On the origin of dynamics

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Abstract

When physics must be based on an axiomatic foundation then the law set of traditional quantum logic is a valid candidate. However, at first sight, these axioms do not treat physical fields and they do not treat dynamics. It only prescribes the static relations that exist between quantum logical propositions that treat static subjects. Amongst these propositions statements exist that describe everything that can be said about the static condition of a given physical item. Such propositions represent that item. Traditional quantum logic is lattice isomorphic to the set of closed subspaces of an infinite dimensional separatable Hilbert space. That is why quantum mechanics is usually done with the aid of Hilbert space features. The representation of a physical field does not fit in a Hilbert subspace. Physical fields have a universe wide range and their presentation would cover all of a complete Hilbert space.

Piron has shown that this Hilbert space can be defined over one of three division rings. The choice comprises the real numbers, the complex numbers and the quaternions. The choice for the quaternions means that manipulations of the Hilbert space, such as Fourier transforms, in general use such multi-dimensional numbers.

According to Helmholtz decomposition theorem, the quaternionic Fourier transform can be divided in a complex longitudinal Fourier transform and a transverse Fourier transform. For quaternionic functions this means that they can be *locally* split into a one-dimensional rotation free part and a two-dimensional divergence free part.

This e-paper indicates that traditional quantum logic can be expanded to extended quantum logic, which includes the influences of physical fields in the form of potential propositions that concern virtual items. Extended quantum logic is lattice isomorphic with the set of subspaces of *a set of* Hilbert spaces. The fields represent the glue between these Hilbert spaces.

In this complicated way the axioms of traditional quantum logic form the constraints of the dynamics of quantum physics. When the dynamics of the universe would be put to a hold, then the axioms of extended quantum logic would describe all static constraints that are put to that universe. Dynamics means that universe steps from one static status quo to the next. After the step the conditions are changed and the static constraints are reestablished. If we find the laws that control the steps, then we have found a complete axiomatic foundation of

physics. Classical physics forms another constraint of dynamical quantum physics. This e-paper studies what happens during the step.

In the process a classification of skew Hilbert fields will be generated that corresponds closely to the Maxwell fields. Further, this e-paper investigates what happens in the infinitesimal steps that nature takes in order to arrive at the next static status quo. In this way the origin of dynamics and the origin of special relativity may be revealed.

Contents

It is undeniable that a Creator exists.
It is easy to give a creator a place in physics
It is our fate to interpret who or what this creator is.
Fourier transforms are abundantly present in physics and have the habit to convert something that is strongly localized into something that is very widespread and vice versa.
Consider what the Fourier transform of the universe will be!

Introduction

The aim of this paper is to build upon a fundament consisting of a minimal set of axioms and then derive as much as is possible from fundamental physics by using only purely mathematical methods.

This e-paper is not about quantum logic. It uses quantum logic because traditional quantum logic defines the static framework in which quantum dynamics takes place. Traditional quantum logic prescribes the relations that exist between quantum logical propositions. Amongst these propositions statements exist that describe everything that can be said about the static condition of a given physical item. Such propositions represent that item.

Traditional quantum logic is lattice isomorphic to the set of closed subspaces of an infinite dimensional separatable Hilbert space. That is why quantum mechanics is usually done with the aid of Hilbert space features. The representation of a physical field does not fit in a Hilbert subspace. Physical fields have a universe wide territory and their presentation would cover all of a complete Hilbert space.

Piron has shown that this Hilbert space can be defined over one of three division rings. The choice comprises the real numbers, the complex numbers and the quaternions. The choice for the quaternions means that manipulations of the Hilbert space, such as the Fourier transforms in general operate on these multi-dimensional numbers. The representations of physical fields are Hilbert fields. Hilbert fields are blurred Hilbert distributions. Hilbert distributions are sets of Hilbert vectors. These vectors may be equipped with one or more hyper complex numbers or with one or more hyper complex functions that represent a local blur. In this way these fields not only touch all Hilbert vectors but they also become differentiable. With the help of a normal operator a Hilbert vector can be converted in a hyper complex function. We call such functions Hilbert functions. As a consequence, the derived theory is largely based on the properties of these multidimensional transforms and on the properties of Hilbert fields and Hilbert functions.

A three-dimensional Fourier transform can be divided in a one-dimensional longitudinal Fourier transform and a two-dimensional transverse Fourier transform. The longitudinal transform works only on the longitudinal part of the function or field that is being transformed. The same holds for the transverse parts and transforms. The division in a longitudinal part and a transverse part of a field has only a local validity. For Hilbert fields this means that they can be locally split in a one-dimensional rotation free part and a two-dimensional divergence free part. This is the subject of the Helmholtz decomposition

theorem. Multi-dimensional Dirac delta functions show the same decomposition as the multi-dimensional Fourier transform.

The fact that this categorization has only local validity and that it is related to an imaginary direction causes that the quaternionic Fourier transform must be considered to operate in a curved coordinate space. The differentiability of quaternionic functions and Hilbert fields also offers this categorization. In addition the functions and fields can be categorized according to their symmetry properties. These categorizations must also cope with a curved coordinate space.

Traditional quantum logic and the Helmholtz decomposition theorem together form a set of laws that define the static relational status quo that would exist in nature when dynamics could be put to a hold. This paper points out that traditional quantum logic can be expanded such that it includes the representation of static physical fields. In this picture the fields represent sets of potential propositions. These propositions are statements that say everything that can be said about the static condition of virtual items. The set of propositions in extended quantum logic is no longer isomorph with the set of closed subspaces of a single Hilbert space. It is isomorph with the closed subspaces of a set of Hilbert spaces. One member of this set of Hilbert spaces is the actual Hilbert space. It contains the representatives of actual physical items. The other members are virtual Hilbert spaces. They contain the representations of virtual physical itemsEach virtual Hilbert space corresponds to an instance of a virtual (traditional) quantum logic. These virtual logics represent potential replacements of the actual traditional quantum logic. A virtual logic differs from other virtual logics in the fact that their propositions have a different configuration in terms of their atomic predicates or in terms of their sub-ordered propositions. In a similar sense they will differ from the actual logic.

The replacement will be made between logics that have a great resemblance with each other and the values of the predicates within equivalent propositions will be the same or close to each other. At each replacement the existing items are annihilated and replacing items are created. It is also possible that annihilation is not followed by creation or that a created item is not the replacement of an annihilated item.

A redefiner, which steps from the actual Hilbert space to a virtual one, implements dynamics. The redefinition step exchanges the actual Hilbert space against a virtual Hilbert space that is taken from the representation of the fields.

Dynamics can be interpreted as a sequence of steps in which each step leads nature from the conditions of one status quo to the conditions of the next status quo. The laws that define the status quo are clear. During these steps several things happen. The laws that govern the dynamics are still obscure. The steps couple the static ingredients into a dynamic mixture. The steps are taken universe wide. A redefiner with a universe wide domain controls these steps. The step counter presents a universe-wide progression parameter. This parameter must not be confused with our common notion of time, but it cannot be denied that it has some relation with it. It certainly has a close relation with the relativistic proper time.

Traditional quantum logic is usually defined via its structure as an orthomodular lattice. This logic only defines part of the *static* skeleton of the frame in which quantum physics operates. It does not state anything about physical fields. Inertia reveals the importance of the gravitation fields. The Helmholtz/Hodge decomposition theorem defines the structure of static physical fields. In that way this theorem plays a similar role as traditional quantum logic. Extended quantum logic encompasses both law sets. These law sets do not specify or even touch the source of *dynamics*. Dynamics couples the static fields. The coupling not only applies to parts of the same field. It also concerns different fields. For example dynamics couples electrostatic fields with magnetostatic fields into dynamical electromagnetic fields and it couples electromagnetic fields with gravitational fields.

Both the propositions about a quantum physical system and physical fields are closely related. However, this relation only gets relevant when dynamics comes into play. Dynamics causes a continuing redefinition of the propositions. This disturbs the current static status quo. When one proposition is changed it interchanges its constituting atomic predicates with other predicates. The change can even involve the exchange of atomic predicates against atomic predicates that are of another type. It is also possible that the configuration of a complex system that consists of simpler components is altered.

The static physical fields can be interpreted as storage of the preconditions for the next step. The physical fields are the representatives of the influences that go together with the sticky resistance of the set of propositions against the changes that occur due to the redefinitions of the propositions that describe physical items. This sticky resistance also occurs in propositions that are sub-ordered to other propositions. Inertia is a feature that shows this resistance explicitly.

The propositions about quantum physical items can be represented by closed subspaces of a Hilbert space. The presence of dynamics means that the relations between these subspaces are not stationary. It is also possible to give the physical fields a representation in Hilbert space. However, it must be clear that quantum physical items and physical fields are not the same stuff. Physical fields cannot be represented by closed Hilbert subspaces. They cover the whole universe. In fact, each field covers the whole Hilbert space. However, their strength may be concentrated around separate excited places and it may diminish with distance.

The actual physical items are distributed in space and are surrounded by potentials that act as a kind of blur. This is why quantum physics has much in common with optics. The blur characterizes the transfer quality of the corresponding field. At not too short distances the blur of electromagnetic fields has the same shape as the blur of gravitational fields. These fields are based on the same Hilbert distributions. The main difference lays in the fact that the charges of electromagnetic fields have the same size but may have different sign such that they may partly compensate each other's influence. The charge (mass) of the gravitational field is always positive, but it may differ in size. The masses of all physical items work together in order to create the immense potential that causes inertia.

It is an elucidating experience to try to implement a complicated quantum logical proposition in the representation of quantum logic in Hilbert space. In that way we may discover how dynamics emerges in this static skeleton. For that reason, we choose as an

example a predicate with quantifiers rather than a clean proposition. In the course of this project it will become clear that there is a way to extend the rather static traditional quantum logic into a dynamic version. The phenomenon of inertia guides our way.

The selected example proposition (♠) is

> "*All items in universe influence each other's position*".

The final conclusion of this experiment is: A well-ordered replacement of atomic predicates in an enveloping proposition appears to occur without strong consequences, but any deviation of a well ordered replacement causes an influence of the complete set of all propositions. This explains the interaction between fields and physical items. A local deviation of the uniformity of the distribution of physical items can still cause a slight influence of neighboring items. At small distances the influences can be large. The influence of fields can be implemented in the Hilbert space. Via an action = reaction game the interaction between fields and Hilbert subspaces form the source of dynamics.

What further happens during the implementation of our example proposition (♠) is completely governed by mathematics. Thus, *for our example* no further extension of quantum logic is needed to transform it into a useful version of dynamic quantum logic.

As number spaces we use the 2^n-ons of Warren Smith rather than the hyper complex numbers based on the Cayley-Dickson construction. Up to the octonions the corresponding number spaces are similar. (See http://www.math.temple.edu/~wds/homepage/nce2.pdf or the appendix). For higher n the 2^n-ons behave in a nicer way. They keep more of their number characteristics. We use the quaternions (n=2) as the number space that is used to define the inner product of the Hilbert space. However, we tolerate operators to have eigenvalues that are higher dimensional 2^n-ons. We also use 2^n-ons in order to set the values of physical fields.

When we use these numbers as eigenvalues or as field values, then we apply their number characteristics as well as their storage capacity. A 2^n-on contains 2^n real numbers. We also tolerate that operators and fields support multiple sign selections, such as the inversion of the real axis and the handedness of external vector products for their eigenvalues. 2^n-ons offer n sign selections and contain n independent imaginary base numbers. Each new independent base number introduces a new sign selection. The sign selections translate into an n-fold hyper complex conjugation and 2^n different skew fields.

With $n > m$, the 2^n-ons act like 2^m-ons in their lower m dimensions. Further, the 2^n-ons contain several subspaces of 2^m-ons. We may use smoothly curved manifolds that are crossed by curves which form trails of 2^n-on numbers and that are locally touched by tangent spaces that can be interpreted as 2^n-on number spaces.

When the members of a set 2^n-ons approach zero, then in their mutual arithmetic actions they are getting more and more the characters of lower dimensional 2^m-ons. In the same sense, when two 2^n-ons approach each other, their mutual arithmetic actions are getting more and more the characteristics the arithmetic of lower dimensional 2^m-ons.

The implementation of the proposition (♠) leads to a story of manipulators and manipulated observables. The number waltz feature ($c=ab/a$) of the 2^n-ons that becomes a noticeable effect for $n>1$ seems to play a significant role in our model. If this model applies to quantum physics, then it may reveal why special relativity exists and brings clearness in the different notions of time that exist in quantum physics. The curvature introduced by the spatial variance of what the longitudinal direction is reveals how the mentioned influences can be implemented as component fields which are defined on a curved coordinate system. This holds for gravitational fields as well as for the other fields such as electromagnetic fields.

Implementing quantum physics in a complex Hilbert space hides these interesting features and diminishes the insight that higher dimensional 2^n-ons can reveal.

The article shows that there is a need to extend traditional quantum logic such that it not only includes the representations of fields but also includes axioms, which specify the dynamic underpinning of quantum physics. In the course of this project several fundamental aspects of physics get uncovered.

Comments

This project is far from finished. Most parts I have rewritten several times. Some ingredients were already included before they are finalized and before they are put at the proper position in the context. I try to make the whole paper consistent with its parts and I try to keep my goal to include nothing that did not follow directly or indirectly from the axioms of traditional quantum logic. I only tolerate mathematics as a valid tool and ingredient. I will not use or accept intuition as a reason to include a subject in the text. I will also not tolerate the usefulness of a concept or its acceptance in the physical community as a valid argument to include that concept. If you encounter places where I did not succeed in that goal, then you may conclude that I still have to work on that section. When the paper gets its final version, then no deviations of my goal should result. However, I might never reach that condition. Then, *you* reader might take over and finish the job. But first think of the possibility that we will succeed. *What does it mean that all of fundamental physics is based on mathematics and on the 25 axioms of traditional quantum logic?!*

References

This e-paper contains no lengthy reference list. References to other documents are usually presented inline and are mostly put in the form of hyperlinks. A sometimes referenced toolkit contains a collection of stuff that otherwise must be grasped from internet. As is done in this article, much of the contents of the toolkit are directly or indirectly obtained from Wikipedia. In that case the text is adapted to the requirements of the papers that use this toolkit. Most texts on internet are based on complex Hilbert spaces, so where necessary I have converted these texts into quaternionic versions.

This paper is prepared with MS Word 2010. This word processor version contains a rather capable equation editor and a large series of fonts including Cambria Math. However, the equation editor does not cooperate with the paragraph indexing in order to automatically enumerate the out of line equations. For that reason equations are enumerated manually and relative to the current paragraph header. References inside that paragraph just use the

equation number. References from outside of the paragraph are hyperlinks that refer to the paragraph header. The hyperlink text will then include the equation number. So, you must move manually to the equation inside the target paragraph.

This paper draws significantly from the book on electromagnetic field theory of Bo Thidé. That book has a different goal and uses different premises. The book does not use the quaternionic field approach as is done here, but its contents easily translate to quaternions. Further its formulation is very precise, it links formulas to physical concepts and most of all it is online: http://www.plasma.uu.se/CED/Book/EMFT_Book.pdf.

Notation note

This paper uses {} in order to indicate a set or a function.
Depending on the context $\{|f_s\rangle\}_s$ means an ordered set of vectors $|f_s\rangle$ where s is the ordering index. In other contexts$\{|f_s\rangle\}_s$ means a vector function $|f(s)\rangle$ where s is the (discrete) parameter. Continuous functions are presented in the normal way.
$f(\{q_j\}_j)$ is a function $f(q_1, q_2, q_3, q_4,\ldots q_n,)$ of the set of parameters $(\{q_j\}_j$, where $j = 1, 2, \ldots, n$. The index constraint n might be infinity.

The appendix and the toolkit contain information about other notation and naming conventions that are used in this paper.

Logic

The set of propositions of traditional quantum logic is lattice isomorphic with the set of closed subspaces of an infinite dimensional separatable Hilbert space **H**. This isomorphism means that quantum logical propositions can be represented by closed subspaces of a Hilbert space. The inner products of that Hilbert space can be defined by using numbers of a 2^n-on number space. Taking $n>2$ for that purpose raises numeric problems with the closure of the subspaces. Traditional quantum logic does not include any axioms that treat dynamics and it does not treat the influences of physical fields. It only specifies stationary relations that are possible between physical items.

In order to discover the emergence of dynamics we will implement a quantum logical proposition in Hilbert space and test its truthfulness. We will introduce in this example proposition physical fields as well as dynamics.

The example proposition is:

All items in universe influence each other's position. (♠)

It can be answered with either yes or no. And, if we succeed, it can be implemented in Hilbert space. So, in that case it is a valid quantum logical proposition.
Proving 'yes' is cumbersome, but the 'no' is hardly less difficult. It requires finding an item of which the position is not influenced by at least one of the other items. For this purpose it is necessary to implement notions of *items*, the *universe*, *influences* and *position* in Hilbert space.

The statement includes quantifiers (*position*) and operational elements (*influence*). The set of axioms of traditional quantum logic does not treat these subjects. As we will see, the *influence* of the *universe* of propositions (*items*) will put particular restrictions to the extension of quantum logic into the realm of a dynamic logic. This restriction is manifested in the occurrence of inertia.

Translated in physical terms inertia means that in contrast to a *uniform movement*, the *acceleration* of an *item* will go together with the *action* of a *physical field*. Notice that we use the words "goes together with" instead of "generates" or "causes".

Translated in logical terms a conclusion of the analysis of inertia runs: "
During a redefinition of a proposition the exchange of atomic predicates in that proposition must be done in well-ordered and controlled steps. Otherwise the *community of propositions* will *influence* the considered proposition."

Again it must be noticed that there is no causal relation between the event of being well-ordered and the event of influencing. The interaction is instantaneous.

When nature's logic is put in axioms, then influences that correspond to physical fields must follow from the axioms. Together with the specification of the origin of dynamics this will then result in a dynamic version of quantum logic.

I assume that this category of logic does not yet exist in mathematics. There exists a version of dynamic operational quantum logic, but it does not cover or mention the effects of the representation of physical fields in logic and it does not specify the *origin of dynamics*.

Atomic predicates

Atomic propositions are statements that are either true or false and which cannot be broken down into other simpler propositions. When an atomic proposition concerns a property, then it may contain the value of that property. We will call that kind of atomic propositions atomic predicates. For example "The speed is 5." The identity or the category of the property is "speed". The value of the property is 5. Its dimension is "meter per second", but that is another atomic statement and it is a fixed statement. This information is part of the type definition of the atomic predicate category "speed".

The atomic predicates form a set with a particular lattice structure. In this set we only consider atomic predicates that are independent of all other atomic predicates. Several choices of such sets exist. A subset consisting of members of a chosen set may be canonical conjugates of members of another set.

In Hilbert space the type definitions of atomic predicates that concern numeric variables are represented by **operators**. The values of the properties in the atomic predicates correspond to the eigenvalues of the operators or they are expectation values. Expectation values are statistically determined via a probability characteristic that characterizes both the operator and a physical item. See Wave function.

Type definitions

Type definitions are propositions that describe and categorize subjects without specifying their variable values. A type definition of a category of atomic predicates specifies the type of property that these propositions treat. If that category is "speed", then the definition also contains the dimension (e.g. meters per second) and the allowed range of the potential values.
When the type definition concerns a more complex object that can act as an individual the definition will be called an **item type** definition. Item type definitions use **atomic predicate types**.

When that item cannot be broken into simpler objects that still can act as an individual, then the type definition is an **elementary type** definition. Elementary type definitions are constructed of type definitions of atomic predicates.
The elementary types form (a rather small) subset of the whole set of type definitions. Elementary types appear to divide into two categories**: bosons** and **fermions**.

If the item is not an elementary type, then its type specification is a **system or sub-system type** definition. A (sub)-system type definition is constructed of elementary item type definitions and atomic predicate types.

The type definitions form a set with a different lattice structure. Its structure is isomorph with the structure of classical logic.

In Hilbert space no representation for item type definitions exists. However, in Hilbert space atomic predicate types are represented by operators.

Items

The first problem that is raised by constructing the representation of proposition (♠) is to determine what in this representation stands for an item. The simplest solution is to attach a subspace of the Hilbert space to the item. The corresponding proposition can be phrased as: "*This is the item*". Something either belongs to the subspace or it is outside that subspace. Everything that can be attributed to the item can also be attributed to this subspace. Each of these propositions belongs to a hierarchy for which the mentioned proposition forms the top. All sub-ordered propositions correspond to subspaces of the item's subspace. In this way the universe of items can be represented by a set of mutual orthogonal subspaces of the Hilbert space. Rays that are spanned by a single Hilbert vector and that are connected with a numeric value can be considered as atomic predicates. Subspaces spanned by such rays that are related to the same type of value can be considered as statements with a wider scope. The rays can be subspace of an items subspace. The subspace that corresponds to a conglomerate of elementary items also represents that conglomerate as an item. The configuration of the subspace that represents an item will change as a function of the parameter that measures the progression of the dynamic behavior of the item. It is possible that not only the values of the atomic predicates change. The types of these atomic predicates may change as well. This happens for example with atomic types that are each other's canonical conjugate. It is also possible that the configuration of the subspace changes more drastically.

A nice extra is the fact that the subspace can be moved around (rotated around the origin) in Hilbert space. In this way it may be possible to implement the dynamics of items. This moving around does not mean that the vectors are moved around. It means that at each step of the move the set of vectors that span the considered subspace is **redefined**. The redefinition corresponds to a redefinition of the corresponding proposition. Thus, redefinition and the laws that govern redefinition convert the static quantum logic into a dynamic version of quantum logic. It will be shown that physical fields play a significant role in this redefinition.

With his bra-ket notation Dirac has provided us with a marvelous symbolism for vectors and even for operators. He did not provide us with symbols for subspaces. However, it is easy to extend his symbolism and indicate a subspace with a set of vectors that spans that subspace. For example $\{|f_s\rangle\}_s$ indicates a set of element vectors $|f_s\rangle$ with enumerator s that span a closed subspace. This set identifies the subspace. Different sets may identify the same closed subspace.

It is sensible to have one vector inside the item's subspace that is considered as characteristic for the location of the representation of the item in Hilbert space. We reserve the name **locator** for this vector. When the item is redefined, that vector may be redefined as well. This characteristic vector can be used to obtain a precise location of the subspace in Hilbert space. The process via which the locator is determined depends on the requirements. The requirements may be set in relation to an operator. For example the vector that corresponds with the expectation value of the operator for that subspace can be chosen as the locator. In

that case the state vector that corresponds with that operator may play the role of the locator. Two or more bosons can share the same locator. Fermions that possess the same property values cannot share the same vector as a locator.

Atomic predicates are not considered to be statements that describe a physical item. The statement "This is the item" forms the top of a hierarchy of statements that all say something about the item. The hierarchy contains statements that define the item's type. Other members of the hierarchy specify the items constituents. Still other statements concern the item's atomic variables that together with the type definition specify the item's identity. For atoms the variables of the subsystems are hidden from the outside of the atom. This means that atoms can be considered as modules.

Vacuum

Multidimensional subspaces exist that do not represent a dynamical item. They can be considered as **vacuum**. It is still possible that the subspace represents a ground state. We will assume that on the average the 'filled' and the vacuum subspaces are evenly distributed over a connected part of the Hilbert space. The phrase "evenly distributed" means that the distance between the representations of items makes sense. Here we do not mean the distance related to the norm of Hilbert vectors, but the coordinate related distance that will be introduced later.

"Vacuum" does not say that these subspaces are empty. It is rather an indication that the subspace does not represent a dynamical object. Instead the subspace may represent a ground state.

Fields

Physical fields are not physical items. Physical fields are represented by Hilbert fields. For each Hilbert field, every member of an orthonormal base of the Hilbert space corresponds to a value of the field. Hilbert fields have much in common with wave functions. In quantum field dynamics the Hilbert fields play a similar role as wave functions do in quantum mechanics. Each elementary type corresponds to a kind of physical field. With each fermion type an anti-type exists.

Sign selections

Four possibilities exist due to the sign selections of the quaternions. One sign selection is covered by the conjugation $a \rightarrow a^*$. The other is caused by the handedness $a \rightarrow a^{\otimes}$. When both combine then the superscript $a \rightarrow a^{\oplus}$ is used. It is also possible to use the extended **quaternionic conjugation**:

$$a^{*} = a^{①} \tag{1}$$

$$a^{\otimes} = a^{②} \tag{2}$$

$$a^{\oplus} = a^{③} \tag{3}$$

$$a^{****} = a^{\otimes\otimes} = a^{④} = a^{⓪} = a \tag{4}$$

This differs from the complex conjugation:

$$a^{**} = \left(a^{①}\right)^{①} = a \tag{5}$$

The effects of the quaternionic conjugation are visible in the base numbers 1, **i**, **j**, **k**:

$$1^{*} = 1 \tag{6}$$

$$\mathbf{i}^{①} = -\mathbf{i} \tag{7}$$

$$\mathbf{i}^{②} = \mathbf{j} \tag{8}$$

$$\mathbf{i}^{③} = -\mathbf{j} \tag{9}$$

$$\mathbf{i}^{④} = \mathbf{i} \tag{10}$$

$$\mathbf{j}^{①} = -\mathbf{j} \tag{11}$$

$$\mathbf{j}^{②} = \mathbf{i} \tag{12}$$

$$\mathbf{j}^{③} = -\mathbf{i} \tag{13}$$

$$\mathbf{j}^{④} = \mathbf{j} \tag{14}$$

$$\mathbf{k}^{①} = -\mathbf{k} \tag{15}$$

$$\mathbf{k}^{②} = \mathbf{k} \tag{16}$$

$$\mathbf{k}^{③} = -\mathbf{k} \tag{17}$$

$$\mathbf{k}^{④} = \mathbf{k} \tag{18}$$

Thus **k** follows the rules of complex conjugation. This renders its direction to a special direction. This direction is called the **longitudinal** direction. The directions of **i** and **j** are transverse directions. Apart from that they are mutual perpendicular and perpendicular to **k**, they have no preferred direction.

These properties hold locally, they are related to other properties of the operators or the fields that carry these properties.

Product rule

We use the quaternionic product rule. It has eight (16-8) versions. When either a or b is fixed, then the product has four versions:

$$ab = a^{⓪}b^{⓪} = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} \tag{1}$$

$$a^{⓪}b^{①} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} + \boldsymbol{a} b_0 - \boldsymbol{a} \times \boldsymbol{b} \tag{2}$$

$$a^{⓪}b^{②} = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} + \boldsymbol{a} b_0 - \boldsymbol{a} \times \boldsymbol{b} \tag{3}$$

$$a^{⓪}b^{③} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} \tag{4}$$

$$a^{①}b^{⓪} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} + \boldsymbol{a} b_0 - \boldsymbol{a} \times \boldsymbol{b} = a^{⓪}b^{①} \tag{5}$$

$$a^{①}b^{①} = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} - \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} \tag{6}$$

$$a^{①}b^{②} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} - \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} \tag{7}$$

$$a^{①}b^{③} = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} - \boldsymbol{a} b_0 - \boldsymbol{a} \times \boldsymbol{b} \tag{8}$$

$$a^{②}b^{⓪} = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} + \boldsymbol{a} b_0 - \boldsymbol{a} \times \boldsymbol{b} = a^{⓪}b^{②} \tag{9}$$

$$a^{②}b^{①} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} = a^{⓪}b^{③} \tag{10}$$

$$a^{②}b^{②} = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} = a^{⓪}b^{⓪} \tag{11}$$

$$a^{②}b^{③} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} = a^{⓪}b^{③} \tag{12}$$

$$a^{③}b^{⓪} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} - \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} = a^{①}b^{②} \tag{13}$$

$$a^{③}b^{①} = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} - \boldsymbol{a} b_0 - \boldsymbol{a} \times \boldsymbol{b} = a^{①}b^{③} \tag{14}$$

$$a^{③}b^{②} = a_0 b_0 + \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} - \boldsymbol{a} b_0 - \boldsymbol{a} \times \boldsymbol{b} \tag{15}$$

$$a^{③}b^{③} = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0 \boldsymbol{b} - \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} = a^{①}b^{①} \tag{16}$$

Products of the form *aa* have two versions.

$$\begin{aligned} aa = a^{⓪}b^{⓪} &= a^{⓪}a^{②} = a^{②}a^{⓪} \\ &= a^{①}a^{③} = a^{③}a^{①} = a^{①}a^{①} = a^{②}a^{②} = a^{③}a^{③} \\ &= a_0 a_0 - \langle \boldsymbol{a}, \boldsymbol{a} \rangle \end{aligned} \tag{17}$$

$$\begin{aligned} a^{⓪}a^{①} = a^{①}a^{⓪} &= a^{⓪}a^{③} = a^{③}a^{⓪} = a^{②}a^{③} \\ &= a^{③}a^{②} = a^{①}a^{②} = a^{②}a^{①} \\ &= a_0 a_0 + \langle \boldsymbol{a}, \boldsymbol{a} \rangle \end{aligned} \tag{18}$$

Product sub-terms

The product *ab* contains two particular sub-terms:

$$a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} \tag{1}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{2}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{i}(a_2 b_3 - a_3 b_2) + \boldsymbol{j}(a_3 b_1 - a_1 b_3) + \boldsymbol{k}(a_1 b_2 - a_2 b_1) \tag{3}$$

The products $a_0\boldsymbol{b}$ and $\boldsymbol{a}b_0$ have two versions.
The product $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ has two versions.
The product $\boldsymbol{a} \times \boldsymbol{b}$ has two versions.
The sum $a_0\boldsymbol{b} + \boldsymbol{a}b_0$ has four versions. If either *a* or *b* is fixed it has two versions.

Operators

The sign selections of operator $\nabla = (\nabla_0, \boldsymbol{\nabla})$ depend on the sign selections of position operator Q, which determines the sign selections for its eigenvalues $q = (q_0, \boldsymbol{q})$.

Normally we consider the sign selection for operators *Q* and ∇ fixed to operators $Q^{⓪}$ and $\nabla^{⓪}$. Sometimes we chose instead operator $\nabla^{①}$.

Quaternionic conjugation is directly connected with the concepts of **parity** and **spin**.

For quaternionic functions symmetry reduces the differences that are produced by conjugation and anti-symmetry stresses the differences. The same holds for operators.

Matrices

Another possibility is to present sign selections by matrices.

$$\alpha_1 = \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{1}$$

$$\alpha_2 = \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{2}$$

$$\alpha_3 = \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{3}$$

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4}$$

Construction

The Cayley-Dickson construction formula enables the generation of a quaternion from two complex numbers:

$$p = a_0 + a_1\mathbf{k} + \mathbf{i}(b_0 + b_1\mathbf{k}) \tag{1}$$

$$q = c_0 + c_1\mathbf{k} + \mathbf{i}(d_0 + d_1\mathbf{k}) \tag{2}$$

$$(a, b)\,(c, d) = (ac - db^{*}; a^{*}d + cb) \tag{3}$$

$$r = pq \tag{4}$$

$$r_0 = a_0c_0 - a_1c_1 - b_0d_0 - b_1d_1 \tag{5}$$

$$r_k = a_0c_1 - a_1c_0 - b_0d_1 + b_1d_0 \tag{6}$$

$$r_i = a_0d_0 + a_1d_1 + b_0c_0 - b_1c_1 \tag{7}$$

$$r_j = -a_1d_0 + a_0d_1 + b_0c_1 + b_1c_0 \tag{8}$$

Quaternionic conjugation ② means the exchange of r_i and r_j.

Colors

In the following text, the consequences for the product of the sign choices of the conjugate ① or is indicated by blue color ±. The consequence for the product of the choice of the handedness ② of the cross product is indicated by red color ±. The mixed conjugate ③ acts accordingly on both colors.

The sign selections split the ring of quaternions in four different realizations.

GPS of Hilbert space

The first step is the introduction of a suitable GPS system in Hilbert space. This can be done by taking an orthonormal base of Hilbert vectors and add quaternion values to them. The number set must be countable. Let us take the rational quaternions as an example. This construction defines a normal operator Q with countable infinite number of eigenvectors |ϙ> and corresponding eigenvalues ϙ. We will use the name **coordinate space** for the eigenspace of the **coordinate operator** Q.

When we speak about the **coordinate distance** between two vectors |f> and |g> in Hilbert space, then we mean the distance between the values of <f|Q f>/<f|f> and <g|Q g>/<g|g>.

Q has an infinite but countable number of eigenvalues. A Location in coordinate space represents a location in on the unit sphere of Hilbert space. Thus in fact the eigenspace of Q must be *treated as an affine space*. The uni-coordinate space is by definition an affine space.

Take the polar decomposition of Q in a unitary part Ụ and a positive operator Ṇ. The eigenspace of Ụ is the **uni-coordinate space**. Like the unit sphere of the Hilbert space, the uni-coordinate space *is an affine space*. Besides of that also no a preferred direction exists in this unit sphere. Subspaces containing an infinite number of eigenvectors of Q correspond to each separate eigenvalue ụ of Ụ.

Position

The original proposition (♠) speaks about the position of the item. The position must be related to something that is available in the Hilbert space. The Hilbert space is defined over a number space. Thus we might attach a number of this field (or a higher 2n-on) to the

subspace that represents the item. That number must represent position. The natural way of attaching numbers to subspaces of a Hilbert space is via the concept of eigenvalues of normal operators. Any symmetry transform of the coordinate operator Q meets the requirements. However, this position does not reflect the change of the longitudinal direction with the location in an analyzed Hilbert field. Thus it may be sensible to allow transformations of the GPS space that produces a correspondingly curved eigenspace of position operator Q. In that case the quaternionic Fourier transform in Q coordinates becomes a wider validity range.

The separable Hilbert space has a countable dimension. It means that the eigenvalues may offer a dense coverage of a connected part of the number space, but it is not a closed coverage. It does not include all limits of all convergent rows. Thus it is sensible to attach a tiny environment of the actual eigenvalue to each eigenvector. In this way the position is expressed in a tiny environment rather than in a single number. At least the position is represented by a single eigenvector and in this way the whole number space is covered by the set of eigenvectors. The eigenvector represents an atomic predicate that represents the position attribute of the considered item. The eigenvector lies inside the subspace that represents the item. The corresponding atomic predicate states that the position of the item lies inside the environment that is represented by the eigenvector.

Now the position is connected to eigenvectors. The physical item is connected to a subspace rather than to a single vector. So we can use the localizer as a more precise indicator of the position of the physical item. On the other hand physical items are characterized by a state. A state is either a wave function or a probability density operator. Each wave function represents a probability amplitude and can be presented by a Hilbert ray. A Hilbert ray is a one dimensional subspace that is spanned by a single Hilbert vector. The locator of the item is such a vector. In that case the wave function indicates the probability of finding the position of the localizer. The probability density operator is a weighted projection operator that is related both to the subspace that represents the item and to the position operator. It represents the probability that after measuring the position the parameter of the density distribution is found as the result.

The fact, that the position operator must be bounded in order to guarantee that its eigenvectors span the whole Hilbert space, is not crucial to our model. It is sufficient when all positions that are connected to items stay in a finite sphere. When the position operator is blurred and its eigenspace is an affine possibly curved space, then this criterion is fulfilled.

The deliberations that the eigenvalues of operators need not be restricted to the hyper complex number space that is used to specify the Hilbert space also hold for the position operator. The positions may be elements of a curved manifold. In this case we will call the position operator relaxed. The curved position space may be seen as the result of the actions of the fields. The fields themselves can be seen either as functions of the observed curved position space or as a functions of the untransformed GPS operator.

Functions and fields

Functions in quaternionic Hilbert space

Every Hilbert vector |f> can be combined with the eigenvectors $\{|q>\}_q$ of a compact normal operator Q to give a corresponding Hilbert function f(q). The values of this function follow from the inner products of the vector with the eigenvectors of the operator, while the corresponding eigenvalues $\{q\}_q$ form the variable of the function.

$$f(q) = < f|q > \tag{1}$$

Due to its definition the Hilbert functions are only defined for an infinite but countable number of parameter values that lay dense in quaternion space. The Hilbert functions are infinitely close to a corresponding differentiable function. Functions that are defined in the way described here will be called sharp Hilbert functions.

A locatable probability distribution can be described by the convolution of a Dirac delta function, which indicates the Hilbert vector that represents the location of the weighted center of that distribution and a function $f(q)$ that describes the distribution relative to that location. In this way a blurred Hilbert vector is defined. This means that a blurred Hilbert vector is defined by the combination of a sharp locator Hilbert vector and a sharp shape Hilbert vector. The blurred Hilbert vector represents a Hilbert probability distribution.

Pure states are characterized by blurred Hilbert vectors.

Differentiation

The operator ∇ is directly related to operator Q. Thus, the sign selections for Q transfer to the sign selections for operator ∇. With a fixed field there exist four nabla operations. With a fixed nabla operator there may exist four results.

$$\mathrm{g}(q) = \nabla f(q) = \nabla_0 f_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) \pm (\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q)) \tag{1}$$

$$\bar{\mathrm{g}}(q) = \nabla^{①} f(q) \tag{2}$$

$$= \nabla_0 f_0(q) \pm \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \pm \nabla_0 \boldsymbol{f}(q) - \boldsymbol{\nabla} f_0(q) \pm (\mp \boldsymbol{\nabla} \times \boldsymbol{f}(q))$$

∇ turns a symmetric field $f(q)$ into an anti-symmetric field $\nabla f(q)$ and an anti-symmetric field into a symmetric field.

The fact that $\nabla f(q) = 0$ means that $f(q)$ is constant or that at location q function $f(q)$ is in a maximum, a minimum, a saddle point or an asymmetric plateau. The consequence of this restriction is:

$$\nabla_0 f_0(q) = \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \tag{3}$$

$$\pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) = \mp (\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q)) \tag{4}$$

The fact that $\nabla^{①} f(q) = 0$ leads to different equations.

$$\nabla_0 f_0(q) = \pm\langle \boldsymbol{\nabla}, \boldsymbol{f}(q)\rangle \tag{4}$$

$$\pm\nabla_0 \boldsymbol{f}(q) - \boldsymbol{\nabla} f_0(q) = \pm\big(\pm\boldsymbol{\nabla} \times \boldsymbol{f}(q)\big) \tag{5}$$

The quaternionic Laplace operator Δ is defined by

$$h(q) = \Delta f(q) = \nabla^{①}\nabla f(q) = \nabla\nabla^{①} f(q) \tag{5}$$

$$= \nabla_0{}^2 f(q) + \boldsymbol{\nabla}^2 f(q)$$

A quaternionic function that fulfills $\Delta f(q) = 0$ is a harmonic function.
A quaternionic function that fulfills $\boldsymbol{\nabla}^2 f(q) = 0$ is a spatial harmonic function.

Distributions in quaternionic Hilbert space

Using a compact normal operator Q and a second distribution operator ρ with the same eigenvectors $\{|q>\}_q$ but with eigenvalues $\{\rho_q\}$ we can generate a Hilbert distribution.

$$\rho(q) = < q|\rho\ q > \tag{1}$$

Operator ρ need not be a compact normal operator. Its spectrum of eigenvalues may be confined to a discrete set of points. Further, its eigenvalues ρ_q can be hyper complex 2^n-ons where n can by larger than 2. In that case $\rho(q)$ is a 2^n-on distribution. A Hilbert distribution is not differentiable. It can be seen as a combination of a set of Dirac delta functions that are multiplied with hyper complex numbers. If all numbers are quaternions, then it is a linear combination of Dirac delta functions.

Discrete distribution

The Hilbert space is separatable. This means that the set of eigenvalues of an operator is countable. Thus a Hilbert distribution $\rho(q)$ is always discrete:

$$\rho(q) = \sum_i q_{E\,i} \cdot \delta(q - q_i) \tag{1}$$

The factors $q_{E\,i}$ are hyper complex 2^n-ons.
However, the distribution can represent a very dense coverage. In that case the distribution can become quasi differentiable.
Convolution with a blurring spread function can render the result (mostly) differentiable. Depending on the blur, the result may still be singular on the definition points of the Hilbert distribution.
The blur may represent a probability distribution.

Hilbert field

By blurring the Hilbert distribution with a suitable spread function, the distribution can be transformed into a mostly continuous function. This converts the Hilbert distribution $\rho(q)$ into a skew Hilbert field $\phi(q)$.

$$\phi(q) = f(q) \circ \rho(q) \tag{1}$$

With a given Hilbert distribution $\rho(q)$, each blurring function $f(q)$ causes a corresponding Hilbert field.

Hilbert fields that correspond to the same Hilbert distribution form a Hilbert field set.
Each Hilbert field in this set belongs to the same subset that consists of different conjugate versions of the same Hilbert field.

A Hilbert field can be categorized according to its:

- Symmetries
- Conjugation
- Corresponding distribution

The differential of a symmetric field of field part is anti-symmetric.
The differential of an anti- symmetric field of field part is symmetric.

Blur function

The blur is a spread function. It is the reason of the significant similarity between optics and quantum physics. On the other hand, the blur is a probability distribution. This is the source of quantum noise.

The blur is related to the ground state.
Hermite functions, which are eigenfunctions of the Fourier transformation, have a Gaussian blur.
Coherent states, which are eigenfunctions of ladder operators have a Poissonian blur.

Quantum shot noise produces a Poisson distribution. When large numbers of quanta are produced the distribution approaches a Gaussian distribution. A binomial process that follow a noise generating Poisson process can be combined with that binomial process into a generalized Poisson process with a lower efficiency. The binomial process represents a weakening effect. Spatial blur can be interpreted as a binomial process. This is because it represents a spatial integration effect. The efficiency of the detection of quanta is characterized with the detective quantum efficiency (DQE) of the detector. Together with the Fourier transform of the spatial spread function this determines the signal to noise ratio in the information stream. The spread has an integrating (smoothing) effect. A sharper spread improves the signal, but also increases the noise.

The blur plays a role when canonical conjugate operators occur together or in sequence. The blur is caused by the inaccuracy of the combination of these operators.
The blur has many functions and interpretations:

- Convolution with a spread function makes a Hilbert distribution differentiable.
- The spread ensures the compactness of corresponding operators. It also reduces the frequency range covered by its Fourier transform.
- The spread function represents a probability distribution.
- The probability distribution is characteristic for the inaccuracy of a category of operators, such as the Fourier transform, the ladder operators and the number operator.
- A spread function characterizes a ground state.
- The spread represent the probability that virtual items exchange roles with actual items.
- The virtual items represent subspaces of virtual Hilbert spaces that are ready to exchange roles with the currently valid Hilbert space.
- The virtual items represent virtual quantum logical propositions that may exchange roles with currently actual propositions.

- The virtual quantum logical propositions are elements of a virtual traditional quantum logic that is ready to exchange roles with the currently actual traditional quantum logic.
- The blur can be interpreted as a spatial quantum noise distribution.
- The blur works as storage of past, present and future conditions.
- The blur can be squeezed in order to reflect the importance of momentum versus position.
- A basic (ground) blur has in each direction a symmetric cut. A odd-times differentiated blur has in one direction an asymmetric cut. An even-times differentiated blur has in each direction a symmetric cut.
- The blur represents the sticky resistance of the universe against unordered changes (= changes of uniform movement in a geodesic). This is proved by the existence of inertia.
- The blur represents the sticky resistance of the collection of all propositions against unordered redefinitions.

In short: Without blur (quantum) physics is impossible!

Vacuum expectation value

The **vacuum expectation value** (also called **condensate** or simply VEV) of an operator is its average, expected value in the vacuum. The vacuum corresponds to a ground state. The vacuum expectation value of an operator O is usually denoted by $\langle O \rangle$.

Harmonic functions

Harmonic functions are suitable spread functions.
For a harmonic function $f(q)$ holds:

$$\Delta f(q) = \nabla\nabla^{①} f(q) = 0 \tag{1}$$

See for example the section Potential.

Conservation of charge

When $\rho_0(q)$ is interpreted as a charge density distribution, then the conservation of charge is given by:

Total change within V = flow into V + production inside V

$$\frac{d}{dt}\int_V \rho_0 \, dV = \oint_S \hat{\boldsymbol{n}}\rho_0 \boldsymbol{v} \, dS + \int_V s_0 \, dV \tag{1}$$

This equation represents the conservation of charge.
Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface S, $\boldsymbol{v}(q)$ is the velocity at which the charge density $\rho_0(q)$ enters volume V and s_0 is the source density inside V
With the help of Gauss theorem this converts into:

$$s_0(q) = \nabla_0\rho_0(q) \pm \langle \boldsymbol{\nabla}, \left(\rho_0(q)\boldsymbol{v}(q) + \boldsymbol{\nabla} \times \boldsymbol{a}(q)\right)\rangle \tag{2}$$

$$= \nabla_0\rho_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q)\rangle$$

This equation represents a balance equation for charge (or mass) density.
Here $\boldsymbol{\rho}(q)$is the current density and $\boldsymbol{a}(q)$ is an arbitrary differentiable vector function.
The blue colored ± indicates quaternionic sign selection through conjugation ① or ③.

The quaternionic charge density is given by:

$$\rho(q) = \rho_0(q) + \boldsymbol{\rho}(q) = \rho_0(q) - \rho_0(q)\boldsymbol{v}(q) \tag{3}$$

The quaternionic source density $s(q)$ is given by:

$$s(q) = s_0(q) + \boldsymbol{s}(q) = \nabla\rho(q) \tag{4}$$

$$\boldsymbol{\nabla} s_0(q) = \nabla_0 \boldsymbol{\nabla}\rho_0(q) \mp \boldsymbol{\nabla}\langle\boldsymbol{\nabla}, \boldsymbol{\rho}(q)\rangle \tag{5}$$

Probability amplitude, probability density and probability current

The probability amplitude ψ(q) can be used to define a probability density and probability current. See http://www.vttoth.com/qt.htm.

$$\rho_0(q) = |\psi(\mathrm{q})|^2 = (\psi^*(\mathrm{q})\psi(\mathrm{q})) \tag{1}$$

$$\boldsymbol{\rho}(q) = \psi^*(\mathrm{q})\,\boldsymbol{\alpha}\,\psi(\mathrm{q}) \tag{2}$$

$$\alpha_1 = \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{3}$$

$$\alpha_2 = \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{4}$$

$$\alpha_3 = \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{5}$$

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{6}$$

Compare Dirac:

$$\frac{\partial\psi}{\partial t} = (-\boldsymbol{\alpha} i\nabla \pm \beta m)\psi \tag{7}$$

Canonical conjugate

The canonical conjugate of the operator Q is the operator P. It is defined by using a complex subspace of a quaternionic number space that is used to specify inner products. It is defined by specifying the function that defines the inner products of the eigenvectors |q> of Q and |p> of P with real eigenvalues q and p.

$$< q|p > = \tilde{f}(p) = < p|q >^* = f^*(q) = exp\,(\boldsymbol{k} \cdot p \cdot q/\hbar) \tag{1}$$

The constant ħ is Planck's constant and relates to the granularity of the eigenspaces. The imaginary base number $\boldsymbol{k}$ belongs to a complex subspace of the quaternionic number space.

Due to its specification, the canonical conjugate operator P can be interpreted as a generator of displacement of the eigenvalues of Q. For this purpose the considered function $f(q)$ must be differentiable.

$$P = \boldsymbol{k} \cdot \hbar \cdot \frac{\partial}{\partial q} \tag{2}$$

$$< q|P\, f > = \boldsymbol{k} \cdot \hbar \cdot \frac{\partial}{\partial q} f(q) \tag{3}$$

The definition leads to the commutator:

$$[P, Q] = PQ - QP = \boldsymbol{k} \cdot \hbar \tag{4}$$

The sign selections of P depend on the sign selections of Q.

Complex Fourier transform

The specification of the canonical conjugate also defines a **complex** Fourier transform.

Let $|f>$ be the generator of a quaternionic function that is generated with the help of the eigenvectors and eigenfunctions of operator Q with canonical conjugate P.

The Fourier transform $F_i = U_{qp}$ converts the base $\{|q>\}_q$ into the base $\{|p>\}_p$. The inverse Fourier transform U_{pq} does the reverse.

$$< q|f > = < f|q >^* = f^*(q) = \sum_p (< q|p > \cdot < p|f >) \tag{1}$$

$$= \Sigma_p < p\ (< p|q >)|f >$$

$$= \Sigma_p < p\ U_{pq}|f >$$

$$= \sum_p < p|U_{qp}\, f >$$

$$< p|f > = \sum_q (< p|q > \cdot < q|f >) \tag{2}$$

$$= \sum_q < q|U_{pq}\, f >$$

The complex Fourier transform of a symmetric (complex) function is a cosine transform. It is a real function.

The complex Fourier transform of an anti-symmetric (complex) function is a sine transform. It is an imaginary function.

Through Fourier transformation the operators P and Q exchange roles.

Heisenberg's uncertainty

The Heisenberg's uncertainty principle is a consequence of the definition of the combination of the canonical conjugate and the definition of the Hilbert field. It means that a small spread of q values goes together with a large spread of p values and vice versa.

$$\Delta q \cdot \Delta p \geq \hbar/2 \tag{1}$$

A **squeezed** coherent state is any state such that the uncertainty principle is saturated. That is:

$$\Delta q \cdot \Delta p = \hbar/2$$

See: http://en.wikipedia.org/wiki/Squeezed_coherent_state.

Quaternionic Fourier transform split

The complex Fourier transform specified above is a **longitudinal quaternionic** Fourier transform $\mathcal{F}_{\mathbf{k}}$ that works on the complex part of the function that is defined via the imaginary base number **k**. The longitudinal Fourier transform concerns the rotation free part of the function or field. This fact concerns a local property, so definition of the longitudinal part of a field or function has only local validity.

$$\mathcal{F}_{\mathbf{k}}\big(f(q)\big) = \mathcal{F}\big((f((q,\mathbf{k})),\mathbf{k})\big) \tag{1}$$

Or

$$\mathcal{F}_{\parallel}\big(f(q)\big) \equiv \mathcal{F}\left(f_{\parallel}(q)\right) \tag{2}$$

It relates to the full quaternionic Fourier transform $\mathcal{F}$

$$\mathcal{F}\big(f(q)\big) = \tilde{f}(p) \tag{3}$$

The inverse Fourier transform runs:

$$\mathcal{F}^{-1}\big(\tilde{f}(p)\big) = f(q) \tag{4}$$

The split in longitudinal and transverse Fourier transforms corresponds to a corresponding split in the multi-dimensional Dirac delta function.

The transverse Fourier transforms

Apart from the longitudinal quaternionic Fourier transform a **transverse quaternionic** Fourier transform exists. Like the longitudinal part, the definition of the transverse part of a function or field has only local validity.

$$\mathcal{F}_{\perp}\big(f(q)\big) \equiv \mathcal{F}\big(f_{\perp}(q)\big) \tag{1}$$

This Fourier transform acts on the resulting subspace of the quaternionic number space.

Like quaternions, quaternionic functions and operators can be split in a longitudinal part and a transverse part.

Transverse plane

The Cayley-Dickson construction, as well as Warren Smith's construction formula shows that the transverse part can be considered as a complex number space multiplied with a fixed imaginary quaternionic base number. The selection of the imaginary base number $\boldsymbol{i}$ is arbitrary as long as it is perpendicular to $\boldsymbol{k}$. The resulting plane is spanned by axes $\boldsymbol{i}$ and $\boldsymbol{ik}$. When base number $\boldsymbol{i}$ is divided away, then a normal complex number space results. When the origin of the new plane is shifted, then any displacement in the new plane means an angular displacement with respect to the first origin. In that case the momentum operator for this plane is in fact an angular momentum operator.

Also here a complex Fourier transform can be defined in a way that is similar to the longitudinal Fourier transform. It must be reckoned that the sign selections for these directions differ.

Fourier modes

There is still another way to split the quaternion space into complex spaces. Let us redefine the longitudinal canonical conjugate and define two transverse canonical conjugates. All three share the real axis. May be this share includes a shift with respect to the origin of the real axis. Similar shifts may exist with respect to the origin of the imaginary axes.

$$< q_k|p_k > = \tilde{f}_k(p_k) = < p_k|q_k >^* = {f_k}^*(q_k) = exp\,(\boldsymbol{k} \cdot (p_0 q_0 + p_k q_k + \delta_k)/\hbar) \quad (1)$$

$$< q_i|p_i > = \tilde{f}_i(p_i) = < p_i|q_i >^* = {f_i}^*(q_i) = exp\,(\boldsymbol{i} \cdot (p_0 q_0 + p_i q_i + \delta_i)/\hbar) \quad (2)$$

$$< q_j|p_j > = \tilde{f}_j(p_j) = < p_j|q_j >^* = {f_j}^*(q_j) = exp(\boldsymbol{j} \cdot (p_0 q_0 + p_j q_j + \delta_j)/\hbar) \quad (3)$$

A **Fourier mode** corresponds to the inner product of an eigenvector of Q and an eigenvector of P. Thus equations 1, 2 and 3 describe shifted Fourier modes The shifts are indicated as $\delta_{\mu..}$. When the blurring function stretches very wide, then in that direction the blurred Hilbert field approaches a Fourier mode.

Fourier modes can be combined. For example two synchronized (same p_0 and q_0) modes can be combined into elliptically polarized waves. When the shifts are equal, then the wave is circular polarized. When the shifts are half a radian apart, then the wave is linearly polarized. The direction of the wave is perpendicular to the two selected imaginary axes.

If we accept that the longitudinal direction is runs along $\boldsymbol{k}$, then for a given Hilbert field ψ, the imaginary direction is a linear combination of directions $\boldsymbol{i}$ and $\boldsymbol{j}$. This direction may vary with q_0.

If we take the field ψ as a guide, then at every instance of q_0 the longitudinal direction $\boldsymbol{k}$ runs in the direction of the divergence $\nabla\psi_0$ of the field ψ_0.

Alternative approach

The following draws from the work of S. Thangavelu.

Let us take the non-abelian group $\mathbb{H}_1$ which is $\mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R}$ with the group law

$$(x, y, t)(x_0, y_0, t_0) = (x + x_0;\ y + y_0;\ t + t_0 + x\, y_0) \tag{1}$$

Then it is clear that $\mathbb{H}_1$ is non-abelian and the Lebesgue measure $dx\, dy\, dt$ is both left and right invariant Haar measure on $\mathbb{H}_1$. With this measure we can form the Hilbert space $L^2(\mathbb{H}_1)$. Let $\Gamma = \mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z}$. Then it is easy to check that Γ is a subgroup of $\mathbb{H}_1$ so that we can form the quotient $M = \Gamma/\mathbb{H}_1$ consisting of all right cosets of Γ. Functions on M are naturally identified with left Γ-invariant functions on $\mathbb{H}_1$. As the Lebesgue measure $dx\, dy\, dt$ is left Γ-invariant we can form $L_2(M)$ using the Lebesgue measure restricted to M. As a set we can identify M with $[0, 1)^3$ and we just think of $L^2(M)$ as $L^2([0, 1)^3)$.

Fourier expansion in the last variable allows us to decompose $L^2(M)$ into a direct sum of orthogonal subspaces. Simply define $\mathcal{H}_k$ to be the set of all $f \in L^2(M)$ which satisfy the condition

$$f(x, y, t + s) = \exp(2\pi i k s)\, f(x, y, t) \tag{2}$$

Then $\mathcal{H}_k$ is orthogonal to $\mathcal{H}_j$ whenever $k \neq j$ and any $f \in L^2(M)$ has the unique expansion

$$f = \sum_{k=-\infty}^{\infty} f_k;\ f_k \in \mathcal{H}_k \tag{3}$$

We are mainly interested in $\mathcal{H}_1$ which is a Hilbert space in its own right. It is interesting to note that functions in $\mathcal{H}_1$ are also invariant under the left action of Γ.

Our next example of a unitary operator is the following. Consider the map $J : \mathcal{H}_1 \to \mathcal{H}_1$ given by

$$J(x, y, t) = (-x, y, t - xy) \tag{4}$$

$$J^{\dagger}(x, y, t) = (x, -y, t - xy) \tag{5}$$

$$J^{\dagger} = J^{-1} \tag{6}$$

$$J^2(x, y, t) = J(-x, \cdot\, y;\ t - xy) = (-x, -y; t) \tag{7}$$

$$J^4 = I \tag{8}$$

$$J(0, 0, t) = (0, 0, t) \tag{9}$$

$$J f(x, y, t) = f(J(x, y;\ t)) = f(-x, \cdot\, y, t - xy) \tag{10}$$

Weil-Brezin transform

Next consider the Weil-Brezin transform V:

$$V\,f(x,y,t) \;=\; exp(2\,\pi\,\boldsymbol{k}\,t)\,\sum_{n} f(x\;+\;n)\,\exp(2\,\pi\,\boldsymbol{k}\,n\,y) \tag{11}$$

$$\int_{y=0}^{1} |V\,f(x,y,t)|^2 dy = \int_{x=0}^{1} \sum_{n=-\infty}^{n=\infty} |f(x\;+\;n)|^2\,dx \tag{12}$$

$$\iiint_{0}^{1} |V\,f(x,y,t)|^2 dx\,dy\,dt = \int_{0}^{1} |\,f(x)|^2 dx \tag{13}$$

V is unitary.
See also Zak transform

Fourier transform

We define the Fourier transform $\mathcal{F}$ by:

$$\mathcal{F} \;=\; V^{\dagger}\,J\,V \tag{14}$$

- $\mathcal{F}^4\,f \;=\; f$; for every $f \;\in\; L^2(\mathbb{R})$
- $\mathcal{F}^2\,f(x) \;=\; f(-x)$; for almost every $x \;\in\; \mathbb{R}$
- $\|\mathcal{F}\,f\|^2 \;=\; \|f\|^2$

For $f \;\in\; L^1(\mathbb{R}) \setminus L^2(\mathbb{R})$ the Fourier transform is given by

$$\mathcal{F}\mathrm{f}(\xi) \;=\; \int_{x\,\in\,\mathbb{R}} \mathrm{f(x)}\cdot\exp(2\,\pi\,\boldsymbol{k}\,x)dx \tag{15}$$

If we further assume that $\mathcal{F}f \;\in\; L^1(\mathbb{R})$ then for almost every x we have

$$\mathrm{f}(\xi) \;=\; \int_{\xi\,\in\,\mathbb{R}} \mathcal{F}\mathrm{f}(\xi)\,\exp(2\,\pi\,\boldsymbol{k}\,x\,\xi)dx \tag{16}$$

Fourier transform eigenfunctions

In this section we confine to a complex part of the Hilbert space.
See http://en.wikipedia.org/wiki/Hermite_polynomials.
There exist two types of Hermite polynomials: (1, 2)

1. The probalist's Hermite polynomials:

$$H_n^{prob}(z) = (-1)^n \exp(½z^2)\,\frac{d^n}{dz^n}\exp(-½z^2).$$

2. The physicist's Hermite polynomials

$$H_n^{phys}(z) = (-1)^n \exp(z^2)\frac{d^n}{dx^n}\exp(-z^2) = \exp(½z^2)\left(z-\frac{d}{dz}\right)\exp(-½z^2)$$

These two definitions are *not* exactly equivalent; either is a rescaling of the other:

$$H_n^{phys}(z) = 2^{n/2}\, H_n^{prob}\left(z\sqrt{2}\right) \tag{3}$$

In the following we focus on the physicist's Hermite polynomials.

The Gaussian function ϕ(z) defined by

$$\varphi(x) \;=\; exp(-\pi\, z^2) \tag{4}$$

is an eigenfunction of $\mathcal{F}$. It means that its Fourier transform has the same form.

As $\mathcal{F}^4 =$ I any λ in its spectrum $\sigma\,(\mathcal{F})$ satisfies $\lambda^4 = 1$: Hence,

$$\sigma\,(\mathcal{F}) \;=\; \{1; -1;\; i; -i\}. \tag{5}$$

We take the Fourier transform of the expansion:

$$exp(-½\, z^2 +\; 2\, z\, c -\; c^2) \;=\; \sum_{n=0}^{\infty} exp(-½\, z^2)\, H_n(z)\, c^n/n! \tag{6}$$

First we take the Fourier transform of the left hand side:

$$\frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\infty} exp(-\boldsymbol{k}\, z\, p_z)\, exp(-½\, z^2 \;+\; 2\, z\, c -\; c^2)\; dz \tag{7}$$

$$= \; exp(-½\, p_z^2 \;-\; 2\, \boldsymbol{k}\, p_z\, c \;+\; c^2)$$

$$= \; \sum_{n=0}^{\infty} exp(-½\, p_z^2)\, H_n(p_z)\, (-\boldsymbol{k}\, c)^n/n!$$

The Fourier transform of the right hand side is given by

$$\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \int_{z=-\infty}^{\infty} exp(-\boldsymbol{k}\, z\, p_z) \cdot exp(-½\, z^2)\, H_n(z)\, c^n/n!\; dz \tag{8}$$

Equating like powers of *c* in the transformed versions of the left- and right-hand sides gives

$$\tfrac{1}{\sqrt{2\pi}} \textstyle\int_{z=-\infty}^{\infty} \exp(-\boldsymbol{k}\, z\, p_z) \cdot exp(-½\, z^2)\, H_n(z)\, c^n/n!\; dz \tag{9}$$

$$= \; (-\boldsymbol{k})^n \cdot \exp(-½\, p_z^2)\; H_n(p_z)\; \frac{c^n}{n!}$$

Let us define the Hermite functions $\psi_n(z)$

$$\psi_n(z) \;\equiv\; \; < z|\psi_n > = \mathrm{c}_n \exp(-½\, z^2)\; H_n(z) \tag{10}$$

$$|\mathcal{F}\,\psi_n> = |\psi_n>\,(-\boldsymbol{k})^n \tag{11}$$

with suitably chosen c_n so as to make

$$\|\psi_n\|^2 = 1 \tag{12}$$

$$\mathrm{c}_n = \frac{1}{\sqrt{2^n n!\,\sqrt{\pi}}} \tag{13}$$

The importance of the Hermite functions lie in the following theorem.

"The Hermite functions ψ_n; $n \in \mathbb{N}$ form an orthonormal basis for $L^2(\mathbb{R})$"

Consider the operator

$$H = -\tfrac{1}{2}\frac{d^2}{dz^2} + \tfrac{1}{2}\,z^2 \tag{14}$$

Apply this to $\psi_n(z)$:

$$H \cdot \psi_n(z) = (\tfrac{1}{2} + n)\,\psi_n(z) \tag{15}$$

Thus, ψ_n is an eigenfunction of H.

Let $f = \psi_{4k+j}$ be any of the Hermite functions. Then we have

$$\sum_{n=-\infty}^{\infty} f(y + n) \cdot \exp\left(-2\,\pi\,\boldsymbol{k}\,x\,(y+n)\right) \tag{16}$$

$$= (-\boldsymbol{k})^j \sum_{n=-\infty}^{\infty} f(x + n)\,exp(2\,\pi\,\boldsymbol{k}\,n\,y)$$

Proof: As

$$\mathcal{F} = V^{\dagger}\,J\,V \tag{17}$$

the equation

$$\mathcal{F}f = (-\boldsymbol{k})^j f \tag{18}$$

translates into

$$JVf(x;\,y;\,t) = (-\boldsymbol{k})^j\,Vf(x;\,y;\,t) \tag{19}$$

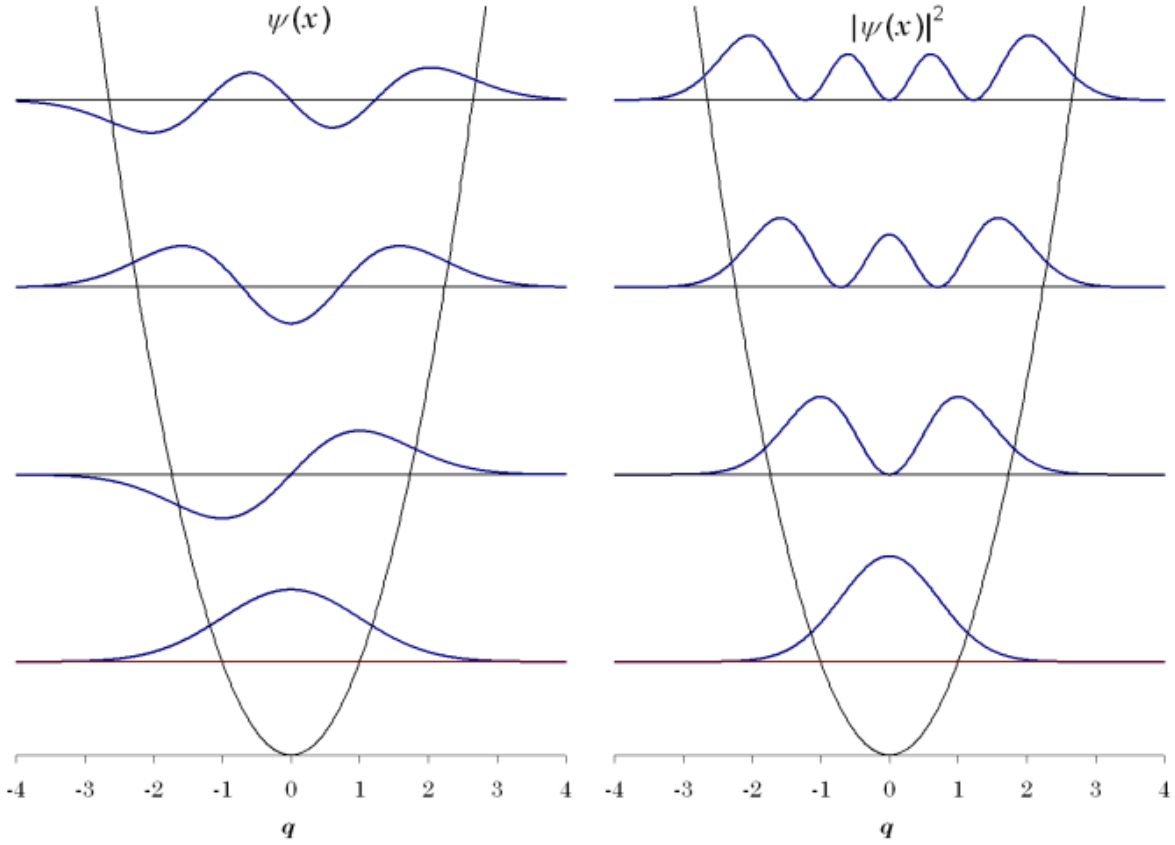
With the definition of V and $t = xy$:

(20)

$$V\, f(x,y,t) \;=\; exp(2\,\pi\,\boldsymbol{k}\,t)\,\sum_{n} f(x\;+\;n)\,exp(2\,\pi\,\boldsymbol{k}\,n\,y)$$

QED.

The vectors $|\psi_n\rangle$ are eigenvectors of the Fourier transform operator with eigenvalues $(-\boldsymbol{k})^n$. The eigenfunctions $\psi_n(x)$ represent eigenvectors $|\psi_n\rangle$ that span the complex Hilbert space $\mathbf{H}_k$.
For higher n the central parts of $\psi_n(x)$ and $|\psi_n(x)|^2$ become a sinusoidal form.



A **coherent state** is a specific kind of state of the quantum harmonic oscillator whose dynamics most closely resemble the oscillating behavior of a classical harmonic oscillator system. The ground state is a squeezed coherent state.

Ladder operator

The Hermite functions ψ_n represent Fock states.

Boson ladder operators are characterized by

$$\mathcal{A}|\psi_n\rangle = \sqrt{n}\,|\psi_{n-1}\rangle \tag{1}$$

$$\mathcal{A}^{\dagger}|\psi_n\rangle = \sqrt{n+1}\,|\psi_{n+1}\rangle \tag{2}$$

$$\mathcal{A} = \frac{1}{\sqrt{2}}\left(c_1\frac{d}{dq} + c_2 q\right) = -\boldsymbol{k}\,P\sqrt{\frac{1}{2\,\hbar\,m\,\omega}} + Q\sqrt{\frac{m\,\omega}{2\,\hbar}}$$

$$\mathcal{A}^\dagger = \frac{1}{\sqrt{2}}\left(-c_1 \frac{d}{dq} + c_2 q\right) = \boldsymbol{k}\, P \sqrt{\frac{1}{2\,\hbar\, m\, \omega}} + Q\sqrt{\frac{m\,\omega}{2\,\hbar}}$$

In the Heisenberg picture, the operators have the following time dependence:

$$\mathcal{A}(t) = \mathcal{A}(t_0)\exp\left(-\boldsymbol{k}\,\omega\,(\mathrm{t} - t_0)\right) \tag{3}$$

$$\mathcal{A}^\dagger(t) = \mathcal{A}^\dagger(t_0)\exp\left(\boldsymbol{k}\,\omega\,(\mathrm{t} - t_0)\right) \tag{4}$$

We can also define an enumeration operator N which has the following property:

$$N\ =\ \mathcal{A}^\dagger\,\mathcal{A} \tag{5}$$

$$N|\psi_n> =\ |\psi_n>\ n \tag{6}$$

In deriving the form of $\mathcal{A}^\dagger$, we have used the fact that the operators X and P_x, which represent observables, are Hermitian. These observable operators can be expressed as a linear combination of the ladder operators as

$$Q(t)\ =\ \sqrt{\frac{\hbar}{2\,m\,\omega}}\ \ (\mathcal{A}^\dagger(t) + \mathcal{A}(t)) \tag{7}$$

$$P(t)\ =\ \boldsymbol{k}\sqrt{½\,\hbar\, m\,\omega}\ \ (\mathcal{A}^\dagger(t) - \mathcal{A}(t)) \tag{8}$$

The $\boldsymbol{Q}$ and $\boldsymbol{P}$ operators obey the following identity, known as the canonical commutation relation:

$$[Q, P] =\ \boldsymbol{k}\,\hbar \tag{9}$$

Using the above, we can prove the identities

$$H\ =\ \hbar\,\omega\left(\mathcal{A}^\dagger\,\mathcal{A}\ +\ ½\right) =\ \hbar\,\omega\,(N\ +\ ½) \tag{10}$$

$$[\mathcal{A}^\dagger, \mathcal{A}]\ =\ 1 \tag{11}$$

Now, let $|f_E>$denote an energy eigenstate with energy E. The inner product of any ket with itself must be non-negative, so

$$<\ \psi_E\,\mathcal{A}|\mathcal{A}\,\psi_E\ > =\ <\ \psi_E E|\,\mathcal{A}^\dagger\,\mathcal{A}\,\psi_E\ >\ \geq\ 0 \tag{12}$$

Expressing $\mathcal{A}^\dagger\mathcal{A}$ in terms of the Hamiltonian H:

$$<\ \psi_E|(H/(\hbar\,\omega)\ -\ ½)\,\psi_E > =\ (E/(\hbar\,\omega)\ -\ ½)\ \geq\ 0 \tag{13}$$

so that

$$E \geq \tfrac{1}{2} \hbar \omega. \tag{14}$$

Note that when $|\mathcal{A}\, \psi_E > = |0>$ (is the zero ket i.e. a ket with length zero), the inequality is saturated, so that

$$E = \tfrac{1}{2} \hbar \omega \tag{15}$$

It is straightforward to check that there exists a state satisfying this condition; it is the ground state

$$|\psi_{ground} > = |\psi_{E_n} >;\ (n = 0) \tag{16}$$

Using the above identities, we can now show that the commutation relations of $\mathcal{A}$ and $\mathcal{A}^\dagger$ with H are:

$$[H, \mathcal{A}] = -\hbar \omega\, \mathcal{A} \tag{17}$$

$$\left[H, \mathcal{A}^\dagger\right] = \hbar \omega\, \mathcal{A}^\dagger \tag{18}$$

Thus, provided $|\mathcal{A}\ \psi_E>$ is not the zero ket,

$$\begin{aligned} |H\, \mathcal{A}\, \psi_E > &= |[H, \mathcal{A}] + \mathcal{A}\, H\, \psi_E > \\ &= |-\hbar \omega\, \mathcal{A} + \mathcal{A}\, E\, \psi_E > \\ &= |-\hbar \omega\, \mathcal{A} + \mathcal{A}\, E\, \psi_E > \\ &= (E - \hbar \omega)\, |\mathcal{A}\, \psi_E > \end{aligned} \tag{19}$$

Similarly, we can show that

$$|H\, \mathcal{A}^\dagger\ \psi_E > = (E + \hbar \omega)\, |\mathcal{A}^\dagger\, \psi_E > \tag{20}$$

In other words, $\mathcal{A}$ acts on an eigenstate of energy E to produce, up to a multiplicative constant, another eigenstate of energy $E - \hbar\,\omega$, and $\mathcal{A}^\dagger$ acts on an eigenstate of energy E to produce an eigenstate of energy $E + \hbar\,\omega$. For this reason, a is called a "lowering operator", and $\mathcal{A}^\dagger\mathcal{A}$ "raising operator". The two operators together are called ladder operators. In quantum field theory, $\mathcal{A}$ and $\mathcal{A}^\dagger$ are alternatively called "annihilation" and "creation" operators because they destroy and create particles, which correspond to our quanta of energy.

Given any energy eigenstate, we can act on it with the lowering operator $\mathcal{A}$, to produce another eigenstate with $\hbar\,\omega$-less energy. By repeated application of the lowering operator, it seems that we can produce energy eigenstates down to $E = -\infty$. However, this would contradict our earlier requirement that $E \geq \hbar\,\omega/2$.

Ground state

Therefore, there must be a ground-state energy eigenstate, which we label $|f_{ground}>$, such that

$$|\mathcal{A}\,\psi_{ground}> = |0>;\ \text{(zero ket)}. \tag{1}$$

In this case, subsequent applications of the lowering operator will just produce zero kets, instead of additional energy eigenstates. Furthermore, we have shown above that

$$|H\,\psi_{ground}> = (½\,\hbar\,\omega)\,|\psi_{ground}> \tag{2}$$

Finally, by acting on $|\psi_{ground}>$ with the raising operator and multiplying by suitable normalization factors, we can produce an infinite set of energy eigenstates

$$\{|\psi_{ground}>, |\psi_{E_1}>, |\psi_{E_2}>, \ldots., |\psi_{E_n}>\}, \tag{3}$$

such that

$$|H\,\psi_{E_n}> = \hbar\,\omega\,(n + ½)\,|\psi_{E_n}> \tag{4}$$

which matches the energy spectrum.
This method can also be used to quickly find the ground state wave function of the quantum harmonic oscillator.
Indeed

$$|\mathcal{A}\,\psi_{ground}> = |0> \tag{5}$$

becomes

$$\psi_{ground}(x) = \psi_0(x) \equiv <x|\,\psi_{ground}> = -\frac{\hbar}{m\,\omega}\frac{d}{dx}\,\psi_0(x) \tag{6}$$

so that

$$d\,\psi_0(x) = \psi_0(x)\,\frac{\hbar}{m\,\omega}\,x\,d\,x \Rightarrow \ln\big(\psi_0(x)\big) = \frac{m\,\omega}{2\hbar}\,x^2 + const \tag{7}$$

After normalization this leads to the following position space representation of the ground state wave function.

$$\psi_0(x) = \sqrt[4]{\frac{m\,\omega}{\pi\hbar}}\ e^{-\frac{m\,\omega}{2\hbar}x^2} \tag{8}$$

Coherent state

A **coherent state** is a specific kind of state of the quantum harmonic oscillator whose dynamics most closely resemble the oscillating behavior of a classical harmonic oscillator system.

The coherent state $|\alpha>$ is defined to be the 'right' eigenstate of the annihilation operator $\mathcal{A}$. Formally, this reads:

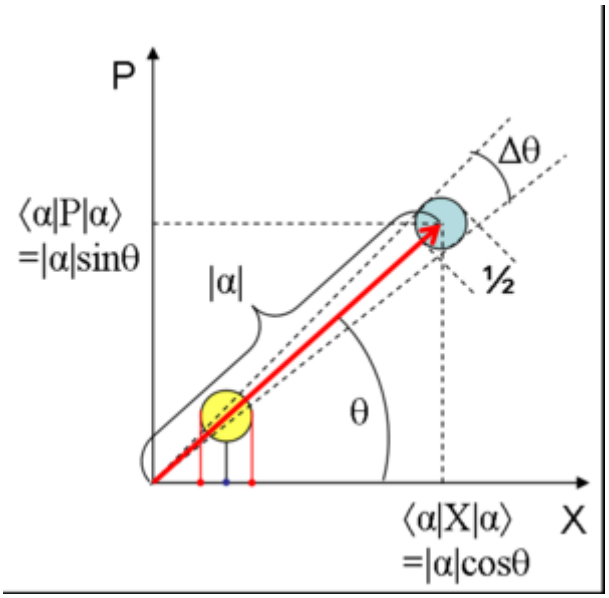
$$|\mathcal{A}\,\alpha> = \alpha|\alpha> \tag{1}$$

Since $\mathcal{A}$ is not Hermitian, α is a hyper complex number that is not necessarily real, and can be represented as

$$\alpha = |\alpha|\exp(\boldsymbol{k}\,\theta) \tag{2}$$

where θ is a real number. $|\alpha|$ is the amplitude and θ is the phase of state $|\alpha>$.
This formula means that a coherent state is left unchanged by the annihilation or the creation of a particle. The eigenstate of the annihilation operator has a Poissonian number distribution A Poisson distribution is a necessary and sufficient condition that all annihilations are statistically independent.

The coherent state's location in the complex plane (phase space) is centered at the position and momentum of a classical oscillator of the same phase θ and amplitude. As the phase increases the coherent state circles the origin and the corresponding disk neither distorts nor spreads. The disc represents Heisenberg's uncertainty. This is the most similar a quantum state can be to a single point in phase space.



Phase space plot of a coherent state. This shows that the uncertainty (blur) in a coherent state is equally distributed in all directions. The horizontal and vertical axes are the X and P quadratures of the field, respectively. Oscillations that are said to be *in quadrature*, if they are separated in phase by π/2 radians. The red dots on the x-axis trace out the boundaries of the quantum noise. Further from the origin the relative contribution of the quantum noise becomes less important.

The representation of the coherent state in the basis of Fock states is:

$$|\alpha> = \exp(-½|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n> = \exp(-½|\alpha|^2)\exp\left(\alpha\mathcal{A}^{\dagger}\right)|0> \tag{3}$$

where |n> are Hermite functions (eigenvectors of the Hamiltonian). This is a Poissonian distribution. The probability of detecting n photons is:

$$\mathcal{P}(n) = \exp(-\langle n \rangle)\frac{\langle n \rangle^n}{n!} \tag{4}$$

Similarly, the average photon number in a coherent state is

$$\langle n \rangle = \langle \mathcal{A}^\dagger \mathcal{A} \rangle = |\alpha|^2 \tag{5}$$

and the variance is

$$(\Delta n)^2 = Var\left(\mathcal{A}^\dagger \mathcal{A}\right) = |\alpha|^2 \tag{6}$$

Squeezing

The squeezing operator can squeeze a state more or less in the direction of either P or Q. The operator is defined as:

$$Sq(z) = \exp\left(½\left(z^* \mathcal{A} + z\mathcal{A}^\dagger\right)\right) \tag{1}$$

$$z = r\exp(\boldsymbol{k}\,\theta) \tag{2}$$

The ground state is a saturated squeezed coherent state where

$$\Delta p = \Delta q \text{ and } \Delta q \cdot \Delta p = \hbar/2 \tag{3}$$

Base transforms

Now we have discovered the following base transforms:
Position⇔momentum:

$$< q|p > = \frac{1}{\sqrt{2\pi\hbar}}\exp(\frac{\boldsymbol{k}\,q\,p}{\hbar}) \tag{1}$$

Position⇔Fock state:

$$< q|n > = \sqrt[4]{\frac{m\omega}{\pi\hbar}}\frac{1}{\sqrt{2^n\,n!}}\exp(-\frac{m\omega}{2\hbar}\,q^2)H_n(q\sqrt{\frac{m\omega}{\hbar}})$$

Fock state⇔coherent state:

$$< n|z > = \frac{1}{\sqrt{n!}}\,z^n\exp(-½|z|^2)$$

Rotational symmetry

In case of rotational symmetry in the imaginary part of quaternion space, the exponential function must be replaced by a Bessel function. The corresponding Fourier transform then becomes a Hankel transform.

Hilbert field equations

Despite the obvious similarity, Hilbert field equations are not Maxwell field equations. First of all, the Hilbert field is a skew field and it carries the properties of the quaternions and the properties of the quaternionic Fourier transform. Next Hilbert fields are mathematical (skew) fields, while Maxwell fields are physical fields. Finally the Hilbert fields consist of a collection of member fields. They are which the members show different behavior.
The law of charge conservation in configuration space runs:

$$s_0(q) = \nabla_0 \rho_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle \tag{1}$$

In Fourier space this becomes:

$$\tilde{s}_0(p) = p_0 \tilde{\rho}_0(p) \mp \langle \boldsymbol{p}, \tilde{\boldsymbol{\rho}}(q) \rangle \tag{2}$$

We reserve the character ***E*** for the imaginary longitudinal static part of the Hilbert fields.
This differs from the approach in Maxwell equations.
We reserve the character ***B*** for the imaginary transverse static part of the Hilbert fields.

The quaternionic divergence is defined by:

$$\nabla f(q) = \nabla_0 f_0(q) \pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q) \right) \tag{3}$$

In Fourier space:

$$p\tilde{f}(q) = p_0 \tilde{f}_0(p) \pm p_0 \tilde{\boldsymbol{f}}(p) + \boldsymbol{p} \tilde{f}_0(p) \mp \langle \boldsymbol{p}, \tilde{\boldsymbol{f}}(p) \rangle \pm \left(\pm \boldsymbol{p} \times \tilde{\boldsymbol{f}}(p) \right) \tag{4}$$

Due to the fact that the Fourier transform transfers a divergence ∇ in a simple multiplication, the split divides differentiable quaternionic functions into a (longitudinal) rotation free part and a (transverse) divergence free part. The longitudinal part is complex. The transverse part is imaginary.

$$f(q) = f_{\parallel}(q) + f_{\perp}(q) \tag{5}$$

$$\nabla f_{\perp}(q) = 0 \tag{6}$$

$$\nabla \times f_{\parallel}(q) = 0 \tag{7}$$

$$\tilde{f}(p) = \tilde{f}_{\parallel}(p) + \tilde{f}_{\perp}(p) \tag{8}$$

$$\langle \hat{\imath}, \tilde{f}_{\perp}(p) \rangle = 0 \tag{9}$$

$$\hat{\imath} \times \tilde{f}_{\parallel}(p) = 0 \tag{10}$$

Special Fourier transform pairs

Functions that keep the same form through Fourier transformation are:

$$f(q) = \exp(-|q|^2) \tag{1}$$

$$f(q) = \frac{1}{|q|} \tag{2}$$

$$f(q) = comb(q) \tag{3}$$

The comb function consists of a set of equidistant Dirac delta functions.

Convolution

Through Fourier transformation a convolution changes into a simple product and vice versa.

$$\mathcal{F}(f(q) \circ g(q)) = \tilde{f}(p) \cdot \tilde{g}(p) \tag{3}$$

Parceval's theorem

Parceval's theorem runs:

$$\int f^*(q) \cdot g(q) \cdot dV_q = \int \tilde{f}^*(p) \cdot \tilde{g}(p) \cdot dV_p \tag{1}$$

This leads to

$$\int |f(q)|^2 \cdot dV_q = \int \left|\tilde{f}(p)\right|^2 \cdot dV_p \tag{2}$$

Source conservation

The law of charge conservation in configuration space runs:

$$s_0(q) = \nabla_0 \rho_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle \tag{1}$$

In configuration space the distribution $\rho(q)$ becomes a potential $\phi(q)$ through convolution with a blur $\phi(q)$.

$$\phi(q) = \rho(q) \circ \varphi(q) \tag{2}$$

In Fourier space the convolution becomes multiplication:

$$\tilde{s}_0(p) = p_0 \tilde{\rho}_0(p) \mp \langle \boldsymbol{p}, \tilde{\boldsymbol{\rho}}(p) \rangle \tag{3}$$

$$\tilde{s}_0(p)\tilde{\varphi}(p) = p_0 \tilde{\rho}_0(p)\tilde{\varphi}(p) \mp \langle \boldsymbol{p}, \tilde{\boldsymbol{\rho}}(q) \rangle \tilde{\varphi}(p)) = p_0 \tilde{\rho}_0(p)\tilde{\varphi}(p) \mp \langle \boldsymbol{p}, \tilde{\boldsymbol{\rho}}(q)\tilde{\varphi}(p) \rangle \tag{4}$$

$$\tilde{\phi}(p) = \tilde{\rho}(p)\tilde{\varphi}(p) \tag{5}$$

$$\tilde{\phi}_0(p) = \tilde{\rho}_0(p)\tilde{\varphi}(p) \tag{6}$$

$$\tilde{\boldsymbol{\phi}}(p) = \tilde{\rho}_0(p)\tilde{\boldsymbol{\varphi}}(p) \tag{7}$$

In configuration space, with $s_0(q) = 0$ this means:

$$\nabla_0 \phi_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q) \rangle = 0 \tag{8}$$

Potential

If there is a static spherically symmetric Gaussian charge density ϱ (*r*):

$$\rho(q) = \frac{Q}{\sqrt{2\pi\sigma^2}^3} \exp(-|q|^2/(2\sigma^2)) \tag{1}$$

where *Q* is the total charge, then the solution ϕ (*r*) of Poisson's equation,

$$\nabla^2 \varphi(q) = -\frac{\rho(q)}{\varepsilon} \tag{2}$$

is given by

$$\varphi(q) = \frac{Q}{4\pi\varepsilon|q|} \operatorname{erf}\left(\frac{|q|}{\sqrt{2\sigma}}\right) \tag{3}$$

where erf(*x*) is the error function.

In fact the quaternionic Poisson's equation represents two separate equations:

$$({\nabla_0}^2 - \boldsymbol{\nabla}^2)\varphi_0(q) = -\frac{\rho_0(q)}{\varepsilon} \tag{4}$$

$$({\nabla_0}^2 - \boldsymbol{\nabla}^2)\boldsymbol{\varphi}(q) = -\frac{\boldsymbol{\rho}(q)}{\varepsilon} \tag{5}$$

Note that, for $|q|$ much greater than σ, the erf function approaches unity and the potential ϕ (*r*) approaches the point charge potential $\frac{Q}{4\pi\varepsilon|q|}$, as one would expect. Furthermore the erf function approaches 1 extremely quickly as its argument increases; in practice for $|q| > 3\sigma$ the relative error is smaller than one part in a thousand.

The definition of the quaternionic potential $\boldsymbol{\phi}(q)$ is based on the convolution of a quaternionic distribution $\rho(q)$ with the real function $\varphi(q)$ See Newton potential and Bertrand's theorem in Wikipedia. The real part $\rho_0(q)$ of the distribution $\rho(q)$ can be interpreted as a charge distribution. The imaginary part $\boldsymbol{\rho}(q)$ can be interpreted as a current distribution.
The convolution blurs the distribution such that the result becomes differentiable.

In configuration space holds:

$$\boldsymbol{\phi}(q) = \rho(q) \circ \frac{1}{|q|}. \tag{6}$$

Reversely, according to Poisson's equation:

$$\rho(q) = = -\Delta\boldsymbol{\phi}(q) \tag{7}$$

The real part of $\phi(q)$ presents a scalar potential. The imaginary part presents a vector potential.

$$\phi(q) = \phi_0(q) + \boldsymbol{\phi}(q) \tag{8}$$

The scalar potential is a blurred charge distribution.

The vector potential is a blurred current distribution.

Current is moving charge.

Mass is a form of charge.

(The selected blurring function has striking resemblance with the ground state of the quantum harmonic oscillator).

In Fourier space holds:

$$\tilde{\phi}(p) = \tilde{\rho}(p) \cdot \frac{1}{|p|}. \tag{9}$$

$$= \tilde{\phi}_0(p) + \tilde{\boldsymbol{\phi}}(p)$$

In Fourier space the frequency spectrum of the Hilbert distribution is multiplied with the Fourier transform of the blurring function. When this falls off when the frequencies go to infinity, then as a consequence the frequency spectrum of the potential is bounded. This is valid independent of the fact that the frequency spectrum of the Hilbert distribution might be unbounded.

The equation for the conservation of charge:

$$s_0(q) = \nabla_0 \rho_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle \tag{10}$$

Translates in the source free case $s_0(q) = 0$ into:

$$\nabla_0 \rho_0(q) = \pm \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle \tag{11}$$

And in the Lorentz Lorentz Gauge condition:

$$\nabla_0 \boldsymbol{\phi}_0(q) = \pm \langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q) \rangle \tag{12}$$

$$\mathfrak{F}(\mathrm{q}) = \nabla \boldsymbol{\phi}(q) \tag{13}$$

$$\mathfrak{F}_0(q) = \nabla_0 \boldsymbol{\phi}_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q) \rangle = 0$$

In the source divergence free case $\boldsymbol{\nabla} s_0(q) = 0$ this means:

$$\nabla_0 \boldsymbol{\nabla} \rho_0(q) = \pm \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(q) \rangle \tag{14}$$

$$\nabla_0 \boldsymbol{\nabla} \boldsymbol{\phi}_0(q) = \pm \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q) \rangle \tag{15}$$

$$\boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q) \rangle = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{\phi}(q) + \boldsymbol{\nabla}^2 \boldsymbol{\phi}(q) \tag{16}$$

Due to the fact that there are other charges present, the divergence of the scalar potential need be in the direction of the current $\boldsymbol{\rho}(q)$, which for a spherical symmetric blur is also in the direction of the vector potential $\boldsymbol{\phi}(q)$. However, a tendency exists to minimize that difference. Thus $\nabla_0 \boldsymbol{\nabla} \boldsymbol{\phi}_0(q)$ is parallel to $\boldsymbol{\phi}(q)$. With other words:

$$\boldsymbol{\phi}(q) \times \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q) \rangle = 0 \tag{17}$$

Reckoning the sign selections for the sign $\pm$ of the conjugation and the handedness $\pm$ of the cross product will provide four different sets of equations. This will provide four different Hilbert fields.

Discrete distribution

If $\rho(q)$ is discrete, such that

$$\rho(q) = \sum_i q_{E\,i} \cdot \delta(q - q_i) \tag{1}$$

where q'_E is a point charge at location q', then the contribution to the field $\boldsymbol{E}(q)$ that is generated by a point charge at location q_i is given by:

$$d\boldsymbol{E}(q) = q_{E\,i} \cdot \frac{q_i - q}{|q_i - q|^3} = -q_{E\,i} \cdot \nabla \cdot \frac{1}{|q_i - q|} \tag{2}$$

Differential potential equations

The gradient and curl of $\boldsymbol{\phi}(q)$ are related. In configuration space holds:

$$\nabla \phi(q) = \nabla_0 \phi_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q) \rangle \pm \nabla_0 \boldsymbol{\phi}(q) \pm \boldsymbol{\nabla} \phi_0(q) \pm \boldsymbol{\nabla} \times \boldsymbol{\phi}(q) \tag{1}$$

$$\mathfrak{E}(q) \equiv -\boldsymbol{\nabla} \phi_0(q) \tag{2}$$

$$\mathfrak{B}(q) \equiv \boldsymbol{\nabla} \times \phi(q) \tag{3}$$

$$\mathfrak{F}(q) = \nabla \phi(q) = \mathfrak{F}_0(q) \mp \mathfrak{E}(q) \pm \mathfrak{B}(q) \pm \nabla_0 \boldsymbol{\phi}(q) \tag{4}$$

$$\mathfrak{F}_0(q) = \nabla_0 \phi_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q) \rangle \tag{5}$$

$$\mathfrak{F}(q) = \mp \mathfrak{E}(q) \pm \mathfrak{B}(q) \pm \nabla_0 \boldsymbol{\phi}(q) \tag{6}$$

$$\langle \boldsymbol{\nabla}, \mathfrak{E}(q) \rangle = -\boldsymbol{\nabla}^2 \phi_0(q) = \rho_0(q) \tag{7}$$

$$\boldsymbol{\nabla} \times \mathfrak{E}(q) = 0; \text{ Rotation free field} \tag{8}$$

$$\langle \boldsymbol{\nabla}, \mathfrak{B}(q) \rangle = 0; \text{ Divergence free } B \text{ field} \tag{9}$$

$$\boldsymbol{\nabla} \times \mathfrak{B}(q) = \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q) \rangle - \boldsymbol{\nabla}^2 \boldsymbol{\phi}(q) = \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q) \rangle + \boldsymbol{\rho}(q) + {\nabla_0}^2 \boldsymbol{\phi}(q) \tag{10}$$

$$\boldsymbol{\nabla} \times \mathfrak{B}(q) = \pm \nabla_0 \boldsymbol{\nabla} \phi_0(q) + \boldsymbol{\rho}(q) + {\nabla_0}^2 \boldsymbol{\phi}(q) \tag{11}$$

$$= \pm \nabla_0 \mathfrak{E}(q) + \boldsymbol{\rho}(q) + {\nabla_0}^2 \boldsymbol{\phi}(q)$$

Since $\nabla_0 \boldsymbol{\phi}(q)$is supposed to be parallel to $\boldsymbol{\nabla}\phi_0(q)$, it is sensible to define $\boldsymbol{E}(q)$as the total field in longitudinal direction:

$$\boldsymbol{E}(q) = -\boldsymbol{\nabla}\phi_0(q) - \nabla_0 \boldsymbol{\phi}(q) = \mathfrak{E}(q) - \nabla_0 \boldsymbol{\phi}(q) \tag{12}$$

And

$$\boldsymbol{B}(q) = \mathfrak{B}(q) \tag{13}$$

With this definition:

$$\boldsymbol{\nabla} \times \boldsymbol{E}(q) = -\nabla_0 \boldsymbol{B}(q) \tag{14}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{B}(q) \rangle = 0 \tag{15}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B}(q) = \boldsymbol{\rho}(q) + \nabla_0 \boldsymbol{E}(q) \tag{16}$$

In Fourier space

In Fourier space holds:

$$\tilde{\mathfrak{F}}(p) = p_0 \tilde{\phi}_0(p) - \langle \boldsymbol{p}, \tilde{\boldsymbol{\phi}}(p) \rangle \pm p_0 \tilde{\boldsymbol{\phi}}(p) \pm \boldsymbol{p}\tilde{\phi}_0(p) \pm \boldsymbol{p} \times \tilde{\boldsymbol{\phi}}(p) \tag{1}$$

$$\tilde{\mathfrak{F}}(p) = p\tilde{\phi}(p) = \mathfrak{F}_0(p) \mp \tilde{\mathfrak{E}}(p) \pm \tilde{\mathfrak{B}}(p) \pm p_0 \tilde{\boldsymbol{\phi}}(p) \tag{2}$$

$$\tilde{\mathfrak{F}}_0(p) = p_0 \tilde{\phi}_0(p) - \langle \boldsymbol{p}, \tilde{\boldsymbol{\phi}}(p) \rangle \tag{3}$$

$$\tilde{\mathfrak{E}}(p) = -\boldsymbol{p}\tilde{\phi}_0(p) \tag{4}$$

$$\tilde{\boldsymbol{E}}(p) = -\boldsymbol{p}\tilde{\phi}_0(p) \pm p_0 \tilde{\boldsymbol{\phi}}(p)$$

$$\tilde{\mathfrak{B}}(p) = \boldsymbol{p} \times \tilde{\boldsymbol{\phi}}(p) \tag{5}$$

$$\tilde{\mathfrak{F}}(p) = \mp \tilde{\mathfrak{E}}(p) \pm \tilde{\mathfrak{B}}(p) \pm p_0 \tilde{\boldsymbol{\phi}}(p) \tag{6}$$

$$\langle \boldsymbol{p}, \tilde{\mathfrak{E}}(p) \rangle = -\boldsymbol{p}^2 \tilde{\phi}_0(p) = \tilde{\rho}_0(p) \tag{7}$$

$$\boldsymbol{p} \times \tilde{\mathfrak{E}}(p) = 0; \text{ Rotation free field} \tag{8}$$

$$\langle \boldsymbol{p}, \tilde{\mathfrak{B}}(p) \rangle = 0; \text{ Divergence free } B \text{ field} \tag{9}$$

$$\boldsymbol{p} \times \tilde{\mathfrak{B}}(p) = \boldsymbol{p}\langle \boldsymbol{p}, \tilde{\boldsymbol{\phi}}(q) \rangle - \boldsymbol{p}^2 \tilde{\boldsymbol{\phi}}(q) = \boldsymbol{p}\langle \boldsymbol{p}, \tilde{\boldsymbol{\phi}}(p) \rangle + \tilde{\boldsymbol{\rho}}(p) \tag{10}$$

$$\boldsymbol{p} \times \tilde{\mathfrak{B}}(p) = \pm p_0 \boldsymbol{p} \tilde{\phi}_0(p) + \tilde{\boldsymbol{\rho}}(p) = \pm p_0 \tilde{\mathfrak{E}}(p) + \tilde{\boldsymbol{\rho}}(p) \tag{11}$$

If the distribution $\rho(q)$ is differentiable, then the same equations that hold for fields $\phi(q)$ and $\tilde{\phi}(p)$ hold for the non-blurred distributions $\rho(q)$ and $\tilde{\rho}(q)$.

Maxwell

First it must be noted that the derived field equations hold for general quaternionic fields.
The resemblance with physical fields hold for electromagnetic fields as well as for gravitational fields and for any fields whose blurring function approximates

$$f(q) \approx \frac{1}{|q|}.$$

In Maxwell equations, $\boldsymbol{E}(\mathbf{r})$ is defined as:

$$\boldsymbol{E}(\boldsymbol{r},t) \equiv -\boldsymbol{\nabla}\phi_0(\boldsymbol{r},t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = \mathfrak{E}(\boldsymbol{r},t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t}$$

Further:

$$\langle \boldsymbol{\nabla}, \boldsymbol{E}(\boldsymbol{r},t)\rangle = -\boldsymbol{\nabla}^2\phi_0(\boldsymbol{r},t) - \frac{\partial\langle \boldsymbol{\nabla}, \boldsymbol{A}(\boldsymbol{r},t)\rangle}{\partial t}$$

$$= \frac{\rho_0(\boldsymbol{r},t)}{\varepsilon_0} - \frac{\partial\langle \boldsymbol{\nabla}, \boldsymbol{A}(\boldsymbol{r},t)\rangle}{\partial t}$$

In Maxwell equations, $\boldsymbol{B}(\boldsymbol{r})$ is defined as:

$$\boldsymbol{B}(\boldsymbol{r},t) \equiv \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r},t) = \mathfrak{B}(\boldsymbol{r},t)$$

Further:

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{B}(\boldsymbol{r},t)\rangle = 0$$

$$\boldsymbol{\nabla} \times \boldsymbol{B}(\boldsymbol{r},t) = \mu_0\left(\boldsymbol{j} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right)$$

Differentiable distribution

If the distribution $\rho(q)$ is differentiable, then the same equations that hold for fields $\phi(q)$ and $\tilde{\phi}(p)$ hold for the non-blurred distributions $\rho(q)$ and $\tilde{\rho}(q)$.
Using:

$$\boldsymbol{B} = \nabla \times \boldsymbol{\phi} = \boldsymbol{i}(\nabla_2\phi_{\|} - \nabla_{\|}\phi_2) + \boldsymbol{j}(\nabla_{\|}\phi_1 - \nabla_1\phi_{\|}) + \boldsymbol{k}(\nabla_1\phi_2 - \nabla_2\phi_1) \quad (1)$$

gives

$$\nabla_0\phi_\parallel(q) = \mp\nabla_\parallel\phi_0(q) \tag{2}$$

$$\nabla_0\phi_1(q) = \mp\left(\nabla_2\phi_\parallel(q) - \nabla_\parallel\phi_2(q)\right) \tag{3}$$

$$\nabla_0\phi_2(q) = \pm\left(\nabla_1\phi_\parallel(q) - \nabla_\parallel\phi_1(q)\right) \tag{4}$$

$$\nabla_0\phi_0(q) = \langle\boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle = \nabla_\parallel\phi_\parallel(q) + \nabla_1\phi_1(q) + \nabla_2\phi_2(q) \tag{5}$$

And correspondingly in Fourier space

$$p_0\tilde{\phi}_\parallel(p) = \mp p_\parallel\tilde{\phi}_0(p) \tag{6}$$

$$p_0\tilde{\phi}_1(p) = \pm\left(p_\parallel\tilde{\phi}_2(p) - p_2\tilde{\phi}_\parallel(p)\right) \tag{7}$$

$$p_0\tilde{\phi}_2(p) = \mp\left(p_\parallel\tilde{\phi}_1(p) - p_1\tilde{\phi}_\parallel(p)\right) \tag{8}$$

$$p_0\tilde{\phi}_0(p) = \langle\boldsymbol{p}, \tilde{\boldsymbol{\phi}}(p)\rangle = p_\parallel\tilde{\phi}_\parallel(p) + p_1\tilde{\phi}_1(p) + p_2\tilde{\phi}_2(p) \tag{9}$$

Conservation laws

Flux vector

The longitudinal direction **k** of $\boldsymbol{E}(q)$ and the direction **i** of $\boldsymbol{B}(q)$ fix two mutual perpendicular directions. This generates curiosity to the significance of the direction $\mathbf{k}\times\mathbf{i}$. With other words what happens with $\boldsymbol{E}(q)\times\boldsymbol{B}(q)$.

The **flux vector** $\mathfrak{S}(q)$ is defined as:

$$\mathfrak{S}(q) \equiv \boldsymbol{E}(q)\times\boldsymbol{B}(q) \tag{1}$$

Conservation of energy

Field energy density

$$\begin{aligned}\langle\boldsymbol{\nabla}, \mathfrak{S}(q)\rangle &= \langle\boldsymbol{B}(q), \boldsymbol{\nabla}\times\boldsymbol{E}(q)\rangle - \langle\boldsymbol{E}(q), \boldsymbol{\nabla}\times\boldsymbol{B}(q)\rangle \\ &= -\langle\boldsymbol{B}(q), \nabla_0\boldsymbol{B}(q)\rangle - \langle\boldsymbol{E}(q), \boldsymbol{\phi}(q)\rangle - \langle\boldsymbol{E}(q), \nabla_0\boldsymbol{E}(q)\rangle \\ &= -½\nabla_0(\langle\boldsymbol{B}(q), \boldsymbol{B}(q)\rangle + \langle\boldsymbol{E}(q), \boldsymbol{E}(q)\rangle) - \langle\boldsymbol{E}(q), \boldsymbol{\phi}(q)\rangle\end{aligned} \tag{1}$$

The **field energy density** is defined as:

$$u_{field}(q) = ½(\langle\boldsymbol{B}(q), \boldsymbol{B}(q)\rangle + \langle\boldsymbol{E}(q), \boldsymbol{E}(q)\rangle) = u_B(q) + u_E(q) \tag{2}$$

$\mathfrak{S}(q)$ can be interpreted as the **field energy current density**.

The continuity equation for field energy density is given by:

$$\nabla_0 u_{field}(q) + \langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle = -\langle \boldsymbol{E}(q), \boldsymbol{\phi}(q) \rangle = -\phi_0(q) \langle \boldsymbol{E}(q), \boldsymbol{v}(q) \rangle \tag{3}$$

This means that $\langle \boldsymbol{E}(q), \boldsymbol{\phi}(q) \rangle$ can be interpreted as a source term.
$\phi_0(q)\boldsymbol{E}(q)$ represents **force** per unit volume.
$\phi_0(q)\langle \boldsymbol{E}(q), \boldsymbol{v}(q) \rangle$ represents **work** per unit volume, or, in other words, the power density. It is known as the Lorentz power density and is equivalent to the time rate of change of the mechanical energy density of the charged particles that form the current $\boldsymbol{\phi}(q)$.

$$\nabla_0 u_{field}(q) + \langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle = -\nabla_0 u_{mechanical}(q) \tag{4}$$

$$\nabla_0 u_{mechanical} = \langle \boldsymbol{E}(q), \boldsymbol{\phi}(q) \rangle = \phi_0(q) \langle \boldsymbol{E}(q), \boldsymbol{v}(q) \rangle \tag{5}$$

$$\nabla_0 u_{field}(q) + \langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle = -\nabla_0 u_{mechanical}(q) \tag{6}$$

$$\nabla_0 \Big(u_{field}(q) + u_{mechanical}(q) \Big) = -\langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle \tag{7}$$

Total change within V = flow into V + production inside V

$$u(q) = u_{field}(q) + u_{mechanical}(q) = u_B(q) + u_E(q) + u_{mechanical}(q) \tag{8}$$

$$U = U_{field} + U_{mechanical} = U_B + U_E + U_{mechanical} = \int_V u \, dV \tag{9}$$

$$\frac{d}{dt} \int_V u \, dV = \oint_S \langle \hat{\boldsymbol{n}}, \mathfrak{S} \rangle dS + \int_V s_0 \, dV \tag{10}$$

Here the source s_0 is zero.

Conservation of linear momentum

Field linear momentum

$\mathfrak{S}(q)$ can also be interpreted as the **field linear momentum density**. The time rate change of the field linear momentum density is:

$$\nabla_0 \mathfrak{S}(q) = \boldsymbol{g}_{field}(q) = \nabla_0 \, \boldsymbol{E}(q) \times \boldsymbol{B}(q) + \boldsymbol{E}(q) \times \nabla_0 \boldsymbol{B}(q) \tag{1}$$

$$= \big(\boldsymbol{\nabla} \times \boldsymbol{B}(q) - \boldsymbol{\rho}(q) \big) \times \boldsymbol{B}(q) - \boldsymbol{E}(q) \times \boldsymbol{\nabla} \times \boldsymbol{E}(q) \tag{2}$$

$$\boldsymbol{G}(\boldsymbol{E}) = \boldsymbol{E} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = \langle \boldsymbol{\nabla E}, \boldsymbol{E} \rangle - \langle \boldsymbol{E}, \boldsymbol{E} \rangle = \tfrac{1}{2} \boldsymbol{\nabla} \langle \boldsymbol{E}, \boldsymbol{E} \rangle - \langle \boldsymbol{E}, \boldsymbol{E} \rangle \tag{3}$$

$$= -\boldsymbol{\nabla}(\mathbf{EE}) + \tfrac{1}{2} \boldsymbol{\nabla} \langle \boldsymbol{E}, \boldsymbol{E} \rangle + \langle \boldsymbol{\nabla}, \boldsymbol{E} \rangle \boldsymbol{E}$$

$$= -\boldsymbol{\nabla}(\mathbf{EE} + \tfrac{1}{2} \mathbf{1_3} \langle \boldsymbol{E}, \boldsymbol{E} \rangle) + \langle \boldsymbol{\nabla}, \boldsymbol{E} \rangle \boldsymbol{E}$$

$$\boldsymbol{G}(\boldsymbol{B}) = \boldsymbol{B} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = -\boldsymbol{\nabla}(\mathbf{BB} + \tfrac{1}{2}\mathbf{1_3}\langle \boldsymbol{B}, \boldsymbol{B}\rangle) + \langle \boldsymbol{\nabla}, \boldsymbol{B}\rangle \boldsymbol{B} \tag{4}$$

$$\boldsymbol{H}(\boldsymbol{B}) = -\boldsymbol{\nabla}(\mathbf{BB} + \tfrac{1}{2}\mathbf{1_3}\langle \boldsymbol{B}, \boldsymbol{B}\rangle) \tag{5}$$

$$\begin{aligned} \nabla_0 \mathfrak{S}(q) &= \boldsymbol{G}(\boldsymbol{B}) + \boldsymbol{G}(\boldsymbol{E}) - \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) \\ &= \boldsymbol{H}(\boldsymbol{E}) + \boldsymbol{H}(\boldsymbol{B}) - \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) + \langle \boldsymbol{\nabla}, \boldsymbol{B}\rangle \boldsymbol{B} + \langle \boldsymbol{\nabla}, \boldsymbol{E}\rangle \boldsymbol{E} \\ &= \boldsymbol{H}(\boldsymbol{E}) + \boldsymbol{H}(\boldsymbol{B}) - \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) - \rho_0(q)\, \boldsymbol{E}(q) \\ &= \boldsymbol{H}(\boldsymbol{E}) + \boldsymbol{H}(\boldsymbol{B}) - \boldsymbol{f}(q) = \boldsymbol{\mathcal{T}}(q) - \boldsymbol{f}(q) \end{aligned} \tag{6}$$

$\mathcal{T}$(q) is the linear momentum flux tensor.
The linear momentum of the field contained in volume V surrounded by surface S is:

$$\boldsymbol{P}_{field} = \int_V \boldsymbol{g}_{field}\, dV = \int_V \rho_0 \boldsymbol{\phi}\, dV + \int_V \langle \nabla \boldsymbol{\phi}, \boldsymbol{E}\rangle\, dV + \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{EA}\rangle dS \tag{7}$$

$$\boldsymbol{f}(q) = \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) + \rho_0(q)\, \boldsymbol{E}(q) \tag{8}$$

Physically, $\boldsymbol{f}(q)$ is the Lorentz force density. It equals the time rate change of the mechanical linear momentum density $\boldsymbol{g}_{mechanical}$.

$$\boldsymbol{g}_{mechanical}(q) = \rho_{0m}(q)\boldsymbol{v}(q) \tag{9}$$

The force acted upon a single particle that is contained in a volume V is:

$$\boldsymbol{F} = \int_V \boldsymbol{f}\, dV = \int_V (\boldsymbol{\rho} \times \boldsymbol{B} + \rho_0\, \boldsymbol{E})\, dV \tag{10}$$

Brought together this gives:

$$\nabla_0 \left(\boldsymbol{g}_{field}(q) + \boldsymbol{g}_{mechanical}(q) \right) = -\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{T}}(q)\rangle \tag{11}$$

This is the continuity equation for linear momentum.
The component $\mathcal{T}_{ij}$ is the linear momentum in the i-th direction that passes a surface element in the j-th direction per unit time, per unit area.

Total change within V = flow into V + production inside V

$$\boldsymbol{g}(q) = \boldsymbol{g}_{field}(q) + \boldsymbol{g}_{mechanical}(q) \tag{12}$$

$$\boldsymbol{P} = \boldsymbol{P}_{field} + \boldsymbol{P}_{mechanical} = \int_V \boldsymbol{g}\, dV \tag{13}$$

$$\frac{d}{dt}\int_V \boldsymbol{g}\, dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{\mathcal{T}} \rangle dS + \int_V \boldsymbol{s_g}\, dV \tag{14}$$

Here the source $\boldsymbol{s_g} = 0$.

Conservation of angular momentum

Field angular momentum

The angular momentum relates to the linear momentum.

$$\boldsymbol{h}(\boldsymbol{q_c}) = (\boldsymbol{q} - \boldsymbol{q_c}) \times \boldsymbol{g}(q) \tag{1}$$

$$\boldsymbol{h}_{field}(\boldsymbol{q_c}) = (\boldsymbol{q} - \boldsymbol{q_c}) \times \boldsymbol{g}_{field}(q) \tag{2}$$

$$\boldsymbol{h}_{mechanical}(q) = (\boldsymbol{q} - \boldsymbol{q_c}) \times \boldsymbol{g}_{mechanical}(q) \tag{3}$$

$$\boldsymbol{\mathcal{K}}(\boldsymbol{q_c}) = (\boldsymbol{q} - \boldsymbol{q_c}) \times \boldsymbol{\mathcal{T}}(q) \tag{4}$$

This enables the balance equation for angular momentum:

$$\nabla_0 \left(\boldsymbol{h}_{field}(\boldsymbol{q_c}) + \boldsymbol{h}_{mechanical}(\boldsymbol{q_c}) \right) = -\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{K}}(\boldsymbol{q_c}) \rangle \tag{5}$$

Total change within V = flow into V + production inside V

$$\boldsymbol{J} = \boldsymbol{J}_{field} + \boldsymbol{J}_{mechanical} = \int_V \boldsymbol{h}\, dV \tag{6}$$

$$\frac{d}{dt}\int_V \boldsymbol{h}\, dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{\mathcal{K}} \rangle dS + \int_V \boldsymbol{s_h}\, dV \tag{7}$$

Here the source $\boldsymbol{s_h} = 0$.

For a localized charge density contained within a volume V holds for the mechanical torque:

$$\begin{aligned} \tau(\boldsymbol{q_c}) &= \int_V (\boldsymbol{q}' - \boldsymbol{q_c}) \times \boldsymbol{f}(q') dV \\ &= \int_V (\boldsymbol{q}' - \boldsymbol{q_c}) \times \left(\rho_0(q') \boldsymbol{E}(q') + \boldsymbol{j}(q') \times \boldsymbol{B}(q') \right) dV \\ &= Q(\boldsymbol{q} - \boldsymbol{q_c}) \times (\boldsymbol{E}(q) + \boldsymbol{v}(q) \times \boldsymbol{B}(q)) \end{aligned} \tag{8}$$

$$\boldsymbol{J}_{field}(\boldsymbol{q_c}) = \boldsymbol{J}_{field}(\boldsymbol{0}) + \boldsymbol{q_c} \times \boldsymbol{P}(q) \tag{9}$$

Using

$$\langle \boldsymbol{\nabla a}, \boldsymbol{b} \rangle = \boldsymbol{n}_\nu \frac{\partial a_\mu}{\partial q_\nu} b_\mu \tag{10}$$

$$\langle \boldsymbol{b}, \boldsymbol{\nabla a} \rangle = \boldsymbol{n}_\mu \frac{\partial a_\mu}{\partial q_\nu} b_\mu \tag{11}$$

holds

$$\begin{aligned} \boldsymbol{J}_{field}(\mathbf{0}) &= \int_V \boldsymbol{q}' \times \mathfrak{S}(q') dV = \int_V \boldsymbol{q}' \times \boldsymbol{E}(q') \times \boldsymbol{\nabla} \times \boldsymbol{\phi}(q')\, dV \\ &= \int_V (\boldsymbol{q}' \times \langle (\boldsymbol{\nabla \phi}), \boldsymbol{E} \rangle - \langle \boldsymbol{q}' \times \boldsymbol{E}, (\boldsymbol{\nabla \phi}) \rangle)\, dV \\ &= \int_V \boldsymbol{q}' \times \langle (\boldsymbol{\nabla \phi}), \boldsymbol{E} \rangle dV \\ &\quad + \int_V \boldsymbol{E} \times \boldsymbol{\phi}\, dV - \int_V \langle \boldsymbol{\nabla}, \boldsymbol{E q}' \times \boldsymbol{\phi} \rangle dV + \int_V (\boldsymbol{q}' \times \boldsymbol{\phi}) \langle \boldsymbol{\nabla}, \boldsymbol{E} \rangle dV \end{aligned} \tag{12}$$

Define the non-local spin term, which does not depend on $\boldsymbol{q}'$ is:

$$\boldsymbol{\Sigma}_{field} = \int_V \boldsymbol{E}(q) \times \boldsymbol{\phi}(q) dV \tag{13}$$

And

$$\boldsymbol{L}_{field}(\mathbf{0}) = \int_V \boldsymbol{q}' \times \langle (\boldsymbol{\nabla \phi}), \boldsymbol{E} \rangle dV + \int_V \boldsymbol{q}' \times \rho_0 \boldsymbol{\phi} dV \tag{14}$$

Using Gauss:

$$\int_V \langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{a} \rangle dS \tag{15}$$

And

$$\rho_0 = \langle \boldsymbol{\nabla}, \boldsymbol{E} \rangle \tag{16}$$

Leads to:

$$\boldsymbol{J}_{field}(\mathbf{0}) = \boldsymbol{\Sigma}_{field} + \boldsymbol{L}_{field}(\mathbf{0}) + \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{E q}' \times \boldsymbol{\phi} \rangle dS \tag{17}$$

States

Where a unique closed Hilbert subspace *represents* a given physical item, its **state** *characterizes* the probabilistic properties of that item. In quantum physics, a quantum state is a set of mathematical variables that as far as is possible describes the corresponding physical item. For example, the set of 4 numbers $\{n, l, m_l, m_s\}$ define part of the state of an electron within a hydrogen atom and are known as the electron's quantum numbers. The observables that determine the state are mutually compatible. The position of the electron within the atom is a hidden property. If two operators are each other's canonical conjugate, then only one of them can participate in the state, or the state must contain an account of the combination of both values. An example of such a combination is the ladder operator.

Quantum states can be either pure or mixed. **Pure states** cannot be described as a mixture of others. **Mixed states** correspond to a random process that blends pure states together. Realizations of elementary types are characterized by pure states.

When performing an observation on a quantum state, the result is generally described by a probability distribution, and the form that this distribution takes is completely determined by the quantum state and the operators that are related to the observation of the quantum state. The result of an observation is only determined probabilistically. In relation to the observables that determine the state a pure state is characterized by a single blurred Hilbert vector and that vector corresponds in relation to these observables to a mathematical object known as a wave function. If a pure state corresponds with an eigenvector of the operator that represents the observation, then the result of the observation equals the corresponding eigenvalue. The probalistic nature of observations reflects a core difference between classical and quantum physics.

Linear combinations (superpositions) of states can describe interference phenomena. A mixed state cannot be characterized by a single blurred Hilbert vector. Instead, it is described by its associated density operator and it is represented by a (blurred) closed Hilbert subspace.

Pure states can be represented by a blurred Hilbert vector.

State definition

In Hilbert space a state, or probability function, is a real function P on the Hilbert subspaces, with the following properties:

1. $P(0) = 0$
2. $\forall_{y \in \mathbf{H}}\{P(y) \geq 0\}$, y is a Hilbert subspace
3. $\sum_j P(x_j) = 1$, where $|x_j>$ form an orthonormal base of **H**, and x_j is the ray spanned by $|x_j>$
4. $P(y) = \sum_{j=1}^{r} P(x_j)$ where x_j are mutually orthogonal rays spanning subspace y

Gleason's theorem states: Given a state P on a space of dimension ≥ 3, there is an Hermitian, non-negative operator ρ on **H**, whose trace is unity, such that

$$\forall_{|x>\in H}\{P(x) = < x|\ \rho\ x >\}, \text{ where } x \text{ is a ray spanned by } |x> \tag{1}$$

With each compact normal operator Q corresponds an orthonormal base of eigenvectors $\{|q>\}_q$ with eigenvalues q. As a consequence a notion of state is attached to each physical item combined with one or more mutually compatible compact normal operators $\{Q_j\}_j$.

Pure state

In particular, if some ray x_0 satisfies $P(x_0) = 1$, then according to Born's rule:

$$\forall_{|x>\in H}\{P(x) = \ |< x_0, x >|^2\} \tag{1}$$

This happens when $|x_0>$ represents a pure state.
The pure state $P(\{q_j\}_j)$ is connected to a **wave function** $\psi(\{q_j\}_j)$, where

$$P(\{q_j\}_j) \ = \ \left|\psi(\{q_j\}_j)\right|^2 \tag{2}$$

and $\{q_j\}_j$ are the eigenvalues of eigenvector$\{|q_j>\}$ of the operators $\{Q_j\}_j$. Two operators A and B are compatible when their commutator is zero:

$$[A, B] \ \equiv \ AB - BA \ = \ 0 \tag{3}$$

If the state is characterized by a *set* of independent properties, then each of these properties corresponds with a corresponding operator. These operators must be normal, but they need not be compact. It must be possible to construct a spectral decomposition for each of the operators. Further, the operators that together determine the state must be mutually compatible. The wave function is then the product of the probability amplitudes that correspond to the separate operators. Thus the resulting wave function is a characteristic that represents the probability amplitudes of a set of mutually compatible observables that correspond to the normal operators that determine the state.

The squared modulus of the probability amplitude is the probability density. The wave function can also be a function of time. Position can be a state characterizing observable. However, spacetime does not occur as an eigenvalue of a single operator. The operators may vary. For example an operator may be replaced by its canonical conjugate. In that case, care must be taken that the operators that form the changed state are still compatible. Thus, even with the same physical item, the wave function is not unique.

For the operator Q with eigenfunctions |q> and eigenvalues q the probability amplitude function $\psi(q)$ is given by the smoothed version of $< \psi|q >$

$$\psi(q) \ \cong < \psi|q > \tag{4}$$

When Q is a compact normal operator then the smoothed version of $\psi(q)$ is a continuous function. Then $\psi(q)$ has a Fourier transform $\phi(p)$, where the operator P with eigenvectors |p>

and eigenvalues p is the canonical conjugate of Q. Like $\psi(q)$, the function $\phi(p)$ is also a function that characterizes the corresponding item and $|\phi>$ is a characterizing vector. The parameters q and p may be quaternionic.

$$\varphi(p) = <\varphi|p> \tag{5}$$

With respect to the correspondence with traditional quantum logic, it is wrong to take any characteristic vector including the locator or any function including the wave function as the *representative* for the item. It is ridiculous to expect that a single vector carries all properties of a complex physical item, such as a DNA molecule or an elephant. The Heisenberg uncertainty relation also offers objections against this single vector based representation.

In quantum mechanics the wave function can be interpreted as the combination of a stationary vector and a progression operator. The progression operator has the form $A \cdot \exp(S/\hbar)$. A is Hermitian and positive. S is anti-Hermitian. This is reflected in the Hamilton-Jacobi equation.

In quantum field theory the fields are replacing the wave function. Thus a field may be interpreted as the amplitude of the probability to find something at the location of the field value. For bosons that something may be interpreted as a virtual particle. For fermions that something may be interpreted as a pair of virtual particles. Each type of virtual particle has its own type of field.

There are some questions left with wave functions:

- Can it be non-zero outside the subspace that represents the physical item?
  - Answer: No
- Is the wave function a regular function?
  - Answer: When only compact normal operators are involved, then the wave function is regular.
- What happens to the representing subspace and to the wave function when a measurement is performed?
  - Answer: The subspace is squeezed into an appropriate configuration and the wave function is adapted to this new form.
- Has a system a wave function?
  - Answer: In general a system must not have a wave function, but it has a density operator.

Probability density

Gleason's theorem states that a probability measure $\mu(P)$ on the lattice $L(\mathbf{H})$ of projections P on closed subspaces of a Hilbert space $\mathbf{H}$, corresponds to a non-negative Hermitian operator ρ with trace 1, such that $\mu(P) = tr(\rho P)$. When the projections P_q correspond to the rays formed by the eigenvectors $|q>$ of operator Q and $\mu_i(P_q)$ corresponds to the considered physical item, then $\mu_i(P_q) = <q, \rho_i q>$ corresponds to the square of the modulus of the wave function $\psi_i(q)$. ρ_i is the probability density operator corresponding to μ_i. The probability measure μ is a regular function. $\mu_i(P_q)$ is zero outside the subspace that represents the considered physical item.

The probability density function $P(q) = |\psi(q)|^2$ of an absolutely continuous random variable q is a function that describes the relative chance for this random variable to occur at a given point in the Q observation space. The probability for a random variable to fall within a given set is given by the integral of its density over the set.

The probability density operator ρ is positive-semi-definite ($\forall_{|f\rangle \in H} \{\langle f|\rho f\rangle \geq 0\}$), self-adjoint ($\rho = \rho^{\dagger}$), and has trace one ($tr(\rho) = 1$). For the operator Q with eigenfunctions $|q\rangle$ and eigenvalues q with probability amplitude $\psi(q)$, the density operator ρ is given by

$$\rho = \sum_q \{|\langle \psi|q\rangle|^2 \cdot |q\rangle\langle q|\} \tag{1}$$

Von Neumann entropy is defined using the density operator of physical items.

The operator A can be decomposed

$$A = \sum_a |a\rangle a \langle a| \tag{2}$$

For the state $|\psi\rangle$ the **expectation value** $\langle A\rangle$ for the observable A is

$$\langle A\rangle \equiv \langle \psi|A\,\psi\rangle = \sum_q \{|\langle \psi|q\rangle|^2 \cdot \langle q|\,Aq\rangle\} = tr(\rho A) \tag{3}$$

A Hilbert field is a blurred Hilbert distribution. The blur may represent a probability distribution. When a Hilbert distribution has only non-zero values for a set of Hilbert vectors and these Hilbert vectors span a Hilbert subspace, then the corresponding Hilbert field represents a density operator.

Observables and field values

In Hilbert space observables are represented by operators. The observed value is represented by an eigenvalue or by the expectation value of the operator that represents the observable. Scalar physical fields have numeric values. Vector and tensor fields consist of vectors and tensors that are constructed using numbers. Both the eigenvalues of operators and the values of fields may be hyper complex 2^n-ons.

Numbers

The Hilbert space can be specified by using a number space that allows the mutual orthogonalization and the closure of subspaces. The real's, the complex numbers and the quaternions can perform that job. Horwitz showed that even the octonions with some trouble can achieve this (see: http://arxiv.org/abs/quant-ph/9602001). The real's, the complex numbers, the quaternions and the octonions are the only normed division algebras and they are the only alternative division algebras. In general the octonions are not associative, but the product of two octonions that belong to the same quaternionic subfield is associative. Neither all quaternions nor all octonions commute. However, within complex subspaces the numbers commute.

We will take the following freedom. The fact that a given number space is used for specifying linear combinations of Hilbert space vectors does not mean that eigenvalues of operators must also be restricted to that same number space. In this sense a Hilbert space

specified over the quaternions may allow eigenvalues of operators that are taken from the octonions or even higher 2^n-ons (see http://www.math.temple.edu/~wds/homepage/nce2.pdf or the toolkit). The problem with higher dimension 2^n-ons is that their number characteristics deteriorate with n. However, as long as the (full) eigenvalues are not used to construct linear combinations of vectors, or to specify the inner products of the Hilbert space, there is no problem. All higher dimensional 2^n-ons contain several subspaces that are lower dimensional 2^m-on number spaces. Further, 2^n-ons behave like 2^m-ons in their lower 2^m dimensions.

In general the elements of curves or curved manifolds are themselves not numbers. So, in general they cannot be used as eigenvalues. However, locally the elements of a curved manifold may resemble numbers of a 2n-on number space. Smoothly curved trails of objects that locally resemble 2^n-ons can be treated with the Frenet-Serret frame toolkit. Number spaces can be **attached** as tangent spaces to smoothly curved manifolds. In that way the elements of the curves and the manifolds obtain number characteristics in a small enough environment. Sequences or sets of operators can locally have eigenvalues that are numbers which can be considered as member of smooth curves or of the tangent space of a curved manifold at that location. In that way the elements of smooth curves or of curved manifolds can be related to the corresponding eigenvalues. 2^n-ons are ideally suited for this purpose. This means that the eigenspaces of the subsequent operators in a trail need not overlap. These eigenspaces are only used locally. When curvature and bending of the operator trail diminish, the dimension of the local number space can be lower. When the curvature and the bending increase, the dimension must be higher. This will be reflected in the dimensionality of the local eigenvalues. Apart from the application as eigenvalues of operators the 2^n-ons are suited as values of physical fields.

We will restrict to the 2^n-ons as extensions of the quaternions. As we stated, the numbers created with the Cayley-Dickson construction are not so well behaved. Alternatives are the use of Clifford algebras, Jordan algebras or Grassmann algebras. We will show that in the Hilbert space the 2^n-ons for $n > 1$ automatically introduce these latter algebras through their number waltz.

The **niners** are the most extensive numbers that still keep a reasonable set of number characteristics. More precisely said the 2^n-ons, even those that have a higher dimension than the octonions, keep reasonable number characteristics in the space spanned by their coordinates that have an index lower than nine. The real numbers, the complex numbers, the quaternions and the octonions completely fall within these boundaries. The above hyperlink describes exactly what characteristics the niners retain.

The subspace of the 2^n-on field that is spanned by the first 2^m dimensions acts as a 2^m-on number space. Thus in a dynamic situation, an octionic operator acts locally as a quaternionic operator. In a smaller or more flat region it acts as a complex operator and at "nano"-locality as a real (or better as an imaginary) operator.

2^n-on construction

The 2^n-ons use the following doubling formula

$$(a, b)\,(c, d) = (a\cdot c - (b\cdot d^*)^*, (b^*\cdot c^*)^* + (b^*\cdot(a^*\cdot((b^{-1})^*\cdot d^*)^*)^*)^*) \qquad (1)$$

Up until the 16-ons the formula can be simplified to

$$(a, b)\,(c, d) = (a\cdot c - b\cdot d^{*},\ c\cdot b + (a^{*}\cdot b^{-1})\cdot(b\cdot d)) \tag{2}$$

Up to the octonions the Cayley Dickson construction delivers the same as the 2^n-on construction. From n>3 the 2^n-ons are 'nicer' than the Cayley Dickson numbers. They keep more useful number characteristics. The 2^{n+1}-ons contain the 2^n-ons as the sub-algebra of elements of the form (a, 0)

Waltz details

The 16-ons lose the continuity of the map x => xy. Also, in general holds $xy\cdot x \neq x\cdot yx$ for 16-ons. However, for all 2^n-ons the base numbers fulfill $e_i e_j\cdot e_i = e_i\cdot e_j e_i$. All 2^n-ons feature a conjugate and an inverse. The inverse only exists for non-zero numbers. The 2^n-ons support the **number waltz**

$$c = a\cdot b a^{-1}. \tag{1}$$

Often the number waltz appears as a unitary number waltz

$$c = u^{*}\cdot b u \tag{2}$$

where u is a unit size number and u^{*} is its conjugate $u\cdot u^{*} = 1$.

In quaternion space the **quaternion waltz** $a\cdot b\cdot a^{-1}$can be written as

$$a\cdot b\cdot a^{-1} = \exp(2\cdot\pi\cdot\tilde{\mathbf{i}}\cdot\phi)\cdot b\cdot\exp(-2\cdot\pi\cdot\tilde{\mathbf{i}}\cdot\phi) \tag{3}$$

$$= b - \underline{\mathbf{b}}_{\perp} + \exp(2\cdot\pi\cdot\tilde{\mathbf{i}}\cdot\phi)\cdot\underline{\mathbf{b}}_{\perp}\cdot\exp(-2\cdot\pi\cdot\tilde{\mathbf{i}}\cdot\phi)$$

$$= b - \underline{\mathbf{b}}_{\perp} + \exp(4\cdot\pi\cdot\tilde{\mathbf{i}}\cdot\phi)\cdot\underline{\mathbf{b}}_{\perp}$$

$$\Delta b = (\exp(4\cdot\pi\cdot\tilde{\mathbf{i}}\cdot\phi) - 1)\cdot\underline{\mathbf{b}}_{\perp} \tag{4}$$

$$= (\cos(4\cdot\pi\cdot\phi) + \tilde{\mathbf{i}}\cdot\sin(4\cdot\pi\cdot\phi) - 1)\cdot\underline{\mathbf{b}}_{\perp}$$

$$= \exp(2\cdot\pi\cdot\tilde{\mathbf{i}}\cdot\phi)\cdot 2\cdot\tilde{\mathbf{i}}\cdot\sin(2\cdot\pi\cdot\phi)\cdot\underline{\mathbf{b}}_{\perp}$$

$$||\Delta b|| = ||2\cdot\sin(2\cdot\pi\cdot\phi)\cdot\underline{\mathbf{b}}_{\perp}|| \tag{5}$$

Another way of specifying the difference is:

$$\Delta b = (a\cdot b - b\cdot a)/a = 2\cdot(\underline{\mathbf{a}}\times\underline{\mathbf{b}})/a \tag{6}$$

$$||\Delta b|| = 2\cdot||\underline{\mathbf{a}}\times\underline{\mathbf{b}}||/||a|| \qquad (7)$$

Infinitesimal number transformation

The number v is close to 1. Thus $v = 1 + \Delta s$. Let us investigate the transform $c = v^*\cdot b\cdot v$.

$$\begin{aligned} c &= (1 + \Delta s^*)\cdot b\cdot(1 + \Delta s) \qquad (1)\\ &= b + \Delta s^*\cdot b + b\cdot\Delta s + \Delta s^*\cdot b\cdot\Delta s\\ &\approx b + \Delta s^*\cdot b + b\cdot\Delta s\\ &= b + \Delta s_0\cdot b + 2\cdot\underline{\mathbf{b}}\times\boldsymbol{\Delta s} \end{aligned}$$

$$\Delta b = \Delta s_0\cdot b + 2\cdot\underline{\mathbf{b}}\times\boldsymbol{\Delta s} \qquad (2)$$

This comes close to the effect of an infinitesimal number waltz, especially when $\Delta s_0 = 0$ In that case $\Delta b_0 = 0$ and $\boldsymbol{\Delta b}$ is perpendicular to $\boldsymbol{\Delta s}$.

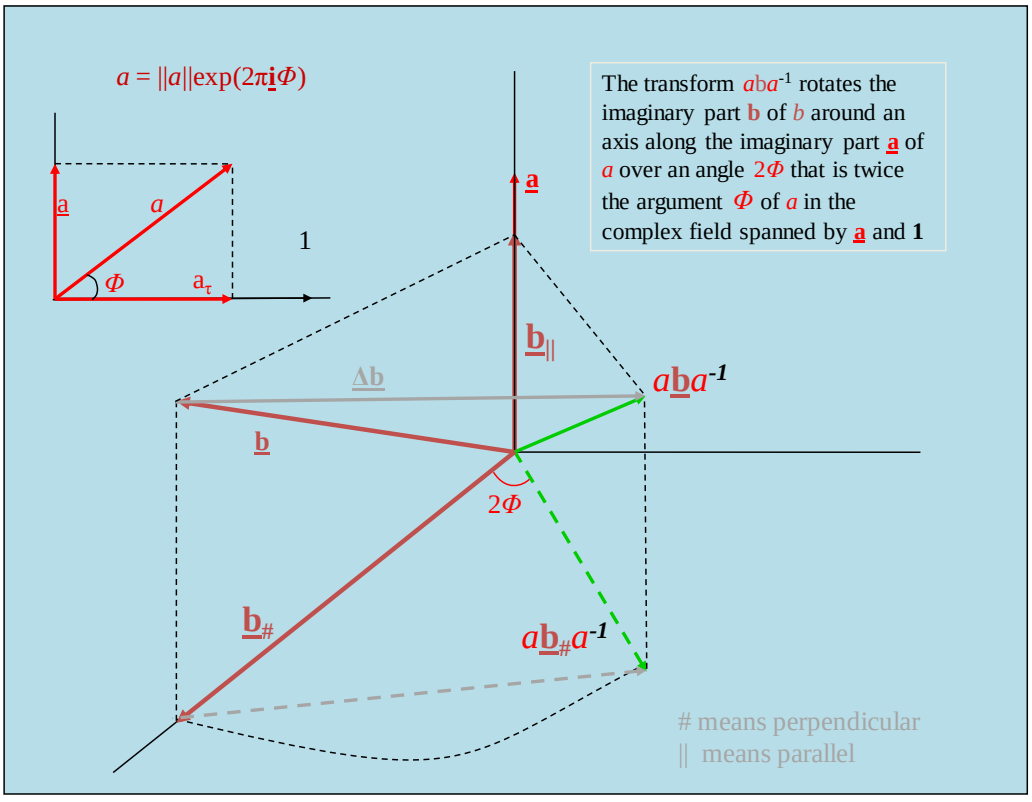


Figure 1. The rotation of a quaternion by a second quaternion.

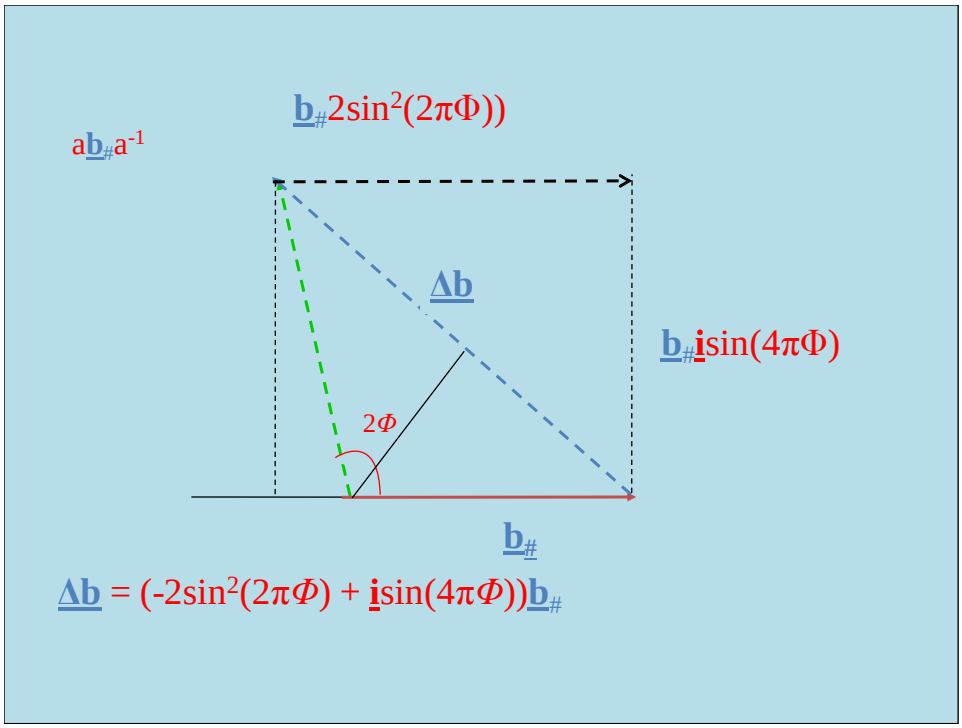


Figure 2: The difference after rotation

For 2^n-ons with n > 1, $a \cdot ba^{-1}$ in general does not equal b. This effect stays unnoticed when quantum mechanics sticks to a complex Hilbert space.

Sign selections

The paper that describes 2^n-ons does not describe the choice for right or left handedness of the external vector product. So, we do it here. The generally accepted convention is to let the handedness depend on the orientation of the underlying $\mathbb{R}^n$ space. However, when numbers are constructed via the Cayley-Dickson construction or the 2^n-on construction then the handedness follows from the applied construction formula. We want to get rid of these restrictions, because we want to give operators the freedom to select the handedness and other sign selections of their eigenvalues.

The 2^n-ons have n independent binary base numbers and n sign selections. The real numbers do not offer a sign selection. The complex numbers offer the selection of the sign of the real or the imaginary axis. This is inherited by all higher 2^n-ons. The quaternions have two independent imaginary base numbers and offer an extra sign selection that represents the handedness of its external product. The octonions have three independent imaginary base numbers and offer an extra sign selection for the handedness in external products that involve this new base number.

Need for spinors

In the number waltz the current manipulator only needs an argument α in order to turn the subject over 2α. This is typical behavior for spinors. Spinors also have a storage place for the handedness of rotations. By using the number waltz and the sign selections the 2^n-ons can perform the same act as the spinors. Spinors are only required when quantum mechanics is restricted to complex Hilbert spaces. Spinors are the carriers of the spin phenomenon. Thus, in our model the sign selections in combination with the number waltz form the carriers of

spin. Because a strange trick is played with the real parts of eigenvalues, the influence of the selection of the sign of the real axis will be revealed later.

The approach taken in this paper might cause a revival of the importance of the hyper complex numbers that turned in oblivion when Gibbs introduced his vector analysis.

Influence

The original proposition (♠) talks about influencing the position of an item. This implies that the position of the item changes due to the mentioned influence. Thus when the influence occurs, the eigenvector that represents the position of the item is exchanged against another eigenvector. That other eigenvector corresponds to another environment inside the eigenspace of the position operator. The new eigenvector takes the role of the old eigenvector and is the new characteristic for the item's position. This replacement may take place inside the subspace, which represents the considered item, or the original eigenvector moves outside the subspace, while the new eigenvector moves in or stays in the subspace. In both cases the eigenvectors of the position operator move with respect to the vector that characterizes the subspace of the item. The movement is relative and takes place inside the Hilbert space. Another possibility is that the eigenvectors stay, but the corresponding eigenvalues change while the Hilbert subspace moves. See Heisenberg picture versus Schrödinger picture.

Thus, there is a way to implement influence in Hilbert space. The influence causes a move of the item's subspace relative to one or more eigenvectors of the position operator. The original proposition (♠) claims that this movement is caused by other items. We must check whether this is true.

If this is true then influences are the motor behind the dynamics of the items.

The universe of items

The original proposition (♠) states that all items influence each other's position. This includes that all items influence the considered item. Part of the items compensates each other's influences on the currently considered item. It will be shown that this holds for the largest part.

Inertia

The influence may decrease with distance according to some function $f(r)$ of the distance r. However the number of contributing items increases with the distance. Depending on function $f(r)$ the most probable result is that the strongest influence comes from the cooperative activity of the most distant items. Due to the enormous number of items in the universe, any variation of the influences of the distant items averages away. This also holds for the density distribution of the items. So there exists a fairly uniform background influence caused by the universe of items. What will happen can be deduced from an equivalent of Denis Sciama's analysis. We will take his analysis as a guide. Sciama's analysis uses a different setting: the (observed) 3D space and coordinate time. (See: http://arxiv.org/abs/physics/0609026v4.pdf). This setting raised critique because the approach involves instantaneous action on large distances. In Sciama's setting this is in conflict with

special relativity. In our setting we do not (yet) encounter special relativity. We use the coordinate space defined by operator Q and the progression parameter t as our setting. A Location in coordinate space represents a location in on the unit sphere of Hilbert space.

The most important aspects of the analysis are:

The total potential at the location of the influenced subject is

$$\Phi = -\int_V \frac{\rho}{r} dV = -\rho \int_V \frac{dV}{r} \tag{1}$$

(See: http://en.wikipedia.org/wiki/Newtonian_potential). This conforms to a Gaussian blur. The integral is taken over the coordinate space volume V. In fact, the integral is taken over the unit sphere of Hilbert space. This is an affine space. The parameter r is the length of the vector from the actor to the location of the subject. The considered subject is located somewhere in the affine coordinate space. All other subjects have positions relative to that considered subject. At large distances, the density ϱ of the contributing items can be considered to be uniformly distributed. Also any variance in strength other than the dependence on r becomes negligible because the differences are blurred and averaged away. In fact we also assumed that the average blur of the distribution of matter in the universe is a Gaussian blur. We take the average of the strength as the significant parameter. We combine it with ρ. Therefore the average of ρ can be taken out of the integral. Thus, apart from its dependence on the average value of ρ, Φ is a huge constant. (Sciama relates Φ to the gravitational constant).As a consequence we can consider the universe as a very large rigid body. If nothing else happens then all influences compensate each other.

In the following equations we use imaginary quaternions rather than 3D vectors. In this way we can avoid the distracting factor i.

If the subject moves relative to the universe with a uniform speed **v**, then a vector potential **A** is generated.

$$\mathbf{A} = -\int_V \frac{\mathbf{v} \cdot \rho}{c \cdot r} dV \tag{2}$$

Both ρ and **v** are independent of r. Together with the constant c they can be taken out of the integral. Thus

$$\mathbf{A} = \Phi \cdot \mathbf{v}/c \tag{3}$$

What we have here is the reverse of the definition of the potential that goes together with a charge distribution. When we defined a Hilbert field we started from a charge distribution and a current distribution and considered the blurred influence of these distributions on the universe. Here we consider the influence of the universe on a local charge or current. For this purpose we use the same volume integrals!

The notions of charge and current correspond to equivalent notions in Noether's theorem. Here we talk about inertia. Thus charge may symbolize mass.

Here a progression parameter t plays the role of manipulator time. Be aware, this is not our usual notion of time.

According to Helmholtz theorem the Hilbert field derived from the above potentials can be split into a divergence free part and a rotation free part. The Helmholtz decomposition theorem only concerns the static versions of the derived field. It is related to the fact that the Fourier transform of a vector field can be split in a longitudinal and a transversal version. There also exists a corresponding split of the multi-dimensional Dirac delta function in a longitudinal and a transversal version. In curved manifolds the Helmholtz decomposition should be replaced by the Hodge decomposition.

A variation of **v** goes together with a variation of **A**. On its turn this goes together with a non-zero field $\dot{\boldsymbol{A}}(\boldsymbol{r}, t)$ which is a **dynamical** part of the derived Hilbert field.

Sciama uses a Maxwell equation to explain the relation between ∂**v**/∂t and $\dot{\boldsymbol{A}}(\boldsymbol{r}, t)$. Our setting differs, but the main reasoning is the same.

$$\boldsymbol{E}(\boldsymbol{r}, t) = -\boldsymbol{\nabla}\Phi(\boldsymbol{r}, t) - \frac{1}{c} \cdot \dot{\boldsymbol{A}}(\boldsymbol{r}, t) \tag{4}$$

$$\widetilde{\boldsymbol{E}}(\boldsymbol{k}, \omega) = -\boldsymbol{k} \cdot \widetilde{\Phi}(\boldsymbol{k}, \omega) - \frac{1}{c} \cdot \omega \widetilde{\boldsymbol{A}}(\boldsymbol{k}, \omega) \tag{5}$$

Remark: As soon as we turn to the dynamic version (4) an extra component $\dot{\boldsymbol{A}}$ of field $\boldsymbol{E}$ appears that corresponds to acceleration ∂**v**/∂t. (See for derivation of Maxwell equations e.g. the online book http://www.plasma.uu.se/CED/Book; formula 3.25)

As already claimed, in our setting the component $\boldsymbol{\nabla}\Phi$ of the field $\boldsymbol{E}$ is negligible. With respect to this component the items compensate each other's influence. This means that if the influenced subject moves with uniform speed **v**, then $\boldsymbol{E} \approx 0$. However, a vector field $\boldsymbol{A}$ is present due to the movement of the considered item. Any acceleration of the considered item goes together with a non-zero $\boldsymbol{E}$ field. In this way the universe of items causes inertia in the form of a force that acts upon the accelerating item's charge.

We have used the coordinate space as a playground to implement an equivalent of Sciama's analysis. The analysis uses the fact that every item in universe causes an influence and that this influence reduces according to $f = -k/r$. (Compare this with Bertrand's theorem in Wikipedia)

A uniform movement in Hilbert space does not generate a reaction of the universe of items. Any alteration of that uniform movement will cause as reaction **a field**. The physical name for this reaction is **action**. It usually gets the symbol **S**. When the path of the item coincides with a **geodesic**, then it can be travelled field free.

It must be noticed that the original analysis of Sciama uses observable position space rather than Hilbert space, coordinate space or action space and it uses a different notion of time. However, the general conclusion stays the same. Sciama's analysis is criticized because it uses infinite speed of information transfer. Since we do not work in observable position space, we do not encounter coordinate time. So for us, this criticism is misplaced. (Coordinate time relates to observations of position.)

The situation with electromagnetic fields is different, because with this field positive and negative charges compensate each other's long range influence. For that reason there exists no electromagnetic background influence. The masses of the gravitational and inertial fields only compensate each other's long range influences through geometrical circumstances. Still they together create gigantic potentials.

Nearby items

Items that are located nearby have a different effect. In general their influence will not have strength equal to the average strength. Further these items are not uniformly distributed. Still their influence depends on inter-distance as $f = -k/r$. As a consequence their influences form a landscape of which the effects will become sensible in the action of the fields that surround the considered item. This landscape will form a curved action space. The considered item will try to follow a geodesic through that curved space.

Rotational inertia

Besides linear inertia there exists rotational inertia. In a non-rotating universe hold near the origin $\mathbf{A} = 0$ and $\Phi = -c^2/G$. We choose units such that c=G=1. In a universe rotating slowly with angular speed ω hold

$$A_x = \omega \cdot y \tag{1}$$

$$A_y = -\omega \cdot x \tag{2}$$

$$A_z = 0 \tag{3}$$

$$\Phi = -\sqrt{1 + (\omega \cdot r)^2} \tag{4}$$

A constant angular movement meets the fields that correspond to a centripetal force.

The field $\boldsymbol{E}$ has the form

$$\boldsymbol{E} = \frac{\omega^2 \boldsymbol{r}}{\sqrt{1 + \omega^2 r^2}} \tag{5}$$

An added uniform speed v meets the fields corresponding to a Coriolis force.

$$\boldsymbol{H} = \boldsymbol{\nabla} \times \boldsymbol{A} = 2 \cdot \boldsymbol{\omega} \tag{6}$$

$$\boldsymbol{v} \times \boldsymbol{H} = 2 \cdot \boldsymbol{v} \times \boldsymbol{\omega} \tag{7}$$

The forces are usually considered as *fictitious* but they are actually caused by inertia. Sciama treats them in section 5 of his paper. Like fields of linear inertia these rotation related fields correspond to actions of the manipulator.

Storage, sign selections and virtual items.

The static fields act as storage media for the location and the speed of the charges of the physical items.

When the values of the fields are stored in hyper complex numbers, then the sign choices for these numbers will also be reflected in these fields. For example the handedness will show in the transverse fields. A right handed and a left handed version of the field will exist. The sign selection of the real part of the hyper complex numbers may also cause extra versions of fields.

The fields can be interpreted as reflections of the presence of virtual items that are ready to exchange roles with actual items.

The proposition

This finding indicates that when our interpretation of Sciama's analysis is correct, the original proposition

All items in universe influence each other's position.

is not generally true. The universe of items does not influence position. It counteracts acceleration of individual items. Position is only influenced in an indirect way and presupposes an observation. If the item moves in a geodesic with uniform speed, then the position changes while the influences of all other items compensate each other. In such cases the summed influence is zero.

We may alter the original proposition (♠). If our analysis is correct, then the proposition

All items in universe influence each other's acceleration.

is true.

The origin of dynamics

If we want to discover the origin of dynamics, we must first determine what the static structure of nature is. We already found an important ingredient of this skeleton: the lattice structure of quantum logic and the corresponding lattice structure of the closed subspaces of a Hilbert space. Both structures are only defined in a static way. Nothing is said about their dynamics. Besides of these static relations the concept of wave functions and density operators offer insight in the probability and information content of these relations.

In the previous part of the paper the next component of the static structure of nature is also investigated: the static structure of the influences. It appears that this structure is identical with the structure of static Hilbert fields. Both the analysis of inertia and the study of Hilbert fields showed the static relation between divergence free fields and rotation free fields. These analyses also showed the influence of dynamics on the coupling of these static fields. The analysis of Hilbert fields explained how these fields change as a function of q_0. Inertia

showed how these fields get coupled when the uniform movement of a physical item is disturbed. Next we try to find the origin of this dynamics.

Extended quantum logic

Wave functions represent the probability of finding a property of realistic items. This property can be the position of the item. In quantum field theory the values of fields are treated in equations of motion in a similar way as the wave functions of realistic items are treated in such equations. When fields can be considered as representations of the probability of finding properties of virtual items, then the fields get an interesting interpretation. In quantum logic the realistic physical item is represented by a proposition in the form of a statement that says everything about that item. For virtual items the new interpretation would mean that in extended quantum logic the virtual items are represented by potential propositions that are ready to become actual propositions.

This would mean that traditional quantum logic id embedded in extended quantum logic that apart from propositions about actual physical items also contains potential proposition that represent virtual particles.

The set of propositions of extended quantum logic is much larger than the set of propositions of quantum logic. It is a set of sets. According to Cantor its cardinality is one level higher and the set is no longer countable.

It may be so that this set is isomorphic with the set of subspaces of a rigged Hilbert space.

Interpretation in logical terms

The results of the analysis of inertia mean that when the redefinition of the set of vectors that belongs to the representation of the item occurs such that this corresponds to a uniform movement of the physical item, then the influences of the universe of items tend to compensate each other. Otherwise, the universe of items reacts with a corresponding field. That field manifests as an action of the current manipulator. Besides of the universe wide response, a local variance in the distribution of items causes a variation in the influences.

It seems that quantum logic and Helmholtz decomposition together define an important part of the static relations that exist in physics. The fields appear to resist the disturbance of the interrelations in the lattice of quantum propositions. In dynamical sense this lattice might step from one static status quo to the next. After a step new conditions are established that again must fulfill the laws that govern the static situation. If this is a proper interpretation, then it is likely that the progression step is taken universe wide. After each step the positions of the physical items relative to the fields have changed, thus when the fields are not uniformly distributed, the items meet a different field configuration. The next step is taken with and due to these new conditions.

Quantum logic only defines a static skeleton in which the dynamics of quantum physics takes place. To make it a dynamic logic, the set of axioms must be extended. The new axioms must state that all propositions influence each other. The influence depends on their mutual (coordinate) distance. In stationary conditions, which include uniform motion, these influences compensate each other. When an atomic predicate that concerns an element of an

ordered set is replaced in a non-ordered fashion, meaning that the distance between the replaced elements does not stay the same, then the universe of all propositions will react such that the influences of the other propositions no longer compensate each other. The disordered influences counteract the disordered replacement.

Besides of that the local variance in the distribution of the propositions, which corresponds to a variance of the distribution of the corresponding physical items, also cause a variation in the influences that propositions have with respect to each other.

In Hilbert space these influences are implemented in the actions of Hilbert fields. In quantum physics the influence appears as a set of physical fields.

Redefiner action

One important step must still be taken. In physics observed spacetime has a Minkowski signature. Further we observe that space corresponds with the imaginary part of a position quaternion for which the real part seems to have no physical meaning. We must find an explanation for these facts. The Minkowski signature defines the following time-like relation between the proper time Δt, the space step Δq and the coordinate time step $\Delta \tau$

$$\Delta t^2 = \Delta \tau^2 - \Delta q^2 / c^2 \tag{1}$$

A possible explanation can be given by the action of the redefiner when the infinitesimal action step is perpendicular to the space step and the coordinate time step is used to close the rectangular triangle. The action step Δs equals Δt. Δt, Δs and Δt are 3D vectors.

$$\Delta \tau = \Delta s + \Delta q / c \tag{2}$$

Dynamics

Schrödinger or Heisenberg picture

For global rotations around its origin the Hilbert unit sphere acts as an affine space. It does not matter whether the eigenvectors of operators or the subspace that represents the item is moved. We can take the picture in which the subspace stays fixed, while the vectors move and the operators change with them. This is the **Heisenberg picture.**
We can also take the picture in which the vectors and operators stay fixed and the subspace moves. This is the **Schrödinger picture**.

We are only interested in the consequences. These are determined by the relative movement, not by the absolute movement. For a given physical item, in both pictures the expectation values of the operators vary in the same way.

Unitary transform

A unitary transform is a bounded normal operator. It has unit sized eigenvalues and to each of these eigenvalues correspond one or more eigenvectors that are mutually orthogonal. Unitary transforms keep the value of inner products untouched. Unitary transforms are completely determined by their vector replacement characteristics, their eigenvectors and the

corresponding eigenvalues. An extra characteristic is for example the smoothness of their eigenspace.

A Fourier transform is an example of a unitary transform. However, it does not leave a single Hilbert vector untouched. There exist Hilbert functions and Hilbert fields that keep their form through Fourier transformation. For that reason they are often called eigenfunctions, but they do not correspond to eigenvectors. The Hermite functions are notorious examples of Fourier "eigenfunctions".

When a unitary transformation U is applied to an arbitrary vector |f>, which is not an eigenvector, then that vector is transferred into another vector |g> = |U f>, which has the same norm. If |f> is an eigenvector of U then |f> is **not** transferred to a different vector, but it is multiplied with the corresponding eigenvalue. Also in this case the norm stays the same.

If a unitary transform is applied to two vectors and one is an eigenvector and the other is not an eigenvector, then the inner product stays the same. The non-eigenvector rotates around the foot of the eigenvector, but keeps its angle with respect to that eigenvector.

Because multidimensional subspaces usually contain one or more eigenvectors of a given unitary transform, the transfer of a multidimensional closed subspace requires a set of parallel unitary transforms. If we take a set of vectors $\{|f_s\rangle\}_s$ that together span a closed subspace, then a set of suitable unitary transforms $\{U_s\}_s$, can in parallel transfer all vectors of this set such that after the transform $|g_s\rangle = |U_s f_s\rangle$ the set $\{|g_s\rangle\}_s$ spans the new subspace.

When a unitary operator U is applied to the eigenvector |q> of an operator Q with eigenvalue q, then the eigenvector is transferred into another vector |U q>. In general |U q> is not another eigenvector of Q. The expectation value for |QU q> is no longer q, but

$$\langle q\, U|QU\, q\rangle = \langle q|U^{\dagger}QU\, q\rangle \tag{1}$$

Or, with other words the operator Q is redefined to $U^{\dagger}QU$.

The norm of the expectation value $\langle f\, U|QU\, f\rangle$ for an arbitrary vector |f> does not depend on U. It only depends on Q and |f>.

Each of the members U_s of the set $\{U_s\}_s$ can be split in a trail. $\{U_{st}\}_t$ The situation sketched above can be refined for any instant t occurring after t=0. We can treat it more generally by chopping the path from $\{|f_s\rangle\}_s$ to $\{|g_{st}\rangle\}_s$ into a **trail** of infinitesimal steps of size Δt that is achieved by a set of infinitesimal transforms $\{U_{st}\}_{st}$, where

$$|g_{st}\rangle = |\prod_s U_{st}\, f_s\rangle \tag{2}$$

and

$$U_{st} \approx 1 + \Delta S_{st} \tag{3}$$

The parameter t acts as the trail progression parameter. It is not identical with our common notion of time. The infinitesimal transforms U_{st} work in parallel as well as in sequence. ΔS_{st} represents the current local infinitesimal action step. It is an imaginary operator.

The trail $\{U_{st}\}_{st}$ causes a redefinition of the operators that have eigenvectors in the considered subspace. The Heisenberg picture conforms to the description with unitary transforms where operators are redefined. When this is done in small steps, then the redefined operator becomes a function of progression parameter t.

Single infinitesimal step

The success of the path integral formalism gives us guidance. We analyze an arbitrary trail consisting of infinitesimal trajectory steps:

$$< q_{t_1}|U_{s_{t_1,t_0}}\, q_{t_0} > = \left\{\prod_{t=t_0}^{t=t_1}(< q_t{}'|p_t >< p_t\, U_{s_t}|q_t >)\right\} < q_{t_1}|p_{t_1} > \tag{1}$$

What happens during a single step when the system moves from position q to $q' = q + \Delta q$? Let us evaluate $< q'|p >< p\, U_{st}|q >$ for a single trajectory. Here U_{st} is an infinitesimal unitary transform. It is a member of the set of parallel unitary transforms that act on a target subspace. In the following text we leave the parallel trajectory index s in U_{st} unspecified. We concentrate on the sequence index t, which represents the progression parameter. The infinitesimal sequence step comprises three sub-steps:

1. Goto Fourier space. This is achieved by step $< q'|p >$.
2. Perform the action. This is done by $< p\ U_t^{\dagger}| = \exp(\Delta \boldsymbol{s}_t) < p|$.
3. Go back to configuration space. This is achieved by $< p|q >$.

The sense behind the first and the last step is a travel to and back from Fourier space, where the differential operators appear as simple multiplications.

First we split $< q'|p >$.

$$< q'|p > = \exp\left(\frac{\boldsymbol{qp}}{\hbar}\right) \exp\left(\frac{\Delta \boldsymbol{q}\, \boldsymbol{p}}{\hbar}\right) \approx \exp\left(\frac{\boldsymbol{qp}}{\hbar}\right)\left(1 + \frac{\Delta \boldsymbol{q}\, \boldsymbol{p}}{\hbar}\right) \tag{2}$$

$$< p|q > = \exp\left(\frac{\boldsymbol{pq}}{\hbar}\right) = < q|p >^* = \exp\left(-\frac{\boldsymbol{qp}}{\hbar}\right) \tag{3}$$

$$< q'|p > \exp(\Delta \boldsymbol{s_t}) < p|q > \approx \exp\left(\frac{\boldsymbol{qp}}{\hbar}\right)\left(1 + \frac{\Delta \boldsymbol{q}\, \boldsymbol{p}}{\hbar}\right)(1 + \Delta \boldsymbol{s_t}) \exp\left(-\frac{\boldsymbol{qp}}{\hbar}\right) \tag{4}$$

$$= \exp\left(\frac{\boldsymbol{qp}}{\hbar}\right) \mathrm{C} \exp\left(-\frac{\boldsymbol{qp}}{\hbar}\right)$$

This is a quaternionic rotation of the central term C, which is close to unity. The quaternionic rotation manipulators stands for the route to Fourier space and the route back from Fourier

space. The central term C stands for what is done during a single step by the action in Fourier space.

$$\mathrm{C} \approx \left(1 + \frac{\Delta \boldsymbol{q}\, \boldsymbol{p}}{\hbar}\right)(1 + \Delta \boldsymbol{s_t}) \approx 1 + \frac{\Delta \boldsymbol{q}\, \boldsymbol{p}}{\hbar} + \Delta \boldsymbol{s_t} = 1 + \Delta \mathbf{C} \tag{5}$$

$$\Delta \mathbf{C} = \frac{\Delta \boldsymbol{q}\, \boldsymbol{p}}{\hbar} + \Delta \boldsymbol{s_t} \tag{6}$$

In configuration space a similar step occurs, but then rotated with the transformation

$$\Delta \mathrm{C}_{config} = \exp\left(\frac{\boldsymbol{qp}}{\hbar}\right) \Delta \mathrm{C}_{Fourier} \exp\left(-\frac{\boldsymbol{qp}}{\hbar}\right) \tag{7}$$

We study the step $\Delta \mathbf{C}$ somewhat deeper. As in the case of Hilbert fields we will split the analysis in a longitudinal part and a transverse part. The longitudinal part treats the part of p that is parallel to Δq. The transverse analysis treats the part of p that is perpendicular to Δq The longitudinal equation is:

$$\Delta \boldsymbol{s}_{\parallel_t} = \frac{\langle \boldsymbol{p}, \Delta \boldsymbol{q} \rangle}{\hbar} + \Delta \mathbf{C}_{\parallel} = \frac{\langle \boldsymbol{p}, \Delta \boldsymbol{q} \rangle}{\hbar} - \mathbf{H}_{\parallel}\, \Delta \tau \tag{8}$$

$$ds_{\parallel_t} = \frac{p_{\parallel}}{\hbar} dq - \mathrm{H}_{\parallel}\, d\tau \tag{9}$$

$$\frac{\partial s_{\parallel_t}}{\partial q} = \frac{p_{\parallel}}{\hbar} \tag{10}$$

$$\frac{\partial s_{\parallel_t}}{\partial \tau} = -\mathrm{H}_{\parallel} \tag{11}$$

$$\dot{s}_{\parallel_t} - \frac{p_{\parallel}}{\hbar} \dot{q} = \mathrm{H}_{\parallel} \tag{12}$$

The transversal equation is:

$$\Delta \boldsymbol{s}_{\perp_t} = \frac{\boldsymbol{p} \times \Delta \boldsymbol{q}}{\hbar} + \Delta \mathbf{C}_{\perp} = \frac{\boldsymbol{p} \times \Delta \boldsymbol{q}}{\hbar} - \mathbf{H}_{\perp} \Delta \tau \tag{13}$$

$$ds_{\perp_t} = \frac{p_{\perp}}{\hbar} dq - \mathrm{H}_{\perp} d\tau \tag{14}$$

$$\frac{\partial s_{\perp_t}}{\partial q_{\perp}} = \frac{p_{\perp}}{\hbar} \tag{15}$$

$$\frac{\partial s_{\perp_t}}{\partial \tau} = -\mathrm{H}_{\perp} \tag{16}$$

(17)

$$\dot{s}_{\perp t} - \frac{p_\perp}{\hbar}\dot{q}_\perp = \mathrm{H}_\perp$$

The trail corresponds to a sum:

$$< q_{t_1}|U_{t_1,t_0}\, q_{t_0} > = \left\{\prod_{t=t_0}^{t=t_1}(< q_t{}'|p_t >< p_t\, U_t|q_t >)\right\} < q_{t_1}|p_{t_1} > \tag{18}$$

$$= 1 + \sum_{t=t_0}^{t=t_1}\left\{\frac{\Delta \boldsymbol{q}\,\boldsymbol{p}}{\hbar} + \Delta \boldsymbol{s_t}\right\}$$

Depending on environmental conditions, the longitudinal direction $\boldsymbol{k}_t$ varies with the trajectory parameter t. The steps Δq_t and Δs_t depend on the step Δt of the trajectory parameter t that is used to chop the unitary transform U_{t_1,t_0}.

Following a trail has much in common with ray tracing in optics. However in optics the use of characteristics that have their base in Fourier analysis seems to be more fruitful than ray tracing. Ray tracing follows the path of a sharp particle, while Fourier analysis is capable of following the life path of a blurred particle and include more of the influences of the environment in the analysis. It is sensible to expect that the advantages of Fourier analysis also hold for wave mechanics. To a certain extent the path integral approach makes also use of Fourier analysis.

Relativity

Einstein's own explanation of the origin of relativity was: "*There is no logical way to the discovery of these elementary laws. There is only the way of intuition.*" Read more in: http://www.time.com/time/magazine/article/0,9171,878733,00.html#ixzz15NlhpWDu

Thus, Einstein never gave a proper explanation for the existence of special relativity. He just provided a set of formulas that work properly. He left us the choice of finding the origin of special relativity or otherwise to follow his intuition. Let us give it a try:

The position operator $\boldsymbol{Q}$ is modified by the unitary operators of the trail into another operator $\boldsymbol{Q}_t$ that has different eigenvectors and different eigenvalues.

$$\boldsymbol{Q}_{t+\Delta t} = U_t \boldsymbol{Q}_t U_t^\dagger \tag{1}$$

$$U_t \approx 1 + \Delta \boldsymbol{S}_t \tag{2}$$

$$U_t^\dagger \approx 1 - \Delta \boldsymbol{S}_t \tag{3}$$

$$\boldsymbol{Q}_{t+\Delta t} \approx \boldsymbol{Q}_t + [\Delta \boldsymbol{S}_t, \boldsymbol{Q}_t] \tag{4}$$

$$\Delta\langle \boldsymbol{q} \rangle = \langle \boldsymbol{Q}_{t+\Delta t} \rangle - \langle \boldsymbol{Q}_t \rangle \approx [\Delta \boldsymbol{S}_t, \boldsymbol{Q}_t] = 2\, \boldsymbol{Q}_t \times \Delta \boldsymbol{S}_t \tag{5}$$

This indicates that the step $\Delta\langle \boldsymbol{q} \rangle$ in the expectation value $\langle \boldsymbol{Q}_t \rangle$ of $\boldsymbol{Q}_t$ is perpendicular to both $\boldsymbol{Q}_t$ and $\Delta \boldsymbol{S}_t$. The steps $\Delta\langle \boldsymbol{q} \rangle$ and $\Delta \boldsymbol{S}_t$ form a right angular triangle with a hypotenuse: $c\, \Delta \boldsymbol{\tau}$, such that:

$$c\, \Delta \boldsymbol{\tau} = \Delta\langle \boldsymbol{q} \rangle + \Delta \boldsymbol{S}_t \tag{6}$$

With $\Delta \boldsymbol{\sigma} = \Delta \boldsymbol{S}_t / c$ the Minkowski signature of a new observable spacetime becomes visible.

$$\Delta \boldsymbol{\tau} = \frac{\Delta\langle \boldsymbol{q} \rangle}{c} + \Delta \boldsymbol{\sigma} \tag{7}$$

$$\Delta \boldsymbol{\sigma} = \Delta \boldsymbol{\tau} - \Delta\langle \boldsymbol{q} \rangle / c \tag{8}$$

$$|\Delta \boldsymbol{\sigma}|^2 = |\Delta \boldsymbol{\tau}|^2 - |\Delta\langle \boldsymbol{q} \rangle|^2 / c^2 \tag{9}$$

Thus, the analysis of what occurs during a single infinitesimal step gives us an indication how relativity enters physics. However, it asks for the introduction of a local notion of time $\boldsymbol{\tau}$ that differs considerably from the (global) progression parameter t. This new parameter is the **coordinate time** t_c.

$$\Delta \boldsymbol{\tau} = \boldsymbol{e}_{\boldsymbol{\tau}}\, \Delta t_c \tag{10}$$

Proper time

In relativity, **proper time** t_p is time measured by a single clock between events that occur at the same place as the clock. It depends not only on the events but also on the motion of the clock between the events. An accelerated clock will measure a proper time between two events that is shorter than the coordinate time measured by a non-accelerated (inertial) clock between the same events.

$$\left|\Delta t_p\right|^2 = |\Delta \boldsymbol{\tau}|^2 - |\Delta\langle \boldsymbol{q} \rangle|^2 / c^2 \tag{1}$$

$$\Delta \boldsymbol{\sigma} = \boldsymbol{e}_{\boldsymbol{\sigma}}\, \Delta t_p \tag{2}$$

Thus, proper time is, upon a proportionality factor, identical with our notion of progression parameter t.

Discussion

We have successfully introduced special relativity into our model.
By introducing relativity the way we did we played a few tricks.

- We neglect the real part of the position observable. It plays no part in dynamics.

- We shift from the global progression parameter t to the local coordinate time t_c.
- We shift from Hilbert space via coordinate space to observed space, thereby losing one dimension.
- We combine the resulting observed space with coordinate time into a Minkowski/Lorentzian space.

As a consequence

- We shift from 2^n-on/Riemannian space to Minkowski/Lorentzian space.
- Most physicists will use Clifford, Jordan and Grassmann algebras rather than 2^n-on algebras.
- With these algebras they can use complex analysis instead of the more complicated 2n-on analysis.
- But if they do so, they are confronted with unintuitive selection features.
- In the new space the quaternion waltz becomes an odd operation.
- Spinors can help in order to cope with these changes.

Can we do without relativity?

Yes.

- Skip coordinate time.
- Use clocks that measure the proper time.

However, you would have to fight existing conventions.

Speed along the live path

For the speed υ_{trail} along the action trail measured in coordinate time units holds:

$$\frac{ds}{dt_c} = v_{trail} = \sqrt{1 - (v_{Qspace}/c)^2} \tag{1}$$

Where

$$v_{Qspace} = dq/dt_c \tag{2}$$

is the speed measured in coordinate time units in the Q space along the observed life path of the item.

Path characteristics

The Frenet-Serret frame is devised for describing curved paths of particles, but we use it here for another purpose.

Let $\{\boldsymbol{\alpha}_{qt}\}_t = \boldsymbol{\alpha}(q,t)$ describe a curved path consisting of infinitesimal steps through a landscape $\{\boldsymbol{\alpha}_q\}_q = \boldsymbol{\alpha}(q)$ of imaginary quaternions $\boldsymbol{\alpha}_{qt}$, such that $||\dot{\boldsymbol{\alpha}}(q(t))|| = 1$ for all t.

The 3D Frenet-Serret frame for the above path is given by:

$$\boldsymbol{T}\big(q(t)\big) := \frac{\partial \boldsymbol{\alpha}\big(q(t)\big)}{\partial t} = \boldsymbol{T}(t) = \dot{\boldsymbol{\alpha}}(t) \tag{1}$$

$$\kappa(t) := ||\dot{\boldsymbol{T}}(t)|| \tag{2}$$

$$\kappa(t) \cdot \boldsymbol{N}(t) := \dot{\boldsymbol{T}}(t) \tag{3}$$

$$\boldsymbol{B}(t) := \boldsymbol{T}(t) \times \boldsymbol{N}(t) \tag{4}$$

$$||\boldsymbol{T}(t)|| = ||\boldsymbol{N}(t)|| = ||\boldsymbol{B}(t)|| = 1 \tag{5}$$

$\boldsymbol{T}(t)$ is the **tantrix** of curve $\boldsymbol{\alpha}(q(\mathrm{t}))$ at instance t.
$\boldsymbol{N}(t)$ is the **principal normal** of curve $\boldsymbol{\alpha}(q(\mathrm{t}))$ at instance t. It is only defined when $\kappa(\mathrm{t}) \neq 0$.
$\boldsymbol{B}(t)$ is the **binormal** of curve $\boldsymbol{\alpha}(q(\mathrm{t}))$ at instance t.
$\boldsymbol{T}(t)$, $\boldsymbol{N}(t)$ and $\boldsymbol{B}(t)$ are imaginary quaternions.
$\kappa(\mathrm{t})$ is the curvature of curve at $\boldsymbol{\alpha}(q(\mathrm{t}))$ at instance t.
$\mathrm{r}(\mathrm{t}) = 1/\kappa(\mathrm{t})$ is the radius of curvature at instance t.
$\tau(\mathrm{t})$ is the torsion of curve $\boldsymbol{\alpha}(q(\mathrm{t}))$ at instance t.

$$\begin{bmatrix} \dot{\boldsymbol{T}}(t) \\ \dot{\boldsymbol{N}}(t) \\ \dot{\boldsymbol{B}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(\mathrm{t}) & 0 \\ -\kappa(\mathrm{t}) & 0 & \tau(\mathrm{t}) \\ 0 & -\tau(\mathrm{t}) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}(t) \\ \boldsymbol{N}(t) \\ \boldsymbol{B}(t) \end{bmatrix} \tag{6}$$

The Frenet-Serret curves have particular characteristics. The path may be curved and curled. The path is completely determined by its tantrix, curvature and torsion given by functions of t. Each coordinate of the quaternionic function $\boldsymbol{\alpha}(q(\mathrm{t}))$ has its own set of characteristics. This means that for a given quaternionic function these characteristics are quaternions rather than real numbers and they are all functions of parameter t.

Path equations

The path equations are given by

$$\dot{\boldsymbol{T}}(t) = \kappa(\mathrm{t}) \cdot \boldsymbol{N}(t) \tag{1}$$

$$\dot{\boldsymbol{N}}(t) = -\kappa(\mathrm{t}) \cdot \boldsymbol{T}(t) + \tau(\mathrm{t}) \cdot \boldsymbol{B}(t) = -\kappa(\mathrm{t}) \cdot \boldsymbol{T}(t) + \tau(\mathrm{t}) \cdot \boldsymbol{T}(t) \times \boldsymbol{N}(t) \tag{2}$$

$$\dot{\boldsymbol{B}}(t) = -\tau(\mathrm{t}) \cdot \boldsymbol{N}(t) = \boldsymbol{T}(t) \times \dot{\boldsymbol{N}}(t) + \dot{\boldsymbol{T}}(t) \times \boldsymbol{N}(t) \tag{3}$$

$$= \tau(\mathrm{t}) \cdot \boldsymbol{T}(t) \times \boldsymbol{B}(t)$$

Curve length

The curve length $l(a, b)$ is defined by:

$$l(a, b) = \int_{x=a}^{x=b} |\dot{\boldsymbol{\alpha}}(q(\mathrm{x}))| \, dx \tag{1}$$

The integration over the square of the modulus delivers the **action** S of the curve.

(2)

$$S(a,b) = \int_{x=a}^{x=b} |\dot{\boldsymbol{\alpha}}(q(\mathrm{x}))|^2 \, dx$$

Reparameterization

The path characteristics κ(t) and τ(t) together with the curve length and the curve action are independent of any reparameterization $s(t)$ of the progression parameter t.
A natural reparameterization is given by $s(t) = l(t_0, t)$.
This turns the curve $\boldsymbol{\alpha}(q(\mathrm{t}))$ into a **natural** curve $\boldsymbol{\gamma}\big(q(\mathrm{s})\big)$:

$$\boldsymbol{\gamma}\big(q(\mathrm{s})\big) = \boldsymbol{\alpha}(q(\mathrm{t})) \tag{1}$$

Curves on a surface which minimize length between the endpoints are called geodesics.
The natural curve corresponds to a geodesic.
The consequence is that in three-dimensional space the corresponding movement obeys the geodesic equation. The Lagrangian is an equivalent of this equation.

Path through field

A **geodesic** on a smooth manifold M with an affine connection ∇ is defined as a curve $\boldsymbol{\gamma}(t)$ such that parallel transport along the curve preserves the tangent vector to the curve, so

$$\nabla_{\dot{\boldsymbol{\gamma}}} \dot{\boldsymbol{\gamma}}(t) = 0 \tag{1}$$

at each point along the curve, where $\dot{\boldsymbol{\gamma}}$ is the derivative with respect to t. More precisely, in order to define the covariant derivative of $\dot{\boldsymbol{\gamma}}$ it is necessary first to extend $\dot{\boldsymbol{\gamma}}$ to a continuously differentiable imaginary Hilbert field in an open set. However, the resulting value of the equation is independent of the choice of extension.

Using local coordinates on M, we can write the **geodesic equation** (using the summation convention) as

$$\frac{d^2 x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu} \cdot \frac{dx^\mu}{dt} \cdot \frac{dx^\nu}{dt} = 0 \tag{2}$$

where $x^\mu(t)$ are the coordinates of the curve $\boldsymbol{\gamma}(t)$ and $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols of the connection ∇. This is just an ordinary differential equation for the coordinates. It has a unique solution, given an initial position and an initial velocity.

From the point of view of classical mechanics, geodesics can be thought of as trajectories of free particles in a manifold. Indeed, the equation $\nabla_{\dot{\boldsymbol{\gamma}}} \dot{\boldsymbol{\gamma}}(t) = 0$ means that the acceleration of the curve has no components in the direction of the surface (and therefore it is perpendicular to the tangent plane of the surface at each point of the curve). So, the motion is completely determined by the bending of the surface. This is also the idea of the general relativity where particles move on geodesics and the bending is caused by the gravity.

Christoffel symbols

If x^i, $i = 1,2,...,n$, is a local coordinate system on a manifold M, then the tangent vectors

$$e_\mu = \frac{\partial}{\partial x_\mu}, \qquad \mu = 1, 2, \ldots, n \tag{1}$$

define a basis of the tangent space of M at each point. The Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ are defined as the unique coefficients such that the equation

$$\nabla_\mu e_\nu = \Gamma^\lambda_{\mu\nu} \cdot e_\lambda \tag{2}$$

holds, where ∇_μ is the Levi-Civita connection on M taken in the coordinate direction e_μ.

The Christoffel symbols can be derived from the vanishing of the covariant derivative of the metric tensor g_{ik}:

$$0 = \nabla_\lambda g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x_\lambda} - g_{\eta\mu} \cdot \Gamma^\eta_{\mu\lambda} - g_{\mu\eta} \cdot \Gamma^\eta_{\nu\lambda} \tag{3}$$

By permuting the indices, and re-summing, one can solve explicitly for the Christoffel symbols as a function of the metric tensor:

$$\Gamma^\mu_{\nu\lambda} = \tfrac{1}{2} \cdot g^{\mu\nu} \cdot \left(\frac{\partial g_{\eta\nu}}{\partial x^\lambda} + \frac{\partial g_{\eta\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\eta}\right) \tag{4}$$

where the matrix $(g^{\mu\nu})$ is an inverse of the matrix $(g_{\mu\nu})$, defined as (using the Kronecker delta, and Einstein notation for summation)

$$g^{\lambda\mu} \cdot g_{\mu\nu} = \delta^\lambda_\nu \tag{5}$$

Although the Christoffel symbols are written in the same notation as tensors with index notation, they are **not** tensors, since they do not transform like tensors under a change of coordinates.

Under a change of variable from $(x^1, \ldots, x^n)$ to $(y^1, \ldots, y^n)$, vectors transform as

$$\frac{\partial}{\partial y^i} = \frac{\partial x^k}{\partial y^i} \cdot \frac{\partial}{\partial x^k} \tag{6}$$

and so

$$\Gamma^k_{ij} = \frac{\partial x^p}{\partial y^i} \cdot \frac{\partial x^q}{\partial y^j} \cdot \Gamma^r_{pq} \cdot \frac{\partial y^k}{\partial x^r} + \frac{\partial y^k}{\partial x^m} \cdot \frac{\partial^2 x^m}{\partial y^i \partial y^j} \tag{7}$$

where the underline denotes the Christoffel symbols in the y coordinate frame. Note that the Christoffel symbol does **not** transform as a tensor, but rather as an object in the jet bundle.

At each point, there exist coordinate systems in which the Christoffel symbols vanish at the point. These are called (geodesic) normal coordinates, and are often used in Riemannian geometry.

The Christoffel symbols are most typically defined in a coordinate basis, which is the convention followed here. However, the Christoffel symbols can also be defined in an arbitrary basis of tangent vectors e_μ by

$$\nabla_{e_\mu} e_\nu = \Gamma^\lambda_{\mu\nu} \cdot e_\lambda \tag{8}$$

The action along the live path

The integrated action S_{ab} is performed over a distance along the action trail or equivalently over a period of coordination time

$$S_{ab} = -\int_a^b m \cdot c^2 \cdot ds + matter\ terms \tag{1}$$

$$= -\int_{\tau_a}^{\tau_b} m \cdot c^2 \cdot \sqrt{1 - \left(\frac{v}{c}\right)^2} \cdot d\tau + matter\ terms$$

$$= \int_{\tau_a}^{\tau_b} \mathcal{L} \cdot d\tau$$

m is the mass of the considered item.
v is the speed in Q space.
$\mathcal{L}$ is the Lagrangian.

The first line of this formula can be considered as an integral along the trail in coordinate space or equivalently over the trail in Hilbert space. The next lines concern integrals over the corresponding path in observed space combined with coordinate time. It must be noticed that these spaces have different signature.

$$\mathcal{L} = -m \cdot c^2 \cdot \frac{ds}{d\tau} + \text{matter terms} \tag{2}$$

In general relativity, the first term generalizes (includes) both the classical kinetic energy and interaction with the Newtonian gravitational potential. It becomes:

$$m \cdot c^2 \cdot \frac{ds}{d\tau} = -m \cdot c \cdot \sqrt{g_{\alpha\beta} \cdot \dot{q}_\alpha \cdot \dot{q}_\beta} \tag{3}$$

$g_{\alpha\beta}$ is the rank 2 symmetric metric tensor which is also the gravitational potential. Notice that a factor of c has been absorbed into the square root.
The matter terms in the Lagrangian $\mathcal{L}$ differ from those in the integrated action S_{ab}.

(4)

$$S_{ab\_matter} = -\int_a^b e \cdot A_\gamma \cdot dq^\gamma + \text{other matter terms}$$

The matter term in the Lagrangian due to the presence of an electromagnetic field is given by:

$$\mathcal{L} = -m \cdot c^2 \cdot \frac{ds}{d\tau} + e \cdot \dot{q^\gamma} \cdot A_\gamma + \text{other matter terms} \quad (5)$$

A_γ is the electromagnetic 4-vector potential.

Redefinition

If we want to use the Schrödinger picture, rather than the Heisenberg picture, then it is better not to use unitary transforms, because they act on the eigenvectors of operators and by doing so, they change the operators. Instead the subspace should be redefined.

Let us suppose that there exists a dynamical equivalent of the traditional quantum logic. The equivalent of a move of a physical item in the lattice of propositions is a redefinition of a subset of the propositions. The redefinition occurs in terms of atomic predicates that describe the properties of the physical items. In the Hilbert space this corresponds with a redefinition of a relevant part of the Hilbert subspace in terms of the eigenvectors that belong to the new eigenvalues.

The redefinition concerns the Hilbert space which represents the current status quo. The step transforms the current version of the Hilbert space into a past version of the Hilbert space and it transforms a future version of the Hilbert space into the new current version. This is interesting in the light that a Hilbert field exists that represent past, current and future versions of the Hilbert fields. For that reason we will call this special Hilbert field the **adventure field**. A transform that controls dynamics converts a future Hilbert space into the new current Hilbert spaces and it converts the current Hilbert spaces into a past Hilbert space. This transform will be called **progression transform**. The local blurs that characterize the adventure field form boundary conditions for the local transfer characteristics of the progression transform. Each item type is surrounded by a characteristic blur.

A progression transform that moves Hilbert subspaces without touching the eigenvectors of normal operators will be called a **redefiner**. The effect of the action of the redefiner on expectation values of operators must be similar to the effect of the trail of parallel unitary transforms treated in the previous paragraphs. While the set of parallel trails of unitary transforms act in the Heisenberg picture, the redefiner acts in the Schrödinger picture. As indicated earlier, the redefiner has an equivalent in the dynamic version of quantum logic.

In order to achieve the same effect as the Heisenberg picture, the Hilbert subspace redefiner must to a large degree have similar properties as the trails of parallel infinitesimal unitary transformations that are used to move the subspace in the Heisenberg picture. The redefinition keeps the inner products of vectors intact. Where unitary transforms rotate vectors around the origin of a Hilbert space, the redefiner takes subspaces of a potential

future Hilbert space in order to redefine them into subspaces of the new current Hilbert space. In contrast to a unitary transform the redefiner does not change the eigenvectors of normal operators. Thus, it leaves the operators untouched. Like the trails of unitary transforms the redefinition works in infinitesimal steps. These infinitesimal actions also form trails. In this way the manipulated subspace can move continuously through Hilbert space. Where the redefiners act on subspaces, the trails of unitary transforms redefine operators.

During this process the subspace may change its configuration. This may include a change that corresponds to the change of type definitions of atomic predicates. The redefiner steps from one stationary situation to the next. The Schrödinger picture conforms to the description with a redefiner. The result for the position of the locator must be the same as it was under the influence of the set of parallel infinitesimal unitary operators in the Heisenberg picture. The redefiner moves the subspace such that the new locator position is similar to the value as was established by the redefined position operator. It means that during the redefiner step the position of the locator undergoes an infinitesimal number transform that is equivalent to the infinitesimal transform that is established by the redefined position operator. That redefinition was caused by the parallel infinitesimal unitary transforms.

Trails

In fact the t step characterizes the redefinition step. The subsequent replacement of vectors and the replacement of the corresponding eigenvalues can be interpreted as a rather continuous movement of the corresponding characteristic subjects. Here we encountered ten different trails.

1. The trail of subsequent manipulators (infinitesimal unitary transforms or infinitesimal redefiners) that each perform an infinitesimal action.
2. The trail of subspaces, which with respect to the manipulators are characteristic for the considered item.
3. The trail of corresponding "action values" of the redefiner.
4. The trails of corresponding "action values" of the unitary transforms.
5. The trail of eigenvectors $|q_t\rangle$
6. The trail of corresponding observables Q_t.
7. The trail of corresponding observed expectation values q_t.
8. The trail of values $\psi(q_t)$ of a wave function.
9. This, on its turn corresponds to a trail of a state in coordinate space
10. And a trail of that state in Hilbert space.

Cycles

It is quite possible that subsequent steps are done in cycles of two or more steps. It is obvious that movements inside an item are cyclic. In ideal circumstances these movements are harmonic.

Redefiner

The concept of dynamic manipulator gives us reason to introduce a **new type of actuator**: the redefiner ℜ. This actuator moves subspaces, but leaves vectors untouched. It works in infinitesimal steps. It is easily interpreted as a function $\Re_t$ of the progression parameter t. Its scope spans the Hilbert space. The effect of each step on an item is similar to the effect of a

set of parallel infinitesimal unitary transforms $\{U_{ts}\}_s$. The current "action value" of the redefiner is a number, which is close to unity. It is an "average" of the "actions values" of the parallel infinitesimal uniforms that are active in the same step. The redefiner accepts 2^n-ons as "action values".

The redefiner has an equivalent in a dynamic quantum logic, where it redefines propositions that concern the same objects as are represented by the closed subspaces of the Hilbert space that are moved by $\Re_t$. There seems to be no objection against the assumption that $\Re_t$ has a global scope. If we take that point of view, then the progression parameter t also has a global scope.

With this interpretation, the redefiner is a universe-wide stepper. It transforms the universe from one static situation to the next static situation. These static situations are governed both by traditional quantum logic and by the Helmholtz/Hodge decomposition theorems. After each step the status quo of subspaces and fields is reestablished. However, after the step the conditions have been changed. After each step the position of the physical item relative to the fields has changed, thus when the fields are not uniformly distributed, the item meets a different field configuration. On the other hand the fields represent the blurs of the individual items. Thus, when the position or the type of the item has changed, then the local configuration of the field has changed. This is the way that macroscopic dynamics takes place in quantum physics.

Optics

The optical Fourier transform (OTF) is an objective imaging quality characteristic for imaging devices in a similar way as the frequency transfer function qualifies the signal transfer function of a linearly operating electronic device. The transfer quality of a chain of linear signal transforming devices is characterized by the product of the frequency transfer functions of the elements of the chain. In a similar way the OTF of a chain of imaging devices is given by the product of the OTF's of the elements of the chain. However, this is a profound simplification of reality. The product rule only holds when the transfer characteristics of the imaging devices are spatially uniform over the complete input field of the separate imaging components. Further, the conditions in which the OTF's of the components are determined must be similar to the conditions in the chain. More in detail, this means that the angular distribution, the chromatic distribution and the homogeneity of the radiation must be identical.

In optics, the image sided spread function equals the convolution of the object sided spread function and the point spread function (PSF, the image of a point). The Fourier transform of the image sided spread function is equal to the product of the Fourier transform of the object sided spread function and the optical Fourier transforms (OTF's) of the imaging devices. When several imaging devices work in sequence, then the total optical transfer function of the imaging system equals the product of the transfer functions of the components.

If we restrict to a static situation and include the "depth" of the image, the static PSF is a three parametric function. Thus the OTF must have the same number of parameters. Like the PSF the three dimensional OTF has a longitudinal component and a two dimensional

transverse component. In most cases only the transverse component is used as an imaging quality characteristic. On-axis the transverse component is rotationally symmetric. Off-axis its modulus, the MTF, is symmetric but not rotationally symmetric. On-axis the MTF is presented as a one parametric curve in which only the positive axis is given. Off-axis the two extremes of the MTF are given. They correspond to radial and tangential directions.

Due to the fact that Hilbert fields are blurred Hilbert distributions, wave mechanics has much in common with wave optics. For each compact normal operator the Hilbert subspace that represents a physical item corresponds to a spread in Hilbert space and a corresponding spread in the eigenspaces of that normal operator. The distribution of this spread is represented in a wave function, or more correctly, in a probability distribution. For example the wave function that has the position as a variable corresponds to the triple consisting of a physical item, its Hilbert subspace representation and the position operator.

After a move of a physical item its position related wave function has much in common with the spread function that characterizes the blur of the image sided pictures in a linear operating imaging system. The physical fields that influence the physical item have an equivalent in the chain of imaging devices that transfers the image.

The product formula for the transfer functions relies on several preconditions. First of all it relies on the fulfillment of the requirement for sufficient spatial uniformity of the transfer. At all places where information is passed, the transfer characteristics must be sufficiently identical. The product formula has only validity in the spatial area where this requirement is fulfilled.

The transfer characteristics will be different for each Fourier component. Their quality will reduce with higher spatial frequencies.

The final result can be computed in longitudinal direction by multiplication. In lateral direction these regions are restricted to areas where the transfer is locally sufficiently uniform. In the summation that is used to compute a sensible average the angular and chromatic distribution of the transferred information play a role. These distributions determine the summation coefficients. The extent of the region in which the considered transfer function is considered valid depends on the accuracy that is required for the result of the computation. Sign selections inside the radiation determine the polarization. Often in optics this feature and its influence is ignored. Coherence plays a role as well, but in practice optical imaging uses either nearly completely incoherent light or nearly completely coherent light.

In wave mechanics the wave function, which is taken just before the item moves, gets the role of the object. After a movement through a region of the fields the wave function has been changed. Its Fourier transform then equals the product of the Fourier transform of the original wave function and the wave transfer functions (WTF's) of the fields that influence the item. If several steps are taken in sequence, then the transfer functions of the passed field pieces must be multiplied in order to get the overall result. This transfer is affected in a similar way by spatial non-uniformity as the optical case.

In cylindrical imaging systems Seidel aberrations take their toll. When the system is folded or when lenses are not perfectly in line, also non-cylindrical influences will influence the imaging quality. The measurement and the specification of the OTF must cope with the spatial non-uniformity of the imaging characteristics of the imaging devices and with the angular and chromatic distribution of the radiation. The OTF also depends on the longitudinal location of the object and where the image is detected. This also occurs with the WTF of physical fields. Both in optics and in wave mechanics the precise locations of the "object" and the "image" are often not well determined. They are defined by spatial distributions in three dimensions. In both cases the angular and chromatic distributions of the contributing radiation influence the transfer. The final result is constituted by the weighted sum of all contributions.

With inhomogeneous (= incoherent) imaging the phases are ignored. These facts indicate the difference between the particle view and the wave view. From optics it is known that the modulation transfer function (MTF) is a proper imaging qualifier for inhomogeneous light imaging. In inhomogeneous imaging the imaging process can be properly described by ray tracing. Ray tracing has much similarities with the application of the path integral. However, ray tracing normally does not use arbitrary paths. In inhomogeneous imaging phases are scrambled. For holographic imaging the phase transfer function (PTF) or the whole OTF is the better measure. With holographic imaging the phases carry the depth information. Feynman's path integral can cover arbitrary paths because, according to Feynman's claim, interference via the phases eliminates the contributions of non-realistic paths. That is why in the path integral the angular distribution of the radiation plays no role.

In optics the image space is often a surface. In optics the OTF depends on the position in the object space. Off axis the OTF is not rotationally symmetric. The OTF also depends on the angular distribution and the chromatic distribution of the radiation. These dependencies also hold for the WTF in wave mechanics.

A longitudinal displacement of the image spread function with respect to the object spread function corresponds to an extra phase term in the longitudinal component of the Fourier transform of the image spread function. A lateral displacement corresponds to an extra phase term in the transverse component of the Fourier transform. In wave mechanics this holds for the respective components of the Fourier transform of the wave function after the move.

The resemblance between optics and wave mechanics becomes striking when the discrete lens pack is replaced by a medium with a continuously varying refraction. In optics this happens with electron optical lenses that are used in imaging with charged particles.

When the point spread function is a function of three-dimensional position, then the OTF is also a three-parametric function of spatial frequency. The MTF is a symmetric function. However, the MTF is not rotationally symmetric (in 2D) or spherical symmetric (in 3D). On its vertical axis the MTF indicates the part of the energy of the radiation that is transferred by a given spatial frequency.

Veiling glare and halo

Due to reflections on refracting surfaces some of the energy of the radiation loses much of its spatial information content. As a consequence the MTF shows a sharp peak near zero spatial frequency. This phenomenon is called **veiling glare**. When the drop is not so fast the phenomenon is called **halo**.

Equivalents of veiling glare and halo can also occur in wave mechanics. In this way spurious radiation and a spurious halo can enter space. This can happen in the form of energy or in the form of matter. Spurious radiation contains no spatial information.

Fields

It is clear that the physical fields play an important role in nature. They form an indispensible ingredient in the establishment of dynamics. Each physical item follows a path through a set of universe wide fields. The static gravitational field, the electrostatic field and the electromagnetic field are all subjected to the Helmholtz decomposition theorem. The difference between the gravitational field and the electromagnetic field is that the masses are non-negative and the electric charges are, apart from a sign, always the same. When the path with respect to a one of these fields corresponds to a unit speed curve then that field executes no action onto that item.

More fields

It is sensible to expect that the gravitation field and the electromagnetic fields make use of the same Hilbert distribution. May be there exist other fields that also share this distribution. Each of these fields will have its own kind of blur. It suffices when only one Hilbert distribution exists that covers the whole Hilbert space.

There exists a list of fields with shorter ranges than the range of the gravitation field and the range of the electromagnetic fields. These are not treated here. If this story is correct, then all these fields have a storage place in the eigenvalues of the manipulators.
The action represented by a complete Lagrangian indicates how fields appear in the argument of a manipulator. See Lagrangian of the world for a complete survey of terms. Mendel Sachs has found a way to bring all terms under the same hood.

Thoughts

The following texts represent collections of thoughts that still have to be brought in proper order and in mutual consistency.

Harmonic oscillating Hilbert field

Take the ingredients of the complex harmonic oscillator and interpret these as similar ingredients of a harmonic oscillating Hilbert field that is based on a Gaussian blur. The blur delivers the conditions of the ground state.

$$\psi_0(r) = \sqrt[4]{\frac{m\,\omega}{\pi\hbar}}\; e^{-\frac{m\,\omega}{2\hbar}r^2} \tag{1}$$

This means that the ground state corresponds with a Gaussian charge distribution. Higher states correspond to a blurred current. We indicate this current as vector potential $\boldsymbol{\varphi}$. Its time derivative $\dot{\boldsymbol{\varphi}}$ is perpendicular to $\boldsymbol{\varphi}$. The other ingredients are P, Q, $\mathcal{A}$ and $\mathcal{A}^\dagger$.

$$Q \Leftrightarrow \boldsymbol{\varphi} = \varphi_x = \sqrt{\frac{\hbar}{2m\omega}}(\mathcal{A}_x + \mathcal{A}_x^\dagger) \tag{2}$$

$$\mathcal{A}_x - \mathcal{A}_x^\dagger = 0 \tag{3}$$

$$P \Leftrightarrow m\dot{\boldsymbol{\varphi}} = m\dot{\varphi}_y = \sqrt{\frac{m\omega\hbar}{2}}(-\mathcal{A}_y + \mathcal{A}_y^\dagger) \tag{4}$$

$$\mathcal{A}_y + \mathcal{A}_y^\dagger = 0 \tag{5}$$

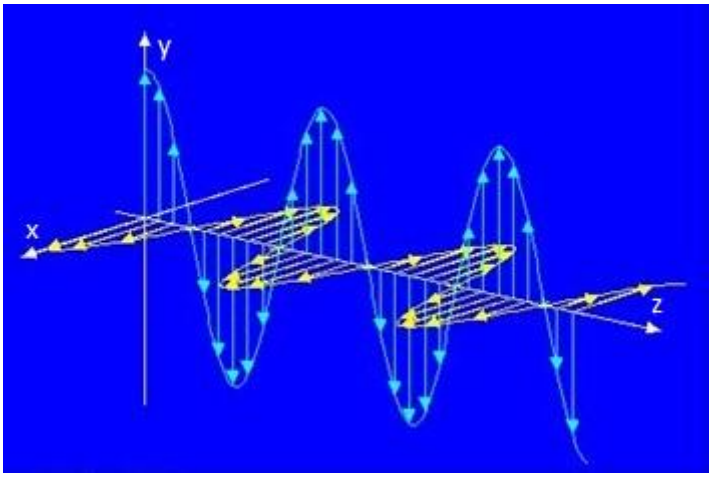
$$\mathcal{A} \Leftrightarrow \boldsymbol{\mathcal{A}} = \boldsymbol{i}\mathcal{A}_x - \boldsymbol{ik}\mathcal{A}_y = \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{\varphi} + \frac{\dot{\boldsymbol{\varphi}}}{\omega}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{i}\varphi_x + \boldsymbol{ik}\frac{\dot{\varphi}_y}{\omega}\right) \tag{6}$$

$$\mathcal{A}^\dagger \Leftrightarrow \boldsymbol{\mathcal{A}}^\dagger = \boldsymbol{i}\mathcal{A}_x^\dagger + \boldsymbol{ik}\mathcal{A}_y^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{\varphi} - \frac{\dot{\boldsymbol{\varphi}}}{\omega}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{i}\varphi_x - \boldsymbol{ik}\frac{\dot{\varphi}_y}{\omega}\right) \tag{7}$$

The $\boldsymbol{\varphi}$ field and the $\dot{\boldsymbol{\varphi}}$ field are mutually perpendicular. If both fields are subjected to a synchronized quantum harmonic oscillation, then an oscillating wave results. We take the same ground state for each of the fields. These ground states correspond to a spherical symmetric Gaussian blur.

When bounds of the cavity are removed or relaxed, then the higher order modes may differ in a phase shift. The sign selections set the eigenvalues of the spin operator. The result is an elliptically polarized wave that moves in directions along $\boldsymbol{\varphi} \times \dot{\boldsymbol{\varphi}}$.

$\boldsymbol{\varphi}$ no longer stands for a single position, but instead for a Gaussian distribution of positions. Similarly $\dot{\boldsymbol{\varphi}}$ does not stand for a single moving particle, but for a moving Gaussian cloud of virtual particles.



Annihilator and creator

The annihilator $\mathcal{A}$ and the creator $\mathcal{A}^\dagger$ are examples of boson operators. This is a consequence of their commutation relations.

$$\mathcal{A} + \mathcal{A}^\dagger = \alpha\, \boldsymbol{\varphi} \tag{1}$$

$$\mathcal{A} - \mathcal{A}^\dagger = \beta\, \dot{\boldsymbol{\varphi}} \tag{2}$$

$$\mathcal{A} = \tfrac{1}{2}\alpha\, \boldsymbol{\varphi} + \tfrac{1}{2}\beta\, \dot{\boldsymbol{\varphi}} \tag{3}$$

$$\mathcal{A}^\dagger = \tfrac{1}{2}\alpha\, \boldsymbol{\varphi} - \tfrac{1}{2}\beta\, \dot{\boldsymbol{\varphi}} \tag{4}$$

$$\left[\mathcal{A}(f), \mathcal{A}^\dagger(g)\right] = < f|g > \tag{5}$$

$$[\mathcal{A}(f), \mathcal{A}(g)] = 0 \tag{6}$$

$$\left[\mathcal{A}^\dagger(f), \mathcal{A}^\dagger(g)\right] = 0 \tag{7}$$

The corresponding fermion operators are:

$$\left\{\mathcal{B}(f), \mathcal{B}^\dagger(g)\right\} = < f|g > \tag{8}$$

$$\{\mathcal{B}(f), \mathcal{B}(g)\} = 0 \tag{9}$$

$$\left\{\mathcal{B}^\dagger(f), \mathcal{B}^\dagger(g)\right\} = 0 \tag{10}$$

The fermion operators can be represented by imaginary quaternionic base numbers:

$$\mathcal{B} + \mathcal{B}^\dagger = \boldsymbol{i} \tag{11}$$

$$\mathcal{B} - \mathcal{B}^\dagger = \boldsymbol{j} \tag{12}$$

$$\mathcal{B} = \tfrac{1}{2}(\boldsymbol{i} + \boldsymbol{j}) \tag{13}$$

$$\mathcal{B}^\dagger = \tfrac{1}{2}(\boldsymbol{i} - \boldsymbol{j}) \tag{14}$$

$$\left(\mathcal{B} + \mathcal{B}^\dagger\right)\left(\mathcal{B} - \mathcal{B}^\dagger\right) = \mathcal{B}\mathcal{B} - \mathcal{B}\mathcal{B}^\dagger + \mathcal{B}^\dagger\mathcal{B} - \mathcal{B}^\dagger\mathcal{B}^\dagger = \mathcal{B}^\dagger\mathcal{B} - \mathcal{B}\mathcal{B}^\dagger = \boldsymbol{i}\,\boldsymbol{j} \tag{15}$$

Spin

The spin term is defined by:

$$\boldsymbol{\Sigma}_{field} = \int_V \boldsymbol{E}(q) \times \boldsymbol{\phi}(q) dV \tag{1}$$

In free space the charge density ρ_0 vanishes and the scalar potential ϕ_0 shows no variance. Only the vector potential $\boldsymbol{\phi}$ may vary with q_0. Thus:

$$\boldsymbol{E} = \boldsymbol{\nabla}\phi_0 - \nabla_0\boldsymbol{\phi} \approx -\nabla_0\boldsymbol{\phi} \tag{2}$$

$$\boldsymbol{\Sigma}_{field} \approx \int_V (\nabla_0\boldsymbol{\phi}(q)) \times \boldsymbol{\phi}(q)dV \tag{3}$$

If $\frac{\phi(q)}{|\phi(q)|}$ can be interpreted as tantrix and $\frac{\nabla_0\boldsymbol{\phi}(q)}{|\nabla_0\boldsymbol{\phi}(q)|}$ can be interpreted as the principle normal, then $\frac{(\nabla_0\boldsymbol{\phi}(q))\times\boldsymbol{\phi}(q)}{|(\nabla_0\boldsymbol{\phi}(q))\times\boldsymbol{\phi}(q)|}$ can be interpreted as the binormal.

Depending on the selected field $\boldsymbol{\Sigma}_{field}$ has two versions that differ in their sign. These versions can be combined in a single operator:

$$\boldsymbol{\Sigma}_{field} = \begin{bmatrix} \boldsymbol{\Sigma}^{+}{}_{field} \\ \boldsymbol{\Sigma}^{-}{}_{field} \end{bmatrix} \tag{4}$$

Spin and dyadic product

As factors of the dyadic product we consider imaginary quaternionic numbers or vectors in $\mathbb{R}_3$. The product corresponds to a matrix. This matrix acts as an operator.

$$u \otimes v \to \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix} \tag{1}$$

The product of quaternions contains sign selections. For the imaginary parts this selection has to do with the handedness of the external product. Dyadic products are well suited to store the product such that the sign selections are stored as well. The sign selection plays its role in the dyad **ij**, which consists of two imaginary base numbers. The dyad **ij** =−**ji**, and **k** can be ± **ij**. Let us apply this to the definition of S_z.

$$s_{z=} - i\hbar \begin{bmatrix} 0 & \boldsymbol{e}_x\boldsymbol{e}_y - \boldsymbol{e}_y\boldsymbol{e}_x & 0 \\ \boldsymbol{e}_y\boldsymbol{e}_x - \boldsymbol{e}_x\boldsymbol{e}_y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = i\hbar \begin{bmatrix} 0 & -2\boldsymbol{e}_x\boldsymbol{e}_y & 0 \\ 2\boldsymbol{e}_x\boldsymbol{e}_y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

This shows that the definition of S_z via the dyadic product reflects the choice in handedness of the external product of $\mathbf{e}_x$ and $\mathbf{e}_y$.

Wave package

The linear momentum is interpretable as a displacement operator. This operator is better treated in Fourier space than in configuration space. In Fourier space a particle becomes a wave package. The Fourier transforms of the fields describe the wave package.
Operator $\boldsymbol{P}$ has eigenfunctions $\tilde{f}(p)$ with eigenvalues p:

$$< q|p > = \tilde{f}(p) = < p|q >^* = f^*(q) = \exp\left(\boldsymbol{k} \cdot \frac{p\,q}{\hbar}\right)) \tag{1}$$

$$< p|\boldsymbol{P}\, p >= p \tag{2}$$

A pure particle can be represented by a single Hilbert vector |f>. Its wave function is given by:

$$\psi(q) = < \psi|q > \tag{3}$$

Or by:

$$\tilde{\psi}(p) = < p|\psi > \tag{4}$$

A mixed particle takes a Hilbert distribution in order to define its presence.

$$\rho(q) = < \rho|q > \tag{5}$$

A blurred Hilbert distribution is a Hilbert field.

$$\phi(q) = \rho(q) \circ \varphi(q) \tag{6}$$

A different type of blur gives a different type of Hilbert field.
The wave functions and particular Hilbert fields represent particles. Their Fourier transforms represent wave packages. A very particular Hilbert field is a probability density that is based on a probability density operator.
A single wave mode represents a plane wave. Look at the linear momentum of the field contained in a volume V surrounded by surface S:

$$\boldsymbol{P}_{field} = \int_V \boldsymbol{g}_{field}\, dV = \int_V \rho_0 \boldsymbol{\phi}\, dV + \int_V \langle \nabla \boldsymbol{\phi}, \boldsymbol{E} \rangle\, dV + \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{E}\boldsymbol{\phi} \rangle dS \tag{7}$$

For each temporal Fourier mode of the field in free space (vanishing charge density ρ_0, no variance of scalar potential ϕ_0), where $\boldsymbol{E\phi}$ falls off rapidly, we can neglect the first and the third term.

$$\boldsymbol{P}_{field} \approx \int_V \langle \boldsymbol{\nabla\phi}, \boldsymbol{E} \rangle\, dV \tag{8}$$

Further:

$$\boldsymbol{E} = \boldsymbol{\nabla}\phi_0 - \nabla_0 \boldsymbol{\phi} \approx -\nabla_0 \boldsymbol{\phi} \tag{9}$$

$$\boldsymbol{P}_{field} \approx -\int_V \langle \boldsymbol{\nabla\phi}, \nabla_0 \boldsymbol{\phi} \rangle\, dV \tag{10}$$

$$= \int_{V_p} \langle \mathbf{p}\tilde{\boldsymbol{\phi}}, \nabla_0 \tilde{\boldsymbol{\phi}} \rangle \, dV_p$$

$$= \int_{V_p} \mathbf{p}\langle \tilde{\boldsymbol{\phi}}, \nabla_0 \tilde{\boldsymbol{\phi}} \rangle \, dV_p$$

$$= \int_{V_p} \omega(p) \, \mathbf{p}\langle \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}} \rangle \, dV_p$$

If the function $\langle \tilde{\boldsymbol{\phi}}(p), \tilde{\boldsymbol{\phi}}(p) \rangle$ gives the probability density for eigenvalue $\boldsymbol{p}$. Then, this gives reason to interpret $\langle \boldsymbol{\phi}(q), \boldsymbol{\phi}(q) \rangle$ as probability density for the position $\boldsymbol{q}$ of the particle.

Fourier mode

A Fourier mode is a single frequency wave. It can be interpreted as a particle or as a train of particles whose charge is blurred by a very wide spread function. The corresponding current is blurred by that same spread function. It means that the divergence along the wave reduces to zero.

Often waves of the same frequency that belong to different mutually perpendicular fields combine to form polarized waves. The waves may differ in their phase shifts. The combination then forms a polarized wave. Depending on the phase difference it may be an elliptical polarized wave, a circular polarized wave or a linearly polarized wave.

Systems

A system is a local assembly of physical items that act as a single physical item. Its state is mixed. When a redefinition of physical items in terms of atomic predicates goes together with influences between items in the form of fields, then a redefinition of a system in terms of its components will certainly also have such effects. The redefinition may take different forms. It may be represented by an emission or absorption of a component or it may be a reshuffling of the components. The simplest case of reshuffling is a permutation of items that belong to the same category. A more complex situation is a periodic movement of one or more components within the realm of a system. In addition each sequence of creation and annihilation is a form of redefinition.

The system has its own characteristic vectors. The wave function may depend on the permutation state of the system. For example for fermions an odd permutation changes the sign of the (position related) wave function. For bosons a permutation does not affect the wave function. Permutations of different categories of components go together with their own type of influence. Thus, there are fermionic fields and there are bosonic fields. Each of these fields has its own type of creation and The annihilation. Being fermion or boson relates to the spin type of the component. The annihilation and creation operators are closely related to the type of components involved and are also closely related to the type of fields involved. The annihilation/creation operators of fermions anti-commute and the annihilation/creation operators of bosons commute.

Entropy

A system is a local assembly of physical items that act as a single physical item. The Density operator ϱ relates to the currently considered observable Q. A pure state is a ray spanned by an eigenvector of the operator Q.

The von Neumann entropy $S(\varrho)$ of a physical system that is characterized by a state $|\psi>$ is given by

$$\rho = \sum_q \{|q> \lambda_q <q|\} = \sum_q \{\lambda_q \cdot \rho_q\} \tag{1}$$

$$\rho_q = |q><q| \tag{2}$$

$$\lambda_q = |<\psi|q>|^2$$

$$S(\rho) = -k_B \cdot \sum_q \{\lambda_q \cdot \ln(\lambda_q)\} \tag{3}$$

The entropy $S(\varrho)$ describes the departure of the system from a pure state. In other words, it measures the degree of mixture (entanglement) of the state $|\psi>$.

Some properties of the von Neumann entropy:

- $S(\varrho)$ is only zero for pure states.
- $S(\varrho)$ is maximal and equal to $\log_2 N$ for a maximally mixed state, N being the dimension of the Hilbert space.
- $S(\varrho)$ is invariant under changes in the basis of ϱ, that is, $S(\varrho) = S(U\varrho U^\dagger)$, with U a unitary transformation.
- $S(\varrho)$ is concave, that is, given a collection of positive numbers λ_q which sum to unity ($\Sigma_q \lambda_q = 1$) and density operators ϱ_q, we have

$$S\left(\sum_q \lambda_q \rho_q\right) \geq \sum_q \lambda_q S(\rho_q) \tag{4}$$

- $S(\varrho)$ is additive. Given two density matrices ϱ_A, ϱ_B describing independent systems A and B, then

$$S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B) \tag{5}$$

Instead, if ϱ_A, ϱ_B are the reduced density operators of the general state ϱ_{AB}, then

$$|S(\rho_A) - S(\rho_B)| \leq S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B) \tag{6}$$

While in Shannon's theory the entropy of a composite system can never be lower than the entropy of any of its parts, in quantum theory this is not the case, i.e., it is possible that $S(\varrho_{AB}) = 0$ while $S(\varrho_A) > 0$ and $S(\varrho_B) > 0$.

Intuitively, this can be understood as follows: In quantum mechanics, the entropy of the joint system can be less than the sum of the entropy of its components because the components may be entangled. The left-hand inequality can be roughly interpreted as saying that entropy can only be canceled by an equal amount of entropy. If system *A* and system *B* have different amounts of entropy, the lesser can only partially cancel the greater, and some entropy must be left over. Likewise, the right-hand inequality can be interpreted as saying that the entropy of a composite system is maximized when its components are uncorrelated, in which case the total entropy is just a sum of the sub-entropies.

- The von Neumann entropy is also *strongly sub-additive.* Given three Hilbert spaces, *A,B,C,*

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}) \tag{7}$$

Isolated systems

With isolated systems we mean systems in a geometrically compound environment where influences from the environment compensate each other, possibly including the influences on the environment that are caused by the system under consideration. This includes e.g. the gravitation field. Internal influences are internally compensated such that they are not felt by other systems. For example the sum of the charges, which are related to electromagnetic fields is zero. It means that the Fourier transforms of the local fields consist of linear combinations of discrete terms. This holds for the electrostatic fields and the magneto-static fields. It holds for rectangular components as well as for polar components. These components are the germs of quanta and are the source of creations and annihilations.
For example consider the vector potential A. Its Fourier transform can be written as:

$$\boldsymbol{A}(\boldsymbol{r},t) = \sum_{k}\sum_{\mu=-1,1}\{\boldsymbol{e}_\mu \cdot a_{\mu\boldsymbol{k}}(t)\cdot\exp\big(i(\boldsymbol{k},\boldsymbol{r})\big) + \bar{\boldsymbol{e}}_\mu \cdot \bar{a}_{\mu\boldsymbol{k}}(t)\cdot\exp\big(-i(\boldsymbol{k},\boldsymbol{r})\big)\} \tag{1}$$

Where $\mathbf{e}_\mu$ are unit sized polarization vectors. They depend on the orthonormal vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ that represent quaternionic imaginary base numbers. The index μ labels the photon spin. The product $\mathbf{e}_\mu \cdot a_\mu$ represents a quaternionic imaginary number. The number i can be interpreted as a base imaginary number in the direction of $\boldsymbol{k}$.

$$\boldsymbol{e}_1 \equiv \frac{-1}{\sqrt{2}}\big(\boldsymbol{e}_x + i\cdot\boldsymbol{e}_y\big) \tag{2}$$

$$\boldsymbol{e}_{-1} \equiv \frac{1}{\sqrt{2}}\big(\boldsymbol{e}_x - i\cdot\boldsymbol{e}_y\big) \tag{3}$$

$$(\boldsymbol{e}_x,\boldsymbol{k}) = 0 \tag{4}$$

$$\big(\boldsymbol{e}_y,\boldsymbol{k}\big) = 0 \tag{5}$$

$$\left[a_\mu(\boldsymbol{k}), a_{\mu'}(\boldsymbol{k}')\right] = 0 \tag{6}$$

$$\left[a^\dagger{}_\mu(\boldsymbol{k}), a^\dagger{}_{\mu'}(\boldsymbol{k}')\right] = 0 \tag{7}$$

$$\left[a_\mu(\boldsymbol{k}), a^\dagger{}_{\mu'}(\boldsymbol{k}')\right] = \delta_{\mu\mu'} \cdot \delta_{\boldsymbol{k}\boldsymbol{k}'} \tag{8}$$

Here the $\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}a_\mu(\boldsymbol{k})$ are the operator equivalents of the coefficients $a_{\mu\boldsymbol{k}}$ and $\omega = c\ |\mathbf{k}| = ck$. This results in:

$$\boldsymbol{A}(\boldsymbol{r},t) = \sum_{\boldsymbol{k},\mu}\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}\left\{\boldsymbol{e}_{\mu(\boldsymbol{k})}\cdot a_\mu(\boldsymbol{k},t)\cdot\exp\left(i(\boldsymbol{k},\boldsymbol{r})\right) + \bar{\boldsymbol{e}}_{\mu(\boldsymbol{k})}\cdot a^\dagger{}_\mu(\boldsymbol{k},t)\cdot\exp\left(-i(\boldsymbol{k},\boldsymbol{r})\right)\right\} \tag{9}$$

$$\boldsymbol{E}(\boldsymbol{r},t) = i\cdot\sum_{\boldsymbol{k},\mu}\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}\left\{\boldsymbol{e}_{\mu(\boldsymbol{k})}\cdot a_\mu(\boldsymbol{k},t)\cdot\exp\left(i(\boldsymbol{k},\boldsymbol{r})\right) - \bar{\boldsymbol{e}}_{\mu(\boldsymbol{k})}\cdot a^\dagger{}_\mu(\boldsymbol{k},t)\cdot\exp\left(-i(\boldsymbol{k},\boldsymbol{r})\right)\right\} \tag{10}$$

$a_\mu(\boldsymbol{k},t)$ is an annihilation operator and $a^\dagger{}_\mu(\boldsymbol{k},t)$ is a creation operator.

$$a^\dagger{}_\mu(\boldsymbol{k},t)\ |n> = |n+1>\sqrt{n+1} \tag{11}$$

$$a^\dagger{}_\mu(\boldsymbol{k},t)\ |0> = |1> \tag{12}$$

$$a_\mu(\boldsymbol{k},t)\ |n> = |n-1>\sqrt{n} \tag{13}$$

$$a_\mu(\boldsymbol{k},t)\ |0> = 0 \tag{14}$$

$$\left[a_\mu(\boldsymbol{k}), \left(a^\dagger{}_\mu(\boldsymbol{k})\right)^n\right] = \left(a_\mu(\boldsymbol{k})\right)^n \tag{15}$$

The Hamiltonian is:

$$\boldsymbol{H}(t) = \hbar\omega\sum_{\boldsymbol{k},\mu}\left\{a_\mu{}^\dagger(\boldsymbol{k},t)\cdot a_\mu(\boldsymbol{k},t) + ½\right\} \tag{16}$$

The number operator N_μ gives the number of quanta:

$$N_\mu(\boldsymbol{k},t) = a_\mu{}^\dagger(\boldsymbol{k},t)\cdot a_\mu(\boldsymbol{k},t) \tag{17}$$

The quanta discussed here are bosons. With the electromagnetic field they are photons. Photons have integer spin 1. With the dyadic product $\otimes$ follows:

$$S_z \equiv -i\hbar\left(\boldsymbol{e}_x\otimes\boldsymbol{e}_y - \boldsymbol{e}_y\otimes\boldsymbol{e}_x\right) \text{ and cyclically for } x\to y\to z\to x \tag{18}$$

$$\left[S_x, S_y\right] = i\hbar S_z \tag{19}$$

$$S_z \cdot \boldsymbol{e}_\mu = \mu \cdot \boldsymbol{e}_\mu \tag{20}$$

Fermions have half integer spin. With fermions the creation and annihilation operators a and $a^\dagger$ have different commutation relations. Instead of commuting, these operators anti-commute.

Measurement

We differentiate between a measurement using a piece of equipment and an observation as is done between items in universe. In the particle view the measuring equipment scrambles the phases. After that scrambling an observation is done. In the wave view the measuring equipment takes care that the phases stay intact, while the amplitudes are ignored during the next observation.

In measurement terms the scramble of the phases is called **de-coherence**. In the same sense the care to keep phases pure and the neglecting of the amplitudes could be called re-coherence. Both actions can be related with the Fourier transforms that convert the wave view into the particle view or visa versa.

Measurement preparation

In a measurement the observation follows after a preparation phase by the measuring equipment. Such a preparation may change the shape of the subspace that represents the item. For example, a preparation for precise position measurement may squeeze the item's subspace such that its range of covered position eigenvectors becomes very short and that its range of covered momentum eigenvectors extends very far. Similarly, when a preparation is made for precise momentum measurement then the item's subspace is squeezed in the other direction, such that it covers a huge range of position eigenvectors. A Fourier transform does not squeeze the item's subspace. It changes the state of the item from position based to momentum based or vice versa.

Squeezing the item's subspace such that its range of covered position eigenvectors becomes very short and that its range of covered momentum eigenvectors extends very far is called **decoherence**. In case of a system it reduces the entanglement of that system.

Hamilton-Jacobi

The Hamilton-Jacobi equation shows how the Hamiltonian relates to the action S of the current manipulator. In this section we consider *t* to be the manipulator time!

$$H \cdot U_t = \tilde{\mathrm{I}}_t \cdot \hbar \cdot \frac{\partial U_t}{\partial t} \tag{1}$$

For the eigenvalues holds

$$\Delta u_t \approx \Delta \boldsymbol{S}_{lt} \cdot u_t \tag{2}$$

Thus, we can put

(3)

$$H \cdot U_t = -\left(\frac{\partial S_t}{\partial t}\right) \cdot U_t$$

$$H = -\left(\frac{\partial S_t}{\partial t}\right) \tag{4}$$

For the expectation values s_t of the action operator S_t holds

$$\tilde{\imath}_t \cdot \hbar \cdot \frac{\Delta s_t}{\Delta t} = \boldsymbol{e}_{t0} + \boldsymbol{e}_{t1} \cdot \Delta t \cdot \frac{\chi_{t1}}{2} + \boldsymbol{e}_{t2} \cdot \Delta t^2 \cdot \frac{\chi_{t1} \cdot \chi_{t2}}{6} \tag{5}$$

$$- \boldsymbol{e}_{t3} \cdot \Delta t^3 \cdot \frac{\chi_{t1} \cdot \chi_{t2} \cdot \chi_{t3}}{24} + \mathcal{O}(\Delta t^3)$$

This derivation is completely independent from the observation of Q. Thus S_t has nothing to do with the Minkowski metric that appears during observations of position.

The Lagrangian

The Lagrangian is equivalent to the local geodesic equation.

The Lagrangian $\mathcal{L}_\tau$ is related with the action s_t.

$$s_t = \int_a^b \mathcal{L}_\tau d\tau \tag{1}$$

The integral is taken over the trail with the observed path. The index t of the action S_t is the trail progression parameter. The integration parameter stands for the coordinate time. The right side of the equation plays in Lorentzian space.

The Euler Lagrange equations explicitly use observations. For that reason the Lagrangian is considered to be a function of the observed q, the velocity $\dot{q}$ and the coordinate time $\boldsymbol{\tau}$. The velocity is measured with the coordinate time.

$$\mathcal{L}_\tau = \mathcal{L}_\tau(\tau, q, \dot{q}) \tag{2}$$

$$\dot{q} = \frac{dq}{d\tau} \tag{3}$$

The Euler-Lagrange equations are:

$$\frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial q_i} - \frac{d}{d\tau}\frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial \dot{q}_i} = 0 \tag{4}$$

for i = x, y, z

When the Lagriangian does not vary with one or more of its parameters, then this corresponds with a symmetry of the system. By Noether's theorem, such symmetries of the

system correspond to conservation laws. In particular, the invariance of the Lagrangian with respect to time τ implies the conservation of energy.)

By partial differentiation of the above Lagrangian, we find:

$$\frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial q_i} = \frac{\partial U}{\partial q_i} = F_i \tag{5}$$

$$\frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial \dot{q}_i} = m \cdot \dot{q}_i = p_i \tag{6}$$

where the force is $\boldsymbol{F} = -\boldsymbol{\nabla} U$ (the negative gradient of the potential, by definition of conservative force), and $\boldsymbol{p}$ is the momentum. By substituting these into the Euler–Lagrange equation, we obtain a system of second-order differential equations for the coordinates on the particle's trajectory,

$$F_i = \frac{d(m\dot{q}_i)}{dt} = m \cdot \ddot{q}_i = \dot{p}_i \tag{7}$$

which is Newton's second law.

The world's action

The action S_t represents the influences that the rest of the world via unitary operator U_t release onto the state $\{|f\rangle_s\}_s$.

In his book about quantum gravity Rovelli writes:
"In the general relativistic parlance 'matter' is anything which is not the gravitational field. As far as current physics knows, the world is made up of the gravitational field, Yang Mills fields, fermion fields and, presumably, scalar fields."
(Carlo Rovelli, book: Quantum gravity, 2004, chapter 2, paragraph 2.1.2)

All these fields give a contribution to the action S.

$$S(e, \omega, A, \psi, \phi) \tag{1}$$

$$= S_{GR}[e, \omega] + S_{matter}[e, \omega, A, \psi, \phi] = S_{GR}[e, \omega] + S_{YM}[e, A] + S_f(e, \omega, A, \psi) + S_{sc}[e, A, \phi)$$

e is the gravitational field.
A(q) is the electromagnetic field.
ω(q) is the spin connection. It is a one form in the Lie algebra of the Lorentz group so(3,1)
ψ(q) is a scalar field, possibly with values in the representation of the Yang Mills group.
ϕ(q) is a field in the spinor representation of the Lorenz group.
A(q) has a non Abelian connection to the Yang Mills group.

The local characteristics of these fields must be represented in the eigenvalue of the current manipulator.

Representing multiple fields

Professor Mendel Sachs recently wrote a few books in which he promotes the inclusion of more terms in the metric than Einstein did. Sachs uses a four vector with quaternionic coefficients in order to specify the metric. Sachs uses all sixteen terms, while Einstein skipped six due to symmetry considerations. The argument of Sachs is that the symmetry is broken due to the characteristics of the quaternion number space. See: http://www.compukol.com/mendel/publications/publications.html.

16-ons contain the required 16 real numbers that can be arranged as a four vector with quaternion coefficients. Sachs still uses the Minkowski metric. So, his view concerns observed space.

Discussion

Macro and micro

The treatise up to so far confines to macroscopic dynamics. Micro dynamics concerns movements that occur inside the representation of small particles. Thus, inside the subspace that represent the particle.
In order to stay inside the item, the internal movements must be periodical. They can be combinations of oscillations and rotations. The harmonics oscillator and the spherical harmonics are well known examples.
The current manipulator can be seen as a very complicated (Fourier?) transform. The eigenfunctions of quantum harmonic movements seem to be governed by the eigenfunctions of this manipulator. Thus micro dynamics occurs via a different process.

Dynamic logic

The current trend in quantum logic development is to add axioms that change the static character of quantum logic in a more dynamic and operational logic. Logic of quantum actions (LQA) adds unitary transforms as the source of dynamics. As we see in this article these transforms are not the real fundamental causes of dynamics. The fields that are exerted by the items are more fundamental causes of dynamics. They represent potential propositions that may be used to redefine the actual propositions. To my knowledge the influences of physical fields are not yet covered by any dynamic logic theory.

Conclusion

Quantum logic is only a partial description of the fundamentals of quantum physics. It only describes the static skeleton in which the quantum dynamics takes place. It does not treat physical fields. However, traditional quantum logic can be extended into a wider logic, such that fields are also included. When this is done, the fields become representatives of past, current and future versions of quantum logics and dynamics can simply be considered as the simultaneous step from a future version, to a current version and from the current version to a past version. An important ruler of quantum dynamics is the influence that is exposed by

the universe of items in the phenomenon inertia. It indicates the laws that govern the exchange of atomic predicates from enveloping propositions. It characterizes the fields as the sticky resistance of the universe of quantum logical propositions against unordered redefinition of their members. This shapes the dynamics of the logic that describes dynamic quantum physics.

The fact that the set of propositions in traditional quantum logic is lattice isomorphic with the set of closed subspaces of an infinite dimensional separatable Hilbert space offers the possibility to study the fields and the dynamics of the propositions with mathematical means. When this study is focused on the infinitesimal steps, the equations of motion are revealed. Blurred Hilbert distributions are carriers of Hilbert fields. Different fields correspond to different blurs. The blurs can be interpreted as probability distributions and as such they are the sources of quantum noise. This means that the blurs also represent the probability of the generation, presence and annihilation of actual and virtual elementary particles. In this light it must be reckoned that these particles are annihilated and (re)generated at each redefinition step.

The dynamics of the life path of an item can be described by a geodesic equation. The live environment can be considered as sets of 2^n-ons that locally resemble quaternions or in a still smaller region resemble complex numbers. These numbers constitute the fields that influence the dynamics of the items. The analysis of the local infinitesimal dynamic step also reveals the origin of special relativity.

There are strong indications that a universe wide clock exists in our model. This is the manipulator time clock. If this is the case, then the redefinitions are universe wide synchronized. There are also strong indications that in our model universe is controlled by a single dynamic redefiner. However, its actions are locally influenced by fields, which are directly connected to the items that are present in this environment.

Microscopic movements are governed by a different process. They are directly controlled by the current manipulator and relate to its eigenfunctions.

Trying to implement a complex quantum logical proposition in Hilbert space is indeed an elucidating experience.

Appendix

History of quantum logic

Around 1930 John von Neuman and Garrett Birkhoff were searching for an acceptable explanation of the results of experiments that showed that the execution of an observation of a very small object can completely destroy the validity of an earlier observation of another observable of that object. The Schrödinger equation that agreed with the dynamic behaviour of the particles already existed. Not much later Heisenberg's matrix formulation became popular as well. Quite soon the conclusion was made that something was fundamentally wrong with the logic behind the behaviour of small particles. These small objects show particle behaviour as well as wave behaviour and they show quantization effects. It was found that the distribution axiom of classical logic had to be changed. Soon it became apparent that the lattice structure of classical logic must be weakened from an orthocomplementary modular form to an orthocomplementary weakly modular lattice. The quantum logic was born. The next step was to find a useful mathematical presentation of this new logic. A historic review of what happened can be found in: "Quantum Theory: von Neumann" vs. Dirac; http://www.illc.uva.nl/~seop/entries/qt-nvd/. It includes extensions of the concept of Hilbert space and application of these concepts to quantum field theory. Another source is: http://www.quantonics.com/Foulis_On_Quantum_Logic.html.

Quantum logic

Elementary particles behave non-classical. They can present themselves either as a particle or as a wave. A measurement of the particle properties of the object destroys the information that was obtained from an earlier measurement of the wave properties of that object.
With elementary particles it becomes clear that that nature obeys a different logic than our old trusted classical logic. The difference resides in the modularity axiom. That axiom is weakened. The classical logic is congruent to an orthocomplemented modular lattice. The quantum logic is congruent to an orthocomplemented weakly modulare lattice. Another name for that lattice is orthomodular lattice.

Lattices

A lattice is a set of elements a, b, c, …that is closed for the connections $\cap$ and $\cup$. These connections obey:

- The set is partially ordered. With each pair of elements a, b belongs an element c, such that $a \subset c$ and $b \subset c$.

- The set is a $\cap$ half lattice if with each pair of elements a, b an element c exists, such that $c = a \cap b$.
- The set is a $\cup$ half lattice if with each pair of elements a, b an element c exists, such that $c = a \cup b$.
- The set is a lattice if it is both a $\cap$ half lattice and a $\cup$ half lattice.

The following relations hold in a lattice:

$$a \cap b = b \cap a \tag{A1}$$

$$(a \cap b) \cap c = a \cap (b \cap c) \tag{A2}$$

$$a \cap (a \cup b) = a \tag{A3}$$

$$a \cup b = b \cup a \tag{A4}$$

$$(a \cup b) \cup c = a \cup (b \cup c) \tag{A5}$$

$$a \cup (a \cap b) = a \tag{A6}$$

The lattice has a partial order inclusion $\subset$:

$$a \subset b \Leftrightarrow a \cap b = a \tag{A7}$$

A complementary lattice contains two elements n and e with each element a an complementary element a′ such that:

$$a \cap a' = n \tag{A8}$$

$$a \cap n = n \tag{A9}$$

$$a \cap e = a \tag{A10}$$

$$a \cup a' = e \tag{A11}$$

$$a \cup e = e \tag{A12}$$

$$a \cup n = a \tag{A13}$$

An orthocomplemented lattice contains two elements n and e and with each element a an element a″ such that:

(A14)

$$a \cup a'' = e$$

$$a \cap a'' = n$$

$$(a'')'' = a \tag{A15}$$

$$a \subset b \Leftrightarrow b'' \subset a'' \tag{A16}$$

e is the unity element; n is the null element of the lattice

A distributive lattice supports the distributive laws:

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \tag{A17}$$

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c) \tag{A18}$$

A modular lattice supports:

$$(a \cap b) \cup (a \cap c) = a \cap (b \cup (a \cap c)) \tag{A19}$$

A weak modular lattice supports instead:

There exists an element d such that

$$a \subset c \Leftrightarrow (a \cup b) \cap c = a \cup (b \cap c) \cup (d \cap c) \tag{A20}$$

where d obeys:

$$(a \cup b) \cap d = d \tag{A21}$$

$$a \cap d = n \tag{A22}$$

$$b \cap d = n \tag{A23}$$

$$[(a \subset g) \text{ and } (b \subset g) \Leftrightarrow d \subset g \tag{A24}$$

In an atomic lattice holds

$$\exists_{p \in L} \forall_{x \in L} \{x \subset p \Rightarrow x = n\} \tag{A25}$$

$$\forall_{a \in L} \forall_{x \in L} \{(a < x < a \cap p) \Rightarrow (x = a \text{ or } x = a \cap p)\} \tag{A26}$$

p is an atom

Both the set of propositions of quantum logic and the set of subspaces of a separable Hilbert space have the structure of an orthomodular lattice. In this respect these sets are congruent. In Hilbert space, an atom is a pure state (a ray spanned by a single vector).

Classical logic has the structure of an orthocomplemented distributive modular and atomic lattice.
Quantum logic has the structure of an orthomodular lattice. That is an orthocomplented weakly modular and atomic lattice. The set of closed subspaces of a Hilbert space also has that structure.

Proposition

In Aristotelian logic a proposition is a particular kind of sentence, one which affirms or denies a predicate of a subject. Propositions have binary values. They are either true or they are false.

Propositions take forms like "*This is a particle or a wave*". In quantum logic "*This is a particle.*" is not a proposition.

In mathematical logic, propositions, also called "propositional formulas" or "statement forms", are statements that do not contain quantifiers. They are composed of well-formed formulas consisting entirely of atomic formulas, the five logical connectives, and symbols of grouping (parentheses etc.). Propositional logic is one of the few areas of mathematics that is totally solved, in the sense that it has been proven internally consistent, every theorem is true, and every true statement can be proved. Predicate logic is an extension of propositional logic, which adds variables and quantifiers.

In Hilbert space a vector is either inside or not inside a closed subspace. A proper quantum logical proposition is "*Vector |f> is inside state s*".

In Hilbert space, an atomic predicate corresponds with a subspace that is spanned be a single vector.

Predicates may accept attributes and quantifiers. The predicate logic is also called first order logic. A dynamic logic can handle the fact that predicates may influence each other when atomic predicates are exchanged.

Observation

In physics, particularly in quantum physics, a system **observable** is a property of the system state that can be determined by some sequence of physical operations. This paper distinguishes between measurements and observations.

- With an observation the state is considered as a linear combination of eigenvectors of the observable. An observation returns the statistical expectation value of the eigenvalue of the observable.
- A measurement transforms the observed state to one of the eigenvectors of the observable. What happens depends on the characteristics of the measuring

equipment. The measurement can be seen as a combination of a transformation and an observation.

Depending on the characteristics of the measuring equipment a measurement and a clean observation can give the same result.

With this interpretation of the concept of observation it is possible to let states observe other states. A state might do a transformation before doing an observation but in general it fails the equipment to arrange that transformation. In nature observations are far more common than measurements.

Quaternion coordinates

This part of the appendix describes candidates for the coordinates on the coordinate sphere.

Polar coordinates

The equivalent to rectangular coordinates in quaternion space is (a_τ, a_x, a_y, a_z)

$$a = a_\tau + \underline{\mathbf{i}}\cdot a_x + \underline{\mathbf{j}}\cdot a_y \pm \underline{\mathbf{i}}\cdot\underline{\mathbf{j}}\cdot a_z \tag{1}$$

The equivalent to polar coordinates in quaternion space is

$$a_\tau = ||a||\cdot\cos(\psi) \tag{2}$$

$$a_x = ||a||\cdot\sin(\psi)\cdot\sin(\theta)\cdot\cos(\phi) \tag{3}$$

$$a_y = ||a||\cdot\sin(\psi)\cdot\sin(\theta)\cdot\sin(\phi) \tag{4}$$

$$a_z = ||a||\cdot\sin(\psi)\cdot\cos(\theta) \tag{5}$$

$\mathrm{Sin}(\psi)$, where $\psi = (0,\pi)$, is known as the (imaginary) amplitude of the quaternion. Angle $\theta = (0,\pi)$ is the (co-)latitude and angle $\phi = (0,2\pi)$ is the longitude.
For any fixed value of ψ, θ and ϕ parameterize a 2-sphere of radius $\sin(\psi)$, except for the degenerate cases, when ψ equals 0 or π, in which case they describe a point.

This suggests the following structure of the argument $\underline{\mathbf{\Lambda}}$

$$a = ||a||\cdot\exp(\tilde{\underline{\mathbf{i}}}\cdot\psi) \tag{6}$$

$$= ||a||\cdot(\cos(\psi) + \tilde{\underline{\mathbf{i}}}\cdot\sin(\psi)) \tag{7}$$

$$= a_\tau + ||a||\cdot\tilde{\underline{\mathbf{i}}}\cdot\sin(\psi) = a_\tau + \underline{\mathbf{a}} \tag{8}$$

The imaginary number $\tilde{\underline{\mathbf{i}}}$ may take any direction.

3 sphere

A 3-sphere is a compact, connected, 3-dimensional manifold without boundary. It is also simply-connected. What this means, loosely speaking, is that any loop, or circular path, on the 3-sphere can be continuously shrunk to a point without leaving the 3-sphere. The Poincaré conjecture proposes that the 3-sphere is the only three dimensional manifold with these properties (up to homeomorphism).

The round metric on the 3-sphere in these coordinates is given by

$$ds^2 = d\psi^2 + \sin^2(\psi)\,(d\theta^2 + \sin^2(\theta).d\phi^2) \tag{1}$$

The volume form is given by

$$dV = \sin^2(\psi)\cdot\sin(\theta)\cdot d\psi \wedge d\theta \wedge d\phi \tag{2}$$

The 3-dimensional volume (or **hyperarea**) of a 3-sphere of radius r is

$$2\cdot\pi^2\cdot r^3 \tag{3}$$

The 4-dimensional **hypervolume** (the volume of the 4-dimensional region bounded by the 3-sphere) is

$$\tfrac{1}{2}\cdot\pi^2\cdot r^4 \tag{4}$$

The 3-sphere has constant positive sectional curvature equal to $1/r^2$.

The 3-sphere has a natural Lie group structure SU(2) given by quaternion multiplication.

The 3-sphere admits nonvanishing vector fields (sections of its tangent bundle). One can even find three linearly-independent and nonvanishing vector fields. These may be taken to be any left-invariant vector fields forming a basis for the Lie algebra of the 3-sphere. This implies that the 3-sphere is parallelizable. It follows that the tangent bundle of the 3-sphere is trivial.

There is an interesting action of the circle group $\mathbb{T}$ on $\mathbb{S}^3$ giving the 3-sphere the structure of a principal circle bundle known as the Hopf bundle. If one thinks of S^3 as a subset of $\mathbf{C}^2$, the action is given by

$$(z_1, z_2)\cdot\lambda = (z_1\cdot\lambda, z_2\cdot\lambda)\ \forall\lambda \in \mathbb{T}. \tag{5}$$

The orbit space of this action is homeomorphic to the two-sphere S^2. Since S^3 is not homeomorphic to $S^2\times S^1$, the Hopf bundle is nontrivial.

Hopf coordinates

Another choice of hyperspherical coordinates, (η, ξ1, ξ2), makes use of the embedding of S3 in $\mathbf{C}^2$. In complex coordinates $(z_1, z_2) \in \mathbf{C}^2$ we write

$$z_1 = \exp(\tilde{\mathbf{i}}\cdot\xi_1)\cdot\sin(\eta) \tag{1}$$

$$z_2 = \exp(\tilde{\mathbf{i}}\cdot\xi_2)\cdot\cos(\eta) \tag{2}$$

Here η runs over the range 0 to π/2, and ξ_1 and ξ_2 can take any values between 0 and 2π. These coordinates are useful in the description of the 3-sphere as the Hopf bundle

$$\mathbb{S}^1 \rightarrow \mathbb{S}^3 \rightarrow \mathbb{S}^2 \tag{3}$$

For any fixed value of η between 0 and π/2, the coordinates (ξ_1, ξ_2) parameterize a 2-dimensional torus. In the degenerate cases, when η equals 0 or π/2, these coordinates describe a circle.

The round metric on the 3-sphere in these coordinates is given by

$$ds^2 = d\eta^2 + \sin^2(\eta)\,(d\zeta_1^2 + \cos^2(\eta).d\,\zeta_2^2) \tag{4}$$

and the volume form by

$$dV = \sin(\eta)\cdot\cos(\eta)\cdot d\eta \wedge d\zeta_1 \wedge d\zeta_2 \tag{5}$$

Stereographic coordinates

Another convenient set of coordinates can be obtained via stereographic projection of $\mathbb{S}^3$ from a pole onto the corresponding equatorial $\mathbb{R}^3$ hyperplane. For example, if we project from the point (−1, 0, 0, 0) we can write a point p in $\mathcal{S}^3$ as

$$p = ((1-||u||^2)/(1+||u||^2),\ 2\cdot\underline{\mathbf{u}}/(1+||u||^2)) = (1+\underline{\mathbf{u}})/(1-\underline{\mathbf{u}}) \tag{1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is a vector in $\mathbb{R}^3$ and $||u||^2 = u_1^2 + u_2^2 + u_3^2$. In the second equality above we have identified p with a unit quaternion and $\mathbf{u} = u_1\cdot\underline{\mathbf{i}} + u_2\cdot\underline{\mathbf{j}} + u_3\cdot\underline{\mathbf{k}}$ with a pure (imaginary) quaternion. (Note that the numerator and denominator commute here even though quaternionic multiplication is generally non-commutative). The inverse of this map takes $p = (x_0, x_1, x_2, x_3)$ in $\mathcal{S}^3$ to

$$\underline{\mathbf{u}} = 1/(1+x_0)\cdot(x_1, x_2, x_3) \tag{2}$$

We could just as well have projected from the point (1, 0, 0, 0) in which case the point p is given by

$$p = ((-1+||v||^2)/(1+||v||^2),\ 2\cdot\underline{\mathbf{v}}/(1+||v||^2)) = (-1+\underline{\mathbf{v}})/(1-\underline{\mathbf{v}}) \tag{3}$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is another vector in $\mathbb{R}^3$. The inverse of this map takes p to

$$\underline{\mathbf{v}} = 1/(1 - x_0)\cdot(x_1, x_2, x_3) \tag{4}$$

Note that the **u** coordinates are defined everywhere but (−1, 0, 0, 0) and the **v** coordinates everywhere but (1, 0, 0, 0). This defines an atlas on $\mathbb{S}^3$ consisting of two coordinate charts or "patches", which together cover all of $\mathbb{S}^3$. Note that the transition function between these two charts on their overlap is given by

$$\underline{\mathbf{v}} = \underline{\mathbf{u}}/||u||^2 \tag{5}$$

and vice-versa.

Group structure

Because the set of unit quaternions is closed under multiplication, $\mathbb{S}^3$ takes on the structure of a group. Moreover, since quaternionic multiplication is smooth, $\mathbb{S}^3$ can be regarded as a real Lie group. It is a non-abelian, compact Lie group of dimension 3. When thought of as a Lie group $\mathbb{S}^3$ is often denoted Sp(1) or U(1, $\mathbb{H}$).

It turns out that the only spheres which admit a Lie group structure are $\mathbb{S}^1$, thought of as the set of unit complex numbers, and $\mathbb{S}^3$, the set of unit quaternions. One might think that $\mathbb{S}^7$, the set of unit octonions, would form a Lie group, but this fails since octonion multiplication is non-associative. The octonionic structure does give $\mathbb{S}^7$ one important property: *parallelizability*. It turns out that the only spheres which are parallelizable are $\mathbb{S}^1$, $\mathbb{S}^3$, and $\mathbb{S}^7$.

By using a matrix representation of the quaternions, $\mathbb{H}$, one obtains a matrix representation of $\mathbb{S}^3$. One convenient choice is given by the Pauli matrices:

$$\left(a_\tau + a_x \cdot \mathbf{i} + a_y \cdot \mathbf{j} + a_z \cdot \mathbf{k}\right) = \begin{bmatrix} a_\tau + \tilde{\imath}\cdot a_x & a_y + \tilde{\imath}\cdot a_z \\ -a_y + \tilde{\imath}\cdot a_z & a_\tau - \tilde{\imath}\cdot a_x \end{bmatrix} \tag{1}$$

This map gives an injective algebra homomorphism from $\mathbb{H}$ to the set of 2×2 complex matrices. It has the property that the absolute value of a quaternion q is equal to the square root of the determinant of the matrix image of q.

The set of unit quaternions is then given by matrices of the above form with unit determinant. This matrix subgroup is precisely the special unitary group SU(2). Thus, $\mathbb{S}^3$ as a Lie group is isomorphic to SU(2).

Using our hyperspherical coordinates (η, ξ_1, ξ_2) we can then write any element of SU(2) in the form

$$\begin{bmatrix} \exp(\tilde{\imath}\cdot\xi_1)\cdot\sin(\eta) & \exp(\tilde{\imath}\cdot\xi_2)\cdot\cos(\eta) \\ -\exp(\tilde{\imath}\cdot\xi_2)\cdot\cos(\eta) & \exp(-\tilde{\imath}\cdot\xi_1)\cdot\sin(\eta) \end{bmatrix} \tag{2}$$

Another way to state this result is if we express the matrix representation of an element of SU(2) as a linear combination of the Pauli matrices. It is seen that an arbitrary element $U \in SU(2)$ can be written as

$$U = \alpha_\tau \cdot 1 + \sum_{n=x,y,z} \alpha_n \cdot \underline{\mathbf{I}}_n \quad (3)$$

The condition that the determinant of U is +1 implies that the coefficients α_n are constrained to lie on a 3-sphere.

Versor

Any **unit quaternion** q can be written as a **versor**:

$$u = \exp(\tilde{\underline{\imath}} \cdot \psi) = \cos(\psi) + \tilde{\underline{\imath}} \cdot \sin(\psi) \quad (1)$$

This is the quaternionic analogue of Euler's formula. Now the unit imaginary quaternions all lie on the unit 2-sphere in Im $\mathbb{H}$ so any such $\tilde{\underline{\imath}}$ can be written:

$$\tilde{\underline{\imath}} = \underline{\mathbf{i}} \cdot \cos(\phi) \cdot \sin(\theta) + \underline{\mathbf{j}} \cdot \sin(\phi) \cdot \sin(\theta) + \underline{\mathbf{k}} \cdot \cos(\theta) \quad (2)$$

Symplectic decomposition

Quaternions can be written as the combination of two complex numbers and an imaginary number k with unit length.

2^n-on construction

The 2^n-ons use the following doubling formula

$$(a, b)\,(c, d) = (a \cdot c - (b \cdot d^*)^*, (b^* \cdot c^*)^* + (b^* \cdot (a^* \cdot ((b^{-1})^* \cdot d^*)^*)^*)^*) \quad (1)$$

Up until the 16-ons the formula can be simplified to

$$(a, b)\,(c, d) = (a \cdot c - b \cdot d^*,\ c \cdot b + (a^* \cdot b^{-1}) \cdot (b \cdot d)) \quad (2)$$

Up to the octonions the Cayley Dickson construction delivers the same as the 2^n-on construction. From n>3 the 2^n-ons are 'nicer'.

2^n-ons

Table of properties of the 2^n-ons. See www.math.temple.edu/~wds/homepage/nce2.ps.

Type	*name*	Lose
1-ons	*Reals.*	
2-ons	*Complex numbers*	$z^* = z$ (the * denotes conjugating); the ordering properties that both {z > 0, -z > 0, or z = 0} and {w > 0, z > 0 implies w + z > 0, wz > 0}.
4-ons	*Quaternions*	commutativity ab = ba; the algebraic closedness property that every univariate polynomial equation has a root.
8-ons	*Octonions*	associativity ab · c = a · bc.
16-ons	(*not Sedenions!*)	right-alternativity x · yy = xy · y; right-cancellation x = xy · y^{-1} ; flexibility x · yx = xy · x; left-linearity (b + c)a = ba + ca; anti-automorphism ab = ba, $(ab)^{-1} = b^{-1} a^{-1}$;

		left-linearity (b + c)a = ba + ca; continuity of the map x → xy; Moufang and Bol identities; diassociativity
32-ons		generalized-smoothness of the map x → xy; right-division properties that xa = b has (generically) a solution x, and the uniqueness of such an x; the "fundamental theorem of algebra" that every polynomial having a unique "asymptotically dominant monomial" must have a root; Trotter's formula: $\lim_{n\to\infty}\left[e^{x/n}e^{y/n}\right]^n = \lim_{n\to\infty}\left(1+\frac{x+y}{n}\right)^n = e^{x+y}$

Type	***name***	**Retain**
2^n-ons		Unique 2-sided multiplicative & additive identity elements 1 & 0; Norm-multiplicativity $\|xy\|^2 = \|x\|^2\cdot\|y\|^2$; Norm-subadditivity $\|a + b\| \le \|a\| + \|b\|$; 2-sided inverse $a^{-1} = a^*/\|a\|^2$ (a # 0); $a^{**} = a$; $(x \pm y)^* = x^* \pm y^*$; $(a^{-1})^{-1} = a$; $(a^*)^{-1} = (a^{-1})^*$; $\|a\|^2 = \|a\|^2 = a^*a$; Left-alternativity yy · x = y · yx; Left-cancellation $x = y^{-1} \cdot yx$; Right-linearity a(b + c) = ab + ac; r^{th} power-associativity $a^n a^m = a^{n+m}$; Scaling s · ab = sa · b = as · b = a · sb = a · bs = ab · s (s real); Power-distributivity $(ra^n + sa^m)b = ra^n b + sa^m b$ (r, s real); Vector product properties of the imaginary part: **ab** - re(**ab**) of the product for pure-imaginary 2^n-ons **a**,**b** regarded as $(2^n - 1)$-vectors; $\langle xa,b\rangle = \langle a,x^*b\rangle$, $\langle xa,xb\rangle = \|x\|^2\cdot\langle a,b\rangle$ and $\langle x,y\rangle = \langle x^*,y^*\rangle$ Numerous weakened associativity, commutativity, distributivity, antiautomorphism, and Moufang and Bol properties including 9-coordinate ``niner'' versions of most of those properties; contains 2^{n-1}-ons as subalgebra.

The most important properties of 2^n-ons

If a,b,x,y are 2^n-ons, n ≥ 0, and s and t are scalars (i.e. all coordinates are 0 except the real coordinate) then

unit: A unique 2^n-on 1 exists, with $1\cdot x = x\cdot 1 = x$.

zero: A unique 2^n-on 0 exists, with $0 + x = x + 0 = x$ and $0\cdot x = x\cdot 0 = 0$.

additive properties: x+y = y+x, (x+y)+z = x+(y+z);

−x exists with x + (−x) = x − x = 0.

norm: $|x|^2 = xx^* = x^*x$.

norm-multiplicativity: $|x|^2\cdot|y|^2 = |x\cdot y|^2$.

scaling: s · x·y = s·x · y = x·s · y = x · s·y = x · y·s.

weak-linearity: (x + s)·y = x·y + s·y and x·(y + s) = x·y + x·s.

right-linearity: x·(y + z) = x·y + x·z.

inversion: If x ≠ 0 then a unique x^{-1} exists, obeying $x^{-1}\cdot x = x\cdot x^{-1} = 1$. It is $x^{-1} = x\cdot|x|^{-2}$.

left-alternativity: x · xy = x^2·y.

left-cancellation: x · x^{-1}·y = y.

effect on inner products: ⟨x·a,b⟩ = ⟨a, x^*·b⟩, ⟨x,y⟩ = ⟨x^*, y^*⟩, ⟨x^*·a, x^{-1}·b⟩ = ⟨a,b⟩, and ⟨x·a,x·b⟩ = $|x|^2$·⟨a,b⟩.

Conjugate of inverse: $(x^{-1})^* = (x^*)^{-1}$.

Near-anticommutativity of unequal basis elements: $e_k^2 = -1$ and $e_k \cdot e_l^* = -e_l \cdot e_k^*$ if $k \neq l$.
(Note: the case $k; l > 0$ shows that unequal pure-imaginary basis elements anticommute.)
Alternative basis elements: $e_k \cdot e_l \cdot e_k = e_k \cdot e_l \cdot e_k$, $e_l \cdot e_k \cdot e_k = e_l \cdot e_k \cdot e_k$, and $e_k \cdot e_k \cdot e_l = e_k \cdot e_k \cdot e_l$.
(However, when n ≥ 4 the 2^n-ons are not flexible i.e. it is not generally true that x·y · x = x · y·x if x and y are 16-ons that are not basis elements. They also are not right-alternative.)
Quadratic identity: If x is a 2^n-on (over any field F with charF ≠ 2), then $x^2 + |x|^2 = 2 \cdot x \text{ re } x$
Squares of imaginaries: If x is a 2^n-on with re x = 0 ("pure imaginary") then $x^2 = -|x|^2$ is nonpositive pure-real.
Powering preserves im$x$ direction

Niners

Niners are 2n-ons whose coordinates with index > 8 are zero. The index starts with 0.
9-**flexibility** xp · x = x · px, px · p = p · xp.
9-**similitude unambiguity** $xp \cdot x^{-1} = x \cdot px^{-1}$, $px \cdot p^{-1} = p \cdot xp^{-1}$.
9-**right-alternativity** $xp \cdot p = x \cdot p^2$, $px \cdot x = p \cdot x^2$.
9-**right-cancellation** $xp^{-1} \cdot p = x$, $px^{-1} \cdot x = p$.
9-**effect on inner products** $\langle x, yp\rangle = \langle xp, y\rangle$, $\langle xp, yp\rangle = |p|^2\langle x, y\rangle$.
9-**left-linearity** (x + y)p = xp + yp, (p + q)x = px + qx.
9-**Jordan-identity** xp · xx = x(p · xx), py · pp = p(y · pp).
9-**coordinate-distributivity** $([x + y]z)_{0;\ldots;8} = (xz + yz)_{0;\ldots;8}$.
9-**coordinate-Jordan-identity** $[xy \cdot xx]_{0;\ldots;8} = [x(y \cdot xx)]_{0;\ldots;8}$.
9-**anticommutativity for orthogonal imaginary 2^n-ons**
If ⟨p, x⟩ = re p = re x = 0 then px = −xp.
9-**reflection** If |a| = 1 and the geometric reflection operator is defined below then $-(\text{refl}[a](y))_{0;\ldots;8} = (a \cdot y^* a)_{0;\ldots;8}$, and $-\{\text{refl}[a](y)\}^*_{0;\ldots;8} = (a^* y \cdot a^*)_{0;\ldots;8}$, and
if either a or y is a niner then −refl[a](y) = a · y*a and −refl[a](y) = a*y · a*.

$$\text{refl}[\vec{x}](\vec{t}) \stackrel{\text{def}}{=} \vec{t} - \frac{2\langle \vec{x}, \vec{t}\rangle}{|\vec{x}|^2}\vec{x} \tag{3}$$

What holds for the niners, also holds for the octonions.

The separable Hilbert space **H**

Notations and naming conventions

$\{f_x\}_x$ means ordered set of f_x . It is a way to define functions.

The use of bras and kets differs slightly from the way Dirac uses them.

|f> is a ket vector, f> is the same ket
<f| is a bra vector, <f is the same bra

A is an operator. |A is the same operator
$A^\dagger$ is the adjoint operator of operator A. A| is the same operator
| on its own, is a nil operator
|A| is a self-adjoint (Hermitian) operator

We will use capitals for operators and lower case for quaternions, eigenvalues, ket vectors, bra vectors and eigenvectors. Quaternions and eigenvalues will be indicated with *italic* characters. Imaginary and anti-Hermitian objects are often underlined and/or indicated in **bold** text.

$\sum_k$ means: sum over all items with index k.
$\int_x$ means: integral over all items with parameter x.

Quaternionic Hilbert space

The Hilbert space is a **linear space**. That means for the elements $|f\rangle$, $|g\rangle$ and $|h\rangle$ and numbers a and b:

Ket vectors

For **ket** vectors hold

$$|f\rangle + |g\rangle = |g\rangle + |f\rangle = |g + f\rangle \tag{1}$$

$$(|f\rangle + |g\rangle) + |h\rangle = |f\rangle + (|g\rangle + |h\rangle) \tag{2}$$

$$|(a + b)\, f\rangle = |f\rangle \cdot a + |f\rangle \cdot b \tag{3}$$

$$(|f\rangle + |g\rangle) \cdot a = |f\rangle \cdot a + |g\rangle \cdot a \tag{4}$$

$$|f\rangle \cdot 0 = |0\rangle \tag{5}$$

$$|f\rangle \cdot 1 = |f\rangle \tag{6}$$

Depending on the number field that the Hilbert space supports, a and b can be real numbers, complex numbers or (real) quaternions.

Bra vectors

The **bra** vectors form the dual Hilbert space $\mathbf{H}_{\mathbf{q}}^{\dagger}$ of $\mathbf{H}_{\mathbf{q}}$.

$$\langle f| + \langle g| = \langle g| + \langle f| = |g + f\rangle \tag{1}$$

$$(\langle f| + \langle g|) + \langle h| = \langle f| + (\langle g| + \langle h|) \tag{2}$$

$$\langle f\,(a + b)\rangle = \langle f| \cdot a + \langle f| \cdot b = a^* \cdot \langle f| + b^* \cdot \langle f| \tag{3}$$

$$(\langle f| + \langle g|) \cdot a = \langle f| \cdot a + \langle g| \cdot a = a^* \cdot \langle f| + a^* \cdot \langle g| \tag{4}$$

$$0 \cdot \langle f| = \langle 0| \tag{5}$$

$$1 \cdot \langle f| = \langle f| \tag{6}$$

Scalar product

The Hilbert space contains a **scalar product** <f | g> that combines $\mathbf{H}$ and $\mathbf{H}^\dagger$ in a direct product that we also indicate with $\mathbf{H}$.
The scalar product <f | g> satisfies:

$$<f \mid g + h> = <f \mid g> + <f \mid h> \tag{1}$$

$$<f \mid \{\mid g>\cdot a\}_g = \{<f \mid g>\}_g \cdot a \tag{2}$$

With each ket vector | g> in $\mathbf{H}$ belongs a bra vector <g | in $\mathbf{H}^\dagger$ such that for all bra vectors <f| in $\mathbf{H}^\dagger$

$$<f \mid g> = <g \mid f>^* \tag{3}$$

$$<f \mid f> = 0 \text{ when } \mid f> = \mid 0> \tag{4}$$

$$<f \mid a\, g> = <f \mid g>\cdot a = <g \mid f>^*\cdot a = <g\ a \mid f>^* = (a^*\cdot<g \mid f>)^* = <f \mid g>\cdot a \tag{5}$$

In general is <f | a g> ≠ <f a | g>. However for real numbers r holds <f | r g> = <f r | g>

Remember that when the number field consists of quaternions, then also <f | g> is a quaternion and a quaternion q and <f | g> do in general not commute.

The scalar product defines a **norm**:

$$\| f \| = \sqrt{(<f \mid f>)} \tag{6}$$

And a **distance**:

$$D(f,g) = \| f - g \| \tag{7}$$

The Hilbert space $\mathbf{H}$ is closed under its norm. Each converging row of elements of converges to an element of this space.

Separable

In mathematics a topological space is called separable if it contains a countable dense subset; that is, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence.
Every continuous function on the separable space $\mathbf{H}$ is determined by its values on this countable dense subset.

Base vectors

The Hilbert space $\mathbf{H}$ is **separable**. That means that there exist a countable row of elements $\{f_n>\}$ that **spans** the whole space.

If $\langle f_n | f_m \rangle = \delta(m,n) = [1 \text{ when } n = m; 0 \text{ otherwise}]$
then $\{|f_n\rangle\}$ forms an **orthonormal base** of the Hilbert space.

A ket base $\{|k\rangle\}$of $\mathbf{H}$ is a minimal set of ket vectors $|k\rangle$ that together span the Hilbert space $\mathbf{H}$.
Any ket vector $|f\rangle$ in $\mathbf{H}$ can be written as a linear combination of elements of $\{|k\rangle\}$.

$$|f\rangle = \sum_k (|k\rangle \cdot \langle k|f\rangle) \quad (1)$$

A bra base $\{\langle b|\}$of $\mathbf{H}^\dagger$ is a minimal set of bra vectors $\langle b|$ that together span the Hilbert space $\mathbf{H}^\dagger$.
Any bra vector $\langle f|$ in $\mathbf{H}^\dagger$ can be written as a linear combination of elements of $\{\langle b|\}$.

$$\langle f| = \sum_b (\langle f|b\rangle \cdot \langle b|) \quad (2)$$

Usually base vectors are taken such that their norm equals 1. Such a base is called an othonormal base.

Operators

Operators act on a subset of the elements of the Hilbert space.

Linear operators

An operator Q is linear when for all vectors $|f\rangle$ and $|g\rangle$ for which Q is defined and for all quaternionic numbers a and b:

$$|Q \cdot a\, f\rangle + |Q \cdot b\, g\rangle = |a \cdot Q\, f\rangle + |b \cdot Q\, g\rangle = |Q\, f\rangle \cdot a + |Q\, g\rangle \cdot b = \quad (1)$$

$$Q(|f\rangle \cdot a + |g\rangle \cdot b) = Q(|a\, f\rangle + |b\, g\rangle) \quad (2)$$

B is **colinear** when for all vectors $|f\rangle$ for which B is defined and for all quaternionic numbers a there exists a quaternionic number c such that:

$$|B \cdot a\, f\rangle = |a \cdot B\, f\rangle = |B\, f\rangle\, c \cdot a \cdot c^{-1} \quad (3)$$

If $|f\rangle$ is an eigenvector of operator A with quaternionic eigenvalue a, then is $|b\, f\rangle$ an eigenvector of A with quaternionic eigenvalue $b \cdot a \cdot b^{-1}$.
$A| = A^\dagger$ is the **adjoint** of the **normal** operator A. $|A$ is the same as A.

$$\langle f\, A|\, g\rangle = \langle f A^\dagger | g\rangle^* \quad (4)$$

$$A^{\dagger\dagger} = A \quad (5)$$

$$(A \cdot B)^\dagger = B^\dagger \cdot A^\dagger \quad (6)$$

$|B|$ is a **self adjoint** operator.

| is a nil operator.

The construct |f><g| acts as a linear operator. |g><f| is its adjoint operator.

$$\sum_n \{|f_n\rangle \cdot a_n \cdot \langle f_n|\}, \tag{7}$$

where a n is real and acts as a density function.

The set of eigenvectors of a normal operator form an orthonormal base of the Hilbert space.
A self adjoint operator has real numbers as eigenvalues.

$\{\langle q|f\rangle\}_q$ is a function f(q) of parameter q.
$\{\langle g|q\rangle\}_q$ is a function g(q) of parameter q.

When possible, we use the same letter for identifying eigenvalues, eigenvalues and the corresponding operator.
So, usually |q> is an eigenvector of a normal operator Q with eigenvalues q.

{q} is the set of eigenvalues of Q.
$\{q\}_q$ is the ordered field of eigenvalues of q.
$\{|q\rangle\}_q$ is the ordered set of eigenvectors of Q.
$\{\langle q|f\rangle\}_q$ is the **Q view** of |f>.

Normal operators

The most common definition of continuous operators is:

> A **continuous** operator is an operator that creates images such that the inverse images of open sets are open.
>
> Similarly, a **continuous** operator creates images such that the inverse images of closed sets are closed.

A normal operator is a continuous linear operator.
A normal operator in $\mathbf{H}_1$ creates an image of $\mathbf{H}_1$ onto $\mathbf{H}_1$**.** It transfers closed subspaces of $\mathbf{H}_1$ into closed subspaces of $\mathbf{H}_1$.

Normal operators represent continuous quantum logical observables.

The normal operators N have the following property.

$$N: \mathbf{H}_1 \Rightarrow \mathbf{H}_1 \tag{1}$$

N commutes with its **(Hermitian) adjoint** $N^\dagger$

$$N \cdot N^\dagger = N^\dagger \cdot N \tag{2}$$

Normal operators are important because the spectral theorem holds for them.

Examples of normal operators are

- **unitary** operators: $U^{\dagger} = U^{-1}$, unitary operators are bounded;
- **Hermitian** operators (i.e., self-adjoint operators): $N^{\dagger} = N$;
- **Anti-Hermitian** or anti-self-adjoint operators: $N^{\dagger} = -N$;
- **Anti-unitary** operators: $I^{\dagger} = -I = I^{-1}$, anti-unitary operators are bounded;
- **positive operators**: $N = MM^{\dagger}$
- **orthogonal projection** operators: $N = N^{\dagger} = N^2$

Spectral theorem

For every compact self-adjoint operator T on a real, complex or quaternionic Hilbert space $\mathbf{H}$, there exists an orthonormal basis of H consisting of eigenvectors of T. More specifically, the orthogonal complement of the kernel (null space) of T admits, either a finite orthonormal basis of eigenvectors of T, or a countable infinite orthonormal basis {en} of eigenvectors of T, with corresponding eigenvalues $\{\lambda_n\} \subset R$, such that $\lambda_n \to 0$. Due to the fact that $\mathbf{H}$ is separable the set of eigenvectors of T can be extended with a base of the kernel in order to form a complete orthonormal base of $\mathbf{H}$.

If T is compact on an infinite dimensional Hilbert space $\mathbf{H}$, then T is not invertible, hence $\sigma(T)$, the spectrum of T, always contains 0. The spectral theorem shows that $\sigma(T)$ consists of the eigenvalues $\{\lambda_n\}$ of T, and of 0 (if 0 is not already an eigenvalue). The set $\sigma(T)$ is a compact subset of the real line, and the eigenvalues are dense in $\sigma(T)$.

A normal operator has a set of eigenvectors that spans the whole Hilbert space $\mathbf{H}$.
In quaternionic Hilbert space a normal operator has quaternions as eigenvalues.

The set of eigenvalues of a normal operator is NOT compact. This is due to the fact that $\mathbf{H}$ is separable. Therefore the set of eigenvectors is countable. As a consequence the set of eigenvalues is countable. Further, the eigenspace of normal operators has no finite diameter.

A continuous bounded linear operator on $\mathbf{H}$ has a compact eigenspace. The set of eigenvalues has a closure and it has a finite diameter.

Eigenspace

The set of eigenvalues $\{q\}$ of the operator Q form the eigenspace of Q

Eigenvectors and eigenvalues

For the eigenvector $|q\rangle$ of normal operator Q holds

$$|Q\ q\rangle = |q\ q\rangle = |q\rangle \cdot q \tag{1}$$

$$\langle q\ Q^{\dagger}| = \langle q\ q^{*}| = q^{*} \cdot \langle q| \tag{2}$$

$$\forall(|f\rangle\ \ni \mathbf{H}) \tag{3}$$

$$[\{\langle f|Q\ q\rangle\}_q = \{\langle f|\ q\rangle \cdot q\}_q = \{\langle q\ Q^{\dagger}|f\rangle^{*}\}_q = \{q^{*} \cdot \langle q\ |f\rangle^{*}\}_q]$$

The eigenvalues of 2^n-on normal operator are 2^n-ons

$$Q = \sum_{j=0}^{n-1} \mathrm{I}_j \cdot Q_i \tag{4}$$

The Q_j are self-adjoint operators.

Generalized Trotter formula

For bounded operators $\{A_j\}$ hold:

$$\lim_{n\to\infty}\left(\prod_{j=1}^{p} e^{A_j/n}\right)^n = \exp\left(\sum_{j=1}^{p} A_j\right) = \lim_{n\to\infty}\left(1+\frac{\sum_{j=1}^{p} A_j}{n}\right)^n \tag{1}$$

In general

$$\exp\left(\sum_{j=1}^{p} A_j\right) \neq \prod_{j=1}^{p} e^{A_j} \tag{2}$$

Unitary operators

For unitary operators holds:

$$\mathrm{U}^{\dagger} = \mathrm{U}^{-1} \tag{1}$$

Thus

$$\mathrm{U}\cdot\mathrm{U}^{\dagger} = \mathrm{U}^{\dagger}\cdot\mathrm{U} = 1 \tag{2}$$

Suppose $U = I + C$ where U is unitary and C is compact. The equations $U\,U^* = U^*U = I$ and $C = U - I$ show that C is normal. The spectrum of C contains 0, and possibly, a finite set or a sequence tending to 0. Since $U = I + C$, the spectrum of U is obtained by shifting the spectrum of C by 1.

The unitary transform can be expressed as:

$$\mathrm{U} = \exp(\tilde{\underline{\mathbf{I}}}\cdot\Phi/\hbar) \tag{3}$$

$$\hbar = \mathrm{h}/(2\cdot\pi) \tag{4}$$

Φ is Hermitian. The constant h refers to the granularity of the eigenspace.
Unitary operators have eigenvalues that are located in the unity sphere of the 2^n-ons field.
The eigenvalues have the form:

$$u = \exp(\underline{\mathbf{i}}\cdot\phi/\hbar) \tag{5}$$

ϕ is real. $\underline{\mathbf{i}}$ is a unit length imaginary number in 2^n-on space. It represents a direction.

u spans a sphere in 2^n-on space. For constant $\underline{\mathbf{i}}$, *u* spans a circle in a complex subspace.

Polar decomposition

Normal operators N can be split into a real operator A and a unitary operator U. U and A have the same set of eigenvectors as N.

$$N = ||N|| \cdot U = A \cdot U \tag{1}$$

$$N = A \cdot U = U \cdot A \tag{2}$$

$$= A \cdot \exp(\tilde{\underline{\mathbf{I}}} \cdot \Phi)/\hbar)$$

$$= \exp(\Phi_r + \tilde{\underline{\mathbf{I}}} \cdot \Phi)/\hbar)$$

Φ_r is a positive normal operator.

Ladder operator

General formulation

Suppose that two operators *X* and *N* have the commutation relation:

$$[N, X] = c \cdot X \tag{1}$$

for some scalar *c*. If $|n\rangle$ is an eigenstate of *N* with eigenvalue equation,

$$|N\ n\rangle = |n\rangle \cdot n \tag{2}$$

then the operator *X* acts on $|n\rangle$ in such a way as to shift the eigenvalue by *c*:

$$\begin{aligned} |N \cdot X\ n\rangle &= |(X \cdot N + [N, X])\ n\rangle \\ &= |(X \cdot N + c \cdot X)\ n\rangle \\ &= |X \cdot N\ n\rangle + |X\ n\rangle \cdot c \\ &= |X\ n\rangle \cdot n + |X\ n\rangle \cdot c \\ &= |X\ n\rangle \cdot (n+c) \end{aligned} \tag{3}$$

In other words, if $|n\rangle$ is an eigenstate of *N* with eigenvalue *n* then $|X\ n\rangle$ is an eigenstate of *N* with eigenvalue $n + c$.
The operator *X* is a *raising operator* for *N* if *c* is real and positive, and a *lowering operator* for *N* if *c* is real and negative.

If *N* is a Hermitian operator then *c* must be real and the Hermitian adjoint of *X* obeys the commutation relation:

$$[N, X^\dagger] = - c \cdot X^\dagger \tag{4}$$

In particular, if X is a lowering operator for N then X^+ is a raising operator for N and vice-versa.

Unit sphere of $\mathbf{H}_\mathbf{i}$

The ket vectors in $\mathbf{H}_\mathbf{i}$ that have their norm equal to one form together the **unit sphere** $\mathbf{\Theta}$ of $\mathbf{H}_\mathbf{i}$. Base vectors are all member of the unit sphere. The eigenvectors of a normal operator are all member of the unit sphere.

The end points of the eigenvectors of a normal operator form a **grid** on the unit sphere $\mathbf{\Theta}$of $\mathbf{H}_\mathbf{i}$.

Closure

The closure of $\mathbf{H}_\mathbf{i}$ means that converging rows of vectors converge to a vector of $\mathbf{H}_\mathbf{i}$.

In general converging rows of eigenvalues of Q do not converge to an eigenvalue of Q.
Thus, the set of eigenvalues of Q is open.
At best the density of the coverage of the set of eigenvalues is comparable with the set of 2^n-ons that have rational numbers as coordinate values.
With other words, compared to the set of real numbers the eigenvalue spectrum of Q has holes.
The set of eigenvalues of operator Q includes 0. This means that Q does not have an inverse.

The rigged Hilbert space $\mathbf{H}$ can offer a solution, but then the direct relation with quantum logic is lost.

Canonical conjugate operator P and quaternionic Fourier transform

The existence of a canonical conjugate represents a stronger requirement on the continuity of the eigenvalues of canonical eigenvalues.
Q has eigenvectors $\{|q\rangle\}_q$ and eigenvalues q.
P has eigenvectors $\{|p\rangle\}_p$ and eigenvalues p.
For each eigenvector $|q\rangle$ of Q we define an eigenvector $|p\rangle$ and eigenvalues p of P such that:

$$\langle q|p\rangle = \langle p|q\rangle^* = \exp(\mathbf{\hat{\imath}}\cdot p\cdot q/\hbar) \qquad (1)$$

$\hbar = h/(2\cdot\pi)$ is a scaling factor. $\langle q|p\rangle$ is a quaternion. $\mathbf{\hat{\imath}}$ is a unit length imaginary quaternion.

This specification also defines a Fourier transform in the complex subspace spanned by $\mathbf{\hat{\imath}}$ and the real axis. This is a unitary transform with eigenvalues that are equal to the value of the inner product $\langle q|p\rangle$. The inverse Fourier transform has eigenvalues that equal $\langle p|q\rangle$. The Fourier transform U_{qp} converts the base $\{|q\rangle\}_q$ into the base $\{|p\rangle\}_p$. The inverse Fourier transform U_{pq} does the reverse.

$$\langle q|f\rangle = \sum_p (\langle q|p\rangle\cdot\langle p|f\rangle) \qquad (2)$$

$$= \sum_p \langle p\,(\langle p|q\rangle)|f\rangle)$$

$$= \sum_p \langle p\,U_{pq}|f\rangle)$$

$= \sum_p \langle p | U_{qp} f \rangle)$

$\langle p | f \rangle = \sum_q (\langle p | q \rangle \cdot \langle q | f \rangle)$ (3)

$= \sum_q \langle q | U_{pq} f \rangle)$

Due to its specification, the operator P can be interpreted as a displacement generator for Q.

$$P = \hat{I} \cdot \hbar \cdot \frac{\partial}{\partial q} \quad (4)$$

The Fourier transform is the source of the existence of the Heisenberg uncertainty relation. This is also shown in the commutator:

$$[P, Q] = PQ - QP = \hat{I} \cdot \hbar \quad (5)$$

In the rigged Hilbert space **H**

$\langle q | f \rangle = \int_p (\langle q | p \rangle \cdot \langle p | f \rangle) \cdot dp$ (6)

$\langle p | f \rangle = \int_q (\langle p | q \rangle \cdot \langle q | f \rangle) \cdot dq$ (7)

Through the definition of the canonical conjugate do P and Q possess a fixed relation. The Fourier transform can be interpreted as the local

Displacement generators

Variance of the scalar product gives:

$\underline{i} \cdot \hbar \cdot \delta \langle q | p \rangle = -p \cdot \langle q | p \rangle \cdot \delta q$ (1)

$\underline{i} \cdot \hbar \cdot \delta \langle p | q \rangle = -q \cdot \langle p | q \rangle \cdot \delta p$ (2)

In the rigged Hilbert space **H** the variance can be replaced by differentiation.
Partial differentiation of the function $\langle q | p \rangle$ gives:

$\underline{i} \cdot \hbar \cdot \partial / \partial q_s \langle q | p \rangle = -p_s \cdot \langle q | p \rangle$ (3)

$\underline{i} \cdot \hbar \cdot \partial / \partial p_s \langle p | q \rangle = -q_s \cdot \langle p | q \rangle$ (4)

Zak transform

(See also http://eom.springer.de/Z/z130030.htm)

Definition

The Weil-Brezin-Zak transform $Z_\alpha(f)$ of a function f is defined by

(1)

$$Z_\alpha[f](t,\omega) = Z_\alpha f(t,\omega) = \sqrt{\alpha} \sum_{k=-\infty}^{\infty} f(\alpha\, t + \alpha\, k)\ \exp(-2\pi i k\omega)$$

Where $\alpha > 0$ and t and ω are real. When α =1, one denotes $Z_\alpha f$ by Zf.

If f represents a signal, then its Zak transform can be considered as a mixed time-frequency representation of f, and it can also be considered as a generalization of the discrete Fourier transform of f in which an infinite sequence of samples in the form $f(\alpha\, t + \alpha\, k)$, k = 0, ±1, ±2,…, is used

Elementary properties.

(2)

1) (linearity): for any complex numbers α and β,

$$Z[\alpha\, f(t) + \beta\, g(t)](t,\omega) = \alpha\, Z[\, f(t)](t,\omega) + \beta\, Z[\, g(t)](t,\omega)$$

2) (translation): for any integer m, (3,4)

$$Z[\, f(t+m)](t,\omega) = \exp(2\,\pi\, \boldsymbol{k}\, m\, \omega)\, Z[\, f](t,\omega)$$

in particular,

$$(Z\, f)(t+1,\omega) = \exp(2\,\pi\, \boldsymbol{k}\ \omega) Z\, f(t,\omega)$$

3) (modulation): (5)

$$Z\,[\exp(2\,\pi\, \boldsymbol{k}\, m\, t) f](t,\omega) = \exp(2\,\pi\, \boldsymbol{k}\, m\, t)(Z\, f)(t,\omega)$$

4) (periodicity): The Zak transform is periodic in $\boldsymbol{w}$with period one, that is, (6)

$$(Z\, f)(t,\omega+1) = (Z\, f)(t,\omega)$$

5) (translation and modulation): By combining 2) and 3) one obtains (7)

$$Z\,[\exp(2\,\pi\, \boldsymbol{k}\, m\, t) f(t+n)](t,\omega) = \exp(2\,\pi\, \boldsymbol{k}\, m\, t)\exp(2\,\pi\, \boldsymbol{k}\, n\, \omega)(Z\, f)(t,\omega)$$

6) (conjugation): (8)

$$\left(Z\, \bar{f}\right)(t,\omega) = \overline{(Z\, f)}(t,-\omega)$$

7) (symmetry): If f is even, then (9,10)

$$(Z\, f)(t,\omega) = (Z\, f)(-t,-\omega)$$

and if f is odd, then

$$(Z\, f)(t,\omega) = -(Z\, f)(-t,-\omega)$$

From 6) and 7) it follows that if f is real-valued and even, then

$$(Z\,f)(t,\omega) = \overline{(Z\,f)}(t,-\omega) = (Z\,f)(-t,-\omega) \tag{11}$$

Because of 2) and 4), the Zak transform is completely determined by its values on the unit square $Q = [0,1] \times [0,1]$.

(12,13)

8) (convolution): Let

$$h(t) = \int_{-\infty}^{\infty} R(t-s)\, f(s) ds$$

then

$$(Z\,h)(t,\omega) = \int_0^1 (Z\,R)(t-s,\omega)\,(Z\,f)(s,\omega)\; ds$$

Analytic properties.

If f is a continuous function such that

$$f(t) = O((1+|t|)^{-1-\epsilon}) \text{ as } t \to \infty \text{ for some } \epsilon > 0 \tag{1}$$

Then $Z\,f$ is continuous on Q. A rather peculiar property of the Zak transform is that if $Z\,f$ is continuous, it must have a zero in Q. The Zak transform is a unitary transformation from $L^2(\mathbb{R})$ onto $L^2(Q)$.

Inversion formulas.

The following inversion formulas for the Zak transform follow easily from the definition, provided that the series defining the Zak transform converges uniformly:

$$f(t) = \int_0^1 (Z\,f)(t,\omega)\, d\omega\,; -\infty < t < \infty \tag{1}$$

$$\tilde{f}(-2\,\pi\,\omega) = \frac{1}{\sqrt{2\,\pi}} \int_0^1 exp(-2\,\pi\,\boldsymbol{k}\,\omega\,t)\,(Z\,f)(t,\omega)\, dt \tag{2}$$

and

$$f(2\,\pi\,\omega) = \frac{1}{\sqrt{2\,\pi}} \int_0^1 exp(-2\,\pi\,\boldsymbol{k}\,x\,t)\,\left(Z\,\tilde{f}\right)(x,t)\, dx \tag{3}$$

where $\tilde{f}$ is the Fourier transform of f, given by

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} f(x)\, \exp(\boldsymbol{k}\,\omega\,x)\, dx$$

Applications.

The Gabor representation problem can be stated as follows: Given $g \in L^2(\mathbb{R})$ and two real numbers, α, β, different from zero, is it possible to represent any function $f \in L^2(\mathbb{R})$ by a series of the form

$$f = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{mn}\; g_{m\beta,n\alpha}\,, \tag{1}$$

where $g_{m\beta,n\alpha}$ are the Gabor functions, defined by:

$$g_{m\beta,n\alpha}(x) = \exp(2\,\pi\,\beta\,x)\,g(x - n\alpha) \tag{2}$$

and c_{mn} are constants? And under what conditions is the representation unique?

Fix a coordinate x in a line $\mathbb{R}$; the family of functions in $\mathbb{R}$

$$e_\lambda\,(x) \;=\; \sqrt[4]{2}\exp\;(-\pi\,(x\;-\;p)^2\;+\;2\,\pi\,\boldsymbol{k}\,\theta\,x) \tag{3}$$

are called Gabor functions. Here λ = (p, θ) is a point in the phase space $\Phi = \mathbb{R} \otimes \mathbb{R}'$

The operators

$$\mathcal{A} \;=\; \frac{1}{2\,\pi}\,\frac{d}{dx}\;+\;x \tag{4}$$

$$\mathcal{A}^\dagger \;=\; -\frac{1}{2\,\pi}\,\frac{d}{dx}\;+\;x \tag{5}$$

in L^2 are adjoint one to another. They are called the *annihilation* and the *creation* operators.

Any Gabor function is an eigenvector of the annihilation operator:

$$\mathcal{A}e\lambda \;=\; \lambda e\lambda \tag{6}$$

where

$$\lambda \;=\; (p,\theta) \tag{7}$$

and

$$\lambda \;=\; p + \boldsymbol{k}\theta \tag{8}$$

For any φ in the domain of the operator a we have

$$Z\,(\mathcal{A}\varphi) \;=\; AZ\varphi \tag{9}$$

$$A \;=\; \frac{1}{2\pi\boldsymbol{k}}\;\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial y}\right)\;+\;y \tag{10}$$

More stuff

More useful stuff is collected in the toolkit

References:
Axiomatic Quantum Theory,W. Lücke, http://arxiv.org/PS_cache/quant-ph/pdf/9510/9510024v2.pdf

An overview of gravity theories: http://arxiv.org/PS_cache/arxiv/pdf/0909/0909.4672v2.pdf.