb{S}^3$, the set of unit quaternions. One might think that $\mathbb{S}^7$, the set of unit octonions, would form a Lie group, but this fails since octonion multiplication is non-associative. The octonionic structure does give $\mathbb{S}^7$ one important property: *parallelizability*. It turns out that the only spheres which are parallelizable are $\mathbb{S}^1$, $\mathbb{S}^3$, and $\mathbb{S}^7$.

By using a matrix representation of the quaternions, $\mathbb{H}$, one obtains a matrix representation of $\mathbb{S}^3$. One convenient choice is given by the Pauli matrices:

$$\left(a_\tau + a_x \cdot \mathbf{i} + a_y \cdot \mathbf{j} + a_z \cdot \mathbf{k}\right) = \begin{bmatrix} a_\tau + \tilde{\imath} \cdot a_x & a_y + \tilde{\imath} \cdot a_z \\ -a_y + \tilde{\imath} \cdot a_z & a_\tau - \tilde{\imath} \cdot a_x \end{bmatrix} \tag{1}$$

This map gives an injective algebra homomorphism from $\mathbb{H}$ to the set of 2×2 complex matrices. It has the property that the absolute value of a quaternion q is equal to the square root of the determinant of the matrix image of q.

The set of unit quaternions is then given by matrices of the above form with unit determinant. This matrix subgroup is precisely the special unitary group SU(2). Thus, $\mathbb{S}^3$ as a Lie group is isomorphic to SU(2).

Using our hyperspherical coordinates (η, ξ_1, ξ_2) we can then write any element of SU(2) in the form

$$\begin{bmatrix} \exp(\tilde{\imath} \cdot \xi_1) \cdot \sin(\eta) & \exp(\tilde{\imath} \cdot \xi_2) \cdot \cos(\eta) \\ -\exp(\tilde{\imath} \cdot \xi_2) \cdot \cos(\eta) & \exp(-\tilde{\imath} \cdot \xi_1) \cdot \sin(\eta) \end{bmatrix} \tag{2}$$

Another way to state this result is if we express the matrix representation of an element of SU(2) as a linear combination of the Pauli matrices. It is seen that an arbitrary element U $\in$ SU(2) can be written as

$$U = \alpha_\tau \cdot 1 + \sum_{n=x,y,z} \alpha_n \cdot \underline{\mathbf{I}}_n \tag{3}$$

The condition that the determinant of U is +1 implies that the coefficients α_n are constrained to lie on a 3-sphere.

Versor

Any **unit quaternion** q can be written as a **versor**:

$$u = \exp(\underline{\tilde{\imath}} \cdot \psi) = \cos(\psi) + \underline{\tilde{\imath}} \cdot \sin(\psi) \tag{1}$$

This is the quaternionic analogue of Euler's formula. Now the unit imaginary quaternions all lie on the unit 2-sphere in Im $\mathbb{H}$ so any such $\underline{\tilde{\imath}}$ can be written:

$$\underline{\tilde{\imath}} = \underline{\mathbf{i}}\cdot\cos(\phi)\cdot\sin(\theta) + \underline{\mathbf{j}}\cdot\sin(\phi)\cdot\sin(\theta) + \underline{\mathbf{k}}\cdot\cos(\theta) \quad (2)$$

Symplectic decomposition

Quaternions can be written as the combination of two complex numbers and an imaginary number k with unit length.

2^n-on construction

The 2^n-ons use the following doubling formula

$$(a, b)\,(c, d) = (a\cdot c - (b\cdot d^*)^*, (b^*\cdot c^*)^* + (b^*\cdot(a^*\cdot((b^{-1})^*\cdot d^*)^*)^*)^*) \quad (1)$$

Up until the 16-ons the formula can be simplified to

$$(a, b)\,(c, d) = (a\cdot c - b\cdot d^*,\ c\cdot b + (a^*\cdot b^{-1})\cdot(b\cdot d)) \quad (2)$$

Up to the octonions the Cayley Dickson construction delivers the same as the 2^n-on construction. From n>3 the 2^n-ons are 'nicer'.

2^n-ons

Table of properties of the 2^n-ons. See www.math.temple.edu/~wds/homepage/nce2.ps.

Type	***name***	**Lose**
1-ons	*Reals.*	
2-ons	*Complex numbers*	$z^* = z$ (the * denotes conjugating); the ordering properties that both {z > 0, -z > 0, or z = 0} and {w > 0, z > 0 implies w + z > 0, wz > 0}.
4-ons	*Quaternions*	commutativity ab = ba; the algebraic closedness property that every univariate polynomial equation has a root.
8-ons	*Octonions*	associativity ab · c = a · bc.
16-ons	(*not Sedenions!*)	right-alternativity x · yy = xy · y; right-cancellation $x = xy\cdot y^{-1}$; flexibility x · yx = xy · x; left-linearity (b + c)a = ba + ca; anti-automorphism ab = ba, $(ab)^{-1} = b^{-1}a^{-1}$; left-linearity (b + c)a = ba + ca; continuity of the map x → xy; Moufang and Bol identities; diassociativity
32-ons		generalized-smoothness of the map x → xy; right-division properties that xa = b has (generically) a solution x, and the uniqueness of such an x; the "fundamental theorem of algebra" that every polynomial having a unique "asymptotically dominant monomial" must have a root; Trotter's formula: $\lim_{n\to\infty}\left[e^{x/n}e^{y/n}\right]^n = \lim_{n\to\infty}\left(1+\frac{x+y}{n}\right)^n = e^{x+y}$

Type	***name***	**Retain**
2^n-ons		Unique 2-sided multiplicative & additive identity elements 1 & 0; Norm-multiplicativity $\lvert xy\rvert^2 = \lvert x\rvert^2\cdot\lvert y\rvert^2$; Norm-subadditivity $\lvert a + b\rvert \le \lvert a\rvert + \lvert b\rvert$; 2-sided inverse $a^{-1} = a^*/\lvert a\rvert^2$ (a # 0);

		$a^{**} = a$; $(x \pm y)^* = x^* \pm y^*$; $(a^{-1})^{-1} = a$; $(a^*)^{-1} = (a^{-1})^*$; $\|a\|^2 = \|a\|^2 = a^* a$; Left-alternativity $yy \cdot x = y \cdot yx$; Left-cancellation $x = y^{-1} \cdot yx$; Right-linearity $a(b + c) = ab + ac$; r^{th} power-associativity $a^n a^m = a^{n+m}$; Scaling $s \cdot ab = sa \cdot b = as \cdot b = a \cdot sb = a \cdot bs = ab \cdot s$ (s real); Power-distributivity $(ra^n + sa^m)b = ra^n b + sa^m b$ (r, s real); Vector product properties of the imaginary part: **ab** - re(**ab**) of the product for pure-imaginary 2^n-ons **a**,**b** regarded as $(2^n - 1)$-vectors; $\langle xa,b\rangle = \langle a,x^*b\rangle$, $\langle xa,xb\rangle = \|x\|^2 \cdot \langle a,b\rangle$ and $\langle x,y\rangle = \langle x^*,y^*\rangle$ Numerous weakened associativity, commutativity, distributivity, antiautomorphism, and Moufang and Bol properties including 9-coordinate ``niner" versions of most of those properties; contains 2^{n-1}-ons as subalgebra.

The most important properties of 2^n-ons

If a,b,x,y are 2^n-ons, $n \geq 0$, and s and t are scalars (i.e. all coordinates are 0 except the real coordinate) then

unit: A unique 2^n-on 1 exists, with $1 \cdot x = x \cdot 1 = x$.

zero: A unique 2^n-on 0 exists, with $0 + x = x + 0 = x$ and $0 \cdot x = x \cdot 0 = 0$.

additive properties: $x+y = y+x$, $(x+y)+z = x+(y+z)$;

$-x$ exists with $x + (-x) = x - x = 0$.

norm: $|x|^2 = xx^* = x^*x$.

norm-multiplicativity: $|x|^2 \cdot |y|^2 = |x \cdot y|^2$.

scaling: $s \cdot x \cdot y = s \cdot x \cdot y = x \cdot s \cdot y = x \cdot s \cdot y = x \cdot y \cdot s$.

weak-linearity: $(x + s) \cdot y = x \cdot y + s \cdot y$ and $x \cdot (y + s) = x \cdot y + x \cdot s$.

right-linearity: $x \cdot (y + z) = x \cdot y + x \cdot z$.

inversion: If $x \neq 0$ then a unique x^{-1} exists, obeying $x^{-1} \cdot x = x \cdot x^{-1} = 1$. It is $x^{-1} = x \cdot |x|^{-2}$.

left-alternativity: $x \cdot xy = x^2 \cdot y$.

left-cancellation: $x \cdot x^{-1} \cdot y = y$.

effect on inner products: $\langle x \cdot a,b\rangle = \langle a, x^* \cdot b\rangle$, $\langle x,y\rangle = \langle x^*, y^*\rangle$, $\langle x^* \cdot a, x^{-1} \cdot b\rangle = \langle a,b\rangle$, and $\langle x \cdot a,x \cdot b\rangle = |x|^2 \cdot \langle a,b\rangle$.

Conjugate of inverse: $(x^{-1})^* = (x^*)^{-1}$.

Near-anticommutativity of unequal basis elements: $e_k^2 = -1$ and $e_k \cdot e_l^* = -e_l \cdot e_k^*$ if $k \neq l$.

(Note: the case k; $l > 0$ shows that unequal pure-imaginary basis elements anticommute.)

Alternative basis elements: $e_k \cdot e_l \cdot e_k = e_k \cdot e_l \cdot e_k$, $e_l \cdot e_k \cdot e_k = e_l \cdot e_k \cdot e_k$, and $e_k \cdot e_k \cdot e_l = e_k \cdot e_k \cdot e_l$.

(However, when $n \geq 4$ the 2^n-ons are not flexible i.e. it is not generally true that $x \cdot y \cdot x = x \cdot y \cdot x$ if x and y are 16-ons that are not basis elements. They also are not right-alternative.)

Quadratic identity: If x is a 2^n-on (over any field F with charF $\neq 2$), then $x^2 + |x|^2 = 2 \cdot x$ re x

Squares of imaginaries: If x is a 2^n-on with re $x = 0$ ("pure imaginary") then $x^2 = -|x|^2$ is nonpositive pure-real.

Powering preserves im$x$ direction

Niners

Niners are 2n-ons whose coordinates with index > 8 are zero. The index starts with 0.

9-**flexibility** $xp \cdot x = x \cdot px$, $px \cdot p = p \cdot xp$.

9-**similitude unambiguity** $xp \cdot x^{-1} = x \cdot px^{-1}$, $px \cdot p^{-1} = p \cdot xp^{-1}$.

9-**right-alternativity** $xp \cdot p = x \cdot p^2$, $px \cdot x = p \cdot x^2$.
9-**right-cancellation** $xp^{-1} \cdot p = x$, $px^{-1} \cdot x = p$.
9-**effect on inner products** $\langle x, yp\rangle = \langle xp, y\rangle$, $\langle xp, yp\rangle = |p|^2\langle x, y\rangle$.
9-**left-linearity** $(x + y)p = xp + yp$, $(p + q)x = px + qx$.
9-**Jordan-identity** $xp \cdot xx = x(p \cdot xx)$, $py \cdot pp = p(y \cdot pp)$.
9-**coordinate-distributivity** $([x + y]z)_{0;\ldots;8} = (xz + yz)_{0;\ldots;8}$.
9-**coordinate-Jordan-identity** $[xy \cdot xx]_{0;\ldots;8} = [x(y \cdot xx)]_{0;\ldots;8}$.
9-**anticommutativity for orthogonal imaginary 2^n-ons**
If $\langle p, x\rangle = \mathrm{re}\, p = \mathrm{re}\, x = 0$ then $px = -xp$.
9-**reflection** If $|a| = 1$ and the geometric reflection operator is defined below then $-(\mathrm{refl}[a](y))_{0;\ldots;8} = (a \cdot y^* a)_{0;\ldots;8}$, and $-\{\mathrm{refl}[a](y)\}^*_{0;\ldots;8} = (a^* y \cdot a^*)_{0;\ldots;8}$, and
if either a or y is a niner then $-\mathrm{refl}[a](y) = a \cdot y^* a$ and $-\mathrm{refl}[a](y) = a^* y \cdot a^*$.

$$\mathrm{refl}[\vec{x}](\vec{t}) \stackrel{\text{def}}{=} \vec{t} - \frac{2\langle \vec{x}, \vec{t}\rangle}{|\vec{x}|^2}\vec{x} \tag{3}$$

What holds for the niners, also holds for the octonions.

Regular quaternionic functions

See: http://www.zipcon.net/~swhite/docs/math/quaternions/analysis.html and http://world.std.com/~sweetser/quaternions/ps/Quaternionic-analysis-memo.pdf).
The differential df is given by

$$df = \frac{\partial f}{\partial q_0} dq_0 + \frac{\partial f}{\partial q_i} dq_i + \frac{\partial f}{\partial q_j} dq_j + \frac{\partial f}{\partial q_k} dq_k \tag{1}$$

A regular function f obeys:

$$\frac{\partial f}{\partial q_0} + \boldsymbol{i}\frac{\partial f}{\partial q_i} + \boldsymbol{j}\frac{\partial f}{\partial q_j} + \boldsymbol{k}\frac{\partial f}{\partial q_k} = 0 \tag{2}$$

In addition the regular function f obeys:

$$\int_C D_q f = 0 \tag{3}$$

where C is any smooth closed 3-manifold in $\mathbb{H}$. D_q is the quaternion representing an element δC of the 3-manifold, its magnitude being equal to the volume of δC and its direction being normal to δC.

$$f(p) = \frac{1}{2\pi^2}\int_{\mathfrak{D}} \left\{ \frac{(q-p)^{-1}}{|q-p|^2} D_q f(q) \right\} \tag{4}$$

where $\mathfrak{D}$ is a domain in $\mathbb{H}$ in which f is regular and p is a point inside $\mathfrak{D}$.

$$D_q = dq_i \wedge dq_j \wedge dq_k - \boldsymbol{i}\, dq_0 \wedge dq_j \wedge dq_k - \boldsymbol{j}\, dq_0 \wedge dq_k \wedge dq_i - \boldsymbol{k}\, dq_0 \wedge dq_i \wedge dq_j \tag{5}$$

$$d(dq \wedge dq\, f) = Dq\, f'(q) \tag{6}$$

$$dq \wedge dq = \boldsymbol{i}\, dq_j \wedge dq_k + \boldsymbol{j}\, dq_k \wedge dq_i + \boldsymbol{k}\, dq_i \wedge dq_j \tag{7}$$

Here $a \wedge b$ is the external vector product between vectors a and b. It is not the quaternionic external product.

$$d(Dq\, f) = 0 \tag{8}$$

$$d\left[\frac{(q-p)^{-1}}{|q-p|^2}\right] Dq\, f(q) = \Delta\left(\frac{1}{|q-p|^2}\right) f(q)\, \mathrm{d}q_0 \wedge \mathrm{d}q_i \wedge \mathrm{d}q_j \wedge \mathrm{d}q_k \tag{9}$$

where Δ is the Laplacian on $\mathbb{R}^4$ and $\mathrm{dq_0} \wedge \mathrm{dq_i} \wedge \mathrm{dq_j} \wedge \mathrm{dq_k}$ is the standard volume 4-form. Since $\frac{1}{|\mathrm{q-p}|^2}$ is the Green's function for the Laplacian in $\mathbb{R}^4$, (4) follows from (9).

$$\Delta f = \frac{\partial f^2}{\partial q_0^2} + \frac{\partial f^2}{\partial q_i^2} + \frac{\partial f^2}{\partial q_j^2} + \frac{\partial f^2}{\partial q_k^2} \tag{10}$$

If f is regular in an open set U, then it has a power series expansion about each point of U. Thus, point-wise differentiability, together with the four real conditions (2) on the sixteen partial derivatives of f, is sufficient to ensure analyticity.

The set of homogeneous regular functions of degree n forms a quaternionic vector space of dimension $(\mathrm{n}+1)(\mathrm{n}+2)/2$;
This is true for any integer n if for negative n it is understood that the functions are defined and regular everywhere except at 0. The functions with negative degree of homogeneity correspond to negative powers of a complex variable, and occur in the quaternionic Laurent series which exists for any regular function which is regular in an open set except at one point.

On the unit sphere in $\mathbb{H}$ the homogeneous regular functions form a group isomorphic to SU(2). The harmonic analysis of these functions bears the same relation to quaternionic analysis as the theory of complex Fourier series does to complex analysis.

Because the quaternions are four-dimensional, there is no counterpart to the geometrical description of complex analytic functions as conformal mappings. The zeros of a quaternionic regular function are not necessarily isolated, and its range is not necessarily open; neither of these sets needs even be a sub-manifold of $\mathbb{H}$.

Definition: A function $f : \mathbb{H} \to \mathbb{H}$ is quaternion-differentiable on the left at q if the limit

$$\frac{df}{dq} = \lim_{h \to 0} \frac{f(q+h) - f(q)}{h} \tag{11}$$

exists.

Theorem: Suppose the function f is defined and quaternion-differentiable on the left throughout a connected open set U. Then on U, f has the form:

$$f(q) = a + q\,b \tag{12}$$

for some $a, b \in \mathbb{H}$.

Even if f is quaternion-differentiable, it will not in general satisfy Cauchy's theorem in the form

$$\int d_q f = 0 \tag{13}$$

where the integral is round a closed curve; in fact the only functions satisfying this equation for all closed curves are the *constant* functions.

Definition 2: A function f : $\mathbb{H} \to \mathbb{H}$ is left-regular at $q \in \mathbb{H}$ if it is **real-differentiable** at q and there exists a quaternion $f_l'(q)$ such that

$$d(dq \wedge dq\, f) = -2D_q f_l'(q) \tag{14}$$

It is right-regular if there exists a quaternion $f_r'(q)$ such that

$$d(f\, dq \wedge dq) = -2f_r'(q)D_q:$$

Clearly, the theory of left-regular functions will be entirely equivalent to the theory of right-regular functions. For definiteness, we will only consider left-regular functions, which we will call simply **regular**. We will write

$$f'(q) = f_l'(q) \tag{15}$$

and call it the **derivative** of f at q.

***Theorem2*:** (about the Cauchy-Riemann-Fueter equations)
A real-differentiable function f is regular at q if and only if

$$\frac{\partial f}{\partial q_0} + \boldsymbol{i}\frac{\partial f}{\partial q_i} + \boldsymbol{j}\frac{\partial f}{\partial q_j} + \boldsymbol{k}\frac{\partial f}{\partial q_k} = 0 \tag{16}$$

This is formula (2)
Theorem 3: A differentiable function f is regular at q if and only if

$$D_q \wedge df = 0 \tag{17}$$

Theorem 4: If f is regular at every point of the 4-parallelepiped C,

$$\int_C D_q f = 0 \tag{18}$$

This is equivalent to formula (3).

$$G(q) = \frac{q^{-1}}{|q|^2} \tag{19}$$

Note that

$$G(q) = -\partial_l \frac{1}{|q|^2} = -\partial_r \frac{1}{|q|^2} \tag{20}$$

It follows that $\bar{\partial}_l \mathrm{G} = 0$, i.e. G is regular except at 0.

***Theorem* 5:** A function which is regular in an open set U is real-analytic in U
This follows from (4).

Theorem 6: (Cauchy's theorem for a differentiable contour)
Suppose f is regular in an open set U, and let C be a differentiable 3-chain in U, which is homologous to 0 in the differentiable singular homology of U, i.e. $C = \partial C'$ for some differentiable 4-chain C' in U. Then

$$\int_c D_q f = 0 \tag{21}$$

In order to state the general form of the integral formula, we need an analogue of the notion of the winding number of a curve round a point in the plane.

Definition 3: Let q be any quaternion, and let C be a closed 3-chain in $\mathbb{H}\backslash\{\mathrm{q}\}$. Then C is homologous to a 3-chain $\mathrm{C}' : \partial \mathrm{I}^4 \rightarrow \mathrm{S}$, where S is the unit sphere with center q.
The wrapping number of C about q is the degree of the map C'.

***Theorem 7*:** (The integral formula for a differentiable contour)
Suppose f is regular in an open set U. Let $p \in U$, and let C be a differentiable 3-chain in $U\backslash\{p\}$ which is homologous, in the differentiable singular homology of $U\backslash\{p\}$, to a 3-chain whose image is ∂B for some ball $B \subset U$. Then

$$\frac{1}{2\pi^2} \int_C \left\{ \frac{(q-p)^{-1}}{|q-p|^2} D_q f(q) = n f(p) \right\} \tag{22}$$

where n is the wrapping number of C about p.

Formulas (21) and (22) also hold for a rectifiable 3-chain C.

Since regular functions are harmonic, they satisfy a maximum-modulus principle and a Liouville theorem. As with functions of a complex variable, Liouville's theorem follows immediately from the Cauchy-Fueter integral formula.

***Theorem 8*:** (Morera's theorem) Suppose that the function f is continuous in an open set U and that

$$\int_c D_q f = 0 \tag{21}$$

for every 4-parallelepiped C contained in U. Then f is regular in U.

***Theorem 9*:** Let u be a real-valued function defined on a star-shaped open set $U \in \mathbb{H}$.

If u is harmonic and has continuous second derivatives, there is a regular function f defined on U such that $Re\ f = u$.

This shows that there are as many regular functions of a quaternion variable as there are harmonic functions of four real variables. However, these functions do not include the simple algebraic functions, such as powers of the variable, which occur as analytic functions of a complex variable.

The separable Hilbert space H

Notations and naming conventions

$\{f_x\}_x$ means ordered set of f_x . It is a way to define functions.

The use of bras and kets differs slightly from the way Dirac uses them.

|f> is a ket vector, f> is the same ket
<f| is a bra vector, <f is the same bra

A is an operator. |A is the same operator
A^+ is the adjoint operator of operator A. A| is the same operator
| on its own, is a nil operator
|A| is a self-adjoint (Hermitian) operator

We will use capitals for operators and lower case for quaternions, eigenvalues, ket vectors, bra vectors and eigenvectors. Quaternions and eigenvalues will be indicated with *italic* characters. Imaginary and anti-Hermitian objects are often underlined and/or indicated in **bold** text.

$\sum_k$ means: sum over all items with index k.
$\int_x$ means: integral over all items with parameter x.

Quaternionic Hilbert space

The Hilbert space is a **linear space**. That means for the elements |f>, |g> and |h> and numbers a and b:

Ket vectors

For **ket** vectors hold

$$|f\rangle + |g\rangle = |g\rangle + |f\rangle = |g + f\rangle \tag{1}$$

$$(|f\rangle + |g\rangle) + |h\rangle = |f\rangle + (|g\rangle + |h\rangle) \tag{2}$$

$$|(a + b)\, f\rangle = |f\rangle \cdot a + |f\rangle \cdot b \tag{3}$$

$$(|f\rangle + |g\rangle) \cdot a = |f\rangle \cdot a + |g\rangle \cdot a \tag{4}$$

$$|f\rangle \cdot 0 = |0\rangle \tag{5}$$

$$|f\rangle \cdot 1 = |f\rangle \tag{6}$$

Depending on the number field that the Hilbert space supports, a and b can be real numbers, complex numbers or (real) quaternions.

Bra vectors

The **bra** vectors form the dual Hilbert space $\mathbf{H}_1^\dagger$ of $\mathbf{H}_1$.

$$\langle f| + \langle g| = \langle g| + \langle f| = |g + f\rangle \tag{1}$$

$$(\langle f| + \langle g|) + \langle h| = \langle f| + (\langle g| + \langle h|) \tag{2}$$

$$\langle f\,(a + b)\rangle = \langle f|\cdot a + \langle f|\cdot b = a^*\cdot\langle f| + b^*\cdot\langle f| \tag{3}$$

$$(\langle f| + \langle g|)\cdot a = \langle f|\cdot a + \langle g|\cdot a = a^*\cdot\langle f| + a^*\cdot\langle g| \tag{4}$$

$$0\cdot\langle f| = \langle 0| \tag{5}$$

$$1\cdot\langle f| = \langle f| \tag{6}$$

Scalar product

The Hilbert space contains a **scalar product** $\langle f|g\rangle$ that combines $\mathbf{H}_1$ and $\mathbf{H}_1^\dagger$ in a direct product that we also indicate with $\mathbf{H}_1$.
The scalar product $\langle f|g\rangle$ satisfies:

$$\langle f|g + h\rangle = \langle f|g\rangle + \langle f|h\rangle \tag{1}$$

$$\langle f|\{|g\rangle\cdot a\}_g = \{\langle f|g\rangle\}_g\cdot a \tag{2}$$

With each ket vector $|g\rangle$ in $\mathbf{H}_1$ belongs a bra vector $\langle g|$ in $\mathbf{H}_1^\dagger$ such that for all bra vectors $\langle f|$ in $\mathbf{H}_1^\dagger$

$$\langle f|g\rangle = \langle g|f\rangle^* \tag{3}$$

$$\langle f|f\rangle = 0 \text{ when } |f\rangle = |0\rangle \tag{4}$$

$$\langle f|a\,g\rangle = \langle f|g\rangle\cdot a = \langle g|f\rangle^*\cdot a = \langle g\,a|f\rangle^* = (a^*\cdot\langle g|f\rangle)^* = \langle f|g\rangle\cdot a \tag{5}$$

In general is $\langle f|a\,g\rangle \neq \langle f\,a|g\rangle$. However for real numbers r holds $\langle f|r\,g\rangle = \langle f\,r|g\rangle$

Remember that when the number field consists of quaternions, then also $\langle f|g\rangle$ is a quaternion and a quaternion q and $\langle f|g\rangle$ do in general not commute.

The scalar product defines a **norm**:

$$||f|| = \sqrt{(<f|f>)} \tag{6}$$

And a **distance**:

$$D(f,g) = ||f - g|| \tag{7}$$

The Hilbert space $\mathbf{H}$ is closed under its norm. Each converging row of elements of converges to an element of this space.

Separable

In mathematics a topological space is called separable if it contains a countable dense subset; that is, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence.
Every continuous function on the separable space $\mathbf{H}$ is determined by its values on this countable dense subset.

Base vectors

The Hilbert space $\mathbf{H}$ is **separable**. That means that there exist a countable row of elements $\{f_n>\}$ that **spans** the whole space.

If $<f_n|f_m> = \delta(m,n) = [1$ when $n = m$; 0 otherwise$]$
then $\{|f_n>\}$ forms an **orthonormal base** of the Hilbert space.

A ket base $\{|k>\}$of $\mathbf{H}$ is a minimal set of ket vectors $|k>$ that together span the Hilbert space $\mathbf{H}$.
Any ket vector $|f>$ in $\mathbf{H}$ can be written as a linear combination of elements of $\{|k>\}$.

$$|f> = \sum_k (|k>\cdot<k|f>) \tag{1}$$

A bra base $\{<b|\}$of $\mathbf{H}^+$ is a minimal set of bra vectors $<b|$ that together span the Hilbert space $\mathbf{H}^+$.
Any bra vector $<f|$ in $\mathbf{H}^+$ can be written as a linear combination of elements of $\{<b|\}$.

$$<f| = \sum_b (<f|b>\cdot<b|) \tag{2}$$

Usually base vectors are taken such that their norm equals 1. Such a base is called an othonormal base.

Operators

Operators act on a subset of the elements of the Hilbert space.

Linear operators

An operator Q is linear when for all vectors $|f>$ and $|g>$ for which Q is defined and for all quaternionic numbers a and b:

$$|Q\cdot a\ f\rangle + |Q\cdot b\ g\rangle = |a\cdot Q\ f\rangle + |b\cdot Q\ g\rangle = |Q\ f\rangle\cdot a + |Q\ g\rangle\cdot b = \quad (1)$$

$$Q\ (|f\rangle\cdot a + |g\rangle\cdot b) = Q\ (|a\ f\rangle + |b\ g\rangle) \quad (2)$$

B is **colinear** when for all vectors $|f\rangle$ for which B is defined and for all quaternionic numbers a there exists a quaternionic number c such that:

$$|B\cdot a\ f\rangle = |a\cdot B\ f\rangle = |B\ f\rangle\ c\cdot a\cdot c^{-1} \quad (3)$$

If $|f\rangle$ is an eigenvector of operator A with quaternionic eigenvalue a, then is $|b\ f\rangle$ an eigenvector of A with quaternionic eigenvalue $b\cdot a\cdot b^{-1}$.
$A| = A^{\dagger}$ is the **adjoint** of the **normal** operator A. $|A$ is the same as A.

$$\langle f\ A|\ g\rangle = \langle fA^{\dagger}|g\rangle^{*} \quad (4)$$

$$A^{\dagger\,\dagger} = A \quad (5)$$

$$(A\cdot B)^{\dagger} = B^{\dagger}\cdot A^{\dagger} \quad (6)$$

$|B|$ is a **self adjoint** operator.
$|$ is a nil operator.

The construct $|f\rangle\langle g|$ acts as a linear operator. $|g\rangle\langle f|$ is its adjoint operator.

$$\sum_n \{|f_n\rangle\cdot a_n\cdot\langle f_n|\}, \quad (7)$$

where a n is real and acts as a density function.

The set of eigenvectors of a normal operator form an orthonormal base of the Hilbert space.
A self adjoint operator has real numbers as eigenvalues.

$\{\langle q|f\rangle\}_q$ is a function f(q) of parameter q.
$\{\langle g|q\rangle\}_q$ is a function g(q) of parameter q.

When possible, we use the same letter for identifying eigenvalues, eigenvalues and the corresponding operator.
So, usually $|q\rangle$ is an eigenvector of a normal operator Q with eigenvalues q.

{q} is the set of eigenvalues of Q.
$\{q\}_q$ is the ordered field of eigenvalues of q.
$\{|q\rangle\}_q$ is the ordered set of eigenvectors of Q.
$\{\langle q|f\rangle\}_q$ is the **Q view** of $|f\rangle$.

Normal operators

The most common definition of continuous operators is:

A **continuous** operator is an operator that creates images such that the inverse images of open sets are open.

Similarly, a **continuous** operator creates images such that the inverse images of closed sets are closed.

A normal operator is a continuous linear operator.
A normal operator in $\mathbf{H}$ creates an image of $\mathbf{H}$ onto $\mathbf{H}$**.** It transfers closed subspaces of $\mathbf{H}$ into closed subspaces of $\mathbf{H}$.

Normal operators represent continuous quantum logical observables.

The normal operators N have the following property.

$$N: \mathbf{H} \Rightarrow \mathbf{H} \tag{1}$$

N commutes with its **(Hermitian) adjoint** $N^{\dagger}$

$$N \cdot N^{\dagger} = N^{\dagger} \cdot N \tag{2}$$

Normal operators are important because the spectral theorem holds for them.
Examples of normal operators are

- **unitary** operators: $U^{\dagger} = U^{-1}$, unitary operators are bounded;
- **Hermitian** operators (i.e., self-adjoint operators): $N^{\dagger} = N$;
- **Anti-Hermitian** or anti-self-adjoint operators: $N^{\dagger} = -N$;
- **Anti-unitary** operators: $I^{\dagger} = -I = I^{-1}$, anti-unitary operators are bounded;
- **positive operators**: $N = MM^{\dagger}$
- **orthogonal projection** operators: $N = N^{\dagger} = N^2$

Spectral theorem

For every compact self-adjoint operator T on a real, complex or quaternionic Hilbert space $\mathbf{H}$, there exists an orthonormal basis of $\mathbf{H}$ consisting of eigenvectors of T. More specifically, the orthogonal complement of the kernel (null space) of T admits, either a finite orthonormal basis of eigenvectors of T, or a countable infinite orthonormal basis {en} of eigenvectors of T, with corresponding eigenvalues $\{\lambda_n\} \subset R$, such that $\lambda_n \to 0$. Due to the fact that $\mathbf{H}$ is separable the set of eigenvectors of T can be extended with a base of the kernel in order to form a complete orthonormal base of $\mathbf{H}$.

If T is compact on an infinite dimensional Hilbert space $\mathbf{H}$, then T is not invertible, hence $\sigma(T)$, the spectrum of T, always contains 0. The spectral theorem shows that $\sigma(T)$ consists of the eigenvalues $\{\lambda_n\}$ of T, and of 0 (if 0 is not already an eigenvalue). The set $\sigma(T)$ is a compact subset of the real line, and the eigenvalues are dense in $\sigma(T)$.

A normal operator has a set of eigenvectors that spans the whole Hilbert space $\mathbf{H}$.
In quaternionic Hilbert space a normal operator has quaternions as eigenvalues.

The set of eigenvalues of a normal operator is NOT compact. This is due to the fact that **H** is separable. Therefore the set of eigenvectors is countable. As a consequence the set of eigenvalues is countable. Further, the eigenspace of normal operators has no finite diameter.

A continuous bounded linear operator on **H** has a compact eigenspace. The set of eigenvalues has a closure and it has a finite diameter.

Eigenspace

The set of eigenvalues $\{q\}$ of the operator Q form the eigenspace of Q

Eigenvectors and eigenvalues

For the eigenvector $|q\rangle$ of normal operator Q holds

$$|Q\ q\rangle = |q\ q\rangle = |q\rangle \cdot q \tag{1}$$

$$\langle q\ Q^{\dagger}| = \langle q\ q^{*}| = q^{*} \cdot \langle q| \tag{2}$$

$$\forall(|f\rangle\ \ э\ \mathbf{H}) \tag{3}$$

$$[\{\langle f|Q\ q\rangle\}_q = \{\langle f|\ q\rangle \cdot q\}_q = \{\langle q\ Q^{\dagger}|f\rangle^{*}\}_q = \{q^{*} \cdot \langle q\ |f\rangle^{*}\}_q]$$

The eigenvalues of 2^n-on normal operator are 2^n-ons

$$Q = \sum_{j=0}^{n-1} \mathrm{I}_j \cdot Q_i \tag{4}$$

The Q_j are self-adjoint operators.

Generalized Trotter formula

For bounded operators $\{A_j\}$ hold:

$$\lim_{n\to\infty}\left(\prod_{j=1}^{p} e^{A_j/n}\right)^n = \exp\left(\sum_{j=1}^{p} A_j\right) = \lim_{n\to\infty}\left(1 + \frac{\sum_{j=1}^{p} A_j}{n}\right)^n \tag{1}$$

In general

$$\exp\left(\sum_{j=1}^{p} A_j\right) \neq \prod_{j=1}^{p} e^{A_j} \tag{2}$$

Unitary operators

For unitary operators holds:

$$U^{\dagger} = U^{-1} \tag{1}$$

Thus

$$U \cdot U^{\dagger} = U^{\dagger} \cdot U = 1 \tag{2}$$

Suppose $U = I + C$ where U is unitary and C is compact. The equations $U\,U^* = U^*U = I$ and $C = U - I$ show that C is normal. The spectrum of C contains 0, and possibly, a finite set or a sequence tending to 0. Since $U = I + C$, the spectrum of U is obtained by shifting the spectrum of C by 1.

The unitary transform can be expressed as:

$$U = \exp(\underline{\tilde{\mathbf{I}}} \cdot \Phi / \hbar) \tag{3}$$

$$\hbar = h/(2 \cdot \pi) \tag{4}$$

Φ is Hermitian. The constant h refers to the granularity of the eigenspace.
Unitary operators have eigenvalues that are located in the unity sphere of the 2^n-ons field. The eigenvalues have the form:

$$u = \exp(\underline{\mathbf{i}} \cdot \phi / \hbar) \tag{5}$$

ϕ is real. $\underline{\mathbf{i}}$ is a unit length imaginary number in 2^n-on space. It represents a direction.
u spans a sphere in 2^n-on space. For constant $\underline{\mathbf{i}}$, u spans a circle in a complex subspace.

Polar decomposition

Normal operators N can be split into a real operator A and a unitary operator U. U and A have the same set of eigenvectors as N.

$$N = ||N|| \cdot U = A \cdot U \tag{1}$$

$$N = A \cdot U = U \cdot A \tag{2}$$

$$= A \cdot \exp(\underline{\tilde{\mathbf{I}}} \cdot \Phi)/\hbar)$$

$$= \exp\,(\Phi_r + \underline{\tilde{\mathbf{I}}} \cdot \Phi)/\hbar)$$

Φ_r is a positive normal operator.

Ladder operator

General formulation

Suppose that two operators X and N have the commutation relation:

$$[N, X] = c \cdot X \tag{1}$$

for some scalar c. If $|n\rangle$ is an eigenstate of N with eigenvalue equation,

$$|N\ n\rangle = |n\rangle \cdot n \tag{2}$$

then the operator X acts on $|n\rangle$ in such a way as to shift the eigenvalue by c:

$$\begin{aligned} |N \cdot X\ n\rangle &= |(X \cdot N + [N, X])\ n\rangle \\ &= |(X \cdot N + c \cdot X)\ n\rangle \\ &= |X \cdot N\ n\rangle + |X\ n\rangle \cdot c \\ &= |X\ n\rangle \cdot n + |X\ n\rangle \cdot c \\ &= |X\ n\rangle \cdot (n+c) \end{aligned} \tag{3}$$

In other words, if $|n\rangle$ is an eigenstate of N with eigenvalue n then $|X\ n\rangle$ is an eigenstate of N with eigenvalue $n + c$.
The operator X is a *raising operator* for N if c is real and positive, and a *lowering operator* for N if c is real and negative.

If N is a Hermitian operator then c must be real and the Hermitian adjoint of X obeys the commutation relation:

$$[N, X^{\dagger}] = - c \cdot X^{\dagger} \tag{4}$$

In particular, if X is a lowering operator for N then $X^{\dagger}$ is a raising operator for N and vice-versa.

Unit sphere of $\mathbf{H}_{\mathbf{I}}$

The ket vectors in $\mathbf{H}_{\mathbf{I}}$ that have their norm equal to one form together the **unit sphere** Θ of $\mathbf{H}_{\mathbf{I}}$. Base vectors are all member of the unit sphere. The eigenvectors of a normal operator are all member of the unit sphere.
The end points of the eigenvectors of a normal operator form a **grid** on the unit sphere $\Theta$of $\mathbf{H}_{\mathbf{I}}$.

Closure

The closure of $\mathbf{H}_{\mathbf{I}}$ means that converging rows of vectors converge to a vector of $\mathbf{H}_{\mathbf{I}}$.

In general converging rows of eigenvalues of Q do not converge to an eigenvalue of Q.
Thus, the set of eigenvalues of Q is open.
At best the density of the coverage of the set of eigenvalues is comparable with the set of 2^n-ons that have rational numbers as coordinate values.
With other words, compared to the set of real numbers the eigenvalue spectrum of Q has holes.
The set of eigenvalues of operator Q includes 0. This means that Q does not have an inverse.

The rigged Hilbert space $\mathbf{H}$ can offer a solution, but then the direct relation with quantum logic is lost.

Canonical conjugate operator P

The existence of a canonical conjugate represents a stronger requirement on the continuity of the eigenvalues of canonical eigenvalues.
Q has eigenvectors $\{|q\rangle\}_q$ and eigenvalues q.
P has eigenvectors $\{|p\rangle\}_P$ and eigenvalues p.
For each eigenvector $|q\rangle$ of Q we define an eigenvector $|p\rangle$ and eigenvalues p of P such that:

$$\langle q|p\rangle = \langle p|q\rangle^* = \exp(\hat{\mathbf{i}}\cdot p\cdot q/\hbar) \tag{1}$$

$\hbar = h/(2\cdot\pi)$ is a scaling factor. $\langle q|p\rangle$ is a quaternion. $\hat{\mathbf{i}}$ is a unit length imaginary quaternion.

Quaternionic Fourier transform

The specification of the canonical conjugate also defines a Fourier transform in the complex subspace spanned by $\hat{\mathbf{i}}$ and the real axis. This is a unitary transform with eigenvalues that are equal to the value of the inner product $\langle q|p\rangle$. The inverse Fourier transform has eigenvalues that equal $\langle p|q\rangle$. The Fourier transform U_{qp} converts the base $\{|q\rangle\}_q$ into the base $\{|p\rangle\}_P$. The inverse Fourier transform U_{pq} does the reverse.

$$\begin{aligned}\langle q|f\rangle &= \textstyle\sum_P (\langle q|p\rangle\cdot\langle p|f\rangle) \\ &= \textstyle\sum_P \langle p\,(\langle p|q\rangle)|f\rangle) \\ &= \textstyle\sum_P \langle p\,U_{pq}|f\rangle) \\ &= \textstyle\sum_P \langle p|U_{qp}\,f\rangle)\end{aligned} \tag{2}$$

$$\begin{aligned}\langle p|f\rangle &= \textstyle\sum_q (\langle p|q\rangle\cdot\langle q|f\rangle) \\ &= \textstyle\sum_q \langle q|U_{pq}\,f\rangle)\end{aligned} \tag{3}$$

Due to its specification, the operator P can be interpreted as a displacement generator for Q.

$$P = \hat{\mathrm{I}}\cdot\hbar\cdot\frac{\partial}{\partial q} \tag{4}$$

The Fourier transform is the source of the existence of the Heisenberg uncertainty relation. This is also shown in the commutator:

$$[P,Q] = PQ - QP = \hat{\mathrm{I}}\cdot\hbar \tag{5}$$

In the rigged Hilbert space $\mathbf{H}$

$$\langle q|f\rangle = \textstyle\int_P (\langle q|p\rangle\cdot\langle p|f\rangle)\cdot dp \tag{6}$$

$$\langle p|f\rangle = \textstyle\int_q (\langle p|q\rangle\cdot\langle q|f\rangle)\cdot dq \tag{7}$$

Through the definition of the canonical conjugate do P and Q possess a fixed relation. The Fourier transform can be interpreted as the local

Invariant functions

A Fourier transform has NO eigenvectors. However, a Fourier transform has functions that apart from a scaling factor are invariant under Fourier transformation. It means that the function keeps the same form. The scaling factor is $1, i, -1, -i$. Erroneously the invariant functions are sometimes called eigenfunctions and the scaling factors are then called eigenvalues. See invariants for the complex Fourier transform and spherical symmetric invariants.

Displacement generators

Variance of the scalar product gives:

$$\underline{\mathbf{i}}\cdot\hbar\cdot\delta\langle q\,|\,p\rangle = -p\cdot\langle q\,|\,p\rangle\cdot\delta q \tag{1}$$

$$\underline{\mathbf{i}}\cdot\hbar\cdot\delta\langle p\,|\,q\rangle = -q\cdot\langle p\,|\,q\rangle\cdot\delta p \tag{2}$$

In the rigged Hilbert space $\mathbf{H}$ the variance can be replaced by differentiation.
Partial differentiation of the function $\langle q\,|\,p\rangle$ gives:

$$\underline{\mathbf{i}}\cdot\hbar\cdot\partial/\partial q_s\,\langle q\,|\,p\rangle = -p_s\cdot\langle q\,|\,p\rangle \tag{3}$$

$$\underline{\mathbf{i}}\cdot\hbar\cdot\partial/\partial p_s\,\langle p\,|\,q\rangle = -q_s\cdot\langle p\,|\,q\rangle \tag{4}$$

Derivation of the one dimensional Euler Lagrange equation

This is taken from Wikipedia.

Equation

The Euler–Lagrange equation is an equation satisfied by a function, q, of a real argument, t, which is a stationary point of the functional

$$S(q) = \int_a^b L\big(t, q(t), \dot{q}(t)\big)\ dt \tag{1}$$

where:
q is the function to be found:

$$\begin{aligned} q\colon [a,b] &\subset \mathbb{R} \to X \\ t &\mapsto v = \dot{q}(t) \end{aligned} \tag{2}$$

such that q is differentiable, $q(a) = x_a$, and $q(b) = x_b$;
$\dot{q}$ is the derivative of q:

$$\begin{aligned} \dot{q}\colon [a,b] &\to T_{q(t)}X \\ t &\mapsto v = \dot{q}(t) \end{aligned} \tag{3}$$

TX being the tangent bundle of X (the space of possible values of derivatives of functions with values in X);
L is a real-valued function with continuous first partial derivatives:

$$L\colon [a,b]\times X\times TX\to\mathbb{R} \qquad t,x,v\mapsto L(t,x,v) \tag{4}$$

The Euler–Lagrange equation, then, is given by

$$L_x\big(t,q(t),\dot{q}(t)\big)-\frac{d}{dt}L_v\big(t,q(t),\dot{q}(t)\big)=\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial v}=0 \tag{5}$$

where L_x and L_v denote the partial derivatives of L with respect to the second and third arguments, respectively.
If the dimension of the space X is greater than 1, this is a system of differential equations, one for each component:

$$\frac{\partial L}{\partial x_i}-\frac{d}{dt}\frac{\partial L}{\partial v_i}=0; i=1,\dots,n \tag{6}$$

Derivation

Given a functional

$$J=\int_a^b F\big(t,y(t),y'(t)\big)\,dt \tag{1}$$

on $C^1([a,b])$ with the boundary conditions $y(a)=A$ and $y(b)=B$, we proceed by approximating the extremal curve by a polygonal line with n segments and passing to the limit as the number of segments grows arbitrarily large.
Divide the interval $[a,b]$ into $n+1$ equal segments with endpoints $t_0=a,t_1,t_2,\dots.,t_n,t_{n+1}=b$ and let $\Delta t=t_k-t_{k-1}$. Rather than a smooth function $y(t)$ we consider the polygonal line with vertices $(t_0,y_0),\dots,(t_{n+1},y_{n+1})$, where $y_0=A$ and $y_{n+1}=B$. Accordingly, our functional becomes a real function of n variables given by

$$J(y_1,\dots.,y_n)\approx\sum_{k=0}^{n}F\left(t_k,y_{k,\frac{y_{k+1}-y_k}{\Delta t}}\right)\Delta t \tag{2}$$

Extremals of this new functional defined on the discrete points $t_0,\dots.,t_{n+1}$ correspond to points where

$$\frac{\partial J(y_1,\dots.,y_n)}{\partial y_m}=0 \tag{3}$$

Evaluating this partial derivative gives

$$\frac{\partial J}{\partial y_m}=F_y\left(t_m,y_{m,\frac{y_{m+1}-y_m}{\Delta t}}\right)\Delta t+F_{y'}\left(t_{m-1},y_{m-1,\frac{y_m-y_{m-1}}{\Delta t}}\right)-F_y\left(t_m,y_{m,\frac{y_{m+1}-y_m}{\Delta t}}\right) \tag{4}$$

Dividing the above equation by Δt gives

$$\frac{\partial J}{\partial y_m \Delta t} = F_y\left(t_m, y_{m,\frac{y_{m+1}-y_m}{\Delta t}}\right) + \frac{F_{y'}\left(t_{m-1}, y_{m-1,\frac{y_m-y_{m-1}}{\Delta t}}\right) - F_y\left(t_m, y_{m,\frac{y_{m+1}-y_m}{\Delta t}}\right)}{\Delta t} \tag{5}$$

and taking the limit as $\Delta t \to 0$ of the right-hand side of this expression yields

$$F_y - \frac{dF_{y'}}{dt} = 0 \tag{6}$$

The left hand side of the previous equation is the functional derivative $\frac{\delta J}{\delta y}$ of the functional J. A necessary condition for a differentiable functional to have an extremum on some function is that its functional derivative at that function vanishes, which is granted by the last equation.

Zak transform

(See also http://eom.springer.de/Z/z130030.htm)

Definition

The Weil-Brezin-Zak transform $Z_\alpha(f)$ of a function f is defined by

$$Z_\alpha[f](t,\omega) = Z_\alpha f(t,\omega) = \sqrt{\alpha} \sum_{k=-\infty}^{\infty} f(\alpha\, t + \alpha\, k)\ \exp(-2\pi i k \omega) \tag{1}$$

Where $\alpha > 0$ and t and ω are real. When α =1, one denotes $Z_\alpha f$ by Zf.

If f represents a signal, then its Zak transform can be considered as a mixed time-frequency representation of f, and it can also be considered as a generalization of the discrete Fourier transform of f in which an infinite sequence of samples in the form $f(\alpha\, t + \alpha\, k)$, k = 0, ±1, ±2,…, is used

Elementary properties.

(2)

1) (linearity): for any complex numbers α and β,

$$Z[\alpha\, f(t) + \beta\, g(t)](t,\omega) = \alpha\, Z[\, f(t)](t,\omega) + \beta\, Z[\, g(t)](t,\omega)$$

2) (translation): for any integer m, (3,4)

$$Z[\, f(t+m)](t,\omega) = \exp(2\,\pi\, \boldsymbol{k}\, m\, \omega)\, Z[\, f](t,\omega)$$

in particular,

$$(Z\, f)(t+1,\omega) = \exp(2\,\pi\, \boldsymbol{k}\ \omega) Z\, f(t,\omega)$$

3) (modulation): (5)

$$Z\,[\exp(2\,\pi\, \boldsymbol{k}\, m\, t) f](t,\omega) = \exp(2\,\pi\, \boldsymbol{k}\, m\, t)(Z\, f)(t,\omega)$$

4) (periodicity): The Zak transform is periodic in $w$with period one, that is, (6)

$$(Z\,f)(t,\omega+1) = (Z\,f)(t,\omega)$$

5) (translation and modulation): By combining 2) and 3) one obtains (7)

$$Z\,[\exp(2\,\pi\,\boldsymbol{k}\,m\,t)f(t+n)](t,\omega) = \exp(2\,\pi\,\boldsymbol{k}\,m\,t)\exp(2\,\pi\,\boldsymbol{k}\,n\,\omega)(Z\,f)(t,\omega)$$

6) (conjugation): (8)

$$\left(Z\,\bar{f}\right)(t,\omega) = \overline{(Z\,f)}(t,-\omega)$$

7) (symmetry): If f is even, then (9,10)

$$(Z\,f)(t,\omega) = (Z\,f)(-t,-\omega)$$

and if f is odd, then

$$(Z\,f)(t,\omega) = -(Z\,f)(-t,-\omega)$$

From 6) and 7) it follows that if f is real-valued and even, then

$$(Z\,f)(t,\omega) = \overline{(Z\,f)}(t,-\omega) = (Z\,f)(-t,-\omega) \tag{11}$$

Because of 2) and 4), the Zak transform is completely determined by its values on the unit square $Q = [0,1] \times [0,1]$. (12,13)

8) (convolution): Let

$$h(t) = \int_{-\infty}^{\infty} R(t-s)\,f(s)ds$$

then

$$(Z\,h)(t,\omega) = \int_0^1 (Z\,R)(t-s,\omega)\,(Z\,f)(s,\omega)\;ds$$

Analytic properties.

If f is a continuous function such that

$$f(t) = O((1+|t|)^{-1-\epsilon}) \text{ as } t \to \infty \text{ for some } \epsilon > 0 \tag{1}$$

Then $Z\,f$ is continuous on Q. A rather peculiar property of the Zak transform is that if $Z\,f$ is continuous, it must have a zero in Q. The Zak transform is a unitary transformation from $L^2(\mathbb{R})$ onto $L^2(Q)$.

Inversion formulas.

The following inversion formulas for the Zak transform follow easily from the definition, provided that the series defining the Zak transform converges uniformly:

$$f(t) = \int_0^1 (Z\,f)(t,\omega)\,d\omega\,; -\infty < t < \infty \tag{1}$$

$$\tilde{f}(-2\,\pi\,\omega) = \frac{1}{\sqrt{2\,\pi}} \int_0^1 exp(-2\,\pi\,\boldsymbol{k}\,\omega\,t)\,(Z\,f)(t,\omega)\,dt \tag{2}$$

and

$$f(2\,\pi\,\omega) = \frac{1}{\sqrt{2\,\pi}} \int_0^1 exp(-2\,\pi\,\boldsymbol{k}\,x\,t)\,\left(Z\,\tilde{f}\right)(x,t)\,dx \tag{3}$$

where $\tilde{f}$ is the Fourier transform of f, given by

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} f(x)\,\exp(\boldsymbol{k}\,\omega\,x)\,dx$$

Applications.

The Gabor representation problem can be stated as follows: Given $g \in L^2(\mathbb{R})$ and two real numbers, α, β, different from zero, is it possible to represent any function $f \in L^2(\mathbb{R})$ by a series of the form

$$f = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{mn}\ g_{m\beta,n\alpha}\,, \tag{1}$$

where $g_{m\beta,n\alpha}$ are the Gabor functions, defined by:

$$g_{m\beta,n\alpha}(x) = \exp(2\,\pi\,\beta\,x)\,g(x - n\alpha) \tag{2}$$

and c_{mn} are constants? And under what conditions is the representation unique?

Fix a coordinate x in a line $\mathbb{R}$; the family of functions in $\mathbb{R}$

$$e_\lambda\,(x)\ =\ \sqrt[4]{2}\,\exp\ (-\pi\,(x\ -\ p)^2\ +\ 2\,\pi\,\boldsymbol{k}\,\theta\,x) \tag{3}$$

are called Gabor functions. Here λ = (p, θ) is a point in the phase space $\Phi = \mathbb{R} \otimes \mathbb{R}'$

The operators

$$\mathcal{A}\ =\ \frac{1}{2\,\pi}\,\frac{d}{dx}\ +\ x \tag{4}$$

$$\mathcal{A}^{\dagger}\ =\ -\frac{1}{2\,\pi}\,\frac{d}{dx}\ +\ x \tag{5}$$

in L^2 are adjoint one to another. They are called the *annihilation* and the *creation* operators.

Any Gabor function is an eigenvector of the annihilation operator:

$$\mathcal{A}e\lambda\ =\ \lambda e\lambda \tag{6}$$

where

$$\lambda = (p, \theta) \tag{7}$$

and

$$\lambda = p + \boldsymbol{k}\theta \tag{8}$$

For any φ in the domain of the operator a we have

$$Z(\mathcal{A}\varphi) = AZ\varphi \tag{9}$$

$$A = \frac{1}{2\pi\boldsymbol{k}} \left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial y}\right) + y \tag{10}$$

Planck limits for all physical observables

This is taken from: http://www.motionmountain.net/research.html .

Basic measures

The basic measures of physics are:

The Planck length, $l_{Pl} = \sqrt{\hbar G/c^3} = 1.6 \cdot 10^{-35}$ m

The Planck time, $t_{Pl} = \sqrt{\hbar G/c^5} = 5.4 \cdot 10^{-44}\, s$

The Planck energy is given by: $E_{Pl} = \sqrt{\hbar c^5/G} = 2.0GJ$

Fundamentals

A large part of modern physics can be summarized in four simple and fundamental statements on motion:

(1)

quantum theory on action:	$W \geqslant \hbar$
thermodynamics on entropy:	$S \geqslant k$
special relativity on speed:	$v \leqslant c$
general relativity on force:	$F \leqslant \frac{c^4}{4G}$

These limits are valid for all physical systems, whether composite or elementary, and for all observers. Note that the limit quantities of special relativity, thermodynamics, quantum theory and general relativity can also be seen as the right-hand sides of the respective indeterminacy relations:

(2)

length l and acceleration a	$\Delta l\, \Delta a \leqslant c^2$
the displacement d and momentum p	$\Delta d\, \Delta p \geqslant \frac{\hbar}{2}$
temperature T and energy U	$\Delta \frac{1}{T} \Delta U \geqslant \frac{k}{2}$

Energy flow E and size l

$$\Delta E \Delta l \leqslant \frac{c^4}{4G}$$

By combining the three fundamental limits, we can obtain limits on a number of physical observables.The following limits are valid generally, for both composite and elementary systems:

time interval:

$$t \geqslant \sqrt{\frac{4G\hbar}{c^5}} = 1.1 \cdot 10^{-43}\text{s}$$

time-distance product:

$$td \geqslant \frac{4G\hbar}{c^4} = 3.5 \cdot 10^{-78}\,\text{ms}$$

acceleration:

$$a \leqslant \sqrt{\frac{c^7}{4G\hbar}} = 2.8 \cdot 10^{51}\,\text{m/s}^2$$

angular frequency:

$$\omega \leqslant 2\pi \sqrt{\frac{c^5}{2G\hbar}} = 5.8 \cdot 10^{43}\,/\text{s}$$

Adding the knowledge that space and time can mix, we get:

distance:

$$d \geqslant \left(\frac{4G\hbar}{c^3}\right)^{1/2} = 3.2 \cdot 10^{-35}\text{m}$$

area:

$$A \geqslant \frac{4G\hbar}{c^3} = 1.0 \cdot 10^{-69}\,\text{m}^2$$

volume:

$$V \geqslant \left(\frac{4G\hbar}{c^3}\right)^{3/2} = 3.4 \cdot 10^{-104}\,\text{m}^3$$

curvature:

$$K \leqslant \frac{c^3}{4G\hbar} = 1.0 \cdot 10^{69}\,/\text{m}^2$$

mass density:

$$\rho \leqslant \frac{c^5}{16G^2\hbar} = 3.2 \cdot 10^{95}\,\text{kg/m}^3$$

Elementary particles

A particle is elementary if the system size l is smaller than any conceivable dimension, thus for elementary particles:

$$l \leqslant \frac{\hbar}{mc}$$

Using this limit, we find the well-known mass, energy and momentum limits, valid only for elementary particles:

$$m \leqslant \sqrt{\frac{\hbar c}{4G}} = 1.1 \cdot 10^{-8}\,\text{kg} = 0.60 \cdot 10^{19}\,\text{GeV/c}^2$$

$$E \leqslant \sqrt{\frac{\hbar c^5}{4G}} = 9.8 \cdot 10^{8}\,\text{J} = 0.60 \cdot 10^{19}\,\text{GeV}$$

$$p \leqslant \sqrt{\frac{\hbar c^3}{4G}} = 3.2\ \mathrm{kgm/s} = 0.60 \cdot 10^{19}\ \mathrm{GeV/c}$$

Virtual particles

Virtual particles do not obey the mentioned limits.

EM limits

Our discussion of limits can be extended to include electromagnetism. Using the (lowenergy) electromagnetic coupling constant α, we get the following limits for physical systems interacting electromagnetically:

electric charge
$$q \geqslant \sqrt{4\pi\varepsilon_0 \alpha c \hbar} = e = 0.16\ aC$$

electric field
$$E \leqslant \sqrt{\frac{c^7}{64\pi\varepsilon_0 \alpha \hbar G^2}} = \frac{c^4}{4Ge} = 1.9 \cdot 10^{62}\ \mathrm{V/m}$$

magnetic field
$$B \leqslant \sqrt{\frac{c^5}{64\pi\varepsilon_0 \alpha \hbar G^2}} = \frac{c^3}{4Ge} = 6.3 \cdot 10^{53}\mathrm{T}$$

Voltage
$$U \leqslant \sqrt{\frac{c^4}{16\pi\varepsilon_0 \alpha G}} = 1e\sqrt{\frac{\hbar c^5}{4G}} = 6.1 \cdot 10^{27}\mathrm{V}$$

Inductance
$$L \geqslant \frac{1}{4\pi\varepsilon_0 \alpha}\sqrt{\frac{4G\hbar}{c^7}} = \frac{1}{e^2}\sqrt{4G\hbar^3 c^5} = 4.4 \cdot 10^{-40}\mathrm{H}$$

With the additional assumption that in nature utmost one particle can occupy one Planck volume, we get

charge density
$$\rho_e \leqslant \sqrt{\frac{\pi\varepsilon_0 \alpha}{16G^3}}\frac{c^5}{\hbar} = e\sqrt{\frac{c^9}{64G^3\hbar^3}} = 4.7 \cdot 10^{84}\ \mathrm{C/m^3}$$

Capacitance
$$C \geqslant 4\pi\varepsilon_0 \alpha \sqrt{\frac{4G\hbar}{c^3}} = e^2\sqrt{\frac{4G}{c^5\hbar}} = 2.6 \cdot 10^{-47} F$$

For the case of a single conduction channel, we get

electric resistance
$$R \geqslant \frac{1}{4\pi\varepsilon_0 \alpha c} = \hbar\, e^2 = 4.1\ k\Omega$$

electric conductivity
$$G \leqslant 4\pi\varepsilon_0 \alpha c = e^2\, \hbar = 0.24\ mS$$

electric current
$$I \leqslant \sqrt{\frac{\pi\varepsilon_0 \alpha c^6}{G}} = e\sqrt{\frac{c^5}{4\hbar G}} = 1.5 \cdot 10^{24}\ A$$

Indeterminacy relations:

C capacity and U potential difference	$\Delta C\ \Delta U \geqslant e$
Electric current I and time t	$\Delta I\ \Delta t \geqslant e$

Derived limits

The ratio of angular momentum D to energy E times length L has the dimensions of inverse speed. Since speeds are limited by the speed of light, we get

$$D_{system} \leqslant \frac{1}{c} LE$$

The action limit

$$W \leqslant LE/c$$

is not exceeded in any physical process.
Since action values in nature are limited from below by ħ, we get a limit for the speed of a system:

$$v_{system} \geqslant \hbar c^2 \frac{1}{LE}$$

This is not a new result; it is just a form of the indeterminacy relation of quantum theory.
Thanks to the connection $W = FLT$ between action$W$, force F, distance L and time T, we can deduce

$$F_{system} \geqslant \frac{\hbar}{2c} \frac{1}{T^2}$$

The power P emitted by a system of size L and mass M is limited by

$$c^3 \frac{M}{L} \geqslant P_{system} \geqslant 2\hbar G \frac{M}{L^3}$$

In 1973 Bekenstein discovered a famous limit that connects the entropy S of a physical system with its size and mass. No system has larger entropy than one bounded by a horizon. The larger the horizon surface, the larger the entropy.

$$\frac{S}{S_{limit}} \leqslant \frac{A}{A_{limit}}$$

which gives

$$S \leqslant k \frac{c^3}{4G\hbar} A$$

where A is the surface of the system. Equality is realized only for black holes.
We assume that the limits for vacuum are opposite to those for matter. We can then write

$$c^2/4G \leqslant M/L$$

for the vacuum.
Using

$$\frac{S}{S_{c.Planck}} \leqslant \frac{M}{M_{c.Planck}} \frac{A}{A_{c.Planck}} \frac{L_{c.Planck}}{L}$$

we get

$$S \leqslant \frac{\pi kc}{\hbar} ML = \frac{2\pi kc}{\hbar} MR$$

This is called Bekenstein's entropy bound.

A lower limit for the temperature T of a thermal system can be found using the idea that the number of degrees of freedom of a system is limited by its surface, or more precisely, by the ratio between the surface and the Planck surface. We get the limit

$$T \geqslant \frac{4G\hbar}{\pi kc} \frac{M}{L^2}$$

Lower limit for the electric field E:

$$E \geqslant 4Ge \frac{M^2}{Q^2 L^2}$$

Lower limit for the magnetic field B:

$$B \geqslant \frac{4Ge}{c} \frac{M^2}{Q^2 L^2}$$

Cosmological limits

Cosmology is characterized via the cosmological constant Λ by the inequality:

$$l \lessapprox \frac{1}{\Lambda}$$

For single particles, the absolute lower speed limit, the cosmological speed limit, is given by:

$$v_{particle} \geqslant \frac{\sqrt{4G\hbar/c}}{L_{universe}} = L_{corr.Planck}\sqrt{\Lambda}\, c \approx 7 \cdot 10^{-53}\, m/s$$

The negative energy volume density $-\Lambda c^4/4\pi G$ corresponds to a force value

$$F = \frac{\Lambda \hbar c}{2\pi} = 4.8 \cdot 10^{-79}\, N$$

This is also the gravitational force between two corrected Planck masses
located at the cosmological distance $\sqrt{\pi/4\Lambda}$.

In nature there is a minimum time interval, $l_{Pl}/c = t_{Pl}$, the Planck time.

A recent prediction derived from the standard model of elementary particles give as an upper limit for the electron dipole moment d_e a value of

$$\frac{|d_e|}{e} < 3 \cdot 10^{-21}\, m$$

The mass m of any elementary particle is constrained by the Planck mass m_{Pl}

$$m < \frac{\hbar}{c\, l_{Pl}} = \sqrt{\frac{\hbar c}{G}} = m_{Pl} = 2.2 \cdot 10^{-8}\, kg = 1.2 \cdot 10^{19}\, GeV/c^2.$$

The maximum possible value for mass density ρ_{Pl} is

$$\rho_{Pl} = \frac{c^5}{G^2\hbar} = 5.2 \cdot 10^{96}\, kg/m^3$$

Within a factor of order one, we find

$$K < \frac{c^3}{G\hbar} = 0.39 \cdot 10^{70}\, m^{-2}$$

as a limit for the surface curvature K in nature. In other words, the universe has never been a point, never had zero age, never had infinite density, and never had infinite curvature.

Limit quality

Nature provides two limits for each observable: a Planck limit and a cosmological limit.
All measurements are limited in precision.
Because of the fundamental limits to measurement precision, the measured values of physical observables do not require the full set of real numbers. In fact, limited precision implies that observables cannot be described by the real numbers.
At Planck scales it is impossible to distinguish between matter and vacuum. Vacuum and matter do not differ at Planck scales. Similarly, at the Planck length it is impossible to distinguish between positive and negative time values: so particles and antiparticles are not clearly distinguished at Planck scales.

The strictest upper limits are those with the smallest exponent for length, and the strictest lower limits are those with the largest exponent of length.
The accuracy of time measurements is limited by the Planck time t_{Pl}.
The accuracy of length measurements is limited by the Planck length $\mathrm{l_{Pl}}$.
All measurements – be they measurements of position, speed, mass or any other observable – are electromagnetic. In other words, all measurements in nature are detection of photons. And in strand theory photon absorption and detection are intimately related to the crossing switch.
All electromagnetic information is communicated by directed information carrying quanta in the form of shot noise. However, secondary information can be derived from the shape of the quantum cloud.

References:

More useful stuff is collected in the toolkit
Axiomatic Quantum Theory, W. Lücke, http://arxiv.org/PS_cache/quant-ph/pdf/9510/9510024v2.pdf
An overview of gravity theories: http://arxiv.org/PS_cache/arxiv/pdf/0909/0909.4672v2.pdf.